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Research article

Primitive decompositions of idempotents of the group algebras of dihedral groups and generalized quaternion groups

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Abstract: In this paper, we introduce a method for computing the primitive decomposition of idempotents in any semisimple finite group algebra, utilizing its matrix representations and Wedderburn decomposition. Particularly, we use this method to calculate the examples of the dihedral group algebras $\mathbb{C}[D_{2n}]$ and generalized quaternion group algebras $\mathbb{C}[Q_{4m}]$. Inspired by the orthogonality relations of the character tables of these two families of groups, we obtain two sets of trigonometric identities. Furthermore, a group algebra isomorphism between $\mathbb{C}[D_8]$ and $\mathbb{C}[Q_8]$ is described, under which the two complete sets of primitive orthogonal idempotents of these group algebras correspond bijectively.

Keywords: idempotent; primitive decomposition; group algebra
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1. Introduction

Given any finite group G and field F, denote as $F[G]$ the group ring of G over F. When char $F \nmid |G|$, $F[G]$ is semisimple by Maschke's theorem. Then, by Wedderburn's structure theorem, $F[G]$ is isomorphic to a direct sum of matrix algebras. The Wedderburn decomposition becomes a key tool for studying group algebra problems [1–5]. For example, Macedo Ferreira et al. dealt with the Wedderburn b-decomposition for alternative baric algebras [1]. Jespers et al. reduced the number of generators for a subgroup of finite indexes in a certain kind of unit group $\mathcal{U}(\mathbb{Z}[G])$ by having a closer look at the Wedderburn decomposition of $\mathbb{Q}[G]$ [3]. Olivieri et al. studied the automorphism group $\mathrm{Aut}(\mathbb{Q}[G])$ of the rational group algebra $\mathbb{Q}[G]$ of a finite metacyclic group G by describing the simple components of the Wedderburn decomposition of $\mathbb{Q}[G]$ [5].

As the main objects discussed throughout our paper, dihedral groups D_{2n} describe 2-dimensional objects that have rotational and reflective symmetry, such as regular polygons, and generalized quaternion groups Q_{4m} generalize the quaternion group Q_8. In physics, the theory of rigid motion analysis and the practical problem of motion control are all related to quaternions, and many applications in physics use the concept and extension of quaternions.

The Wedderburn decomposition of group algebras of these two families of groups has already attracted much attention. For instance, Giraldo Vergara and Brochero Martínez gave an elementary proof of the Wedderburn decomposition of rational quaternion and dihedral group algebras [6]. Giraldo Vergara used the classification of groups of order $\leq$ 32 and also computed the Wedderburn decomposition of their rational group algebras in order to classify the rational group algebras of dimension $\leq$ 32 [7]. Bakshi et al. calculated a complete set of primitive central idempotents and the Wedderburn decomposition of the rational group algebra of a finite metabelian group [8]. Brochero Martínez showed explicitly the primitive central idempotents of $F_q[D_{2n}]$ and an isomorphism between the group algebra $F_q[D_{2n}]$ and its Wedderburn decomposition when every prime factor of n divides $q-1$ [9]. Gao and Yue focused on the algebraic structure of the generalized quaternion group algebras $F_q[Q_{4m}]$ over finite field F_q [10].

Additionally, the study of primitive orthogonal idempotents of group algebras has ignited much interest. For many classes of groups, such as nilpotent, monomial, and supersolvable groups, a complete description of the idempotents of their group algebras was obtained by Berman (see e.g., [11]). For example, Berman, in 1995, constructed the minimal central idempotents of the group ring $R(G, F)$ in terms of the central idempotents of $R(H, F)$ when G is an abelian extension of a group H. Furthermore, the complete system of minimal idempotents of $R(G, F)$ was given in terms of such a system for $R(H, F)$ when G/H is cyclic [12]. After that, he characterized a complete system of primitive orthogonal idempotents of $F[G]$ for any solvable group G of class M_1 by calculating linear characters of its subgroups, where F is any field of characteristic prime to $|G|$ containing a primitive root of unity of $|G|$ [13].

After nearly 40 years, a method somewhat different but closely related to Berman's in calculating primitive orthogonal idempotents of these group algebras was proposed. In 2004, Olivieri et al. gave a character-free method to describe the primitive central idempotents of $\mathbb{Q}[G]$ when G is a monomial group [14]. Later, an explicit and character-free construction of a complete set of primitive orthogonal idempotents of $\mathbb{Q}[G]$ was provided in [15] for any finite nilpotent group G (see also [16] for the case over finite fields) and in [17] for any finite strongly monomial group G such that there exists a complete and non-redundant set of strong Shoda pairs with trivial twistings. See also [18, Chapter 13] for an overall introduction to this topic.

In this paper, after calculating the primitive central idempotents of $\mathbb{C}[D_{2n}]$ and $\mathbb{C}[Q_{4m}]$ via irreducible characters, we further consider their primitive decompositions of idempotents. Note that dihedral groups D_{2n} and generalized quaternion groups Q_{4m} are not only supersolvable groups but also strongly monomial groups. Their primitive decompositions of idempotents can certainly be obtained using Berman's method from [13]. Also, a complete set of primitive orthogonal idempotents for any dihedral group can be constructed via strong Shoda pairs, though this is questionable for all generalized quaternion groups [17, § 4]. In contrast, here, the computation of primitive decompositions of idempotents mainly depends on matrix representations of groups and Wedderburn decompositions of group algebras (Lemma 2.1). Such an approach is theoretically applicable to any semisimple group algebra over an arbitrary field whenever a complete set of its non-equivalent irreducible matrix representations have been obtained. In particular, it is directly available to examples of dihedral groups and generalized quaternion groups.

On the other hand, given two primitive decompositions of idempotents of two isomorphic group algebras, it seems challenging to obtain a specific algebra isomorphism between them that makes the two complete sets of primitive orthogonal idempotents correspond to each other. Here, we solve one small but nontrivial case by establishing an explicit isomorphism between $\mathbb{C}[D_8]$ and $\mathbb{C}[Q_8]$, which respects the list of primitive orthogonal idempotents we previously found. Indeed, there are plenty of results for the group algebras of D_8 and Q_8. For example, Bagiński studied group algebras of 2-groups of maximal class over fields of characteristic 2, showing that $F_2[D_8]$ and $F_2[Q_8]$ are not isomorphic as rings [19]. Coleman discussed group rings over the complex and real number fields and over the ring of integers in [20], where it was demonstrated that $\mathbb{C}[Q_8] \cong \mathbb{C}[D_8]$, but $\mathbb{R}[Q_8] \ncong \mathbb{R}[D_8]$ and $\mathbb{Z}[Q_8] \ncong \mathbb{Z}[D_8]$. As $\mathbb{R}$ is a field extension of $\mathbb{Q}$, it also implies that $\mathbb{Q}[Q_8] \ncong \mathbb{Q}[D_8]$. Tambara and Yamagami pointed out that Q_8 and D_8 have the same representation ring, but non-isomorphic representation categories as tensor categories [21].

Here is the layout of the paper. In Sections 2 and 3, the primitive central idempotents of dihedral groups and generalized quaternion groups are calculated by their irreducible characters. Furthermore, primitive decompositions of idempotents corresponding to their two-dimensional representations are analyzed. In Section 4, two sets of general trigonometric identities reflecting the orthogonality relations of irreducible characters of dihedral groups and generalized quaternion groups are given. In Section 5, a group algebra isomorphism between $\mathbb{C}[Q_8]$ and $\mathbb{C}[D_8]$ is described, which also provides a correspondence between their primitive orthogonal idempotents previously established.

