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Research article

Explicit formulae for Bernoulli numbers

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Abstract: By examining the connection coefficients, we systematically review and extend (with an extra integer parameter) several double sum expressions for the Bernoulli numbers. New summation formulae are also established explicitly.

Keywords: Bernoulli number; harmonic number; Stirling number of the second kind; binomial coefficient; connection coefficient
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1. Introduction and motivation

The Bernoulli numbers are defined by the exponential generating function:

$$\frac{y}{e^y - 1} = \sum_{n=0}^{\infty} B_n \frac{y^n}{n!}.$$

For their wide applications to classical analysis (cf. Stromberg [47, Chapter 7], number theory (cf. Apostol [3, §12.12]), combinatorics [44, §3.4] and numerical mathematics (cf. Arfken [5]), these numbers appear frequently in the mathematical literature (see Comtet [17, §1.14], Graham et al. [26, §6.5] and Hansen [27, §50]).

There exist numerous interesting properties (cf. [2, 4, 8, 13, 24]); in particular, about recurrence relations (cf. [1, 21, 30, 37]), reciprocities (cf. [12, 25, 41, 42, 49]), convolutions (cf. [14–16, 22, 34]), multiple sums (cf. [10, 11, 19, 32]) and combinatorial applications (cf. [6, 31, 35, 38]). A few of them are recorded here as examples:

- Arithmetic sums

$$\sum_{k=1}^{m} k^n = \sum_{j=0}^{n} \frac{(m+1)^{j+1}}{n+1}\binom{n+1}{j+1} B_{n-j}.$$

- Binomial recurrences

$$\sum_{k=0}^{n}\binom{n}{k}B_k = (-1)^n B_n, \qquad n \geq 0;$$
$$\sum_{k=1}^{n-2}\binom{n}{k}kB_k = -n^2 B_{n-1}, \qquad n \geq 3;$$
$$\sum_{k=0}^{n}\binom{n}{k}\frac{B_k}{2+n-k} = \frac{B_{n+1}}{n+1}, \qquad n \geq 0.$$

- Convolutions of the Miki type (cf. [12, 22, 34, 36, 40])

$$\sum_{k=1}^{\ell-1}\frac{B_k B_{\ell-k}}{k} - \sum_{k=2}^{\ell-1}\binom{\ell}{k}\frac{B_k B_{\ell-k}}{k} = H_\ell B_\ell,$$
$$\sum_{k=0}^{\ell} B_k B_{\ell-k} - 2\sum_{k=2}^{\ell}\binom{\ell+1}{k+1}\frac{B_k B_{\ell-k}}{k+2} = (\ell+1)B_\ell.$$

- Riemann zeta series

$$\sum_{k=1}^{\infty}\frac{1}{k^{2n}} = (-1)^{n-1}\frac{2^{2n-1}B_{2n}}{(2n)!}\pi^{2n},$$
$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^{2n}} = (-1)^n\frac{(1-2^{2n-1})B_{2n}}{(2n)!}\pi^{2n},$$
$$\sum_{k=1}^{\infty}\frac{1}{(2k-1)^{2n}} = (-1)^n\frac{(1-2^{2n})B_{2n}}{2(2n)!}\pi^{2n}.$$

The aim of this paper is to examine and review systematically explicit formulae of Bernoulli numbers. Among known double sum expressions, the simplest one reads as

$$B_n = \sum_{j=0}^{n}\sum_{k=0}^{j}\frac{(-1)^k}{j+1}\binom{j}{k}k^n = \sum_{k=0}^{n}k^n\sum_{j=k}^{n}\frac{(-1)^k}{j+1}\binom{j}{k}.$$

We observe that when the upper limit n is replaced by $m \geq n$, the last formula is still valid. Letting $\Omega(m,k)$ be the connection coefficients

$$\Omega(m,k) = \sum_{j=k}^{m}\frac{(-1)^k}{j+1}\binom{j}{k},$$

we have the following formula (see Theorem 1) with an extra parameter m:

$$B_n = \sum_{k=0}^{m}k^n\Omega(m,k), \quad \text{where} \quad m \geq n.$$

Three remarkable formulae can be highlighted in anticipation as exemplification (see Eqs (2.20), (3.14) and (4.1)), where for $m = n$, the last two identities resemble those found respectively by Bergmann [7] and Gould [24, Eq (1.4)]:

$$n + B_n = \sum_{k=0}^{m}(k+2)^n \sum_{j=k}^{m} \frac{(-1)^k}{j+1}\binom{j}{k}, \qquad m \ge n \ge 2;$$

$$B_n = \sum_{k=1}^{m} k^n \sum_{j=k}^{m} \frac{(-1)^{j-1}}{j(j+1)}\binom{m}{j}, \qquad m \ge n \ge 2;$$

$$B_n = \sum_{k=1}^{m} k^n \sum_{j=k}^{m}(-1)^{k-1}\binom{j-1}{k-1}H_j, \qquad m > n \ge 1.$$

The rest of the paper will be organized as follows. In the next section, we shall establish several representation formulae of B_n by parameterizing known double sums with m. Then in Section 3, by examining equivalent expressions for $\Omega(m,k)$ and another connection coefficient $\omega(m,k)$, we shall prove further explicit formulae for Bernoulli numbers. Finally, the paper will end in Section 4, where more summation formulae will be shown.

Throughout the paper, we shall frequently make use of the following notations. Let $\mathbb{N}$ be the set of natural numbers with $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. For $n \in \mathbb{N}_0$ and an indeterminate x, the shifted factorial is defined by

$$(x)_0 \equiv 1 \quad \text{and} \quad (x)_n = \prod_{k=0}^{n-1}(x+k) \quad \text{for} \quad n \in \mathbb{N}.$$

The harmonic numbers H_n are given by the partial sums $H_n = \sum_{k=1}^{n} \frac{1}{k}$, where $n \in \mathbb{N}$.

2. Double sum representations

Denote by $[y^n]\phi(y)$ the coefficient of y^n in the formal power series $\phi(y)$. Recall the exponential generating function of the Bernoulli numbers

$$\frac{y}{e^y - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} y^n.$$

Expanding this function into the series

$$\frac{y}{e^y-1} = \frac{\ln\{1+(e^y-1)\}}{e^y-1} = \sum_{j=0}^{\infty} \frac{(-1)^j}{j+1}(e^y-1)^j,$$

we get, for $m \ge n$, the following expression

$$B_n = n![y^n]\sum_{j=0}^{m} \frac{(-1)^j}{j+1}(e^y-1)^j = n!\sum_{j=0}^{m}(-1)^j \frac{j!}{j+1}[y^n]\frac{(e^y-1)^j}{j!} = \sum_{j=0}^{m}(-1)^j \frac{j!}{j+1} S_2(n,j),$$

where $S_2(n, j)$ is the Stirling number of the second kind

$$S_2(n,j) = n![y^n]\frac{(e^y-1)^j}{j!} = \frac{1}{j!}\sum_{k=0}^{j}(-1)^{j-k}\binom{j}{k}k^n. \tag{2.1}$$

Therefore, we have established, by substitution, the following explicit formula.

