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Research article

Ostrowski and Hermite-Hadamard type inequalities via $(\alpha - s)$ exponential type convex functions with applications

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Abstract: Integral inequalities involving exponential convexity are significant in both theoretical and applied mathematics. In this paper, we establish a new Hermite-Hadamard type inequality for the class of exponentially convex functions by using the concept of $(\alpha - s)$ exponentially convex function. Additionally, using the well-known Hermite-Hadamard and Ostrowski inequalities, we establish several new integral inequalities. These newly obtained results contain several well-known results as special cases. Finally, new estimations for the trapezoidal formula have been provided, illustrating the practical applications of the research.

Keywords: convex function; exponential type s-convexity; Hermite Hadamard inequality; Ostrowski inequality; Holder's inequality
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1. Introduction and preliminaries

The association between convex function and theories of inequality is very strong. Convexity is an enormous subject that involves the theory of convex functions. Convexity is a very powerful function feature. It is referred to as a natural characteristic of functions. Furthermore, its minimizing property distinguishes it as distinct, original, and advantageous. It is important in optimization theory, calculus of variation, and probability theory; see [1–3].

The idea of convex functions has placed a major influence in modern mathematics. Since then, we have observed that a lot of research articles and books have been dedicated to this field in the last number of years. Due to their numerous applications in classical calculus [4], fractal sets [5], interval valued [6], time scale calculus [7], quantum calculus [8], fractional calculus [9], stochastic [10] etc. Moreover, inequalities have an incredible mathematical model.

Several integral inequalities have been established over time by various researchers. Convexity is one of the most significant notions in the literature for establishing inequalities and their applications. The integral inequalities associated with convex functions include Olsen inequalities [11], Opial inequalities [12], Gagliardo-Nirenberg-type inequalities [13], Simpson type inequalities [14], Hardy-type inequalities [15], Hermite-Hadamard-Fejer type inequalities [16], and Ostrowski inequalities [17]. Similarly, multiple prominent integral inequalities have been discovered in the literature, but the most notable integral inequality is the Hermite-Hadamard, and it has a significant part in the study of convex function. In that sense, it could be said that the Hermite-Hadamard inequality serves as one of the core findings for convex functions with a natural geometrical interpretation and has a lot of applications that attract much interest in elementary mathematics.

Let $\Psi : \Im \subseteq \mathrm{R} \to \mathrm{R}$ be a function that is both integrable and convex. It follows that

$$\Psi\left(\frac{\tau_1+\tau_2}{2}\right) \leq \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\Psi(x)dx \leq \frac{\Psi(\tau_1)+\Psi(\tau_2)}{2} \tag{1.1}$$

holds for all $\tau_1, \tau_2 \in I$ with $\tau_1 < \tau_2$.

Currently, the field of integral inequalities is rapidly advancing, mainly due to its diverse range of practical applications. Guessab et al. [18–21] applied convexity to find out estimate errors and approximate convex polytopes. Many researchers have done a lot of work to refine and expand the Hermite-Hadamard inequality for many other types of functions, such as p-functions [22], s-convex functions [23], quasi-convex functions [24], log-convex functions [25], and m-convex functions [26].

Later, based on the theory of s-convex functions, Fitzpatrick and Dragomir [27] formulated the following variant of the Hermite-Hadamard inequality:

$$2^{s-1}\Psi\left(\frac{\tau_1+\tau_2}{2}\right) \leq \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\Psi(x)dx \leq \frac{\Psi(\tau_1)+\Psi(\tau_2)}{s+1}. \tag{1.2}$$

Avci et al. [28] introduced a new formula for integrals that works with differentiable mappings. They then used this formula to find inequalities similar to the Hermite-Hadamard type for mappings where the first derivatives show s-convexity. Furthermore, Iscan [29] extended the inequality further with an improved formulation of the Hermite-Hadamard inequality that involves harmonic convex functions. In [30], Toplu et al. devised a novel concept of generalized convexity termed the n-polynomial convex function together with the inequalities accompanying it. Yao et al. [31] investigated the idea of modified (p, h)-convex functions that combine p-convexity with modified h-convexity. In [32], Tunc et al. further extended the idea of convex functions by introducing the tgs-convex function. They used this novel notion to illustrate the Hermite-Hadamard inequality in both classical and fractional integrals. A. Bakht et al. [33] introduced and investigated a new form of convex mapping known as α-exponential type convexity, which presented several algebraic properties associated with this newly introduced convexity.

Iscan [34] proposed a variant of the Hermite-Hadamard inequality that incorporates exponential convexity and integral inequalities and is given by

$$\frac{1}{2(\sqrt{e}-1)}\Psi\left(\frac{\tau_1+\tau_2}{2}\right) \leq \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\Psi(x)dx \leq (e-2)\left[\Psi(\tau_1)+\Psi(\tau_2)\right]. \tag{1.3}$$

Convexity has extended the Hermite-Hadamard inequality in various ways. Khan et al. [35] introduced novel Hermite-Hadamard fractional inequalities for m-polynomial convex along with harmonically convex functions. Also, they discussed several practical applications of these results, including error estimation for trapezoidal formulas. Furthermore, Iscan et al. [23] used Holder-Iscan inequality and enhanced power mean inequality to study several related inequalities. Korus et al. [36] used iterated integrals to refine the Hermite-Hadamard inequality for s-convex and convex functions. M. Tariq et al. [37] define and investigate generalized exponential type convex functions, namely exponentially s-convex functions. S. Sahoo et al. [38] introduce the idea and concept of m-polynomial p-harmonic exponential-type convex functions. S. Butt et al. [39] obtain some refinements of the trapezium type inequality for functions whose first derivative in absolute value at certain power n-polynomial exponential type p-convex.

Motivation seeking from the above-mentioned literature and research. The present paper discusses the notion of $(\alpha - s)$ exponential type convex function; while employing this concept, we obtain new variants of the conventional Hermite-Hadamard and Ostrowski type inequalities. This paper provides a comprehensive generalization of the results presented in [33] and in [34]. While studying exponential convexity, a number of researchers are showing intense curiosity in information theory, big data analysis, and deep learning. As a consequence of this, we believe that the present research is going to draw researchers for more developments in the aforementioned fields.

This paper's structure is as follows: Section 2 explores and defines the algebraic features of the $(\alpha - s)$ exponential type convex function. In Section 3, for an $(\alpha - s)$ exponential type convex function, Hermite-Hadamard inequality is derived. Section 4 deals with an Ostrowski-type inequality for the aforementioned function. Section 5 discusses novel trapezoidal formula estimations and their practical applications. The final portion contains the conclusion.

Definition 1.1. *[27] On an interval $[a, b]$, a function Ψ is assumed to be convex if for any pair of points $\tau_1, \tau_2 \in [a, b]$, and for any υ such that $0 \leq \upsilon \leq 1$, we have*

$$\Psi[\upsilon\tau_1 + (1 - \upsilon)\tau_2] \leq \upsilon\Psi(\tau_1) + (1 - \upsilon)\Psi(\tau_2) \tag{1.4}$$

holds.

Definition 1.2. *[34] A function $\Psi : \Im \to \mathbb{R}$ is supposed to be exponential convex function, if*

$$\Psi(\upsilon\tau_1 + (1 - \upsilon)\tau_2) \leq (e^{1-\upsilon} - 1)\Psi(\tau_2) + (e^{\upsilon} - 1)\Psi(\tau_1) \tag{1.5}$$

holds for all $\tau_1, \tau_2 \in \Im$ and $\upsilon \in [0, 1]$.

Definition 1.3. *[40] A function $\Psi : \Im \subseteq \mathbb{R} \to \mathbb{R}$ is supposed to be s-convex in the fourth sense if following inequality*

$$\Psi[\upsilon\tau_1 + (1 - \upsilon)\tau_2] \leq \upsilon^{\frac{1}{s}}\Psi(\tau_1) + (1 - \upsilon)^{\frac{1}{s}}\Psi(\tau_2) \tag{1.6}$$

holds for all $\tau_1, \tau_2 \in \Im$, $\upsilon \in [0, 1]$, and $s \in (0, 1]$.

Definition 1.4. *[41] A function $\Psi : \Im \subseteq \mathbb{R} \to \mathbb{R}$ is termed an exponential type convex function if the inequality*

$$\Psi(\upsilon\tau_1 + (1 - \upsilon)\tau_2) \leq \upsilon\frac{\Psi(\tau_1)}{e^{\alpha\tau_1}} + (1 - \upsilon)\frac{\Psi(\tau_2)}{e^{\alpha\tau_2}} \tag{1.7}$$

hold true for a fixed $\alpha \in \mathbb{R}$, $\tau_1, \tau_2 \in \Im$, and $\upsilon \in [0, 1]$.

