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Research article

Stability analysis of Caputo fractional time-dependent systems with delay using vector lyapunov functions

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Abstract: In this study, we investigate the stability and asymptotic stability properties of Caputo fractional time-dependent systems with delay by employing vector Lyapunov functions. Utilizing the Caputo fractional Dini derivative on Lyapunov-like functions, along with a new comparison theorem and differential inequalities, we derive and prove sufficient conditions for the stability and asymptotic stability of these complex systems. An example is included to showcase the method's practicality and to specifically illustrate its advantages over scalar Lyapunov functions. Our results improves, extends, and generalizes several existing findings in the literature.

Keywords: stability; asymptotic stability; Caputo derivative; vector Lyapunov function; fractional delay differential equation
Mathematics Subject Classification: 34A08, 34A34, 34D20, 34K37

1. Introduction

Fractional calculus extends traditional differentiation and integration concepts to non-integer orders, and it has gained considerable academic interest in recent decades due to its efficacy in modeling various real-world systems. Fractional derivatives are instrumental in describing mechanical and electrical properties of materials, as well as the behaviors of gases, liquids, and minerals across diverse fields. For foundational understanding, refer to the monographs [1–4] and their cited references.

Fractional time-dependent systems with delays have gained prominence for their enhanced accuracy in capturing memory and hereditary behaviors. Studies have explored the existence and uniqueness of solutions for fractional differential systems, both with and without delays, in works such as [5–8]. For instance, Deng et al. [9] derived stability criteria for fractional differential systems with multiple time delays by employing the Laplace transform to convert fractional differential equations (FDEs) into algebraic equations in the Laplace domain, analyzing stability based on the poles of the resulting transfer function.

Cermak et al. [10] investigated the stability of solutions to FDEs with constant delays using Lyapunov functional methods and fractional calculus tools. They derived stability conditions based on the nonlinear function f, the delay τ, and the fractional order α. Their results indicate that if specific conditions on f and τ are met, the zero solution of the FDE is asymptotically stable. They also explored the asymptotic behavior of solutions, providing estimates for the rate of decay over time, which depends on the fractional order α. Specifically, for $0 < \alpha < 1$, the solution decays at a rate proportional to $t^{-\alpha}$ as $t \to \infty$.

Tuan [11] focused on the stability analysis of nonlinear delay fractional differential equations (DFDEs) by developing a linearized stability theorem that extends classical results to FDEs with delays. This theorem offers conditions under which equilibrium solutions of such systems are asymptotically stable, providing a robust framework for analyzing equilibrium solutions in systems with fractional dynamics and time delays. Similarly, Li and Wang [12] addressed the stability analysis of fractional delay differential equations (FDDEs) by exploring delayed Mittag-Leffler type matrix functions to determine conditions for convergence to an equilibrium point within a finite time. Thanh [13] proposed new criteria for finite-time stability of systems with singular FDEs and time-varying delays. Using a Lyapunov-Krasovskii functional designed to handle fractional orders and time-varying delays, stability conditions are expressed in terms of linear matrix inequalities (LMIs), which offer a convenient computational framework.

Following the discussion so far on stability, its critical importance in the dynamics of systems, especially those with feedback control, deserves further attention. For linear fractional systems, various reliable methods have been established to maintain stability (see [14–17]). Meanwhile, Lyapunov stability theory provides a strong foundation for analyzing nonlinear systems. In particular, Lyapunov's second method, or direct method, is highly effective because it allows for stability assessment without requiring the explicit solution of the system's differential equations, making it a versatile tool for stability analysis (see [18–21]).

In [22], Argawal et al. identified three types of fractional derivatives of Lyapunov functions used in stability analysis of time-dependent systems with delay: the Dini fractional derivative, Caputo fractional Dini derivative, and Dini fractional derivative. The Caputo fractional derivative is commonly used and is defined as:

$$ {}_{t_0}^{C}D_t^{\alpha} V(t, g(t)) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} (t-\xi)^{-\alpha} \frac{d}{d\xi}\Big(V(s, g(\xi))\Big) d\xi, \quad t \in \mathbb{R}_+, \alpha \in (0,1). $$