Theorem 1 ($m,\ n \in \mathbb{N}_0$ with $m \geq n$)**.**

$$B_n = \sum_{j=0}^{m} \sum_{k=0}^{j} \frac{(-1)^k}{j+1} \binom{j}{k} k^n = \sum_{k=0}^{m} k^n \sum_{j=k}^{m} \frac{(-1)^k}{j+1} \binom{j}{k}.$$

This fundamental result will be the starting point for us to examine explicit formulae of Bernoulli numbers. When $m = n + 1$, the corresponding formula can be located in Jordan [29, §78, Page 236]. Instead, its $m = n$ case is well-known; that can be found in Cook [18], Gould [23, Eq (1)] and [24, Eq (1.3)], Higgens [28], Quaintance and Gould [43, Eq (15.2)].

2.1. Connection coefficients $\Omega(m, k)$

Define the coefficients $\Omega(m, k)$ by

$$\Omega(m, k) := \sum_{j=k}^{m} \frac{(-1)^k}{j+1} \binom{j}{k}. \tag{2.2}$$

We can rewrite the basic formula in Theorem 1 as

$$B_n = \sum_{k=0}^{m} k^n \Omega(m, k), \quad \text{where} \quad m \geq n. \tag{2.3}$$

Lemma 2. *For $m \geq k$, the coefficients $\Omega(m, k)$ satisfy the recurrence relations:*

$$\Omega(m, k) - \Omega(m-1, k) = \frac{(-1)^k}{m+1} \binom{m}{k}, \tag{2.4}$$

$$\Omega(m, k) - \Omega(m, k-1) = \frac{(-1)^k}{k} \binom{m+1}{k}. \tag{2.5}$$

Proof. The first one (2.4) follows directly from the definition of $\Omega(m, k)$. The second one can be done as follows

$$\begin{aligned}
\Omega(m, k) - \Omega(m, k-1) &= \sum_{j=k}^{m} \frac{(-1)^k}{j+1} \binom{j}{k} + \sum_{j=k-1}^{m} \frac{(-1)^k}{j+1} \binom{j}{k-1} \\
&= \sum_{j=k-1}^{m} \frac{(-1)^k}{j+1} \left\{ \binom{j}{k} + \binom{j}{k-1} \right\} \\
&= \sum_{j=k-1}^{m} \frac{(-1)^k}{j+1} \binom{j+1}{k} \\
&= \frac{(-1)^k}{k} \sum_{j=k-1}^{m} \binom{j}{k-1},
\end{aligned}$$

which confirms (2.5) after having evaluated the binomial sum

$$\sum_{j=k-1}^{m} \binom{j}{k-1} = \binom{m+1}{k}.$$

□

In addition, the lower triangular matrix

$$\Omega_m := [\Omega(i,j)]_{1\le i,j\le m}$$

with diagonal elements

$$\Omega(k,k) = \frac{(-1)^k}{k+1}$$

is invertible. We can determine its inverse explicitly by

$$\Omega_m^{-1} = \left[(-1)^j\binom{i}{j}\frac{1+2i+ij}{1+j}\right]_{1\le i,j\le m}.$$

2.2. Formulae of Fekih-Ahmed [20]

Similar to the formula in Theorem 1, we have the following three variants.

Proposition 3 ($m,\ n \in \mathbb{N}$)**.**

$$B_n = \sum_{k=1}^m k^{n+1}\sum_{j=k}^m \frac{(-1)^{k-1}}{j^2}\binom{j}{k}, \qquad m > n \ge 2; \tag{2.6}$$

$$B_n = \sum_{k=1}^m k^{n}\sum_{j=k}^m \frac{(-1)^{k-1}}{j(j+1)}\binom{j}{k}, \qquad m \ge n \ge 2; \tag{2.7}$$

$$B_n = \sum_{k=1}^m k^{n}\sum_{j=k}^m (-1)^k\frac{2j+1}{j(j+1)}\binom{j}{k}, \qquad m \ge n \ge 2. \tag{2.8}$$

Among these formulae, the first two reduce, for $m = n+1$ and $m = n$, to Fekih-Ahmed [20, Eqs (5) and (6)], respectively.

Proof. According to (2.5), we have, for $m \ge n > 1$, the following equalities:

$$\begin{aligned} B_n &= \sum_{k=1}^m k^n\Omega(m,k) \\ &= \sum_{k=1}^m (-1)^k\binom{m+1}{k}k^{n-1} + \sum_{k=1}^m k^n\Omega(m,k-1) \\ &= (-1)^m(m+1)^{n-1} + \sum_{k=1}^m k^n \sum_{i=k-1}^m \frac{(-1)^{k-1}}{i+1}\binom{i}{k-1}. \end{aligned}$$

Simplifying the last line

$$B_n = \sum_{k=1}^{m+1} k^n \sum_{i=k-1}^m \frac{(-1)^{k-1}}{i+1}\binom{i}{k-1}, \quad (m \ge n \ge 2) \tag{2.9}$$

and then reformulating it as

$$B_n = \sum_{k=1}^{m+1} k^{n+1} \sum_{i=k-1}^m \frac{(-1)^{k-1}}{(i+1)^2}\binom{i+1}{k},$$

we confirm (2.6) under the replacements "$m \to m-1$" and "$i \to j-1$".

Observe further that for $m \geq n$, the double sum vanishes

$$\sum_{k=1}^{m} k^n \sum_{j=k}^{m} \frac{(-1)^k}{j}\binom{j}{k} = 0, \quad (m \geq n \geq 2), \tag{2.10}$$

which is justified by the finite differences

$$\sum_{k=1}^{m}(-1)^k k^{n-1} \sum_{j=k}^{m}\binom{j-1}{k-1} = \sum_{k=1}^{m}(-1)^k\binom{m}{k}k^{n-1} = 0.$$

Then (2.7) and (2.8) follow respectively from the difference and sum between Eq (2.10) and that in Theorem 1. □

Besides (2.10) and the formula in Theorem 1, we also have the following counterparts.

Theorem 4 ($m,\ n \in \mathbb{N}$ with $m \geq n \geq 1$)**.**

$$\sum_{k=1}^{m} k^n \sum_{j=k}^{m} \frac{(-1)^k}{j+2}\binom{j}{k} = \begin{cases} B_n, & n \equiv_2 0; \\ B_{n+1}, & n \equiv_2 1. \end{cases} \tag{2.11}$$

$$\sum_{k=1}^{m} k^n \sum_{j=k}^{m} \frac{(-1)^k}{j+3}\binom{j}{k} = \begin{cases} B_n + \frac{1}{2}B_{n+2}, & n \equiv_2 0; \\ \frac{3}{2}B_{n+1}, & n \equiv_2 1. \end{cases} \tag{2.12}$$

Proof. In accordance with the binomial relation

$$\binom{j}{k} = \binom{j+1}{k} - \binom{j}{k-1}, \tag{2.13}$$

we can write the sum in (2.11) as "$A + B$", where

$$A = \sum_{k=0}^{m} k^n \sum_{j=k}^{m} \frac{(-1)^k}{j+2}\binom{j+1}{k},$$
$$B = \sum_{k=0}^{m} k^n \sum_{j=k}^{m} \frac{(-1)^{k-1}}{j+2}\binom{j}{k-1}.$$

Replacing the summation index j by $i-1$ in "A", we can reformulate it as follows:

$$\begin{aligned} A &= \sum_{k=0}^{m} k^n \sum_{i=k+1}^{m+1} \frac{(-1)^k}{i+1}\binom{i}{k} \\ &= \sum_{k=0}^{m+1} k^n \sum_{i=k}^{m+1} \frac{(-1)^k}{i+1}\binom{i}{k} - \sum_{k=0}^{m+1}(-1)^k \frac{k^n}{k+1}. \end{aligned}$$