Definition 1.5. *[42] A function $\Psi : \Im \subseteq \mathbb{R} \to \mathbb{R}$ is termed an exponentially s-convex in the second sense if the inequality*

$$\Psi(\upsilon\tau_1 + (1-\upsilon)\tau_2) \leq \upsilon^s \frac{\Psi(\tau_1)}{e^{\alpha\tau_1}} + (1-\upsilon)^s \frac{\Psi(\tau_2)}{e^{\alpha\tau_2}} \tag{1.8}$$

hold true for a fixed $\alpha \in \mathbb{R}$, $\tau_1, \tau_2 \in \Im$, $\upsilon \in [0,1]$ and $0 < s \leq 1$.

Definition 1.6. *[33] A function $\Psi : \Im \subseteq \mathbb{R} \to \mathbb{R}$ is termed an α-exponential type convex function if the inequality*

$$\Psi(\upsilon\tau_1 + (1-\upsilon)\tau_2) \leq (e^{\upsilon} - 1)\frac{\Psi(\tau_1)}{e^{\alpha\tau_1}} + (e^{(1-\upsilon)} - 1)\frac{\Psi(\tau_2)}{e^{\alpha\tau_2}} \tag{1.9}$$

hold true for a fixed $\alpha \in \mathbb{R}$, $\tau_1, \tau_2 \in \Im$, and $\upsilon \in [0,1]$.

2. $(\alpha - s)$ exponential type convex function and its properties

We now give the concept of $(\alpha - s)$ exponential type convex function and explore some of their related results in the form of Hermite-Hadamard and Ostrowski type inequalities. This new class offers interesting insights and applications in the trapezoidal formula, which we will study in Section 5.

Definition 2.1. *A non-negative function $\Psi : \Im \subseteq \mathbb{R} \to \mathbb{R}$ is termed an $(\alpha - s)$ exponential type convex function if the inequality*

$$\Psi(\upsilon\tau_1 + (1-\upsilon)\tau_2) \leq (e^{\frac{\upsilon}{s}} - 1)\frac{\Psi(\tau_1)}{e^{\alpha\tau_1}} + (e^{\frac{1-\upsilon}{s}} - 1)\frac{\Psi(\tau_2)}{e^{\alpha\tau_2}} \tag{2.1}$$

hold true for a fixed $\alpha \in \mathbb{R}$, $0 < s \leq 1$, $\tau_1, \tau_2 \in \Im$, and $\upsilon \in [0,1]$.

Remark 2.1. *(i). By setting $s = 1$ in (2.1), we obtain the α-exponential type convexity studied by A.Bakht in [33].*

(ii). By setting $s = 1$ and $\alpha = 0$ in (2.1), we obtain the exponential type convexity studied by Iscan in [34].

We examine the class of $(\alpha - s)$ exponential type convex functions and also explore how these functions relate to other types of convex functions. This investigation reveals how these different classes of functions are interconnected.

Lemma 2.1. *For $0 < s \leq 1$ and $\upsilon \in [0,1]$, the subsequent inequalities are established*
$e^{\frac{\upsilon}{s}} - 1 \geq \upsilon^{\frac{1}{s}}$,
$e^{\frac{1-\upsilon}{s}} - 1 \geq (1-\upsilon)^{\frac{1}{s}}$.

Proof. Consider the function

$$\Psi(\upsilon) = e^{\frac{\upsilon}{s}} - 1 - \upsilon^{\frac{1}{s}}.$$

We need to show that $\Psi(\upsilon) \geq 0$ for $\upsilon \in [0,1]$ and $0 < s \leq 1$.
At $\upsilon = 0$:

$$\Psi(0) = e^0 - 1 - 0^{\frac{1}{s}} = 0.$$

The derivative of $\Psi(\upsilon)$ is

$$\Psi'(\upsilon) = \frac{1}{s}e^{\frac{\upsilon}{s}} - \frac{1}{s}\upsilon^{\frac{1}{s}-1}.$$

For $\upsilon \geq 0$, since $e^{\frac{\upsilon}{s}}$ grows faster than $\upsilon^{\frac{1}{s}-1}$, we have

$$\Psi'(\upsilon) \geq 0.$$

Thus, $\Psi(\upsilon)$ is non-decreasing for $\upsilon \in [0, 1]$ and $0 < s \leq 1$.

□

Proposition 2.1. *A function that is s-convex in the fourth sense is known as* $(\alpha - s)$ *exponential type convex.*

Proof. By applying Lemma 2.1, for $s \in (0, 1]$, $\upsilon \in [0, 1]$, and $\alpha = 0$, it follows that

$$\Psi(\upsilon\tau_1 + (1-\upsilon)\tau_2) \leq \upsilon^{\frac{1}{s}}\Psi(\tau_1) + (1-\upsilon)^{\frac{1}{s}}\Psi(\tau_2) \leq (e^{\frac{\upsilon}{s}} - 1)\frac{\Psi(\tau_1)}{e^{\alpha\tau_1}} + (e^{\frac{1-\upsilon}{s}} - 1)\frac{\Psi(\tau_2)}{e^{\alpha\tau_2}}.$$

□

Theorem 2.1. *Let* $\Psi, \Phi : [\tau_1, \tau_2] \longrightarrow \mathbb{R}$. *If* Ψ, Φ *are functions with* $(\alpha - s)$*-exponential type convexity, then*
(i) The function $\Psi + \Phi$ *is also an* $(\alpha - s)$ *exponential type convex function.*
(ii) For any non-negative $\kappa \geq 0$, $\kappa\Psi$ *is a function of* $(\alpha - s)$ *exponential type convexity.*

Proof. (i) Assume that Ψ is a function of $(\alpha - s)$ exponential type convexity.

$$\begin{aligned}
&(\Psi + \Phi)(\upsilon\tau_1 + (1-\upsilon)\tau_2) = \Psi(\upsilon\tau_1 + (1-\upsilon)\tau_2) + \Phi(\upsilon\tau_1 + (1-\upsilon)\tau_2) \\
&\leq (e^{\frac{\upsilon}{s}} - 1)\frac{\Psi(\tau_1)}{e^{\alpha\tau_1}} + (e^{\frac{(1-\upsilon)}{s}} - 1)\frac{\Psi(\tau_2)}{e^{\alpha\tau_2}} + (e^{\frac{\upsilon}{s}} - 1)\frac{\Phi(\tau_1)}{e^{\alpha\tau_1}} + (e^{\frac{1-\upsilon}{s}} - 1)\frac{\Phi(\tau_2)}{e^{\alpha\tau_2}} \\
&= (e^{\frac{\upsilon}{s}} - 1)\left[\frac{\Psi(\tau_1) + \Phi(\tau_1)}{e^{\alpha\tau_1}}\right] + (e^{\frac{1-\upsilon}{s}} - 1)\left[\frac{\Psi(\tau_2) + \Phi(\tau_2)}{e^{\alpha\tau_2}}\right] \\
&= (e^{\frac{\upsilon}{s}} - 1)\frac{\Big(\Psi + \Phi\Big)(\tau_1)}{e^{\alpha\tau_1}} + (e^{\frac{1-\upsilon}{s}} - 1)\frac{\Big(\Psi + \Phi\Big)(\tau_2)}{e^{\alpha\tau_2}}.
\end{aligned}$$

(ii) Assume that Ψ is a function of $(\alpha - s)$ exponential type convexity. For any non-negative $\kappa \geq 0$, then

$$\begin{aligned}
&(\kappa\Psi)(\upsilon\tau_1 + (1-\upsilon)\tau_2) \leq \kappa\left[(e^{\frac{\upsilon}{s}} - 1)\frac{\Psi(\tau_1)}{e^{\alpha\tau_1}} + (e^{\frac{1-\upsilon}{s}} - 1)\frac{\Psi(\tau_2)}{e^{\alpha\tau_2}}\right] \\
&= (e^{\frac{\upsilon}{s}} - 1)\frac{\kappa\Psi(\tau_1)}{e^{\alpha\tau_1}} + (e^{\frac{1-\upsilon}{s}} - 1)\frac{\kappa\Psi(\tau_2)}{e^{\alpha\tau_2}} \\
&= (e^{\frac{\upsilon}{s}} - 1)\frac{\Big(\kappa\Psi\Big)(\tau_1)}{e^{\alpha\tau_1}} + (e^{\frac{1-\upsilon}{s}} - 1)\frac{\Big(\kappa\Psi\Big)(\tau_2)}{e^{\alpha\tau_2}}.
\end{aligned}$$

□

Theorem 2.2. *Suppose* $\Phi : \mathfrak{I} \to \mathbb{J}$ *is an exponential s-convex function in the fourth sense. Also assume that* $\Psi : \mathbb{J} \to \mathbb{R}$ *is a non-decreasing function of* $(\alpha - s)$ *exponential type convex. Then the composition* $\Psi \circ \Phi : \mathfrak{I} \to \mathbb{R}$ *is also a function of* $(\alpha - s)$ *exponential type convex.*