2. A primitive decomposition of idempotents of $\mathbb{C}[D_{2n}]$

2.1. *Conjugacy classes of D_{2n}*

Let D_{2n} be the dihedral group of order $2n$, i.e.,

$$D_{2n} = \{r, s \,|\, r^n = s^2 = \mathbf{1},\ srs = r^{-1}\}.$$

When n is an odd number, namely $n = 2m + 1$, D_{2n} has the following conjugacy classes:

$$[\mathbf{1}] = \{\mathbf{1}\},\ [r^i] = \{r^{\pm i} \,|\, 1 \le i \le m\},\ [s] = \{s, rs, \ldots, r^{n-1}s\}.$$

When n is an even number, namely $n = 2m$, D_{2n} has the following conjugacy classes:

$$[\mathbf{1}] = \{\mathbf{1}\},\ [r^m] = \{r^m\},\ [r^i] = \{r^{\pm i} \,|\, 1 \le i \le m-1\},\ [s] = \{r^{2k}s \,|\, 0 \le k \le m-1\},\ [r^s] = \{r^{2k+1}s \,|\, 0 \le k \le m-1\}.$$

2.2. *Character table of D_{2n}*

(i) $n = 2m + 1$. We look at the one-dimensional representations first. Note that $D_{2n}/\langle r\rangle \cong \langle s\rangle$, which is abelian, hence the derived subgroup $D'_{2n} \subseteq \langle r\rangle$. Clearly, $s^{-1}r^{-1}sr = r^2 \in D'_{2n}$, thus we have $D'_{2n} \supseteq \langle r^2\rangle$. Note that $r^{2m} = r^{-1} \in \langle r^2\rangle$, therefore $\langle r^2\rangle = \langle r\rangle$. Then, $D'_{2n} = \langle r\rangle$. As a result, D_{2n} has two one-dimensional representations and $D_{2n}/\langle r\rangle \cong C_2$, where C_2 is the cyclic group of order 2.

Next, we introduce these two-dimensional irreducible representations of D_{2n} from its natural geometric description [22, Part I, 5.3]. We can set up a rectangular coordinate system, where the origin is the center of a regular n-sided polygon, and the angular bisector in the first and third quadrants is one of the symmetry axes of the regular n-sided polygon. Since D_{2n} is a permutation group of regular

n-sided polygons, the matrices of r, s with respect to the standard basis can be given. Then, we have the following natural representations:

$$\rho_k(r) = \begin{pmatrix} \cos\frac{2k\pi}{n} & -\sin\frac{2k\pi}{n} \\ \sin\frac{2k\pi}{n} & \cos\frac{2k\pi}{n} \end{pmatrix}, \quad \rho_k(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad 1 \le k \le m, \tag{2.1}$$

which are m mutually non-equivalent two-dimensional irreducible representations of D_{2n}. Thus, when n is an odd number, we set $\theta = \dfrac{2\pi}{n}$, and list the character table of D_{2n} (Table 1):

Table 1. Irreducible characters of D_{2n} ($n = 2m + 1$).

	$\mathbf{1}$	s	r	r^2	r^3	$\cdots$	r^{m-1}	r^m
	(1)	(n)	(2)	(2)	(2)	$\cdots$	(2)	(2)
χ_1	1	1	1	1	1	$\cdots$	1	1
χ_2	1	−1	1	1	1	$\cdots$	1	1
χ_{ρ_1}	2	0	$2\cos\theta$	$2\cos 2\theta$	$2\cos 3\theta$	$\cdots$	$2\cos(m-1)\theta$	$2\cos m\theta$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
χ_{ρ_m}	2	0	$2\cos m\theta$	$2\cos 2m\theta$	$2\cos 3m\theta$	$\cdots$	$2\cos(m-1)m\theta$	$2\cos m^2\theta$

(ii) $n = 2m$. Similarly, $\langle r^2\rangle$ is a normal subgroup of D_{2n} as $sr^2s^{-1} = r^{-2} \in \langle r^2\rangle$, and $|D_{2n}/\langle r^2\rangle| = 4$, then $D_{2n}/\langle r^2\rangle$ is abelian, and thus $D'_{2n} \subseteq \langle r^2\rangle$. Clearly, $r^2 = s^{-1}r^{-1}sr \in D'_{2n}$, we also have $D'_{2n} \supseteq \langle r^2\rangle$, so $D'_{2n} = \langle r^2\rangle$. As a result, D_{2n} has four one-dimensional representations and $D_{2n}/\langle r^2\rangle \cong C_2 \times C_2$.

If n is an even number, we can also obtain $m - 1$ pairwise non-equivalent two-dimensional irreducible representations of D_{2n}:

$$\rho_k(r) = \begin{pmatrix} \cos\frac{2k\pi}{n} & -\sin\frac{2k\pi}{n} \\ \sin\frac{2k\pi}{n} & \cos\frac{2k\pi}{n} \end{pmatrix}, \quad \rho_k(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad 1 \le k \le m-1. \tag{2.2}$$

Thus, when n is an even number, we set $\theta = \dfrac{2\pi}{n}$, and list the character table of D_{2n} (Table 2):

Table 2. Irreducible characters of D_{2n} ($n = 2m$).

	$\mathbf{1}$	s	sr	r	r^2	$\ldots$	r^{m-1}	r^m
	(1)	(m)	(m)	(2)	(2)	$\ldots$	(2)	(1)
χ_1	1	1	1	1	1	$\ldots$	1	1
χ_2	1	1	−1	−1	1	$\cdots$	$(-1)^{m-1}$	$(-1)^m$
χ_3	1	−1	1	−1	1	$\cdots$	$(-1)^{m-1}$	$(-1)^m$
χ_4	1	−1	−1	1	1	$\cdots$	1	1
χ_{ρ_1}	2	0	0	$2\cos\theta$	$2\cos 2\theta$	$\cdots$	$2\cos(m-1)\theta$	−2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\chi_{\rho_{m-1}}$	2	0	0	$2\cos(m-1)\theta$	$2\cos 2(m-1)\theta$	$\cdots$	$2\cos(m-1)^2\theta$	$2(-1)^{m-1}$

2.3. A primitive decomposition of idempotents

Theorem 2.1. *(Wedderburn structure theorem). Let F be any field such that* char $F \nmid |G|$*. Then*

$$F[G] \overset{\varphi}{\cong} M_{n_1}(D_1) \oplus \cdots \oplus M_{n_s}(D_s)$$

as algebras, where D_k is a division F-algebra, and each matrix algebra $M_{n_k}(D_k)$ uniquely determines an irreducible representation ρ_k of G up to equivalence, and n_k is equal to its dimension over D_k for $k = 1, \ldots, s$.

According to Theorem 2.1, we obtain the following useful lemma.