Recalling Theorem 1, we deduce the expression

$$A = B_n - \sum_{k=0}^{m+1}(-1)^k \frac{k^n}{k+1}. \tag{2.14}$$

When $n > 1$, applying first the partial fractions and then making the replacement "$j \to i-1$", we can rewrite "B" as

$$\begin{aligned}B &= \sum_{k=1}^{m} k^{n+1} \sum_{j=k}^{m} \frac{(-1)^{k-1}}{(j+1)(j+2)} \binom{j+1}{k}\\ &= \sum_{k=1}^{m} k^{n+1} \sum_{j=k}^{m} \frac{(-1)^{k-1}}{j+1} \binom{j+1}{k} + \sum_{k=1}^{m} k^{n+1} \sum_{j=k}^{m} \frac{(-1)^{k}}{j+2} \binom{j+1}{k}\\ &= \sum_{k=1}^{m} k^{n+1} \sum_{i=k+1}^{m+1} \frac{(-1)^{k}}{i+1} \binom{i}{k} - \sum_{k=1}^{m} k^{n+1} \sum_{i=k+1}^{m+1} \frac{(-1)^{k}}{i} \binom{i}{k}.\end{aligned}$$

By appealing to Theorem 1 and (2.10), we reduce the last expression to the following one:

$$B = B_{n+1} + \sum_{k=1}^{m+1} (-1)^k k^n - \sum_{k=1}^{m+1} (-1)^k \frac{k^{n+1}}{k+1} = B_{n+1} + \sum_{k=1}^{m+1} (-1)^k \frac{k^n}{k+1}. \tag{2.15}$$

Putting together (2.14) and (2.15), we find that

$$A + B = B_n + B_{n+1},$$

which is equivalent to the expression in (2.11).

Similarly, by making use of (2.13), we can write the sum (2.12) as "$C + D$", where

$$\begin{aligned}C &= \sum_{k=1}^{m} k^{n} \sum_{j=k}^{m} \frac{(-1)^{k}}{j+3} \binom{j+1}{k},\\ D &= \sum_{k=1}^{m} k^{n} \sum_{j=k}^{m} \frac{(-1)^{k-1}}{j+3} \binom{j}{k-1}.\end{aligned}$$

Replacing the summation index j by $i-1$ in "C", we can manipulate it as follows:

$$C = \sum_{k=1}^{m} k^{n} \sum_{i=k+1}^{m+1} \frac{(-1)^{k}}{i+2} \binom{i}{k} = \sum_{k=1}^{m+1} k^{n} \sum_{i=k}^{m+1} \frac{(-1)^{k}}{i+2} \binom{i}{k} - \sum_{k=1}^{m+1} (-1)^k \frac{k^n}{k+2}.$$

Evaluating the first sum by (2.11), we obtain the expression

$$C = B_n + B_{n+1} - \sum_{k=1}^{m+1} (-1)^k \frac{k^n}{k+2}. \tag{2.16}$$

We can analogously treat "D" as follows:

$$\begin{aligned}D &= \sum_{k=1}^{m} k^{n+1} \sum_{j=k}^{m} \frac{(-1)^{k+1}}{(j+1)(j+3)} \binom{j+1}{k}\\ &= \frac{1}{2} \sum_{k=1}^{m} k^{n+1} \sum_{j=k}^{m} \frac{(-1)^{k}}{j+3} \binom{j+1}{k} - \frac{1}{2} \sum_{k=1}^{m} k^{n+1} \sum_{j=k}^{m} \frac{(-1)^{k}}{j+1} \binom{j+1}{k}\end{aligned}$$

$$=\frac{1}{2}\sum_{k=1}^{m}k^{n+1}\sum_{i=k+1}^{m+1}\frac{(-1)^k}{i+2}\binom{i}{k}-\frac{1}{2}\sum_{k=1}^{m}k^{n+1}\sum_{i=k+1}^{m+1}\frac{(-1)^k}{i}\binom{i}{k}.$$

Applying (2.10) and (2.11), we can further simplify the last expression

$$\begin{aligned}D&=\frac{B_{n+1}+B_{n+2}}{2}-\frac{1}{2}\sum_{k=1}^{m+1}(-1)^k\frac{k^{n+1}}{k+2}+\frac{1}{2}\sum_{k=1}^{m+1}(-1)^k k^n\\&=\frac{B_{n+1}+B_{n+2}}{2}+\sum_{k=1}^{m+1}(-1)^k\frac{k^n}{k+2}.\end{aligned}\tag{2.17}$$

Finally, putting (2.16) and (2.17), we arrive at

$$C+D=B_n+\frac{3B_{n+1}+B_{n+2}}{2},$$

which proves the second identity (2.12). □

2.3. Four variants with k being shifted

Furthermore, there are four similar sums that can be expressed in closed forms in Bernoulli numbers.

Theorem 5 ($m,\ n\in\mathbb{N}$)**.**

$$\sum_{k=0}^{m}(k+1)^n\sum_{i=k}^{m}\frac{(-1)^k}{i+1}\binom{i}{k}=B_n,\qquad m\ge n\ge 2;\tag{2.18}$$

$$\sum_{k=0}^{m}(k+1)^n\sum_{i=k}^{m}\frac{(-1)^k}{i+2}\binom{i}{k}=B_{n+1},\qquad m\ge n\ge 1;\tag{2.19}$$

$$\sum_{k=0}^{m}(k+2)^n\sum_{i=k}^{m}\frac{(-1)^k}{i+1}\binom{i}{k}=n+B_n,\qquad m\ge n\ge 2;\tag{2.20}$$

$$\sum_{k=0}^{m}(k+2)^n\sum_{i=k}^{m}\frac{(-1)^k}{i+2}\binom{i}{k}=1-B_n+B_{n+1},\quad m\ge n\ge 1.\tag{2.21}$$

Proof. The first one (2.18) is deduced from (2.9) under the replacement "$k\to k+1$".

By splitting the sum in (2.19) into two, then making the replacements "$i\to j-1,\ k\to k-1$" for the former and "$k\to k-1$" for the latter, we can confirm the second identity (2.19) as follows:

$$\begin{aligned}&\sum_{k=0}^{m}(k+1)^n\sum_{i=k}^{m}\frac{(-1)^k}{i+2}\binom{i}{k}\\&=\sum_{k=0}^{m}(k+1)^n\sum_{i=k}^{m}\frac{(-1)^k}{i+2}\left\{\binom{i+1}{k+1}-\binom{i}{k+1}\right\}\\&=\sum_{k=1}^{m+1}k^n\sum_{j=k}^{m+1}\frac{(-1)^{k-1}}{j+1}\binom{j}{k}+\sum_{k=1}^{m+1}k^n\sum_{i=k-1}^{m}\frac{(-1)^k}{i+2}\binom{i}{k}\end{aligned}$$

$$= \sum_{k=1}^{m} k^n \sum_{i=k}^{m} \frac{(-1)^k}{i+2}\binom{i}{k} - \sum_{k=1}^{m+1} k^n \sum_{j=k}^{m+1} \frac{(-1)^k}{j+1}\binom{j}{k}$$
$$= (B_n + B_{n+1}) - B_n = B_{n+1}.$$

According to (2.13), we can write the third sum in (2.20) as "$E + F$", where

$$E = \sum_{k=0}^{m} (k+2)^n \sum_{i=k}^{m} \frac{(-1)^k}{i+1}\binom{i+1}{k+1},$$
$$F = \sum_{k=0}^{m} (k+2)^n \sum_{i=k}^{m} \frac{(-1)^{k-1}}{i+1}\binom{i}{k+1}.$$