Proof. For $\alpha \in \mathbb{R}$, $0 < s \leq 1$, $\tau_1, \tau_2 \in \Im$, and $\upsilon \in [0, 1]$. we have

$$\begin{aligned}
&(\Psi o \Phi)(\upsilon\tau_1 + (1-\upsilon)\tau_2) = \Psi(\Phi(\upsilon\tau_1 + (1-\upsilon)\tau_2)) \\
&\leq \Psi\left(\upsilon^{\frac{1}{s}} \frac{\Phi(\tau_1)}{e^{\alpha\tau_1}} + (1-\upsilon)^{\frac{1}{s}} \frac{\Phi(\tau_2)}{e^{\alpha\tau_2}}\right) \\
&\leq (e^{\frac{\upsilon}{s}} - 1)\Psi\left(\frac{\Phi(\tau_1)}{e^{\alpha\tau_1}}\right) + (e^{\frac{1-\upsilon}{s}}\Psi\left(\frac{\Phi(\tau_2)}{e^{\alpha\tau_2}}\right) \\
&= (e^{\frac{\upsilon}{s}} - 1)\frac{(\Psi o \Phi)(\tau_1)}{e^{\alpha\tau_1}} + (e^{\frac{1-\upsilon}{s}} - 1)\frac{(\Psi o \Phi)(\tau_2)}{e^{\alpha\tau_2}}.
\end{aligned}$$

□

3. Hermite-Hadamard type inequality for $(\alpha - s)$ exponential type convex function

The fundamental purpose of the following portion is to formulate Hermite-Hadamard type inequalities. These inequalities are specifically applicable for a function of $(\alpha - s)$ exponential type convexity.

Theorem 3.1. *Suppose $\Psi : [\tau_1, \tau_2] \longrightarrow \mathbb{R}$ is a function of $(\alpha - s)$ exponential type convexity. If $\tau_1 < \tau_2$ and $\Psi \in L[\tau_1, \tau_2]$, subsequently the given Hermite-Hadamard type inequality is satisfied.*

$$\frac{1}{2(e^{\frac{1}{2s}} - 1)}\Psi(\frac{\tau_1 + \tau_2}{2}) \leq \frac{1}{\tau_2 - \tau_1}\int_{\tau_1}^{\tau_2} \frac{\Psi(x)}{e^{\alpha x}}dx \leq A(\upsilon)\frac{\Psi(\tau_1)}{e^{\alpha\tau_1}} + B(\upsilon)\frac{\Psi(\tau_2)}{e^{\alpha\tau_2}}, \tag{3.1}$$

where

$$A(\upsilon) = \int_0^1 \frac{(e^{\frac{\upsilon}{s}} - 1)}{e^{\alpha(\upsilon\tau_1 + (1-\upsilon)\tau_2)}}d\upsilon, B(\upsilon) = \int_0^1 \frac{(e^{\frac{1-\upsilon}{s}} - 1)}{e^{\alpha(\upsilon\tau_1 + (1-\upsilon)\tau_2)}}d\upsilon.$$

Proof. Since

$$\Psi\left(\frac{\tau_1 + \tau_2}{2}\right) = \Psi\left(\frac{(\upsilon\tau_1 + (1-\upsilon)\tau_2) + (\upsilon\tau_2 + (1-\upsilon)\tau_1)}{2}\right)$$

assume that

$$\tau_1 = \upsilon\tau_1 + (1-\upsilon)\tau_2, \qquad \tau_2 = \upsilon\tau_2 + (1-\upsilon)\tau_1,$$

$$\Psi\left(\frac{\tau_1 + \tau_2}{2}\right) = \Psi\left(\frac{1}{2}(\upsilon\tau_1 + (1-\upsilon)\tau_2) + \frac{1}{2}(\upsilon\tau_2 + (1-\upsilon)\tau_1)\right). \tag{3.2}$$

Using (2.1) upon (3.2) gives the following:

$$\Psi\left(\frac{\tau_1 + \tau_2}{2}\right) \leq (e^{\frac{1}{2s}} - 1)\frac{\Psi(\upsilon\tau_1 + (1-\upsilon)\tau_2)}{e^{\alpha(\upsilon\tau_1 + (1-\upsilon)\tau_2)}} + (e^{\frac{1}{2s}} - 1)\frac{\Psi(\upsilon\tau_2 + (1-\upsilon)\tau_1)}{e^{\alpha(\upsilon\tau_2 + (1-\upsilon)\tau_1)}}. \tag{3.3}$$

Integrate the expression in (3.3) with respect to υ from 0 to 1. Apply a change of variable, we have

$$\frac{1}{2(e^{\frac{1}{2s}} - 1)}\Psi\left(\frac{\tau_1 + \tau_2}{2}\right) \leq \left[\frac{1}{\tau_2 - \tau_1}\int_{\tau_1}^{\tau_2} \frac{\Psi(u)}{e^{\alpha u}}du\right]. \tag{3.4}$$

Using (2.1),

$$\frac{\Psi(\upsilon\tau_1 + (1-\upsilon)\tau_2)}{e^{\alpha(\upsilon\tau_1 + (1-\upsilon)\tau_2)}} \leq \frac{(e^{\frac{\upsilon}{s}} - 1)\frac{\Psi(\tau_1)}{e^{\alpha\tau_1}}}{e^{\alpha(\upsilon\tau_1 + (1-\upsilon)\tau_2)}} + \frac{(e^{\frac{\upsilon}{s}} - 1)\frac{\Psi(\tau_2)}{e^{\alpha\tau_2}}}{e^{\alpha(\upsilon\tau_1) + (1-\upsilon)\tau_2)}}. \tag{3.5}$$

Integrate the expression in (3.5) with respect to υ from 0 to 1, and we have

$$\frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\frac{\Psi(u)}{e^{\alpha u}}du \le \frac{\Psi(\tau_1)}{e^{\alpha\tau_1}}\int_0^1 \frac{(e^{\frac{\upsilon}{s}}-1)}{e^{\alpha(\upsilon\tau_1+(1-\upsilon)\tau_2)}}d\upsilon + \frac{\Psi(\tau_2)}{e^{\alpha\tau_2}}\int_0^1 \frac{(e^{\frac{1-\upsilon}{s}}-1)}{e^{\alpha(\upsilon\tau_1+(1-\upsilon)\tau_2)}}d\upsilon. \tag{3.6}$$

From (3.4) and (3.6), we obtain (3.1). □

Remark 3.1. *From above Theorem* 3.1*, we obtained Theorem 4 in [33] by letting* $\alpha = 0$ *and* $s = 1$.

This section aims to examine several estimates that improve the Hermite-Hadamard (H-H) inequality for functions with an $(\alpha - s)$ exponential type convex first derivative at a specific power. Agarwal and Dragomir used the subsequent lemma in [24].

Lemma 3.1. *On* $\mathfrak{I}^o$*, assume that* $\Psi : \mathfrak{I} \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ *be a differentiable function. Consider* $\tau_1, \tau_2 \in \mathfrak{I}^o$*, where* $\tau_1 < \tau_2$*. The following identity holds if* $\Psi' \in L[\tau_1, \tau_2]$*:*

$$\frac{\Psi(\tau_1)+\Psi(\tau_2)}{2} - \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\Psi(x)dx = \frac{\tau_2-\tau_1}{2}\int_0^1 (1-2\upsilon)\Psi'(\upsilon\tau_1+(1-\upsilon)\tau_2)d\upsilon. \tag{3.7}$$

Theorem 3.2. *Assume that* Ψ *be a differentiable function mapping from the interval* $\mathfrak{I}$*, which is a subset of* $\mathbb{R}$*, to* $\mathbb{R}$*. The function* Ψ *is defined on the interior of* $\mathfrak{I}$*. Consider two points* τ_1 *and* τ_2 *within this interval, where* $\tau_1 < \tau_2$*, and let* $\Psi' \in L[\tau_1, \tau_2]$*. Suppose that* $|\Psi'|$ *is a function with* $(\alpha - s)$*-exponential type convexity over this interval. For* $0 < s \le 1$ *and* $0 \le \nu \le 1$*, the subsequent inequality holds*

$$\begin{aligned}&\left|\frac{\Psi(\tau_1)+\Psi(\tau_2)}{2} - \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\Psi(x)dx\right| \\ \le&\frac{\tau_2-\tau_1}{2}\left((4e^{\frac{1}{2s}}-2e^{\frac{1}{s}}-2)s^2+(e^{\frac{1}{s}}-1)s-\frac{1}{2}\right)\left[\left|\frac{\Psi'(\tau_1)}{e^{\alpha\tau_1}}\right|+\left|\frac{\Psi'(\tau_2)}{e^{\alpha\tau_2}}\right|\right].\end{aligned} \tag{3.8}$$

Proof. From Lemma 3.1, we have

$$\begin{aligned}\left|\frac{\Psi(\tau_1)+\Psi(\tau_2)}{2} - \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\Psi(x)dx\right| &= \frac{\tau_2-\tau_1}{2}\left|\int_0^1 (1-2\upsilon)\Psi'(\upsilon\tau_1+(1-\upsilon)\tau_2)d\upsilon\right| \\ &\le \frac{\tau_2-\tau_1}{2}\int_0^1 \left|1-2\upsilon\right|\left|\Psi'(\upsilon\tau_1+(1-\upsilon)\tau_2)\right|d\upsilon.\end{aligned}$$