However, this derivative has limitations, as it requires the use of the Razumikhin criterion over the entire delay interval and differentiable Lyapunov functions. For studying stability characteristics, primarily quadratic Lyapunov functions are used (see [23]). The Dini fractional derivative does not have this drawback, maintaining the concept of fractional derivatives due to its memory property, and is defined as:

$$ {}^{C}D_{+}^{\alpha}V(t,\phi(0),\phi) = \limsup_{h\to 0^+} \frac{1}{h^\alpha}\left\{V(t,\phi(0)) - \sum_{l=1}^{[\frac{t-t_0}{h}]}(-1)^l\binom{\alpha}{l}V(t-lh,\phi(0)-h^\alpha f(t,\omega(0)))\right\}, \quad (1.1) $$
$$ t\in\mathbb{R}_+, \alpha\in(0,1), f\in C[R_+\times\mathbb{R}^n,\mathbb{R}^n], $$

and the Caputo fractional Dini derivative is defined as:

$$ {}_{t_0}^{C}D_{+}^{\alpha}g(t) = \lim_{h\to 0^+}\frac{1}{h^\alpha}\left\{g(t)-g(t_0)-\sum_{l=1}^{[\frac{t-t_0}{h}]}(-1)^{l+1}\binom{\alpha}{l}\Big[g(t-lh)-g(t_0)\Big]\right\}, \quad \alpha\in(0,1). \quad (1.2) $$

The Caputo fractional Dini derivative has been used to analyze various stability types of Caputo fractional time-dependent systems with and without delay (see [22,24–28]). Scalar Lyapunov functions may not fully capture interactions among dimensions. Vector Lyapunov functions, on the other hand, offer greater flexibility and precision in constructing stability criteria for complex systems, providing a more detailed analysis of subsystems and their interactions. They are particularly useful for examining nonlinear systems where interactions can be intricate and nonlinear relationships are prevalent (see [29–32]).

Let $\mathbb{R}_+ = [0,\infty)$ and assume that $t_0 \geq 0 \in \mathbb{R}_+$. Let $J_0 = [-\gamma, 0]$, $J = [-\gamma,\infty), \gamma > 0$ and $I = [t_0, T]$ be intervals in $\mathbb{R}$. Let $\mathfrak{D}^N = C(J_0,\mathbb{R}^N)$ be the space of all continuous maps on J_0, where $\mathbb{R}^N$ is the N-dimensional Euclidean vector space endowed with norm $||.||$. For any $\phi\in\mathfrak{D}^N$, we define the norm of ϕ by

$$ ||\phi||_0 = \sup_{s\in J_0}||\phi(s)||. $$

In this paper, we consider the retarded Caputo fractional time-dependent system of the form

$$ \begin{cases} {}^{C}D^{\alpha}g(t) = f(t,g(t),g_t),\ t\geq t_0, \\ g_{t_0} = \omega_0, \end{cases} \quad (1.3) $$

where ${}^{C}D^{\alpha}$ denotes the Caputo fractional derivative of order $\alpha\in(0,1)$, $t\in J$, $g\in\mathbb{R}^N$, $\omega_0\in\mathfrak{D}^N$, and $f\in C(\mathbb{R}\times B_\rho\times\mathfrak{D}^N,\mathbb{R}^N)$. Here, $g_t\in\mathfrak{D}^N$ represents the history of the state from time $t-\gamma$ to the present time t, defined by $g_t(s) = g(t+s)$, $s\in J_0$. In other words, $g_t = \{g(\tau):\tau\leq t\}$ represents the trajectory of the solution in the past.

We assume that the following conditions hold:

(1) The function f guarantees that for any initial condition $(t_0,\omega_0)\in\mathbb{R}_+\times\mathfrak{D}^N$, the system (1.3) possesses a solution $g(t_0,\omega_0)(t)\in C^q([t_0,T],\mathbb{R}^N)$.
(2) $f(t,0,0) = 0$ for $t\geq t_0$.

We will utilize comparison results for the Caputo fractional time-varying system of the form

$$ \begin{cases} {}^{C}D^{\alpha}u(t) = \zeta(t,u,u_t),\ t\geq t_0, \\ u_{t_0} = \theta_0, \end{cases} \quad (1.4) $$

where $u \in \mathbb{R}^n$, $\zeta \in C[\mathbb{R}_+ \times \mathbb{R}^n \times \mathfrak{D}^n, \mathbb{R}^n]$, $\mathfrak{D}^n = C(J_0, \mathbb{R}^n)$ and $\zeta(t, 0, 0) \equiv 0$. The function ζ ensures that for any initial values $(t_0, \theta_0) \in \mathbb{R}_+ \times \mathfrak{D}^n$, the system (1.4) with the given initial condition has a solution $u(t_0, \theta_0)(t) \in C^\alpha([t_0, T], \mathbb{R}^n)$.