The sum "E" can be evaluated by

$$E = \sum_{k=0}^{m} (-1)^k \frac{(k+2)^n}{k+1} \sum_{i=k}^{m} \binom{i}{k}$$
$$= \sum_{k=0}^{m} (-1)^k \frac{(k+2)^n}{k+1}\binom{m+1}{k+1}$$
$$= \sum_{i=0}^{n} \binom{n}{i} \sum_{k=0}^{m} (-1)^k \binom{m+1}{k+1}(k+1)^{i-1}.$$

Because the above inner sum results in zero for $2 \le i \le n$, there remain, for "E", only two terms corresponding to $i = 1$ and $i = 0$:

$$E = n + \sum_{k=0}^{m} \frac{(-1)^k}{k+1}\binom{m+1}{k+1} = n + H_{m+1}. \tag{2.22}$$

The sum "F" can be restated, under "$k \to k-1$", as

$$F = \sum_{k=1}^{m+1} (k+1)^n \sum_{i=k-1}^{m} \frac{(-1)^k}{i+1}\binom{i}{k}$$
$$= \sum_{k=1}^{m} (k+1)^n \sum_{i=k}^{m} \frac{(-1)^k}{i+1}\binom{i}{k}$$
$$= \sum_{k=0}^{m} (k+1)^n \sum_{i=k}^{m} \frac{(-1)^k}{i+1}\binom{i}{k} - \sum_{i=0}^{m} \frac{1}{i+1}.$$

Evaluating the former sum by (2.18), we find that

$$F = B_n - H_{m+1}. \tag{2.23}$$

Then the third identity (2.20) follows by putting (2.22) and (2.23) together.

Finally, for (2.21), write that sum as "$G + H$", where

$$G = \sum_{k=0}^{m} (k+2)^n \sum_{i=k}^{m} \frac{(-1)^k}{i+2}\binom{i+1}{k+1},$$

$$H=\sum_{k=0}^{m}(k+2)^n\sum_{i=k}^{m}\frac{(-1)^{k-1}}{i+2}\binom{i}{k+1}.$$

By making use of (2.18) and (2.7), we can evaluate

$$\begin{aligned}
G&=\sum_{k=1}^{m+1}(k+1)^n\sum_{j=k}^{m+1}\frac{(-1)^{k-1}}{j+1}\binom{j}{k}=H_{m+2}-B_n,\\
H&=\sum_{k=0}^{m}(k+2)^{n+1}\sum_{i=k}^{m}\frac{(-1)^{k-1}}{(i+1)(i+2)}\binom{i+1}{k+2}\\
&=\sum_{k=2}^{m+1}k^{n+1}\sum_{j=k}^{m+1}\frac{(-1)^{k+1}}{j(j+1)}\binom{j}{k}\\
&=B_{n+1}-\sum_{j=1}^{m+1}\frac{1}{j+1}=1+B_{n+1}-H_{m+2}.
\end{aligned}$$

It follows consequently that

$$G+H=1-B_n+B_{n+1},$$

which coincides with the right-hand side of (2.21). □

3. Further explicit expressions

In this section, we shall first prove a binomial identity. Then it will be utilized to derive equivalent expressions for the connection coefficients $\Omega(m,k)$ and $\omega(m,k)$. These equivalent forms will be useful in proving further explicit formulae for Bernoulli numbers.

3.1. A binomial sum identity

We begin with the following binomial identity.

Theorem 6. *For two indeterminates x, y and $m, k \in \mathbb{N}_0$ with $m \geq k$, the following algebraic identity holds:*

$$\begin{aligned}
&\sum_{j=k}^{m}\frac{(x-j)_k}{(1+y)_k(1-x+y+j)}\\
&=\sum_{i=k}^{m}\left\{\frac{(x-i)_k}{(1+y+m-i)_{k+1}}+\frac{(x-k)_{k+1}}{(1-x+y+i)(1+y-k+i)_{k+1}}\right\}\\
&=\sum_{i=k}^{m}\left\{\frac{(x-m+i-k)_k}{(1+y-k+i)_{k+1}}+\frac{(x-k)_{k+1}}{(1-x+y+i)(1+y-k+i)_{k+1}}\right\}.
\end{aligned}$$

Proof. We prove the theorem by examining the double sum

$$S:=\sum_{i=k}^{m}\sum_{j=i}^{m}\Lambda(i,j),\quad\text{where}\quad\Lambda(i,j):=\frac{(k+1)(x-i)_k}{(y-i+j)_{k+2}}.$$

For the given λ'_j-sequence below, it is routine to check its difference

$$\lambda'_j = \frac{(x-i)_k}{(y-i+j)_{k+1}}, \qquad \lambda'_j - \lambda'_{j+1} = \Lambda(i,j).$$

Therefore, we can manipulate S by telescoping as follows:

$$\mathrm{S} = \sum_{i=k}^{m}\sum_{j=i}^{m}\Lambda(i,j) = \sum_{i=k}^{m}\sum_{j=i}^{m}\{\lambda'_j - \lambda'_{j+1}\} = \sum_{i=k}^{m}\{\lambda'_i - \lambda'_{m+1}\},$$

which can be restated as

$$\mathrm{S} = \sum_{i=k}^{m}\left\{\frac{(x-i)_k}{(y)_{k+1}} - \frac{(x-i)_k}{(1+y-i+m)_{k+1}}\right\}. \tag{3.1}$$

Alternatively, for another λ''_i-sequence, we have

$$\lambda''_i := \frac{(x-i)_{k+1}}{(1+y-i+j)_{k+1}}, \qquad \lambda''_i - \lambda''_{i+1} = (1-x+y+j)\Lambda(i,j).$$

Hence, we can reformulate S analogously as follows:

$$\mathrm{S} = \sum_{j=k}^{m}\sum_{i=k}^{j}\Lambda(i,j) = \sum_{j=k}^{m}\sum_{i=k}^{j}\frac{\lambda''_i - \lambda''_{i+1}}{1-x+y+j} = \sum_{j=k}^{m}\frac{\lambda''_k - \lambda''_{j+1}}{1-x+y+j},$$

which can be rewritten explicitly

$$\mathrm{S} = \sum_{j=k}^{m}\left\{\frac{(x-k)_{k+1}}{(1-x+y+j)(1+y-k+j)_{k+1}} - \frac{(x-j-1)_{k+1}}{(1-x+y+j)(y)_{k+1}}\right\}. \tag{3.2}$$

By relating (3.1) to (3.2), we derive the equality

$$\begin{aligned}
&\sum_{i=k}^{m}\frac{(x-i)_k}{(1+y-i+m)_{k+1}} + \sum_{j=k}^{m}\frac{(x-k)_{k+1}}{(1-x+y+j)(1+y-k+j)_{k+1}}\\
=&\sum_{i=k}^{m}\frac{(x-i)_k}{(y)_{k+1}} + \sum_{j=k}^{m}\frac{(x-j-1)_{k+1}}{(1-x+y+j)(y)_{k+1}}\\
=&\sum_{j=k}^{m}\frac{(x-j)_k}{(1-x+y+j)(1+y)_k},
\end{aligned}$$

which is equivalent to the expression in the theorem. □

When $x = y = 0$, Theorem 6 reduces to the crucial identity used by Komatsu and Pita-Ruiz [31, Eq (27)].

$$\sum_{j=k}^{m}\frac{m+1}{j+1}\binom{j}{k} = \sum_{i=k}^{m}\frac{\binom{m+1}{i-k}}{\binom{m}{i}} = \sum_{i=k}^{m}\frac{\binom{m+1}{m-i}}{\binom{m}{i-k}}.$$

From this, we deduce the equivalent expressions below

$$\Omega(m,k) = \sum_{i=k}^{m} \frac{(-1)^k}{m+1} \frac{\binom{m+1}{i-k}}{\binom{m}{i}} = \sum_{i=k}^{m} \frac{(-i)_k}{(1+m-i)_{k+1}}. \tag{3.3}$$

This leads us to the following formula which reduces, for $m = n$, to Munch [39] (cf. Gould [24, Eq (1.8)], Quaintance and Gould [43, Eq (15.7)]).