Lemma 2.1. *For any semisimple finite group algebra $F[G]$, let e_{ρ_k} be the primitive central idempotent of $F[G]$ corresponding to ρ_k. The group homomorphism $\rho_k : G \to GL(n_k, D_k)$ can be linearly extended to the following algebra homomorphism*

$$F[G] \overset{\varphi}{\cong} M_{n_1}(D_1) \oplus \cdots \oplus M_{n_s}(D_s) \overset{p_k}{\to} M_{n_k}(D_k),$$

which is an isomorphism when restricted on $F[G]e_{\rho_k}$. In particular, the preimages of the matrix units $E_{11}, \ldots, E_{n_k,n_k}$ of $M_{n_k}(D_k)$ under this isomorphism provide a primitive decomposition of e_{ρ_k} in $F[G]$. Here, we denote p_k the natural projection.

Also, it is well-known that all primitive central idempotents of the semisimple group algebra $F[G]$ of a finite group G can be obtained by its character table (see e.g., [23, Theorem 3.6.2]), namely

$$e_\chi = \frac{1}{|G|} \sum_{g \in G} \chi(\mathbf{1})\chi(g^{-1})g, \quad \forall \chi \in \mathrm{Irr}(G). \tag{2.3}$$

Applying Eq (2.3) to Tables 1 and 2 of dihedral group D_{2n}, we immediately have

Proposition 2.1. *Let D_{2n} be the dihedral group of order $2n$. The primitive central idempotents corresponding to the one-dimensional irreducible representations of D_{2n} are as follows.*

(i) When n is an odd number, namely $n = 2m + 1$,

$$e_1 = \frac{1}{4m+2}\Big(\sum_{l=1}^{2m+1} r^l + \sum_{l=1}^{2m+1} r^l s\Big),$$
$$e_2 = \frac{1}{4m+2}\Big(\sum_{l=1}^{2m+1} r^l - \sum_{l=1}^{2m+1} r^l s\Big).$$

(ii) When n is an even number, namely $n = 2m$,

$$e_1 = \frac{1}{4m}\Big(\sum_{l=1}^{2m} r^l + \sum_{l=1}^{2m} r^l s\Big),$$
$$e_2 = \frac{1}{4m}\Big[\mathbf{1} + \sum_{l=1}^{2m}(-1)^l \cdot r^l s + \sum_{l=1}^{m-1}(-1)^l \cdot (r^l + r^{-l}) + (-1)^m \cdot r^m\Big],$$

$$e_3 = \frac{1}{4m}[\mathbf{1} + \sum_{l=1}^{2m}(-1)^{l+1} \cdot r^l s + \sum_{l=1}^{m-1}(-1)^l \cdot (r^l + r^{-l}) + (-1)^m \cdot r^m],$$
$$e_4 = \frac{1}{4m}(\sum_{l=1}^{2m} r^l - \sum_{l=1}^{2m} r^l s).$$

In order to obtain a primitive decomposition of idempotents of $\mathbb{C}[D_{2n}]$, we mainly need to deal with its primitive idempotents corresponding to two-dimensional irreducible representations.

Theorem 2.2. *Let D_{2n} be the dihedral group of order $2n$. We have the following primitive decomposition $e_{\rho_k} = e'_{\rho_k} + e''_{\rho_k}$ of the primitive central idempotent e_{ρ_k} corresponding to the two-dimensional irreducible representation $(\mathbb{C}^2, \rho_k)$ of D_{2n} defined in Eqs (2.1) and (2.2) for $k = 1, \ldots, \lfloor (n-1)/2 \rfloor$.*

(i) When n is an odd number, namely $n = 2m+1$,

$$e_{\rho_k} = \frac{2}{2m+1}\sum_{l=1}^{2m+1} \cos lk\theta \cdot r^l,$$
$$e'_{\rho_k} = \frac{1}{2m+1}(\mathbf{1} + \sum_{l=1}^{2m} \cos lk\theta \cdot r^l + \sum_{l=1}^{2m} \sin lk\theta \cdot r^l s),$$
$$e''_{\rho_k} = \frac{1}{2m+1}(\mathbf{1} + \sum_{l=1}^{2m} \cos lk\theta \cdot r^l - \sum_{l=1}^{2m} \sin lk\theta \cdot r^l s),$$

with $\theta = \dfrac{2\pi}{n}$ and $1 \le k \le m$;

(ii) When n is an even number, namely $n = 2m$,

$$e_{\rho_k} = \frac{1}{m}\sum_{l=1}^{2m} \cos lk\theta \cdot r^l,$$
$$e'_{\rho_k} = \frac{1}{2m}(\mathbf{1} + \sum_{l=1}^{2m-1} \cos lk\theta \cdot r^l + \sum_{l=1}^{2m-1} \sin lk\theta \cdot r^l s),$$
$$e''_{\rho_k} = \frac{1}{2m}(\mathbf{1} + \sum_{l=1}^{2m-1} \cos lk\theta \cdot r^l - \sum_{l=1}^{2m-1} \sin lk\theta \cdot r^l s),$$

with $\theta = \dfrac{2\pi}{n}$ and $1 \le k \le m-1$.

Proof. Under the group homomorphism $\rho_k : D_{2n} \to GL(2, \mathbb{C}), k = 1, \ldots, \lfloor (n-1)/2 \rfloor$, we have

$$r \mapsto \begin{pmatrix} \cos \frac{2k\pi}{n} & -\sin \frac{2k\pi}{n} \\ \sin \frac{2k\pi}{n} & \cos \frac{2k\pi}{n} \end{pmatrix}, \quad s \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{1} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore,

$$rs \mapsto \begin{pmatrix} -\sin \frac{2k\pi}{n} & \cos \frac{2k\pi}{n} \\ \cos \frac{2k\pi}{n} & \sin \frac{2k\pi}{n} \end{pmatrix}, \quad \cos \frac{2k\pi}{n} s - rs \mapsto \begin{pmatrix} \sin \frac{2k\pi}{n} & 0 \\ 0 & -\sin \frac{2k\pi}{n} \end{pmatrix}.$$

Thus,

$$\sin\frac{2k\pi}{n}\mathbf{1}-(\cos\frac{2k\pi}{n}s-rs)\mapsto\begin{pmatrix}0 & 0\\ 0 & 2\sin\frac{2k\pi}{n}\end{pmatrix},$$

$$\sin\frac{2k\pi}{n}\mathbf{1}+(\cos\frac{2k\pi}{n}s-rs)\mapsto\begin{pmatrix}2\sin\frac{2k\pi}{n} & 0\\ 0 & 0\end{pmatrix}.$$

Clearly, $0<\dfrac{2k\pi}{n}<\pi$, we have

$$\frac{1}{2\sin\frac{2k\pi}{n}}(\sin\frac{2k\pi}{n}\mathbf{1}-\cos\frac{2k\pi}{n}s+rs)\mapsto\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix},$$

$$\frac{1}{2\sin\frac{2k\pi}{n}}(\sin\frac{2k\pi}{n}\mathbf{1}+\cos\frac{2k\pi}{n}s-rs)\mapsto\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}.$$

By Lemma 2.1, we know that $F[G]e_{\rho_k}\cong M_{n_k}(F)$ as algebras, and thus

$$\begin{aligned}e'_{\rho_k}&=e_{\rho_k}\cdot\frac{1}{2\sin\frac{2k\pi}{n}}(\sin\frac{2k\pi}{n}\mathbf{1}-\cos\frac{2k\pi}{n}\cdot s+rs)\\&=\frac{1}{2}e_{\rho_k}\cdot(\mathbf{1}-\cot k\theta\cdot s+\csc k\theta\cdot rs),\\e''_{\rho_k}&=\frac{1}{2}e_{\rho_k}\cdot(\mathbf{1}+\cot k\theta\cdot s-\csc k\theta\cdot rs).\end{aligned}$$