Using $(\alpha - s)$ exponential type convexity for Ψ', we obtain

$$\begin{aligned}&\left|\frac{\Psi(\tau_1)+\Psi(\tau_2)}{2} - \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\Psi(x)dx\right| \\ \le&\ \frac{\tau_2-\tau_1}{2}\int_0^1 |1-2\upsilon|\left[\left(e^{\frac{\upsilon}{s}}-1\right)\left|\frac{\Psi'(\tau_1)}{e^{\alpha\tau_1}}\right|+\left(e^{\frac{1-\upsilon}{s}}-1\right)\left|\frac{\Psi'(\tau_2)}{e^{\alpha\tau_2}}\right|\right]d\upsilon \\ =&\ \frac{\tau_2-\tau_1}{2}\left[\left|\frac{\Psi'(\tau_1)}{e^{\alpha\tau_1}}\right|\int_0^1 |(1-2\upsilon)|\left(e^{\frac{\upsilon}{s}}-1\right)d\upsilon+\left|\frac{\Psi'(\tau_2)}{e^{\alpha\tau_2}}\right|\int_0^1 |(1-2\upsilon)|(e^{\frac{1-\upsilon}{s}}-1)d\upsilon\right] \\ =&\ \frac{\tau_2-\tau_1}{2}\left((4e^{\frac{1}{2s}}-2e^{\frac{1}{s}}-2)s^2+(e^{\frac{1}{s}}-1)s-\frac{1}{2}\right)\left[\left|\frac{\Psi'(\tau_1)}{e^{\alpha\tau_1}}\right|+\left|\frac{\Psi'(\tau_2)}{e^{\alpha\tau_2}}\right|\right].\end{aligned} \tag{3.9}$$

Since

$$\int_0^1 \left|1-2\upsilon\right|\left(e^{\frac{\upsilon}{s}}-1\right)d\upsilon = \int_0^1 \left|1-2\upsilon\right|\left(e^{\frac{1-\upsilon}{s}}-1\right)d\upsilon = (4e^{\frac{1}{2s}}-2e^{\frac{1}{s}}-2)s^2+(e^{\frac{1}{s}}-1)s-\frac{1}{2}. \quad (3.10)$$

By substituting (3.10) in (3.9), we obtain (3.8). □

Remark 3.2. *(i) Let $s = 1$ in above Theorem 3.2 to obtain Theorem 5 in [33].*
(ii) Let $\alpha = 0$ and $s = 1$ to obtain Theorem 4.1 in [34].

Theorem 3.3. *Let Ψ be a differentiable function mapping from the interval $\Im$, which is a subset of $\mathbb{R}$, to $\mathbb{R}$. The function Ψ is defined on the interior of $\Im$. Consider two points τ_1 and τ_2 within this interval, where $\tau_1 < \tau_2$, and let $q \geq 1$. Suppose that Ψ' is integrable over the interval $[\tau_1, \tau_2]$, and that $|\Psi'|^q$ is a convex function of $(\alpha - s)$ exponential type over this interval. For $0 < s \leq 1$ and $0 \leq \nu \leq 1$, the subsequent inequality is satisfied:*

$$\begin{aligned}&\left|\frac{\Psi(\tau_1)+\Psi(\tau_2)}{2}-\frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\Psi(x)dx\right|\\ \leq&\frac{\tau_2-\tau_1}{2}\left(-1+se^{\frac{1}{s}}-s\right)^{\frac{1}{q}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left|\frac{\Psi'(\tau_1)}{e^{\alpha\tau_1}}\right|^q+\left|\frac{\Psi'(\tau_2)}{e^{\alpha\tau_2}}\right|^q\right]^{\frac{1}{q}},\end{aligned} \quad (3.11)$$

whenever $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By Lemma 3.1, we have

$$\begin{aligned}&\left|\frac{\Psi(\tau_1)+\Psi(\tau_2)}{2}-\frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\Psi(x)dx\right|\\ =&\frac{\tau_2-\tau_1}{2}\left|\int_0^1(1-2\upsilon)\Psi'(\upsilon\tau_1+(1-\upsilon)\tau_2)d\upsilon\right|\\ \leq&\frac{\tau_2-\tau_1}{2}\int_0^1\left|1-2\upsilon\right|\left|\Psi'(\upsilon\tau_1+(1-\upsilon)\tau_2)\right|d\upsilon.\end{aligned}$$

Applying Holder's integral inequality, we find

$$\begin{aligned}&\frac{\tau_2-\tau_1}{2}\int_0^1\left|1-2\upsilon\right|\left|\Psi'(\upsilon\tau_1+(1-\upsilon)\tau_2)\right|d\upsilon\\ \leq&\frac{\tau_2-\tau_1}{2}\left(\int_0^1\left|1-2\upsilon\right|^pd\upsilon\right)^{\frac{1}{p}}\left(\int_0^1\left|\Psi'(\upsilon\tau_1+(1-\upsilon)\tau_2)\right|^qd\upsilon\right)^{\frac{1}{q}}.\end{aligned} \quad (3.12)$$

Given that $|\Psi|^q$ is $(\alpha - s)$ exponential type convex function, hence

$$\begin{aligned}&\int_0^1\left|\Psi'(\upsilon\tau_1+(1-\upsilon)\tau_2)\right|^qd\upsilon\\ \leq&\int_0^1\left[\left(e^{\frac{\upsilon}{s}}-1\right)\left|\frac{\Psi'(\tau_1)}{e^{\alpha\tau_1}}\right|^q+\left(e^{\frac{1-\upsilon}{s}}-1\right)\left|\frac{\Psi'(\tau_2)}{e^{\alpha\tau_2}}\right|^q\right]d\upsilon\\ =&\left(-1+se^{\frac{1}{s}}-s\right)\left[\left|\frac{\Psi'(\tau_1)}{e^{\alpha\tau_1}}\right|^q+\left|\frac{\Psi'(\tau_2)}{e^{\alpha\tau_2}}\right|^q\right].\end{aligned} \quad (3.13)$$

Since

$$\int_0^1 \left(e^{\frac{\upsilon}{s}} - 1\right) d\upsilon = \int_0^1 \left(e^{\frac{1-\upsilon}{s}} - 1\right) d\upsilon = -1 + se^{\frac{1}{s}} - s. \tag{3.14}$$

$$\int_0^1 \left|1 - 2\upsilon\right|^p d\upsilon = \frac{1}{p+1}. \tag{3.15}$$

By substituting (3.13)–(3.15) in (3.12), we obtained (3.11). □

Remark 3.3. *(i) Let $s = 1$ in above Theorem 3.3 to obtain Theorem 6 in [33].*
(ii) Let $\alpha = 0$ and $s = 1$ in above Theorem 3.3 to obtain Theorem 4.2 in [34].

Theorem 3.4. *Let Ψ be a differentiable function mapping from the interval $\Im$, which is a subset of $\mathbb{R}$, to $\mathbb{R}$. The function Ψ is defined on the interior of $\Im$. Consider two points τ_1 and τ_2 within this interval, where $\tau_1 < \tau_2$, and let $q \geq 1$. Suppose that Ψ' is integrable over the interval $[\tau_1, \tau_2]$, and that $|\Psi'|^q$ is a function of $(\alpha - s)$-exponential type convexity over this interval. For $0 < s \leq 1$ and $0 \leq \nu \leq 1$, the subsequent inequality is satisfied:*

$$\begin{aligned}
&\left|\frac{\Psi(\tau_1) + \Psi(\tau_2)}{2} - \frac{1}{\tau_2 - \tau_1}\int_{\tau_1}^{\tau_2} \Psi(x)dx\right| \\
\leq &\frac{\tau_2 - \tau_1}{2^{2-\frac{1}{q}}}\left[\left((4e^{\frac{1}{2s}} - 2e^{\frac{1}{s}} - 2)s^2 + (e^{\frac{1}{s}} - 1)s - \frac{1}{2}\right)\right]^{\frac{1}{q}}\left[\left|\frac{\Psi'(\tau_1)}{e^{\alpha\tau_1}}\right|^q + \left|\frac{\Psi'(\tau_2)}{e^{\alpha\tau_2}}\right|^q\right].
\end{aligned} \tag{3.16}$$

Proof. From Lemma 3.1, we have

$$\begin{aligned}
&\left|\frac{\Psi(\tau_1) + \Psi(\tau_2)}{2} - \frac{1}{\tau_2 - \tau_1}\int_{\tau_1}^{\tau_2} \Psi(x)dx\right| \\
&= \frac{\tau_2 - \tau_1}{2}\left|\int_0^1 (1 - 2\upsilon)\Psi'(\upsilon\tau_1 + (1-\upsilon)\tau_2)d\upsilon\right| \\
&\leq \frac{\tau_2 - \tau_1}{2}\int_0^1 \left|1 - 2\upsilon\right|\left|\Psi'(\upsilon\tau_1 + (1-\upsilon)\tau_2)\right|d\upsilon.
\end{aligned}$$