Theorem 7 ($m \geq n \geq 1$)**.**

$$B_n = \sum_{k=1}^{m} k^n \sum_{i=k}^{m} \frac{(-i)_k}{(1+m-i)_{k+1}} = \sum_{k=1}^{m} k^n \sum_{i=k}^{m} \frac{(-1)^k}{m+1} \frac{\binom{m+1}{i-k}}{\binom{m}{i}}.$$

3.2. Connection coefficients $\omega(m,k)$

Consider the difference

$$\begin{aligned} &\sum_{k=1}^{m} k^n \sum_{i=k}^{m} \left\{ \frac{(1-i)_{k-1}}{(2+m-i)_k} - \frac{(-i)_k}{(1+m-i)_{k+1}} \right\} \\ =& \sum_{k=1}^{m} k^n \sum_{i=k}^{m} \frac{(1-i)_{k-1}}{(1+m-i)_{k+1}} (m+1) \\ =& \sum_{k=1}^{m} k^n \sum_{i=k}^{m} \frac{\lambda_i - \lambda_{i+1}}{k} \text{ where } \lambda_i = \frac{(1-i)_k}{(2+m-i)_k} \\ =& \sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k} k^{n-1} = 0, \qquad m \geq n > 1. \end{aligned}$$

We infer that the formula in Theorem 7 is equivalent to the following one.

Theorem 8 ($m \geq n \geq 2$)**.**

$$B_n = \sum_{k=1}^{m} k^n \sum_{i=k}^{m} \frac{(-1)^{k-1}}{m+1} \frac{\binom{m+1}{i-k}}{\binom{m}{i-1}}. \tag{3.4}$$

Its special case $m = n$ can be found in Gould [24, Eq (1.9)], Quaintance and Gould [43, Eq (15.5)] and Shanks [45]. However, the formula produced by Komatsu and Pita–Ruiz [31, Eq (2)] is incorrect.

The last formula can be rewritten as

$$B_n = \sum_{k=1}^{m} k^n \omega(m,k), \tag{3.5}$$

where the connection coefficients are defined by

$$\omega(m,k) = \sum_{i=k}^{m} \frac{(1-i)_{k-1}}{(2+m-i)_k} = \sum_{i=k}^{m} \frac{(-1)^{k-1}}{m+1} \frac{\binom{m+1}{i-k}}{\binom{m}{i-1}}. \tag{3.6}$$

It is obvious that the matrix of the connection coefficients

$$\omega_m := [\omega(i,j)]_{1\le i,j\le m}$$

with diagonal entries

$$\omega(k,k) = \frac{(-1)^{k+1}}{k(k+1)}$$

is lower triangular and invertible. It is not difficult to check that its inverse is given explicitly by

$$\omega_m^{-1} = \left[(-1)^{j+1}\binom{i}{j}(2i-j+ij)\right]_{1\le i,j\le m}.$$

Observing further that

$$\omega(m,k) = \sum_{j=k-1}^{m-1} \frac{(-j)_{k-1}}{(1+m-j)_k},$$

we can prove the following interesting lemma.

Lemma 9. *The connection coefficients satisfy the properties:*

- *Relations between* $\Omega(m,k)$ *and* $\omega(m,k)$

$$\begin{aligned}\omega(m,k) &= \Omega(m,k-1) - \frac{(-m)_{k-1}}{k!},\\ \Omega(m,k) &= \omega(m,k+1) + \frac{(-m)_k}{(k+1)!}.\end{aligned} \tag{3.7}$$

- *Recurrence relations:* $m,k \in \mathbb{N}$

$$\omega(m,k) - \omega(m,k-1) = \frac{(-1)^{k-1}}{k(k-1)}\binom{m}{k-1}, \tag{3.8}$$

$$\omega(m,k) - \omega(m-1,k) = \frac{(-1)^{k-1}}{m(m+1)}\binom{m}{k}. \tag{3.9}$$

- *Equivalent expression:* $m,k \in \mathbb{N}$

$$\omega(m,k) = \sum_{j=k}^{m} \frac{(-1)^{k-1}}{j(j+1)}\binom{j}{k}. \tag{3.10}$$

Proof. The recurrence relations (3.8) and (3.9) follow by combining (3.7) with (2.4) and (2.5). The equivalent expression displayed in (3.10) is obtained by iterating (3.9), which has already appeared in (2.7). □

The connection coefficients $\Omega(m,k)$ and $\omega(m,k)$ are related to the harmonic numbers in the following manners.

Proposition 10 (Equivalent expressions: $m, k \in \mathbb{N}$)**.**

$$\Omega(m,k) = H_{m+1} + \sum_{i=1}^{k} \frac{(-1)^i}{i}\binom{m+1}{i} = \sum_{i=k}^{m} \frac{(-1)^i}{i+1}\binom{m+1}{i+1}, \tag{3.11}$$

$$\omega(m,k) = H_{m+1} - 1 + \sum_{i=1}^{k-1} \frac{(-1)^i}{i(i+1)}\binom{m}{i} = \sum_{i=k}^{m} \frac{(-1)^{i-1}}{i(i+1)}\binom{m}{i}. \tag{3.12}$$

Proof. By iterating the relation (2.5) k-times, we find that

$$\Omega(m,k) = \Omega(m,0) + \sum_{i=1}^{k} \frac{(-1)^i}{i}\binom{m+1}{i}$$

which becomes the first expression in (3.11) since $\Omega(m,0) = H_{m+1}$. The second expression in (3.11) follows by the inverse pair

$$H_n = \sum_{i=1}^{n} \frac{(-1)^{i-1}}{i}\binom{n}{i} \quad \text{and} \quad \frac{1}{n} = \sum_{i=1}^{n} (-1)^{i-1}\binom{n}{i}H_i.$$

Analogously, by iterating the relation (3.8) k-times, we have that

$$\omega(m,k) = \omega(m,1) + \sum_{i=1}^{k-1} \frac{(-1)^i}{i(i+1)}\binom{m}{i},$$

which gives the first expression in (3.12) since $\omega(m,1) = H_{m+1} - 1$. The second expression in (3.12) follows by another inverse pair

$$H_{m+1} - 1 = \sum_{i=1}^{m} \frac{(-1)^{i-1}}{i(i+1)}\binom{m}{i} \text{ and } \frac{1}{m(m+1)} = \sum_{i=1}^{m} (-1)^{i-1}\binom{m}{i}(H_{i+1} - 1).$$

□

We have therefore the following four explicit formulae.