We can verify that

$$e_{\rho_k}=e'_{\rho_k}+e''_{\rho_k},\quad e'_{\rho_k}\cdot e''_{\rho_k}=0,\quad e'_{\rho_k}\cdot e'_{\rho_k}=e'_{\rho_k},\quad e''_{\rho_k}\cdot e''_{\rho_k}=e''_{\rho_k}.$$

(i) If $n=2m+1$, the primitive central idempotents e_{ρ_k} are given as follows by Eq (2.3) and the character table of D_{2n}:

$$e_{\rho_k}=\frac{2}{2m+1}\sum_{l=1}^{2m+1}\cos lk\theta\cdot r^l,\quad 1\le k\le m.$$

Thus,

$$e'_{\rho_k}=\frac{1}{2m+1}(\mathbf{1}+\sum_{l=1}^{2m}\cos lk\theta\cdot r^l+\sum_{l=1}^{2m}\sin lk\theta\cdot r^ls),\quad 1\le k\le m.$$

Similarly,

$$e''_{\rho_k}=\frac{1}{2m+1}(\mathbf{1}+\sum_{l=1}^{2m}\cos lk\theta\cdot r^l-\sum_{l=1}^{2m}\sin lk\theta\cdot r^ls),\quad 1\le k\le m.$$

(ii) If $n=2m$, the primitive central idempotents of D_{2n} are given by

$$e_{\rho_k}=\frac{1}{m}\sum_{l=1}^{2m}\cos lk\theta\cdot r^l,\quad 1\le k\le m-1.$$

Therefore,

$$e'_{\rho_k} = \frac{1}{2m}(\mathbf{1} + \sum_{l=1}^{2m-1} \cos lk\theta \cdot r^l + \sum_{l=1}^{2m-1} \sin lk\theta \cdot r^l s), \quad 1 \le k \le m-1.$$

Similarly,

$$e''_{\rho_k} = \frac{1}{2m}(\mathbf{1} + \sum_{l=1}^{2m-1} \cos lk\theta \cdot r^l - \sum_{l=1}^{2m-1} \sin lk\theta \cdot r^l s), \quad 1 \le k \le m-1.$$

□

Example 2.1. *Let D_8 be a dihedral group with order 8. Then, $m = 2$, $k = 1, n = 4$, there is a primitive decomposition of idempotents as follows.*

$$\begin{aligned} e_{\rho_1} &= \frac{1}{2}(\mathbf{1} - r^2), \\ e'_{\rho_1} &= \frac{1}{4}(\mathbf{1} - r^2 + rs - r^3 s), \\ e''_{\rho_1} &= \frac{1}{4}(\mathbf{1} - r^2 - rs + r^3 s). \end{aligned}$$

3. A primitive decomposition of idempotents of $\mathbb{C}[Q_{4m}]$

3.1. Conjugacy classes of Q_{4m}

Let Q_{4m} be the generalized quaternion group of order $4m$, i.e.,

$$Q_{4m} = \{a, b \,|\, a^{2m} = \mathbf{1},\ a^m = b^2,\ b^{-1}ab = a^{-1}\}.$$

Q_{4m} has the following conjugacy classes:

$$[\mathbf{1}] = \{\mathbf{1}\},\ [a^m] = \{a^m\},\ [a^r] = \{a^{\pm r} \,|\, 1 \le r \le m-1\},\ [b] = \{a^{2k}b \,|\, 0 \le k \le m-1\},\ [ab] = \{a^{2k-1}b \,|\, 0 \le k \le m-1\}.$$

3.2. Character table of Q_{4m}

The derived subgroup $Q'_{4m} = \langle a^2 \rangle$. Indeed, $\langle a^2 \rangle$ is a normal subgroup of Q_{4m}, and $|Q_{4m}/\langle a^2 \rangle| = 4$, hence $Q_{4m}/\langle a^2 \rangle$ is abelian and $\langle a^2 \rangle \supseteq Q'_{4m}$. Clearly, $b^{-1}a^{-1}b = a$, thus $b^{-1}a^{-1}ba = a^2 \in Q'_{4m}$, as $\langle a^2 \rangle \subseteq Q'_{4m}$.

As $|Q_{4m}/\langle a^2 \rangle| = 4$, $Q_{4m}/\langle a^2 \rangle \cong C_4$ or $Q_{4m}/\langle a^2 \rangle \cong C_2 \times C_2$, and Q_{4m} has four irreducible one-dimensional representations. Also, it has $m-1$ mutually non-equivalent two-dimensional irreducible representations [24, Exs. 17.6, 18.3, 23.5]. We recall these two-dimensional irreducible representations of Q_{4m} as follows.

Let $\varepsilon := e^{\pi i/m} \in \mathbb{C}$ with $i := \sqrt{-1}$. For each k with $1 \le k \le m-1$, denote matrices

$$A_k = \begin{pmatrix} \varepsilon^k & 0 \\ 0 & \varepsilon^{-k} \end{pmatrix}, \quad B_k = \begin{pmatrix} 0 & 1 \\ (-1)^k & 0 \end{pmatrix},$$

which satisfy the following relations:

$$A_k^{2m} = I, \quad A_k^m = B_k^2, \quad B_k^{-1}A_kB_k = A_k^{-1}.$$

Hence, it follows that

$$\rho_k : Q_{4m} \to GL(2, \mathbb{C}) \tag{3.1}$$

defined by

$$a \mapsto A_k, \quad b \mapsto B_k$$

is a group homomorphism, and we obtain a representation $(\mathbb{C}^2, \rho_k)$ of Q_{4m}.

(i) When m is an odd number, as $2 \nmid m$, we know that $b^2 = a^m \notin Q'_{4m}$, hence the order of b cannot be 2. Then, b is of order 4, so $Q_{4m}/\langle a^2 \rangle \cong C_4$. We set $\vartheta = \dfrac{\pi}{m}$, and list the character table of Q_{4m} (Table 3):

Table 3. Irreducible characters of Q_{4m} $(2 \nmid m)$.

	$\mathbf{1}$	a	a^2	$\cdots$	a^{m-1}	a^m	b	ab
	(1)	(2)	(2)	$\cdots$	(2)	(1)	(m)	(m)
χ_1	1	1	1	$\cdots$	1	1	1	1
χ_2	1	1	1	$\cdots$	1	1	-1	-1
χ_3	1	-1	1	$\cdots$	$(-1)^{m-1}$	-1	i	$-i$
χ_4	1	-1	1	$\cdots$	$(-1)^{m-1}$	-1	$-i$	i
χ_{ρ_1}	2	$2\cos\vartheta$	$2\cos 2\vartheta$	$\cdots$	$2\cos(m-1)\vartheta$	-2	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\chi_{\rho_{m-1}}$	2	$2\cos(m-1)\vartheta$	$2\cos 2(m-1)\vartheta$	$\cdots$	$2\cos(m-1)^2\vartheta$	$2(-1)^{m-1}$	0	0

(ii) When m is an even number, as $2 \mid m$, we have $b^2 = a^m \in Q'_{4m}$. Therefore, $Q_{4m}/\langle a^2 \rangle \cong C_2 \times C_2$. We set $\vartheta = \dfrac{\pi}{m}$, and list the character table of Q_{4m} (Table 4):

Table 4. Irreducible characters of Q_{4m} $(2 \mid m)$.