Employing power mean inequality, we find

$$\begin{aligned}
&\frac{\tau_2 - \tau_1}{2}\int_0^1 \left|1 - 2\upsilon\right|\left|\Psi'(\upsilon\tau_1 + (1-\upsilon)\tau_2)\right|d\upsilon \\
&\leq \frac{\tau_2 - \tau_1}{2}\left(\int_0^1 \left|1 - 2\upsilon\right|d\upsilon\right)^{1-\frac{1}{q}}\left(\int_0^1 \left|1 - 2\upsilon\right|\left|\Psi'(\upsilon\tau_1 + (1-\upsilon)\tau_2)\right|^q d\upsilon\right)^{\frac{1}{q}}.
\end{aligned} \tag{3.17}$$

Given that $|\Psi|^q$ is $(\alpha - s)$ exponential type convex function, hence

$$\begin{aligned}
&\int_0^1 \left|1 - 2\upsilon\right|\left|\Psi'(\upsilon\tau_1 + (1-\upsilon)\tau_2)\right|^q d\upsilon \\
&\leq \int_0^1 \left|1 - 2\upsilon\right|\left[\left(e^{\frac{\upsilon}{s}} - 1\right)\left|\frac{\Psi'(\tau_1)}{e^{\alpha\tau_1}}\right|^q + \left(e^{\frac{1-\upsilon}{s}} - 1\right)\left|\frac{\Psi'(\tau_2)}{e^{\alpha\tau_2}}\right|^q\right]d\upsilon \\
&= \left((4e^{\frac{1}{2s}} - 2e^{\frac{1}{s}} - 2)s^2 + (e^{\frac{1}{s}} - 1)s - \frac{1}{2}\right)\left[\left|\frac{\Psi'(\tau_1)}{e^{\alpha\tau_1}}\right|^q + \left|\frac{\Psi'(\tau_2)}{e^{\alpha\tau_2}}\right|^q\right].
\end{aligned} \tag{3.18}$$

Since

$$\int_0^1 \Big|1-2\upsilon\Big| d\upsilon = \frac{1}{2}. \tag{3.19}$$

Simply putting (3.18) and (3.19) in (3.17) and finally we obtain (3.16). □

Remark 3.4. *(i) Let $s = 1$ in above Theorem 3.4 to obtain Theorem 7 in [33].*
(ii) Let $\alpha = 0$ and $s = 1$ in above Theorem 3.4 to obtain Theorem 4.3 in [34].

4. Refinement of Ostrowski type inequality for $(\alpha - s)$ exponential type convexity

In this section, we proposed various innovations to deal with Ostrowski-type inequality. These improvements apply to differentiable function of $(\alpha - s)$ exponential type convexity. Dragomir and Cerone used this lemma within the results they introduced in [43].

Lemma 4.1. *Assume that Ψ is a differentiable function mapping from the interval $\Im$, which is a subset of $\mathbb{R}$, to $\mathbb{R}$. The function Ψ is defined on the interior of $\Im$. Consider two points τ_1 and τ_2 in the interior of $\Im$, such that $\tau_1 < \tau_2$, assuming $\Psi' \in L[\tau_1, \tau_2]$, and subsequently this identity is valid:*

$$\begin{aligned}&\Psi(z) - \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\Psi(x)dx \\ =&\frac{(z-\tau_1)^2}{\tau_2-\tau_1}\int_0^1 \upsilon\Psi'(\upsilon z+(1-\upsilon)\tau_1)d\upsilon - \frac{(\tau_2-z)^2}{\tau_2-\tau_1}\int_0^1 \upsilon\Psi'(\upsilon z+(1-\upsilon)\tau_2)d\upsilon,\end{aligned} \tag{4.1}$$

for all $z \in [\tau_1, \tau_2]$.

Theorem 4.1. *Assume that Ψ is a differentiable function mapping from the interval $\Im$, which is a subset of $\mathbb{R}$, to $\mathbb{R}$. The function Ψ is defined on the interior of $\Im$. Consider two points τ_1 and τ_2 in the interior of $\Im$, such that $\tau_1 < \tau_2$, assuming $\Psi' \in L[\tau_1, \tau_2]$. If $|\Psi'|$ is a convex function of $(\alpha - s)$-exponential type on the interval $[\tau_1, \tau_2]$ and satisfies $|\Psi'| \leq K$ for all z in $[\tau_1, \tau_2]$, and subsequently this inequality is valid for each z in $[\tau_1, \tau_2]$, $0 \leq \nu \leq 1$ and $0 < s \leq 1$:*

$$\left|\Psi(z) - \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\Psi(x)dx\right| \leq \frac{K\left(se^{\frac{1}{s}}-s-1\right)}{(\tau_2-\tau_1)}\Big[(z-\tau_1)^2+(\tau_2-z)^2\Big]. \tag{4.2}$$

Proof. Employing Lemma 4.1, and noting that $|\Psi'|$ is an $(\alpha - s)$ exponential type convex function that fulfills $|\Psi'| \leq K$,

$$\begin{aligned}&\left|\Psi(z) - \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\Psi(x)dx\right| \\ \leq&\frac{(z-\tau_1)^2}{\tau_2-\tau_1}\int_0^1 \upsilon\Big|\Psi'(\upsilon z+(1-\upsilon)\tau_1)\Big|d\upsilon + \frac{(\tau_2-z)^2}{\tau_2-\tau_1}\int_0^1 \upsilon\Big|\Psi'(\upsilon z+(1-\upsilon)\tau_2)\Big|d\upsilon \\ \leq&\frac{(z-\tau_1)^2}{\tau_2-\tau_1}\int_0^1 \upsilon\Big\{(e^{\frac{\upsilon}{s}}-1)\frac{|\Psi'(z)|}{e^{\alpha z}} + (e^{\frac{1-\upsilon}{s}}-1)\frac{|\Psi'(\tau_1)|}{e^{\alpha\tau_1}}\Big\}d\upsilon \\ &+ \frac{(\tau_2-z)^2}{\tau_2-\tau_1}\int_0^1 \upsilon\Big\{(e^{\frac{\upsilon}{s}}-1)\frac{|\Psi'(z)|}{e^{\alpha z}} + (e^{\frac{1-\upsilon}{s}}-1)\frac{|\Psi'(\tau_2)|}{e^{\alpha\tau_2}}\Big\}d\upsilon\end{aligned}$$

$$\leq\frac{(z-\tau_1)^2}{\tau_2-\tau_1}\left[\frac{|\Psi'(z)|}{e^{\alpha z}}\int_0^1 \upsilon(e^{\frac{\upsilon}{s}}-1)d\upsilon+\frac{|\Psi'(\tau_1)|}{e^{\alpha\tau_1}}\int_0^1 \upsilon(e^{\frac{1-\upsilon}{s}}-1)d\upsilon\right]$$
$$+\frac{(\tau_2-z)^2}{\tau_2-\tau_1}\left[\frac{|\Psi'(z)|}{e^{\alpha z}}\int_0^1 \upsilon(e^{\frac{\upsilon}{s}}-1)d\upsilon+\frac{|\Psi'(\tau_2)|}{e^{\alpha\tau_2}}\int_0^1 \upsilon(e^{\frac{1-\upsilon}{s}}-1)d\upsilon\right]$$
$$\leq\frac{K(z-\tau_1)^2}{\tau_2-\tau_1}\left\{\left(1-e^{\frac{1}{s}}\right)s^2+se^{\frac{1}{s}}-\frac{1}{2}+\left(e^{\frac{1}{s}}-1\right)s^2-s-\frac{1}{2}\right\}$$
$$+\frac{K(\tau_2-z)^2}{\tau_2-\tau_1}\left\{\left(1-e^{\frac{1}{s}}\right)s^2+se^{\frac{1}{s}}-\frac{1}{2}+\left(e^{\frac{1}{s}}-1\right)s^2-s-\frac{1}{2}\right\}$$
$$\leq\frac{K(z-\tau_1)^2}{\tau_2-\tau_1}\left[se^{\frac{1}{s}}-s-1\right]+\frac{K(\tau_2-z)^2}{\tau_2-\tau_1}\left[se^{\frac{1}{s}}-s-1\right]$$
$$\leq\frac{K\left(se^{\frac{1}{s}}-s-1\right)}{(\tau_2-\tau_1)}\left[(z-\tau_1)^2+(\tau_2-z)^2\right].$$

□

Remark 4.1. *Let $s=1$ in above Theorem 4.1 to obtain Theorem 8 in [33].*

Corollary 4.1. *(1). By assuming $z=\frac{\tau_1+\tau_2}{2}$ in Theorem 4.1, it yields the subsequent mid-point inequality:*

$$\left|\Psi(\frac{\tau_1+\tau_2}{2})-\frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\Psi(x)dx\right|\leq\frac{K(\tau_2-\tau_1)}{2}\left[se^{\frac{1}{s}}-s-1\right]. \tag{4.3}$$