Theorem 11 ($m,\ n \in \mathbb{N}_0$)**.**

$$B_n = \sum_{i=0}^{m} \frac{(-1)^i}{i+1}\binom{m+1}{i+1}\sum_{k=0}^{i} k^n, \qquad m \geq n \geq 0; \tag{3.13}$$

$$B_n = \sum_{i=1}^{m} \frac{(-1)^{i-1}}{i(i+1)}\binom{m}{i}\sum_{k=1}^{i} k^n, \qquad m \geq n \geq 2; \tag{3.14}$$

$$B_n = \sum_{i=0}^{m} \frac{(-1)^i}{i+1}\binom{m+1}{i+1}\sum_{k=0}^{i+1} k^n, \qquad m \geq n \geq 2; \tag{3.15}$$

$$B_n = (-1)^n \frac{n!\chi(m=n)}{n+1} + \sum_{i=0}^{m} \frac{(-1)^i}{i+1}\binom{m}{i+1}\sum_{k=0}^{i} k^n, \quad m \geq n \geq 0. \tag{3.16}$$

In the last line, χ stands for the logical function with $\chi(\text{true}) = 1$ and $\chi(\text{false}) = 0$. When $m = n$, the first identity recovers Bergmann [7] (cf. Gould [23, Eq (5)], [24, Eq (1.10)]), and the two variants (3.15) and (3.16) reduce to Gould [24, Eqs (1.11) and (1.12)]. In addition, we remark that when $m > n$, (3.15) is substantially the same as (3.13).

Proof. The first two identities follow directly by (3.11) and (3.12). In comparison with (3.13), the third identity (3.15) is equivalent to

$$\sum_{i=0}^{m}(-1)^i\binom{m+1}{i+1}(i+1)^{n-1} = \sum_{j=1}^{m+1}(-1)^{j-1}\binom{m+1}{j}j^{n-1} = 0,$$

because for $m \geq n > 1$, the order $m + 1$ of the differences is higher than the polynomial degree $n - 1$.

By means of the binomial relation

$$\binom{n+1}{i+1} = \binom{n}{i+1} + \binom{n}{i},$$

we can rewrite (3.13) as

$$B_n = \sum_{i=0}^{m}\frac{(-1)^i}{i+1}\binom{m}{i+1}\sum_{k=0}^{i}k^n + \sum_{i=0}^{m}\frac{(-1)^i}{i+1}\binom{m}{i}\sum_{k=0}^{i}k^n.$$

Recall that the power sum $\sum_{k=0}^{j-1}k^n$ results in a polynomial of degree $n + 1$ in j with the leading coefficient being equal to $\frac{1}{n+1}$. We can evaluate the second sum above in closed form as follows:

$$\begin{aligned}\sum_{i=0}^{m}\frac{(-1)^i}{i+1}\binom{m}{i}\sum_{k=0}^{i}k^n =& \frac{(-1)^m}{m+1}\sum_{j=0}^{m}(-1)^{1+m-j}\binom{m+1}{j}\sum_{k=0}^{j-1}k^n\\ =& \frac{(-1)^m}{m+1}\Delta^{m+1}\frac{x^{n+1}}{n+1}\Big|_{x=0}\\ =&(-1)^n\frac{n!}{n+1}\chi(m=n).\end{aligned}$$

This confirms identity (3.16) and completes the proof of Theorem 11. □

3.3. Partial fractions

Consider the partial fraction decomposition

$$\frac{(x)_\ell}{(1+m+x)_{\ell+1}} = \sum_{j=0}^{\ell}\frac{(-1)^{j+\ell}}{1+m+x+j}\binom{\ell}{j}\binom{1+m+j}{\ell}.$$

In the above equation, letting "$x \to -i,\ \ell \to k$" and "$x \to 1 - i,\ \ell \to k - 1$", we have, respectively, the two equalities:

$$\frac{(-i)_k}{(1+m-i)_{k+1}} = \sum_{j=0}^{k}\frac{(-1)^{j+k}}{1+m-i+j}\binom{k}{j}\binom{1+m+j}{k},$$

$$\frac{(1-i)_{k-1}}{(2+m-i)_k} = \sum_{j=1}^{k} \frac{(-1)^{j+k}}{1+m-i+j}\binom{k-1}{j-1}\binom{m+j}{k-1}.$$

Substituting them into (3.3) and (3.6), respectively, then manipulating the double sums by exchanging the summation order, we derive the following equivalent expressions in terms of harmonic numbers.

Proposition 12 ($m,\ k \in \mathbb{N}_0$)**.**

$$\Omega(m,k) = \sum_{j=0}^{k}(-1)^{j+k}\binom{k}{j}\binom{1+m+j}{k}\{H_{1+m-k+j} - H_j\}, \quad m \geq k \geq 0;$$

$$\omega(m,k) = \sum_{j=1}^{k}(-1)^{j+k}\binom{k-1}{j-1}\binom{m+j}{k-1}\{H_{1+m-k+j} - H_j\}, \quad m \geq k \geq 1.$$

Consequently, we find two further explicit formulae involving harmonic numbers.

Theorem 13 ($m,\ n \in \mathbb{N}$)**.**

$$B_n = \sum_{k=1}^{m} k^n \sum_{j=0}^{k}(-1)^{j+k}\binom{k}{j}\binom{1+m+j}{k}\{H_{1+m-k+j} - H_j\}, \quad m \geq n \geq 1;$$

$$B_n = \sum_{k=1}^{m} k^n \sum_{j=1}^{k}(-1)^{j+k}\binom{k-1}{j-1}\binom{m+j}{k-1}\{H_{1+m-k+j} - H_j\}, \quad m \geq n \geq 2.$$

4. More summation formulae

We are going to review, finally in this section, more summation formulae involving harmonic numbers and Stirling numbers of the second kind by intervening with an extra integer parameter m.

4.1. Harmonic numbers

There exist two formulae expressing B_n in terms of harmonic numbers, that are quite different from those in Theorem 13.

Theorem 14 ($m > n \geq 1$)**.**

$$\begin{aligned} B_n &= \sum_{i=1}^{m} H_i \sum_{k=1}^{i}(-1)^{k-1}\binom{i-1}{k-1}k^n \\ &= \sum_{k=1}^{m} k^n \sum_{i=k}^{m}(-1)^{k-1}\binom{i-1}{k-1}H_i, \end{aligned} \tag{4.1}$$

$$\begin{aligned} B_n &= \frac{2}{n+1}\sum_{i=1}^{m} \frac{H_i}{i+1} \sum_{k=1}^{i}(-1)^{k-1}\binom{i}{k}k^{n+1} \\ &= \frac{2}{n+1}\sum_{k=1}^{m} k^{n+1} \sum_{i=k}^{m}(-1)^{k-1}\binom{i}{k}\frac{H_i}{i+1}. \end{aligned} \tag{4.2}$$

When $m = n$, the second formula (4.2) recovers Gould [24, Eq (1.4)].

Proof. Recalling the generating function

$$\sum_{i=1}^{\infty} H_i y^i = -\frac{\ln(1-y)}{1-y},$$

we can proceed with

$$\begin{aligned} B_n = n![x^n]\frac{x}{e^x-1} &= n![x^n]\frac{e^x}{1-e^x} \times \frac{-\ln\{1-(1-e^x)\}}{1-(1-e^x)} \\ &= n![x^n]\sum_{i=1}^{m} H_i e^x (1-e^x)^{i-1} \qquad \boxed{m > n} \\ &= n![x^n]\sum_{i=1}^{m} H_i \sum_{k=1}^{i} (-1)^{k-1}\binom{i-1}{k-1} e^{kx}, \end{aligned}$$

which results in the first formula (4.1).