	$\mathbf{1}$	a	a^2	$\cdots$	a^{m-1}	a^m	b	ab
	(1)	(2)	(2)	$\cdots$	(2)	(1)	(m)	(m)
χ_1	1	1	1	$\cdots$	1	1	1	1
χ_2	1	1	1	$\cdots$	1	1	-1	-1
χ_3	1	-1	1	$\cdots$	$(-1)^{m-1}$	1	1	-1
χ_4	1	-1	1	$\cdots$	$(-1)^{m-1}$	1	-1	1
χ_{ρ_1}	2	$2\cos\vartheta$	$2\cos 2\vartheta$	$\cdots$	$2\cos(m-1)\vartheta$	-2	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\chi_{\rho_{m-1}}$	2	$2\cos(m-1)\vartheta$	$2\cos 2(m-1)\vartheta$	$\cdots$	$2\cos(m-1)^2\vartheta$	$2(-1)^{m-1}$	0	0

3.3. A primitive decomposition of idempotents

First applying Eq (2.3) to Tables 3 and 4 of generalized quaternion group Q_{4m}, we have

Proposition 3.1. *Let Q_{4m} be the generalized quaternion group of order $4m$. The primitive central idempotents corresponding to the one-dimensional irreducible representations of Q_{4m} are as follows.*

(i) *When m is an odd number,*

$$e_1 = \frac{1}{4m}(\sum_{l=1}^{2m} a^l + \sum_{l=1}^{2m} a^l b),$$
$$e_2 = \frac{1}{4m}(\sum_{l=1}^{2m} a^l - \sum_{l=1}^{2m} a^l b),$$
$$e_3 = \frac{1}{4m}[\mathbf{1} + i\sum_{l=1}^{2m}(-1)^l \cdot a^l b + \sum_{l=1}^{m-1}(-1)^l \cdot (a^l + a^{-l}) - a^m],$$
$$e_4 = \frac{1}{4m}[\mathbf{1} + i\sum_{l=1}^{2m}(-1)^{l+1} \cdot a^l b + \sum_{l=1}^{m-1}(-1)^l \cdot (a^l + a^{-l}) - a^m].$$

(ii) *When m is an even number,*

$$e_1 = \frac{1}{4m}(\sum_{l=1}^{2m} a^l + \sum_{l=1}^{2m} a^l b),$$
$$e_2 = \frac{1}{4m}(\sum_{l=1}^{2m} a^l - \sum_{l=1}^{2m} a^l b),$$
$$e_3 = \frac{1}{4m}[\mathbf{1} + \sum_{l=1}^{2m}(-1)^l \cdot a^l b + \sum_{l=1}^{m-1}(-1)^l \cdot (a^l + a^{-l}) + a^m],$$
$$e_4 = \frac{1}{4m}[\mathbf{1} + \sum_{l=1}^{2m}(-1)^{l+1} \cdot a^l b + \sum_{l=1}^{m-1}(-1)^l \cdot (a^l + a^{-l}) + a^m].$$

For other primitive idempotents corresponding to two-dimensional irreducible representations of Q_{4m}, we have

Theorem 3.1. *Let Q_{4m} be the generalized quaternion group of order $4m$. Then, we have the following primitive decomposition $e_{\rho_k} = e'_{\rho_k} + e''_{\rho_k}$ of the primitive central idempotent e_{ρ_k} corresponding to the two-dimensional irreducible representation $(\mathbb{C}^2, \rho_k)$ of Q_{4m} defined in Eq (3.1) for $k = 1, \ldots, m-1$.*

(i) *When k is an odd number,*

$$e_{\rho_k} = \frac{1}{m}\sum_{l=1}^{2m} \cos lk\vartheta \cdot a^l,$$
$$e'_{\rho_k} = -\frac{1}{2mi\sin k\vartheta}\sum_{l=1}^{2m}(\varepsilon^k a^{m+l} - a^{m+l-1})\cos lk\vartheta,$$
$$e''_{\rho_k} = -\frac{1}{2mi\sin k\vartheta}\sum_{l=1}^{2m}(a^{m+l-1} - \varepsilon^{-k} a^{m+l})\cos lk\vartheta,$$

with $\vartheta = \dfrac{\pi}{m}$ *and* $1 \le k \le m-1$*;*

(ii) When k is an even number,

$$\begin{aligned} e_{\rho_k} &= \frac{1}{m}\sum_{l=1}^{2m}\cos lk\vartheta \cdot a^l, \\ e'_{\rho_k} &= \frac{1}{2mi\sin k\vartheta}\sum_{l=1}^{2m}(\varepsilon^k a^{m+l} - a^{m+l-1})\cos lk\vartheta, \\ e''_{\rho_k} &= \frac{1}{2mi\sin k\vartheta}\sum_{l=1}^{2m}(a^{m+l-1} - \varepsilon^{-k} a^{m+l})\cos lk\vartheta, \end{aligned}$$

with $\vartheta = \dfrac{\pi}{m}$ *and* $1 \le k \le m-1$.

Proof. (i) When k is an odd number, under the group homomorphism $\rho_k : Q_{4m} \to GL(2, \mathbb{C})$, we have

$$a \mapsto \begin{pmatrix} \varepsilon^k & 0 \\ 0 & \varepsilon^{-k} \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then,

$$ab \mapsto \begin{pmatrix} 0 & \varepsilon^k \\ -\varepsilon^{-k} & 0 \end{pmatrix}, \quad \varepsilon^k b \mapsto \begin{pmatrix} 0 & \varepsilon^k \\ -\varepsilon^k & 0 \end{pmatrix}, \quad \varepsilon^{-k} b \mapsto \begin{pmatrix} 0 & \varepsilon^{-k} \\ -\varepsilon^{-k} & 0 \end{pmatrix}.$$

Therefore,

$$\varepsilon^k b - ab \mapsto \begin{pmatrix} 0 & 0 \\ \varepsilon^{-k} - \varepsilon^k & 0 \end{pmatrix}, \quad \varepsilon^{-k} b - ab \mapsto \begin{pmatrix} 0 & \varepsilon^{-k} - \varepsilon^k \\ 0 & 0 \end{pmatrix}.$$

As $\varepsilon^{-k} - \varepsilon^k \neq 0$, it implies that

$$\begin{aligned} \frac{1}{\varepsilon^{-k} - \varepsilon^k}(\varepsilon^k b - ab) &\mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \\ \frac{1}{\varepsilon^{-k} - \varepsilon^k}(\varepsilon^{-k} b - ab) &\mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

As a result, we have

$$\begin{aligned} \frac{b}{\varepsilon^{-k} - \varepsilon^k}(\varepsilon^k b - ab) &\mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \\ -\frac{b}{\varepsilon^{-k} - \varepsilon^k}(\varepsilon^{-k} b - ab) &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Now, one can compute the primitive central idempotents e_{ρ_k} via the character table, and then get their desired primitive decompositions by the similar argument as in the proof of Theorem 2.2.