(2). By assuming $z=\tau_1$ in Theorem 4.1, it leads to the subsequent inequality:

$$\left|\Psi(\tau_1)-\frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\Psi(x)dx\right|\leq K(\tau_2-\tau_1)\left[se^{\frac{1}{s}}-s-1\right]. \tag{4.4}$$

(3). By assuming $z=\tau_2$ in Theorem 4.1, it leads to the subsequent inequality:

$$\left|\Psi(\tau_2)-\frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\Psi(x)dx\right|\leq K(\tau_2-\tau_1)\left[se^{\frac{1}{s}}-s-1\right]. \tag{4.5}$$

Remark 4.2. *By choosing $s=1$ in above Corollary 4.1, we obtain Corollary 4.1 in [33].*

Theorem 4.2. *Suppose that Ψ is a differentiable function mapping from the interval $\Im$, which is a subset of $\mathbb{R}$, to $\mathbb{R}$. The function Ψ is defined on the interior of $\Im$. Consider two points τ_1 and τ_2 in the interior of $\Im$, such that $\tau_1<\tau_2$. Additionally, suppose $\Psi'\in L[\tau_1,\tau_2]$ and consider $q>1$ such that $1-\frac{1}{p}=q^{-1}$. If $|\Psi'|$ is a convex function of $(\alpha-s)$-exponential type on the interval $[\tau_1,\tau_2]$ and satisfies $|\Psi'|\leq K$ for all z in $[\tau_1,\tau_2]$, and subsequently this inequality is valid for each z in $[\tau_1,\tau_2]$, $0\leq\nu\leq 1$ and $0<s\leq 1$:*

$$\left|\Psi(z)-\frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\Psi(x)dx\right|$$
$$\leq\frac{2^{\frac{1}{q}}K}{\tau_2-\tau_1}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[(z-\tau_1)^2\left(\frac{\left(se^{\frac{1}{s}}-s-1\right)}{e^{\alpha z}}+\frac{\left(se^{\frac{1}{s}}-s-1\right)}{e^{\alpha\tau_1}}\right)^{\frac{1}{q}}\right. \tag{4.6}$$
$$\left.+(\tau_2-z)^2\left(\frac{\left(se^{\frac{1}{s}}-s-1\right)}{e^{\alpha z}}+\frac{\left(se^{\frac{1}{s}}-s-1\right)}{e^{\alpha\tau_2}}\right)^{\frac{1}{q}}\right].$$

Proof. Utilizing Lemma 4.1 along with Holder's inequality, we know that $|\Psi'(z)|^q \leq K$ and $|\Psi'|^q$ demonstrates $(\alpha - s)$-exponential type convexity. Consequently, we derive

$$\begin{aligned}
&\left|\Psi(z) - \frac{1}{\tau_2 - \tau_1}\int_{\tau_1}^{\tau_2}\Psi(x)dx\right| \\
\leq& \frac{(z-\tau_1)^2}{\tau_2-\tau_1}\int_0^1 \upsilon\left|\Psi'(\upsilon z + (1-\upsilon)\tau_1)\right|d\upsilon + \frac{(\tau_2-z)^2}{\tau_2-\tau_1}\int_0^1 \upsilon\left|\Psi'(\upsilon z + (1-\upsilon)\tau_2)\right|d\upsilon \\
\leq& \frac{(z-\tau_1)^2}{\tau_2-\tau_1}\left(\int_0^1 \upsilon d\upsilon\right)^{\frac{1}{p}}\left(\int_0^1 \left|\Psi'(\upsilon z + (1-\upsilon)\tau_1)\right|d\upsilon\right)^{\frac{1}{q}} \\
&+ \frac{(\tau_2-z)^2}{\tau_2-\tau_1}\left(\int_0^1 \upsilon d\upsilon\right)^{\frac{1}{p}}\left(\int_0^1 \left|\Psi'(\upsilon z + (1-\upsilon)\tau_2)\right|d\upsilon\right)^{\frac{1}{q}} \\
\leq& \frac{(z-\tau_1)^2}{\tau_2-\tau_1}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\int_0^1 \left(e^{\frac{\upsilon}{s}}-1\right)\frac{|\Psi'(z)|^q}{e^{\alpha z}}d\upsilon + \int_0^1 \left(e^{\frac{1-\upsilon}{s}}-1\right)\frac{|\Psi'(\tau_1)|^q}{e^{\alpha\tau_1}}d\upsilon\right)^{\frac{1}{q}} \\
&+ \frac{(\tau_2-z)^2}{\tau_2-\tau_1}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\int_0^1 \left(e^{\frac{\upsilon}{s}}-1\right)\frac{|\Psi'(z)|^q}{e^{\alpha z}}d\upsilon + \int_0^1 \left(e^{\frac{1-\upsilon}{s}}-1\right)\frac{|\Psi'(\tau_2)|^q}{e^{\alpha\tau_2}}d\upsilon\right)^{\frac{1}{q}} \\
\leq& \frac{(2K^q)^{\frac{1}{q}}(z-\tau_1)^2}{\tau_2-\tau_1}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{\left(se^{\frac{1}{s}}-s-1\right)}{e^{\alpha z}} + \frac{\left(se^{\frac{1}{s}}-s-1\right)}{e^{\alpha\tau_1}}\right)^{\frac{1}{q}}\right] \\
&+ \frac{(2K^q)^{\frac{1}{q}}(\tau_2-z)^2}{\tau_2-\tau_1}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{\left(se^{\frac{1}{s}}-s-1\right)}{e^{\alpha z}} + \frac{\left(se^{\frac{1}{s}}-s-1\right)}{e^{\alpha\tau_2}}\right)^{\frac{1}{q}}\right] \\
\leq& \frac{2^{\frac{1}{q}}K(z-\tau_1)^2}{\tau_2-\tau_1}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{\left(se^{\frac{1}{s}}-s-1\right)}{e^{\alpha z}} + \frac{\left(se^{\frac{1}{s}}-s-1\right)}{e^{\alpha\tau_1}}\right)^{\frac{1}{q}}\right] \\
&+ \frac{2^{\frac{1}{q}}K(\tau_2-z)^2}{\tau_2-\tau_1}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{\left(se^{\frac{1}{s}}-s-1\right)}{e^{\alpha z}} + \frac{\left(se^{\frac{1}{s}}-s-1\right)}{e^{\alpha\tau_2}}\right)^{\frac{1}{q}}\right] \\
\leq& \frac{2^{\frac{1}{q}}K}{\tau_2-\tau_1}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[(z-\tau_1)^2\left(\frac{\left(se^{\frac{1}{s}}-s-1\right)}{e^{\alpha z}} + \frac{\left(se^{\frac{1}{s}}-s-1\right)}{e^{\alpha\tau_1}}\right)^{\frac{1}{q}}\right. \\
&\left. + (\tau_2-z)^2\left(\frac{\left(se^{\frac{1}{s}}-s-1\right)}{e^{\alpha z}} + \frac{\left(se^{\frac{1}{s}}-s-1\right)}{e^{\alpha\tau_2}}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

□

Remark 4.3. *Choosing $s = 1$ results in Theorem 9, which is presented in [33].*

Corollary 4.2. *Assuming $\alpha = 0$ in Theorem 4.6, we have*

$$
\begin{aligned}
&\left|\Psi(z)-\frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\Psi(x)dx\right| \\
\leq&\frac{2^{\frac{1}{q}}K}{\tau_2-\tau_1}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[(z-\tau_1)^2\left(se^{\frac{1}{s}}-s-1\right)^{\frac{1}{q}}+(\tau_2-z)^2\left(se^{\frac{1}{s}}-s-1\right)^{\frac{1}{q}}\right].
\end{aligned}
\tag{4.7}
$$

Corollary 4.3. *(1). By assuming $z=\frac{\tau_1+\tau_2}{2}$ in Corollary 4.2, yields the subsequent mid-point inequality:*

$$
\left|\Psi(\frac{\tau_1+\tau_2}{2})-\frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\Psi(x)dx\right|\leq 2^{\frac{1}{q}-1}K(\tau_2-\tau_1)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(se^{\frac{1}{s}}-s-1\right)^{\frac{1}{q}}. \tag{4.8}
$$

(2). By assuming $z=\tau_1$ in Corollary 4.2, yields the subsequent inequality:

$$
\left|\Psi(\tau_1)-\frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\Psi(x)dx\right|\leq 2^{\frac{1}{q}}K(\tau_2-\tau_1)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(se^{\frac{1}{s}}-s-1\right)^{\frac{1}{q}}. \tag{4.9}
$$

(3). By assuming $z=\tau_2$ in Corollary 4.2, yields the subsequent inequality:

$$
\left|\Psi(\tau_2)-\frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\Psi(x)dx\right|\leq 2^{\frac{1}{q}}K(\tau_2-\tau_1)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(se^{\frac{1}{s}}-s-1\right)^{\frac{1}{q}}. \tag{4.10}
$$