The second formula (4.2) is a variant of (4.1), which can be verified by making use of another generating function

$$\sum_{i=1}^{\infty} \frac{H_i}{i+1} y^{i+1} = \frac{\ln^2(1-y)}{2}.$$

In fact, we can similarly extract the coefficient as follows:

$$\begin{aligned} B_n &= n![x^n]\frac{x}{e^x-1} \\ &= n![x^n]\frac{2\ln^2\{1-(1-e^x)\}}{2x(e^x-1)} \\ &= -2n![x^{n+1}]\sum_{i=1}^{m} \frac{H_i}{i+1}(1-e^x)^i \qquad \boxed{m > n} \\ &= -2n![x^{n+1}]\sum_{i=1}^{m} \frac{H_i}{i+1}\sum_{k=1}^{i}(-1)^k\binom{i}{k} e^{kx} \\ &= \frac{2}{n+1}\sum_{i=1}^{m} \frac{H_i}{i+1}\sum_{k=1}^{i}(-1)^{k-1}\binom{i}{k} k^{n+1}. \end{aligned}$$

□

4.2. Stirling numbers of the second kind

Here we offer three formulae containing the Stirling numbers of the second kind, extending those by Shirai [46, Theorem 6 and Corollary 7] (cf. Morrow [38, Eq (1.5)]).

Theorem 15 ($\lambda \neq 0$ and $m,\ n \in \mathbb{N}$ with $m \geq n \geq 0$)**.**

$$B_n = \sum_{j=0}^{m} (-1)^j \frac{\binom{m+1}{j+1}}{\binom{n+j}{j}} S_2(n+j, j), \tag{4.3}$$

$$B_n = \frac{n}{n-1} \sum_{j=1}^{m+1} (-1)^j \frac{\binom{m+2}{j+1}}{\binom{n+j-1}{j}} S_2(n+j, j), \tag{4.4}$$

$$B_n = \frac{n}{n-1} \sum_{j=1}^{m+2} (-1)^j \frac{\lambda + j + 1}{\lambda} \frac{\binom{m+3}{j+1}}{\binom{n+j-1}{j}} S_2(n+j, j). \tag{4.5}$$

When $m = n$, the first formula (4.3) can be found in Gould [23, Eq (11)], [24, Eq (1.5)] (see also Luo [33], Quaintance and Gould [43, Eq (15.10)], Shirai [46, Theorem 6]). Two variants (4.4) and (4.5) with $m = n$ and $\lambda = n + \frac{2}{3}$ are due to Shirai [46, Corollary 7].

Proof. Recall the exponential generating function

$$\frac{(e^x - 1)^j}{j!} = \sum_{n=0}^{\infty} \frac{x^{n+j}}{(n+j)!} S_2(n+j, j).$$

We can extract the coefficient

$$\begin{aligned} B_n = n![x^n] \frac{x}{e^x - 1} &= n![x^n] \frac{1}{1 - \{1 - \frac{e^x - 1}{x}\}} \\ &= n![x^n] \sum_{i=0}^{m} \Big\{1 - \frac{e^x - 1}{x}\Big\}^i \qquad \boxed{m \geq n} \\ &= n! \sum_{i=0}^{m} \sum_{j=0}^{i} (-1)^j \binom{i}{j} [x^{n+j}](e^x - 1)^j \\ &= n! \sum_{j=0}^{m} (-1)^j [x^{n+j}](e^x - 1)^j \sum_{i=j}^{m} \binom{i}{j}. \end{aligned}$$

Evaluating the last sum

$$\sum_{i=j}^{m} \binom{i}{j} = \binom{m+1}{j+1},$$

we get the expression

$$\begin{aligned} B_n &= n! \sum_{j=0}^{m} (-1)^j \binom{m+1}{j+1} [x^{n+j}](e^x - 1)^j \\ &= \sum_{j=0}^{m} (-1)^j \frac{\binom{m+1}{j+1}}{\binom{n+j}{j}} S_2(n+j, j), \end{aligned}$$

which proves the first formula (4.3).

The second sum (4.4) can be evaluated as follows:

$$\begin{aligned}
&\sum_{j=1}^{m+1}(-1)^j\frac{n}{n-1}\frac{\binom{m+2}{j+1}}{\binom{n+j-1}{j}}S_2(n+j,j)\\
=&n![x^n]\sum_{j=1}^{m+1}(-1)^j\binom{m+2}{j+1}\frac{n+j}{n-1}\Big(\frac{e^x-1}{x}\Big)^j\\
=&n![x^n]\sum_{j=1}^{m+1}(-1)^j\binom{m+2}{j+1}\Big(1+\frac{j+1}{n-1}\Big)\Big(\frac{e^x-1}{x}\Big)^j\\
=&B_n+\frac{n!(m+2)}{n-1}[x^n]\sum_{j=0}^{m+1}(-1)^j\binom{m+1}{j}\Big(\frac{e^x-1}{x}\Big)^j\\
=&B_n+\frac{n!(m+2)}{n-1}[x^n]\Big(1-\frac{e^x-1}{x}\Big)^{m+1},
\end{aligned}$$

because the last coefficient vanishes, thanks to the fact that $m+1>n$ and the constant term of $(1-\frac{e^x-1}{x})$ is equal to zero.

Finally, writing

$$\frac{\lambda+j+1}{\lambda}=1+\frac{j+1}{\lambda},$$

we can reformulate the third sum in (4.5) as

$$\begin{aligned}
&\sum_{j=1}^{m+2}(-1)^j\frac{n(\lambda+j+1)}{(n-1)\lambda}\frac{\binom{m+3}{j+1}}{\binom{n+j-1}{j}}S_2(n+j,j)\\
=&\frac{n}{n-1}\sum_{j=1}^{m+2}(-1)^j\frac{\binom{m+3}{j+1}}{\binom{n+j-1}{j}}S_2(n+j,j)\\
+&\frac{n(m+3)}{(n-1)\lambda}\sum_{j=1}^{m+2}(-1)^j\frac{\binom{m+2}{j}}{\binom{n+j-1}{j}}S_2(n+j,j).
\end{aligned}$$

The first term on the right equals B_n in view of (4.4). The second term on the right vanishes again, which is justified analogously as follows:

$$\begin{aligned}
&\sum_{j=1}^{m+2}(-1)^j\frac{\binom{m+2}{j}}{\binom{n+j-1}{j}}S_2(n+j,j)\\
=&(n-1)!\sum_{j=1}^{m+2}(-1)^j(n+j)\binom{m+2}{j}[x^n]\Big(\frac{e^x-1}{x}\Big)^j\\
=&n(n-1)![x^n]\sum_{j=0}^{m+2}(-1)^j\binom{m+2}{j}\Big(\frac{e^x-1}{x}\Big)^j\\
+&(m+2)(n-1)![x^n]\sum_{j=1}^{m+2}(-1)^j\binom{m+1}{j-1}\Big(\frac{e^x-1}{x}\Big)^j
\end{aligned}$$

$$= n(n-1)![x^n]\Big(1-\frac{e^x-1}{x}\Big)^{m+2}$$
$$-(m+2)(n-1)![x^n]\Big(1-\frac{e^x-1}{x}\Big)^{m+1}\Big(\frac{e^x-1}{x}\Big).$$

□

4.3. Two formulae of Todorov [48]

Theorem 16 ($m \geq n \geq 2$)**.**

$$B_n = \frac{n}{2^n-1}\sum_{i=1}^{m}(-1)^{i-1}\frac{i!}{2^{i+1}}S_2(n-1,i) \tag{4.6}$$
$$= \frac{n}{2^n-1}\sum_{k=1}^{m}k^{n-1}\sum_{i=k}^{m}\frac{(-1)^{k-1}}{2^{i+1}}\binom{i}{k},$$
$$B_n = \frac{n}{2^n-1}\sum_{k=1}^{m}k^{n-1}\sum_{i=k}^{m}\frac{(-1)^{k-1}}{2^{i}}\binom{i-1}{k-1}. \tag{4.7}$$

The special case $m = n-1$ of the first formula (4.6) was found by Worptzky [50] (see also Carlitz [9, Eq (6)], Garabedian [21], Gould [23, Eq (2)], [24, Eq (3.22)]). Instead, the variant (4.7) reduces, for $m = n$, to Carlitz [9, Eq (5)].