(ii) When k is an even number, by similar arguments as in the case when k is an odd number. □

Example 3.1. *Let* Q_8 *be a generalized quaternion group of order* 8, *then* $k = 1$, $m = 2$. *Therefore,*

$$\begin{aligned} e_{\rho_1} &= \frac{1}{2}(\mathbf{1} - a^2), \\ e'_{\rho_1} &= -\frac{1}{4i}(a^3 + i \cdot a^2 - a - i \cdot \mathbf{1}), \\ e''_{\rho_1} &= -\frac{1}{4i}(-a^3 + i \cdot a^2 + a - i \cdot \mathbf{1}). \end{aligned}$$

4. Trigonometric identities

Here, we find the following two sets of trigonometric identities covering the orthogonality relations in the character tables of dihedral groups and generalized quaternion groups.

Proposition 4.1. *For any $n \geq 1$ and $1 \leq k \leq n-1$, and any angle θ that is not an integer multiple of 2π, we have*

(i) $\displaystyle\sum_{r=0}^{n-1}(-1)^r \cos\frac{rk\pi}{n} = \begin{cases} 1, & n+k \text{ odd}, \\ 0, & n+k \text{ even}; \end{cases}$

(ii) $\displaystyle\sum_{r=1}^{n}\cos r\theta = \frac{\sin(\frac{\theta}{2}+n\theta)}{2\sin\frac{\theta}{2}} - \frac{1}{2}.$

Proof. (i) Note that

$$\cos(\frac{rk\pi}{n} \pm \frac{k\pi}{2n}) = \cos\frac{rk\pi}{n}\cos\frac{k\pi}{2n} \mp \sin\frac{rk\pi}{n}\sin\frac{k\pi}{2n}$$

imply the following product-to-sum identity

$$\cos\frac{rk\pi}{n}\cos\frac{k\pi}{2n} = \frac{1}{2}[\cos\frac{(2r+1)k\pi}{2n} + \cos\frac{(2r-1)k\pi}{2n}].$$

As a result, we have

$$\begin{aligned}\sum_{r=0}^{n-1}(-1)^r\cos\frac{rk\pi}{n}\cos\frac{k\pi}{2n} &= \cos\frac{k\pi}{2n} + \sum_{r=1}^{n-1}(-1)^r\cdot\frac{1}{2}[\cos\frac{(2r+1)k\pi}{2n} + \cos\frac{(2r-1)k\pi}{2n}] \\ &= \cos\frac{k\pi}{2n} + \frac{1}{2}\sum_{r=1}^{n-1}(-1)^r\cos\frac{(2r+1)k\pi}{2n} + \frac{1}{2}\sum_{r=0}^{n-2}(-1)^{r+1}\cos\frac{(2r+1)k\pi}{2n} \\ &= \cos\frac{k\pi}{2n} + \frac{1}{2}(-1)^{n-1}\cos\frac{(2n-1)k\pi}{2n} - \frac{1}{2}\cos\frac{k\pi}{2n} \\ &= \frac{1}{2}\cos\frac{k\pi}{2n} + \frac{1}{2}(-1)^{n-1}\cos(k\pi - \frac{k\pi}{2n}) \\ &= \frac{1}{2}[1+(-1)^{n+k-1}]\cos\frac{k\pi}{2n}.\end{aligned}$$

Since $\cos\frac{k\pi}{2n} \neq 0$ for any $1 \leq k \leq n-1$, we see that

$$\sum_{r=0}^{n-1}(-1)^r\cos\frac{rk\pi}{n} = \frac{1}{2}[1+(-1)^{n+k-1}] = \begin{cases} 1, & n+k \text{ odd}, \\ 0, & n+k \text{ even}. \end{cases}$$

(ii) Similarly by product-to-sum identities, we see that

$$\begin{aligned}2\sin\frac{\theta}{2}\sum_{r=1}^{n}\cos r\theta &= 2\sin\frac{\theta}{2}\cos\theta + \cdots + 2\sin\frac{\theta}{2}\cos n\theta \\ &= \sin\frac{3\theta}{2} - \sin\frac{\theta}{2} + \cdots + \sin(n\theta+\frac{\theta}{2}) - \sin(n\theta-\frac{\theta}{2})\end{aligned}$$

$$= \sin(n\theta + \frac{\theta}{2}) - \sin\frac{\theta}{2}.$$

Since θ is not an integer multiple of 2π, we obtain that

$$\sum_{r=1}^{n} \cos r\theta = \frac{\sin(\frac{\theta}{2} + n\theta)}{2\sin\frac{\theta}{2}} - \frac{1}{2}.$$

□

Next, we clarify how these identities are connected to the character tables of dihedral groups and generalized quaternion groups.

Example 4.1. *Using the first orthogonality relation in the character tables of D_{2n} in Table 1 when $n = 2m+1$ and $\theta = \frac{2\pi}{2m+1}$, we have*

$$\langle \chi_1, \chi_{\rho_k} \rangle = \frac{1}{4m+2}[2 + 4\sum_{r=1}^{m} \cos kr\theta] = 0, \quad 1 \le k \le m.$$

The resulting identities

$$\sum_{r=1}^{m} \cos kr\theta = -\frac{1}{2}, \quad 1 \le k \le m,$$

and the identities due to $\langle \chi_1, \chi_{\rho_k} \rangle = 0$ in Table 2 are all special cases of Prop. 4.1 (ii). Additionally,

$$\langle \chi_{\rho_a}, \chi_{\rho_b} \rangle = \frac{1}{4m+2}[4 + 8\sum_{r=1}^{m} \cos ar\theta \cos br\theta] = 0, \quad 1 \le a, b \le m,\ a \ne b.$$

That is,

$$\sum_{r=1}^{m} \cos ar\theta \cos br\theta = -\frac{1}{2},$$

which can also be deduced by Prop. 4.1 (ii).

Example 4.2. *Using the first orthogonality relation in the character tables of Q_{4m} in Tables 3 and 4, when m is an odd number,*

$$\langle \chi_3, \chi_{\rho_k} \rangle = \frac{1}{4m}[2 + 4\sum_{r=1}^{m-1}(-1)^r \cos\frac{kr\pi}{m} + 2(-1)^{k+1}] = 0, \quad 1 \le k \le m-1.$$

When m is an even number,

$$\langle \chi_3, \chi_{\rho_k} \rangle = \frac{1}{4m}[2 + 4\sum_{r=1}^{m-1}(-1)^r \cos\frac{kr\pi}{m} + 2(-1)^{k}] = 0, \quad 1 \le k \le m-1.$$

That means

$$\sum_{r=1}^{m-1}(-1)^r \cos\frac{kr\pi}{m} = \begin{cases} 0, & m+k \text{ odd}, \\ -1, & m+k \text{ even}, \end{cases}$$

equivalent to Prop 4.1 (i). The identities by $\langle \chi_3, \chi_{\rho_k} \rangle = 0$ *in Table 2 are the same. Also, we have*

$$\langle \chi_{\rho_a}, \chi_{\rho_b} \rangle = \frac{1}{4m}[4 + 8\sum_{r=1}^{m-1} \cos\frac{ar\pi}{m}\cos\frac{br\pi}{m} + 4(-1)^{a+b}] = 0, \quad 1 \leq a, b \leq m-1,\ a \neq b.$$

That is,

$$\sum_{r=1}^{m-1} \cos\frac{ar\pi}{m}\cos\frac{br\pi}{m} = \begin{cases} 0, & a+b \text{ odd}, \\ -1, & a+b \text{ even}, \end{cases}$$

which can also be deduced by Prop. 4.1 (ii).