Remark 4.4. *Setting $s=1$ in above Corollary 4.3 yields Corollary 4.1 in [33].*

Theorem 4.3. *Let Ψ be a differentiable function mapping from the interval $\Im$, which is a subset of $\mathbb{R}$, to $\mathbb{R}$. The function Ψ is defined on the interior of $\Im$. Consider two points τ_1 and τ_2 in the interior of $\Im$, such that $\tau_1<\tau_2$. Additionally, suppose $\Psi'\in L[\tau_1,\tau_2]$. If $|\Psi'|$ is a convex function of $(\alpha-s)$ exponential type on the interval $[\tau_1,\tau_2]$ and satisfies $|\Psi'|\leq K$ for all z in $[\tau_1,\tau_2]$, and subsequently this inequality is valid for each z in $[\tau_1,\tau_2]$, $0\leq\nu\leq 1$ and $0<s\leq 1$:*

$$
\begin{aligned}
&\left|\Psi(z)-\frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\Psi(x)dx\right| \\
\leq&\frac{K}{(\tau_2-\tau_1)2^{1-\frac{1}{q}}}\left[(z-\tau_1)^2\left(\left(\frac{(1-e^{\frac{1}{s}})s^2+se^{\frac{1}{s}}-\frac{1}{2}}{e^{\alpha z}}\right)+\left(\frac{(e^{\frac{1}{s}}-1)s^2-s-\frac{1}{2}}{e^{\alpha\tau_1}}\right)\right)^{\frac{1}{q}}\right. \\
&\left.+(\tau_2-z)^2\left(\left(\frac{(1-e^{\frac{1}{s}})s^2+se^{\frac{1}{s}}-\frac{1}{2}}{e^{\alpha z}}\right)+\left(\frac{(e^{\frac{1}{s}}-1)s^2-s-\frac{1}{2}}{e^{\alpha\tau_2}}\right)\right)^{\frac{1}{q}}\right].
\end{aligned}
\tag{4.11}
$$

Proof. By employing Lemma 4.1 as well as the power mean inequality, also assuming that $|\Psi'|^q$ is convex function of $(\alpha - s)$ exponential type with $|\Psi(z)| \leq K$, we obtain the following result:

$$
\begin{aligned}
&\left|\Psi(z) - \frac{1}{\tau_2 - \tau_1}\int_{\tau_1}^{\tau_2}\Psi(x)dx\right| \\
\leq &\frac{(z-\tau_1)^2}{\tau_2-\tau_1}\int_0^1 \upsilon\left|\Psi'(\upsilon z + (1-\upsilon)\tau_1)\right|d\upsilon + \frac{(\tau_2-z)^2}{\tau_2-\tau_1}\int_0^1 \upsilon\left|\Psi'(\upsilon z + (1-\upsilon)\tau_2)\right|d\upsilon \\
\leq &\frac{(z-\tau_1)^2}{\tau_2-\tau_1}\left(\int_0^1 \upsilon d\upsilon\right)^{1-\frac{1}{q}}\left(\int_0^1 \upsilon\left|\Psi'(\upsilon z + (1-\upsilon)\tau_1)\right|d\upsilon\right)^{\frac{1}{q}} \\
&+ \frac{(\tau_2-z)^2}{\tau_2-\tau_1}\left(\int_0^1 \upsilon d\upsilon\right)^{1-\frac{1}{q}}\left(\int_0^1 \upsilon\left|\Psi'(\upsilon z + (1-\upsilon)\tau_2)\right|d\upsilon\right)^{\frac{1}{q}} \\
\leq &\frac{(z-\tau_1)^2}{\tau_2-\tau_1}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1 \upsilon\left(e^{\frac{\upsilon}{s}}-1\right)\frac{|\Psi'(z)|^q}{e^{\alpha z}}d\upsilon + \int_0^1 \upsilon\left(e^{\frac{1-\upsilon}{s}}-1\right)\frac{|\Psi'(\tau_1)|^q}{e^{\alpha\tau_1}}d\upsilon\right)^{\frac{1}{q}} \\
&+ \frac{(\tau_2-z)^2}{\tau_2-\tau_1}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1 \upsilon\left(e^{\frac{\upsilon}{s}}-1\right)\frac{|\Psi'(z)|^q}{e^{\alpha z}}d\upsilon + \int_0^1 \upsilon\left(e^{\frac{1-\upsilon}{s}}-1\right)\frac{|\Psi'(\tau_2)|^q}{e^{\alpha\tau_2}}d\upsilon\right)^{\frac{1}{q}} \\
\leq &\frac{K(z-\tau_1)^2}{\tau_2-\tau_1}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left(\int_0^1 \frac{\upsilon\left(e^{\frac{\upsilon}{s}}-1\right)}{e^{\alpha z}}d\upsilon + \int_0^1 \frac{\upsilon\left(e^{\frac{1-\upsilon}{s}}-1\right)}{e^{\alpha\tau_1}}d\upsilon\right)^{\frac{1}{q}}\right] \\
&+ \frac{K(\tau_2-z)^2}{\tau_2-\tau_1}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left(\int_0^1 \frac{\upsilon\left(e^{\frac{\upsilon}{s}}-1\right)}{e^{\alpha z}}d\upsilon + \int_0^1 \frac{\upsilon\left(e^{\frac{1-\upsilon}{s}}-1\right)}{e^{\alpha\tau_2}}d\upsilon\right)^{\frac{1}{q}}\right] \\
\leq &\frac{K(z-\tau_1)^2}{\tau_2-\tau_1}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left(\left(\frac{(1-e^{\frac{1}{s}})s^2 + se^{\frac{1}{s}} - \frac{1}{2}}{e^{\alpha z}}\right) + \left(\frac{(e^{\frac{1}{s}}-1)s^2 - s - \frac{1}{2}}{e^{\alpha\tau_1}}\right)\right)^{\frac{1}{q}}\right] \\
&+ \frac{K(\tau_2-z)^2}{\tau_2-\tau_1}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left(\left(\frac{(1-e^{\frac{1}{s}})s^2 + se^{\frac{1}{s}} - \frac{1}{2}}{e^{\alpha z}}\right) + \left(\frac{(e^{\frac{1}{s}}-1)s^2 - s - \frac{1}{2}}{e^{\alpha\tau_2}}\right)\right)^{\frac{1}{q}}\right] \\
\leq &\frac{K}{(\tau_2-\tau_1)2^{1-\frac{1}{q}}}\left[(z-\tau_1)^2\left(\left(\frac{(1-e^{\frac{1}{s}})s^2 + se^{\frac{1}{s}} - \frac{1}{2}}{e^{\alpha z}}\right) + \left(\frac{(e^{\frac{1}{s}}-1)s^2 - s - \frac{1}{2}}{e^{\alpha\tau_1}}\right)\right)^{\frac{1}{q}}\right. \\
&\left. + (\tau_2-z)^2\left(\left(\frac{(1-e^{\frac{1}{s}})s^2 + se^{\frac{1}{s}} - \frac{1}{2}}{e^{\alpha z}}\right) + \left(\frac{(e^{\frac{1}{s}}-1)s^2 - s - \frac{1}{2}}{e^{\alpha\tau_2}}\right)\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

□

Remark 4.5. *By putting $s = 1$ in above Theorem* 4.3*, we recover Theorem 10 in [33].*

Corollary 4.4. *Assuming $\alpha = 0$ in Theorem* 4.3*, we obtained*

$$
\begin{aligned}
&\left|\Psi(z)-\frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\Psi(x)dx\right| \\
\leq&\frac{K}{(\tau_2-\tau_1)2^{1-\frac{1}{q}}}\left[(z-\tau_1)^2\left((1-e^{\frac{1}{s}})s^2+se^{\frac{1}{s}}-\frac{1}{2}+(e^{\frac{1}{s}}-1)s^2-s-\frac{1}{2}\right)^{\frac{1}{q}}\right. \\
&\left.+(\tau_2-z)^2\left((1-e^{\frac{1}{s}})s^2+se^{\frac{1}{s}}-\frac{1}{2}+(e^{\frac{1}{s}}-1)s^2-s-\frac{1}{2}\right)^{\frac{1}{q}}\right].
\end{aligned}
\tag{4.12}
$$

Corollary 4.5. *(1). By assuming* $z=\frac{\tau_1+\tau_2}{2}$ *in Corollary* 4.4*, yields the subsequent mid-point inequality:*

$$
\left|\Psi(\frac{\tau_1+\tau_2}{2})-\frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\Psi(x)dx\right|\leq\frac{K(\tau_2-\tau_1)}{4.2^{1-\frac{1}{q}}}\left[\left(se^{\frac{1}{s}}-(1+s)\right)^{\frac{1}{q}}+\left(se^{\frac{1}{s}}-(1+s)\right)^{\frac{1}{q}}\right]. \tag{4.13}
$$