Proof. By making use of the algebraic identity

$$\mathcal{F}(x) := \Big\{\frac{2x}{e^{2x}-1}-1\Big\}-\Big\{\frac{x}{e^x-1}-1\Big\} = \frac{-x}{1+e^x},$$

we can extract the coefficient

$$\frac{2^n-1}{n!}B_n = [x^n]\mathcal{F}(x) = [x^n]\frac{-x}{1+e^x}$$
$$= -[x^{n-1}]\frac{1}{2-(1-e^x)}$$
$$= -[x^{n-1}]\sum_{i=1}^{m}\frac{(-1)^i}{2^{i+1}}(e^x-1)^i \qquad \boxed{1+m\geq n>0}$$
$$= \sum_{i=1}^{m}\frac{(-1)^{i-1}}{2^{i+1}}\frac{i!}{(n-1)!}S_2(n-1,i),$$

which proves the first identity (4.6). Analogously, we have

$$\frac{2^n-1}{n!}B_n = [x^n]\mathcal{F}(x) = [x^{n-1}]\frac{-1}{1+e^x}$$
$$= [x^{n-1}]\Big\{\frac{1}{2(1-\frac{1-e^x}{2})}-\frac{1}{1-\frac{1-e^x}{2}}\Big\}$$
$$= \sum_{i=1}^{m}\frac{(-1)^{i-1}}{2^i}[x^{n-1}]\Big\{(e^x-1)^i+(e^x-1)^{i-1}\Big\} \qquad \boxed{m\geq n-1}$$

$$= \sum_{i=1}^{m} \frac{(-1)^{i-1}}{2^i (n-1)!} \Big\{ i! S_2(n-1, i) + (i-1)! S_2(n-1, i-1) \Big\}.$$

By invoking the expression

$$S_2(n, i) = \sum_{j=1}^{i} \frac{(-1)^{i+j}}{i!} \binom{i}{j} j^n,$$

we can further manipulate the last sum

$$\begin{aligned} \frac{2^n - 1}{n} B_n &= \sum_{i=1}^{m} \frac{(-1)^{k-1}}{2^i} \sum_{k=1}^{i} \left\{ \binom{i}{k} - \binom{i-1}{k} \right\} k^{n-1} \\ &= \sum_{i=1}^{m} \frac{(-1)^{k-1}}{2^i} \sum_{k=1}^{i} \binom{i-1}{k-1} k^{n-1} \\ &= \sum_{i=1}^{m} \frac{(-1)^{k-1}}{i 2^i} \sum_{k=1}^{i} \binom{i}{k} k^n. \end{aligned}$$

This confirms the second formula (4.7). □

The formulae in Theorem 16 imply the following results, that recover, for $m = n - 1$ and $m = n$, the two formulae due to Todorov [48, Eqs (8) and (9)] (see also Gould [24, Eq (3.23)] for the former one).

Theorem 17 ($m \geq n \geq 2$)**.**

$$B_n = \frac{n}{2^{m+1}(2^n - 1)} \sum_{j=0}^{m} \sum_{k=0}^{j} (-1)^{k-1} \binom{m+1}{j+1} k^{n-1}, \tag{4.8}$$

$$B_n = \frac{n}{2^m(2^n - 1)} \sum_{j=1}^{m} \sum_{k=1}^{j} (-1)^{k-1} \binom{m}{j} k^{n-1}. \tag{4.9}$$

Proof. By means of the binomial transform

$$\begin{aligned} \sum_{i=k}^{m} \binom{i}{k} 2^{m-i} &= \sum_{i=k}^{m} \binom{i}{k} \sum_{j=i}^{m} \binom{m-i}{j-i} \\ &= \sum_{j=k}^{m} \sum_{i=k}^{j} \binom{i}{k} \binom{m-i}{m-j} \\ &= \sum_{j=k}^{m} \binom{m+1}{j-k} \\ &= \sum_{j=k}^{m} \binom{m+1}{j+1}, \end{aligned}$$

we can reformulate the sum in (4.6) as

$$
\begin{aligned}
B_n &= \frac{n}{2^n-1}\sum_{k=0}^{m}(-1)^{k-1}\frac{k^{n-1}}{2^{m+1}}\sum_{i=k}^{m}\frac{2^{m+1}}{2^{i+1}}\binom{i}{k}\\
&= \frac{n}{2^n-1}\sum_{k=0}^{m}(-1)^{k-1}\frac{k^{n-1}}{2^{m+1}}\sum_{j=k}^{m}\binom{m+1}{j+1}\\
&= \frac{n}{2^{m+1}(2^n-1)}\sum_{j=0}^{m}\sum_{k=0}^{j}(-1)^{k-1}\binom{m+1}{j+1}k^{n-1},
\end{aligned}
$$

which confirms the first identity (4.8). The second one (4.8) can be done analogously by applying another binomial transform

$$
\begin{aligned}
\sum_{i=k}^{m}\binom{i-1}{k-1}2^{m-i} &= \sum_{i=k}^{m}\binom{i-1}{k-1}\sum_{j=i}^{m}\binom{m-i}{j-i}\\
&= \sum_{j=k}^{m}\sum_{i=k}^{j}\binom{i-1}{k-1}\binom{m-i}{m-j}\\
&= \sum_{j=k}^{m}\binom{m}{j-k} = \sum_{j=k}^{m}\binom{m}{j}
\end{aligned}
$$

to (4.6) and then manipulate the double sum as follows:

$$
\begin{aligned}
B_n &= \frac{n}{2^n-1}\sum_{k=1}^{n}k^n\sum_{i=k}^{n}\frac{(-1)^{k-1}}{i2^i}\binom{i}{k}\\
&= \frac{n}{2^n-1}\sum_{k=1}^{m}k^{n-1}\frac{(-1)^{k-1}}{2^m}\sum_{i=k}^{m}\binom{i-1}{k-1}2^{m-i}\\
&= \frac{n}{2^n-1}\sum_{k=1}^{m}k^{n-1}\frac{(-1)^{k-1}}{2^m}\sum_{j=k}^{m}\binom{m}{j}\\
&= \frac{n}{2^m(2^n-1)}\sum_{j=1}^{m}\sum_{k=1}^{j}(-1)^{k-1}\binom{m}{j}k^{n-1}.
\end{aligned}
$$

□

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Conflict of interest

Prof. Wenchang Chu is the Guest Editor of special issue "Combinatorial Analysis and Mathematical Constants" for AIMS Mathematics. Prof. Wenchang Chu was not involved in the editorial review and the decision to publish this article. The authors declare no conflicts of interest.

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