5. A group algebra isomorphism between $\mathbb{C}[Q_8]$ and $\mathbb{C}[D_8]$

In this section, we would like to specifically describe a group algebra isomorphism between $\mathbb{C}[Q_8]$ and $\mathbb{C}[D_8]$, offering a correspondence between two complete sets of their primitive orthogonal idempotents given in Prop. 2.1, Theorem 2.2 and Prop. 3.1, Theorem 3.1.

Theorem 5.1. *There is an algebra isomorphism*

$$\psi : \mathbb{C}[Q_8] \to \mathbb{C}[D_8]$$

mapping any $\alpha = x_0 \cdot \mathbf{1} + x_1 \cdot a^2 + x_2 \cdot a + x_3 \cdot a^3 + x_4 \cdot b + x_5 \cdot a^2 b + x_6 \cdot ab + x_7 \cdot a^3 b$ *to*

$$\psi(\alpha) = x_0 \cdot \mathbf{1} + x_7 \cdot r + x_1 \cdot r^2 + x_6 \cdot r^3 + \frac{1}{2}(x_2 + x_3 - ix_4 + ix_5) \cdot s + \frac{1}{2}(-ix_2 + ix_3 + x_4 + x_5) \cdot rs$$
$$+ \frac{1}{2}(x_2 + x_3 + ix_4 - ix_5) \cdot r^2 s + \frac{1}{2}(ix_2 - ix_3 + x_4 + x_5) \cdot r^3 s,$$

with $i := \sqrt{-1}$ *and* $x_i \in \mathbb{C}$.

Proof. We note that the generalized quaternion group Q_{4m} and the dihedral group D_{2n} have the same character table when $n = 2m$ and $2 \mid m$. In particular, the smallest case Q_8 and D_8 have the same values in the first column, and consequently $\mathbb{C}[Q_8] \cong \mathbb{C}[D_8]$ as algebras by Lemma 2.1.

The primitive central idempotents corresponding to the two-dimensional irreducible representations of $\mathbb{C}[Q_8]$ and $\mathbb{C}[D_8]$ are

$$\frac{1}{2}(\mathbf{1} - a^2), \quad \frac{1}{2}(\mathbf{1} - r^2).$$

Under any algebra isomorphism from $\mathbb{C}[Q_8]$ to $\mathbb{C}[D_8]$, we must have

$$\mathbf{1} \mapsto \mathbf{1}, \quad a^2 \mapsto r^2.$$

On the other hand, by Prop. 3.1, all primitive central idempotents corresponding to the four one-dimensional representations of $\mathbb{C}[Q_8]$ are as follows:

$$e_1 = \frac{1}{8}(\mathbf{1} + a^2 + a + a^3 + b + a^2 b + ab + a^3 b),$$

$$e_2 = \frac{1}{8}(\mathbf{1} + a^2 + a + a^3 - b - a^2b - ab - a^3b),$$

$$e_3 = \frac{1}{8}(\mathbf{1} + a^2 - a - a^3 + b + a^2b - ab - a^3b),$$

$$e_4 = \frac{1}{8}(\mathbf{1} + a^2 - a - a^3 - b - a^2b + ab + a^3b).$$

That is,

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} \mathbf{1} + a^2 \\ a + a^3 \\ b + a^2b \\ ab + a^3b \end{pmatrix}.$$

By Prop. 2.1, all primitive central idempotents corresponding to the four one-dimensional representations of $\mathbb{C}[D_8]$ are as follows:

$$e_1 = \frac{1}{8}(\mathbf{1} + r + r^2 + r^3 + s + rs + r^2s + r^3s),$$

$$e_2 = \frac{1}{8}(\mathbf{1} - r + r^2 - r^3 + s - rs + r^2s - r^3s),$$

$$e_3 = \frac{1}{8}(\mathbf{1} - r + r^2 - r^3 - s + rs - r^2s + r^3s),$$

$$e_4 = \frac{1}{8}(\mathbf{1} + r + r^2 + r^3 - s - rs - r^2s - r^3s).$$

Namely,

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} \mathbf{1} + r^2 \\ s + r^2s \\ rs + r^3s \\ r + r^3 \end{pmatrix}.$$

Therefore, we can assume that our desired algebra isomorphism $\psi : \mathbb{C}[Q_8] \to \mathbb{C}[D_8]$ satisfies

$$\begin{aligned} ab + a^3b &\mapsto r + r^3, \\ a + a^3 &\mapsto s + r^2s, \\ b + a^2b &\mapsto rs + r^3s. \end{aligned}$$

Furthermore, since $\psi(\xi_1\xi_2) = \psi(\xi_1)\psi(\xi_2)$ for any $\xi_1, \xi_2 \in \mathbb{C}[Q_8]$, the map ψ also satisfies:

$$\psi(abab) = \psi(b^2) = \psi(a^2) = r^2 = \psi(ab)^2,$$
$$\psi(ab + a^3b) = \psi(ab)(\mathbf{1} + r^2) = r + r^3 = r(\mathbf{1} + r^2).$$

That is,

$$\begin{cases} \psi(ab)^2 = r^2, \\ \psi(ab) - r \in (\mathbf{1} - r^2), \end{cases}$$

as the principal ideal $(\mathbf{1}-r^2)$ is the annihilator of $\mathbf{1}+r^2$. Similarly,

$$\begin{cases}\psi(a)^2 = r^2,\\ \psi(a) - s \in (\mathbf{1}-r^2),\end{cases} \qquad \begin{cases}\psi(b)^2 = r^2,\\ \psi(b) - rs \in (\mathbf{1}-r^2).\end{cases}$$

Therefore, we can set

$$\begin{cases}\psi(ab) = r + (k_1\cdot\mathbf{1} + k_2\cdot r + k_3\cdot rs + k_4\cdot s)(\mathbf{1}-r^2),\\ \psi(a) = s + (k_9\cdot\mathbf{1} + k_{10}\cdot r + k_{11}\cdot rs + k_{12}\cdot s)(\mathbf{1}-r^2),\\ \psi(b) = rs + (k_5\cdot\mathbf{1} + k_6\cdot r + k_7\cdot rs + k_8\cdot s)(\mathbf{1}-r^2),\end{cases}$$

with $k_1,\dots,k_{12}\in\mathbb{C}$, and obtain the following system of equations,

$$\begin{cases}k_1k_3 = 0, k_1k_4 = 0,\\ 4k_1k_2 + 2k_1 = 0,\\ 2k_1^2 + 2k_3^2 + 2k_4^2 - 2k_2^2 - 2k_2 = 0;\\ k_5k_6 = 0, k_5k_8 = 0,\\ 4k_5k_7 + 2k_5 = 0,\\ 2k_5^2 + 2k_7^2 + 2k_8^2 - 2k_6^2 + 1 + 2k_7 = 0;\\ k_9k_{10} = 0, k_9k_{11} = 0,\\ 4k_9k_{12} + 2k_9 = 0,\\ 2k_9^2 + 2k_{11}^2 + 2k_{12}^2 - 2k_{10}^2 + 1 + 2x_{12} = 0.\end{cases}$$

Note that there is more than one solution for this system of equations, and any one of these solutions must also satisfy:

$$\begin{cases}\psi(a)\psi(b) = \psi(ab),\\ \psi(b)\psi(ab) = \psi(a),\\ \psi(ab)\psi(a) = \psi(b).\end{cases}$$