(2). In above Corollary 4.4*, choosing* $z=\tau_1$*, yields the subsequent inequality:*

$$
\left|\Psi(\tau_1)-\frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\Psi(x)dx\right|\leq\frac{K(\tau_2-\tau_1)}{2^{1-\frac{1}{q}}}\left(se^{\frac{1}{s}}-(1+s)\right)^{\frac{1}{q}}. \tag{4.14}
$$

(3). In above Corollary 4.4*, choosing* $z=\tau_2$ *yields the subsequent inequality:*

$$
\left|\Psi(\tau_2)-\frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\Psi(x)dx\right|\leq\frac{K(\tau_2-\tau_1)}{2^{1-\frac{1}{q}}}\left(se^{\frac{1}{s}}-(1+s)\right)^{\frac{1}{q}}. \tag{4.15}
$$

5. Applications

Given a partition d of the interval $[\tau_1,\tau_2]$ with $d:\tau_1=w_0<w_1<\cdots<w_{m-1}<w_m=\tau_2$, Then the formula associated with the trapezoidal rule is

$$
T(\Psi,d)=\sum_{n=0}^{m-1}\frac{\Psi(w_n)+\Psi(w_{n+1})}{2}(w_{n+1}-w_n).
$$

It has been clear that if on the open interval (τ_1,τ_2), $\Psi:[\tau_1,\tau_2]\to\mathbb{R}$ is twice differentiable and $M=\max_{w\in(\tau_1,\tau_2)}|\Psi(w)|<\infty$, then

$$
\int_{\tau_1}^{\tau_2}\Psi(w)dw=T(\Psi,d)+R(\Psi,d). \tag{5.1}
$$

The error term $R(\Psi,d)$ is valid for this inequality

$$
|R(\Psi,d)|\leq\frac{M}{12}\sum_{n=0}^{m-1}(w_{n+1}-w_n)^3. \tag{5.2}
$$

In case the second derivative of Ψ is unbounded or does not exist, in that case (5.1) is not applicable. According to Dragomir and Wang [44–46], $R(\Psi,d)$ can be determined using only the first derivative. This method provides various practical advantages.

Dragomir and Wang [44–46] established that the calculation of $R(\Psi,d)$ can be simplified to using only the first derivative. This finding opens up several practical applications.

Proposition 5.1. *Let Ψ be a differentiable function mapping from the interval $\Im$, which is a subset of $\mathbb{R}$, to $\mathbb{R}$. The function Ψ is defined on the interior of $\Im$. Consider two points τ_1 and τ_2 in the interior of $\Im$, such that $\tau_1 < \tau_2$. If $|\Psi|$ is a convex function of $(\alpha - s)$ exponential type on the interval $[\tau_1, \tau_2]$, it follows for each partition d within the interval $[\tau_1, \tau_2]$, Eq* (5.1) *holds the following:*

$$\begin{aligned}\left|R(\Psi, d)\right| &\le \frac{1}{2}\sum_{n=0}^{m-1}(\tau_{n+1}-\tau_n)^2\left((4e^{\frac{1}{2s}}-2e^{\frac{1}{s}}-2)s^2+(e^{\frac{1}{s}}-1)s-\frac{1}{2}\right)\left[\left|\frac{\Psi'(\tau_n)}{e^{\alpha\tau_n}}\right|+\left|\frac{\Psi'(\tau_{n+1})}{e^{\alpha\tau_{n+1}}}\right|\right]\\ &\le Max\left[\left|\frac{\Psi'(\tau_n)}{e^{\alpha\tau_n}}\right|,\left|\frac{\Psi'(\tau_{n+1})}{e^{\alpha\tau_{n+1}}}\right|\right]\left((4e^{\frac{1}{2s}}-2e^{\frac{1}{s}}-2)s^2+(e^{\frac{1}{s}}-1)s-\frac{1}{2}\right)\sum_{n=0}^{m-1}(\tau_{n+1}-\tau_n)^2.\end{aligned} \tag{5.3}$$

Proof. Applying Theorem 3.2 on the sub interval $[\tau_n, \tau_{n+1}]$ (n = 0, 1, . . . , m-1) for any partition d, we have

$$\tau_1 = \tau_n,\ \tau_2 = \tau_{n+1},$$

$$\begin{aligned}&\left|\frac{\Psi(\tau_n)+\Psi(\tau_{n+1})}{2}-\frac{1}{\tau_{n+1}-\tau_n}\int_{\tau_n}^{\tau_{n+1}}\Psi(x)dx\right|\\ \le&\frac{\tau_{n+1}-\tau_n}{2}\left((4e^{\frac{1}{2s}}-2e^{\frac{1}{s}}-2)s^2+(e^{\frac{1}{s}}-1)s-\frac{1}{2}\right)\left[\left|\frac{\Psi'(\tau_n)}{e^{\alpha\tau_n}}\right|+\left|\frac{\Psi'(\tau_{n+1})}{e^{\alpha\tau_{n+1}}}\right|\right].\end{aligned} \tag{5.4}$$

Summing over the interval of n between 0 and $m-1$ yields

$$\begin{aligned}&\left|T(\Psi, d)-\int_{\tau_1}^{\tau_2}\Psi(\tau)d\tau\right|\\ \le&\frac{1}{2}\sum_{n=0}^{m-1}(\tau_{n+1}-\tau_n)^2\left((4e^{\frac{1}{2s}}-2e^{\frac{1}{s}}-2)s^2+(e^{\frac{1}{s}}-1)s-\frac{1}{2}\right)\left[\left|\frac{\Psi'(\tau_n)}{e^{\alpha\tau_n}}\right|+\left|\frac{\Psi'(\tau_{n+1})}{e^{\alpha\tau_{n+1}}}\right|\right]\\ \le&Max\left[\left|\frac{\Psi'(\tau_1)}{e^{\alpha\tau_1}}\right|,\left|\frac{\Psi'(\tau_2)}{e^{\alpha\tau_2}}\right|\right]\left((4e^{\frac{1}{2s}}-2e^{\frac{1}{s}}-2)s^2+(e^{\frac{1}{s}}-1)s-\frac{1}{2}\right)\sum_{n=0}^{m-1}(\tau_{n+1}-\tau_n)^2.\end{aligned} \tag{5.5}$$

□

Remark 5.1. *Putting $s = 1$ in above Proposition* 5.1*, we obtain Proposition 2 in [33].*

Proposition 5.2. *Let Ψ be a differentiable function mapping from the interval $\Im$, which is a subset of $\mathbb{R}$, to $\mathbb{R}$. The function Ψ is defined on the interior of $\Im$. Consider two points τ_1 and τ_2 in the interior of $\Im$, such that $\tau_1 < \tau_2$. Assuming that $|\Psi|$ is a function of $(\alpha - s)$ exponential type convexity over the interval $[\tau_1, \tau_2]$, and given that $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then for each partition d within the interval $[\tau_1, \tau_2]$, under the framework of* (5.1)*, it concludes:*

$$\begin{aligned}\left|R(\Psi, d)\right| &\le \frac{\left(-1+se^{\frac{1}{s}}-s\right)^{\frac{1}{q}}}{2}\sum_{n=0}^{m-1}(\tau_{n+1}-\tau_n)^2\left[\left|\frac{\Psi'(\tau_n)}{e^{\alpha\tau_n}}\right|+\left|\frac{\Psi'(\tau_{n+1})}{e^{\alpha\tau_{n+1}}}\right|\right]^{\frac{1}{q}}\\ \le&Max\left[\left|\frac{\Psi'(\tau_n)}{e^{\alpha\tau_n}}\right|,\left|\frac{\Psi'(\tau_{n+1})}{e^{\alpha\tau_{n+1}}}\right|\right]\frac{\left(-1+se^{\frac{1}{s}}-s\right)^{\frac{1}{q}}}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\sum_{n=0}^{m-1}(\tau_{n+1}-\tau_n)^2.\end{aligned} \tag{5.6}$$

Proof. By applying Theorem 3.3 and employing reasoning analogous to that in Proposition 5.1, we obtained the desired result. This method facilitated an effective derivation of the result.

□

Remark 5.2. *By letting* $s = 1$ *in above Proposition* 5.2*, we obtain Proposition 3 in [33].*

6. Conclusions

This paper examines $(\alpha - s)$ exponential type convex functions. These functions extend the conventional exponential-type convex functions. This research proves the Hermite-Hadamard inequality for $(\alpha - s)$ exponential-type convex functions. New Ostrowski-type inequalities are also established by using this function. This research explores various applications of these findings. This innovative approach may help to discover different types of integral inequalities. To the best of our knowledge, these results are original and have not been documented before. Convex functions are used in numerous mathematical fields. We anticipate that our findings could be useful in fields like convex analysis, special functions, quantum analysis, and quantum mechanics. It would be interesting to derive analogous inequalities for functions of two or more variables. This would offer additional insights into the broader applicability of this research.

Author contributions

Attazar Bakht: Writting-original draft, Methodology, Investigation, Conceptualization; Matloob Anwar: Writing-review & editing, validation, Supervision, Investigation

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Conflict of interest

The authors declare they have no conflicts of interest.

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