However, these three additional equalities fail to hold simultaneously for any solution in which k_1, k_5, and k_9 are not all zero. Instead, we find the solution below satisfying all these equations:

$$k_1 = k_3 = k_4 = 0, \quad k_2 = -1, \quad k_5 = k_6 = 0,$$
$$k_7 = -\frac{1}{2}, \quad k_8 = -\frac{i}{2}, \quad k_9 = k_{10} = 0, \quad k_{11} = -\frac{i}{2}, \quad k_{12} = -\frac{1}{2}.$$

That is,

$$\begin{aligned} ab &\mapsto r^3,\\ a &\mapsto \frac{1}{2}(r^2s + s - i\cdot rs + i\cdot r^3s),\\ b &\mapsto \frac{1}{2}(rs + r^3s - i\cdot s + i\cdot r^2s).\end{aligned}$$

Then

$$\begin{aligned} a^3b &\mapsto r, \\ a^2b &\mapsto \frac{1}{2}(rs + r^3s + i\cdot s - i\cdot r^2s), \\ a^3 &\mapsto \frac{1}{2}(r^2s + s + i\cdot rs - i\cdot r^3s). \end{aligned}$$

Now, we specifically verify that the stated linear map $\psi : \mathbb{C}[Q_8] \to \mathbb{C}[D_8]$ is an algebra isomorphism as desired:

$$\begin{aligned} \psi(a)^2 &= \frac{1}{4}(r^2s + s - irs + ir^3s)^2 \\ &= \frac{1}{4}(\mathbf{1} + r^2 - ir + ir^3 + r^2 + \mathbf{1} - ir^3 + ir - ir^3 - ir - \mathbf{1} + r^2 + ir + ir^3 + r^2 - \mathbf{1}) \\ &= r^2 = \psi(a^2); \\ \psi(a)^3 &= \psi(a^2)\psi(a) = r^2\cdot\frac{1}{2}(r^2s + s - irs + ir^3s) = \frac{1}{2}(r^2s + s + irs - ir^3s) = \psi(a^3); \\ \psi(a)^4 &= \psi(a^2)^2 = (r^2)^2 = r^4 = \mathbf{1}; \\ \psi(a)\psi(b) &= \frac{1}{4}(r^2s + s - irs + ir^3s)(rs + r^3s - is + ir^2s) \\ &= \frac{1}{4}(r + r^3 - ir^2 + i\mathbf{1} + r^3 + r - i\mathbf{1} + ir^2 - i\mathbf{1} - ir^2 - r + r^3 + ir^2 + i\mathbf{1} + r^3 - r) \\ &= r^3 = \psi(ab); \\ \psi(a)^2\psi(b) &= \psi(a^2)\psi(b) = r^2\cdot\frac{1}{2}(rs + r^3s - is + ir^2s) = \frac{1}{2}(r^3s + rs - ir^2s + is) = \psi(a^2b); \\ \psi(a)^3\psi(b) &= \psi(a^2)\psi(ab) = r^2\cdot r^3 = r = \psi(a^3b); \\ \psi(b)^2 &= \frac{1}{4}(rs + r^3s - is + ir^2s)^2 \\ &= \frac{1}{4}(\mathbf{1} + r^2 - ir + ir^3 + r^2 + \mathbf{1} - ir^3 + ir - ir^3 - ir - \mathbf{1} + r^2 + ir + ir^3 + r^2 - \mathbf{1}) \\ &= r^2 = \psi(a^2); \\ \psi(a)\psi(b)\psi(a) &= \psi(ab)\psi(a) = r^3\cdot\frac{1}{2}(r^2s + s - irs + ir^3s) = \frac{1}{2}(rs + r^3s - is + ir^2s) = \psi(b). \end{aligned}$$

□

According to Prop. 2.1, Theorem 2.2 and Prop. 3.1, Theorem 3.1, we have two complete sets of primitive orthogonal idempotents of $\mathbb{C}[Q_8]$ and $\mathbb{C}[D_8]$, respectively. There are primitive idempotents $e_1, \ldots, e_4$ corresponding to one-dimensional irreducible representations of Q_8, and Example 3.1 has calculated the primitive decomposition of idempotents that is given by ρ_1 for the unique two-dimensional irreducible representation of Q_8 up to equivalence.

$$\begin{aligned} e_{\rho_1} &= \frac{1}{2}(\mathbf{1} - a^2) = e'_{\rho_1} + e''_{\rho_1}, \\ e'_{\rho_1} &= -\frac{1}{4i}(a^3 + i\cdot a^2 - a - i\cdot\mathbf{1}), \\ e''_{\rho_1} &= -\frac{1}{4i}(-a^3 + i\cdot a^2 + a - i\cdot\mathbf{1}). \end{aligned}$$

There are primitive idempotents $\bar{e}_1, \ldots, \bar{e}_4$ corresponding to one-dimensional irreducible representations of D_8, and we see by Example 2.1 that the unique two-dimensional irreducible representation ρ_1 of D_8 up to equivalence provides

$$\begin{aligned}\bar{e}_{\rho_1} &= \frac{1}{2}(\mathbf{1} - r^2) = \bar{e}'_{\rho_1} + \bar{e}''_{\rho_1},\\ \bar{e}'_{\rho_1} &= \frac{1}{4}(\mathbf{1} - r^2 + rs - r^3 s),\\ \bar{e}''_{\rho_1} &= \frac{1}{4}(\mathbf{1} - r^2 - rs + r^3 s).\end{aligned}$$

Here, we use bar notation to distinguish the complete set of primitive orthogonal idempotents of $\mathbb{C}[Q_8]$ from that of $\mathbb{C}[D_8]$.

The proof of Theorem 5.1 has shown that $\psi(e_i) = \bar{e}_i$ for $1 \leq i \leq 4$. Now, we further check that

$$\begin{aligned}\psi(e'_{\rho_1}) &= \psi\left(-\frac{1}{4i}(a^3 + i \cdot a^2 - a - i \cdot \mathbf{1})\right)\\ &= \frac{1}{4}(\mathbf{1} - r^2 - rs + r^3 s) = \bar{e}''_{\rho_1},\\ \psi(e''_{\rho_1}) &= \psi\left(-\frac{1}{4i}(-a^3 + i \cdot a^2 + a - i \cdot \mathbf{1})\right)\\ &= \frac{1}{4}(\mathbf{1} - r^2 + rs - r^3 s) = \bar{e}'_{\rho_1}.\end{aligned}$$

Question. In general, we wonder how to find algebra isomorphisms between $\mathbb{C}[Q_{4m}]$ and $\mathbb{C}[D_{2n}]$ when $n = 2m$ and $2 \mid m$, making a one-to-one correspondence between the two complete sets of their primitive orthogonal idempotents given in this paper.

6. Conclusions

Overall, we obtain the formulas for the primitive decompositions of idempotents of the dihedral group algebras $\mathbb{C}[D_{2n}]$ and generalized quaternion group algebras $\mathbb{C}[Q_{4m}]$. Then we present two sets of trigonometric identities by the orthogonality relations of the character tables of these two families of groups. Additionally, we explicitly describe a group algebra isomorphism between $\mathbb{C}[D_8]$ and $\mathbb{C}[Q_8]$.

Author contributions

Lilan Dai: Writing-original draft and editing, conceptualization, software, methodology; Yunnan Li: Topic selection, writing-review and editing, funding acquisition, methodology, supervision. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

The authors declare that there is no conflict of interest.

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