This paper's primary goal is to use vector Lyapunov functions to examine the stability characteristics of Caputo fractional time-dependent systems with delay. This study utilizes the definition of the Caputo fractional Dini derivative for Lyapunov-like functions as introduced in [22, 25], along with the application of the comparison theorem and differential inequalities.

2. Preliminaries

In this paper, we adopt the Caputo (C) definition for fractional derivative, which is expressed as follows:

$$ {}_{t_0}^{C}D_t^\alpha g(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} (t-\xi)^{n-\alpha-1} g^{(n)}(\xi) d\xi, \; t \geq t_0. $$

It is important to note that the Caputo approach has the advantage that the initial conditions for fractional differential equations using the Caputo derivative are expressed in the same form as those for integer-order differentiation, which have well-established physical meanings. There exist various definitions for fractional derivatives. Among the widely used definitions is the Grunwald-Letnikov (GL) fractional derivative, which is expressed as:

$$ {}_{t_0}^{GL}D_t^\alpha g(t) = \lim_{h \to 0^+} \frac{1}{h^\alpha} \sum_{l=0}^{[\frac{(t-t_0)}{h}]} (-1)^l \binom{\alpha}{l} g(t - lh), \; t \geq t_0. $$

The Riemann-Liouville (RL) fractional derivative is of the form:

$$ {}_{t_0}^{RL}D_t^\alpha g(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{t_0}^{t} (t-\xi)^{n-\alpha-1} g(\xi) d\xi, \; t \geq t_0. $$

In all the definitions given above, we have that $n - 1 < \alpha < n, \alpha > 0$, where n is a natural number and $\Gamma(\cdot)$ represents the gamma function. In most applications, the order of α is often less than 1, so that $\alpha \in (0, 1)$. For simplicity of notation, we will use ${}^{C}D^\alpha$ instead of ${}_{t_0}^{C}D_t^\alpha$ so that the Caputo fractional derivative of order α of the function $g(t)$ is given as

$$ {}^{C}D^\alpha g(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} (t-\xi)^{\alpha-1} g'(\xi) d\xi, \; t \geq t_0. \tag{2.1} $$

In this paper, we define the following sets:

$$ \begin{aligned} B_\rho &= \{g \in \mathbb{R}^N : \|g\| < \rho, \; \rho > 0\}, \\ S_\rho &= \{g \in \mathbb{R}^n : \|g\| < \rho, \; \rho > 0\}, \\ C_\rho &= \{\omega \in \mathfrak{D}^N : \|\omega\|_0 < \rho, \; \rho > 0\}. \end{aligned} $$

Remark 2.1. *In the definitions mentioned above and throughout this paper, $n \leq N$.*

Definition 2.1. *[2] The Grunwald-Letnikov (GL) fractional Dini derivative is given by*

$$ {}_{t_0}^{GL}D_+^{\alpha}g(t) = \limsup_{h\to 0^+} \frac{1}{h^{\alpha}} \sum_{l=0}^{[\frac{(t-t_0)}{h}]} (-1)^l \binom{\alpha}{l} g(t-lh),\ t \geq t_0. $$

Definition 2.2. *A function $V(t, g_t) : J \times C_{\rho} \to \mathbb{R}_+^N$ is considered a vector Lyapunov function for* (1.3) *if it is continuous on $J \times C_{\rho}$, satisfies $V(t, 0) = 0$, and is locally Lipschitz continuous with respect to the second argument.*

Definition 2.3. *[22, 25] Let $(t_0, \omega_0) \in \mathbb{R}_+ \times C[J_0, B_{\rho}]$ represent the initial condition of the initial value problem (IVP)* (1.3) *with $f \in C(\mathbb{R} \times B_{\rho} \times \mathfrak{D}^N, \mathbb{R}^N)$. The Caputo fractional Dini Derivative of the Lyapunov function $V(t, g_t)$ is defined as*

$$ \begin{aligned} {}^{C}D_+^{\alpha}V(t,\omega(0),\omega) = \ & \limsup_{h\to 0^+} \frac{1}{h^{\alpha}} \Big\{ V(t,\omega(0)) - V(t_0,\omega_0(0)) - \sum_{l=1}^{[\frac{t-t_0}{h}]} (-1)^{l+1} \binom{\alpha}{l} \\ & \times \Big[V(t-lh, \omega(0) - h^{\alpha} f(t,\omega(0))) - V(t_0, \omega_0(0)) \Big] \Big\}, \end{aligned} \tag{2.2} $$

where it is understood that $\omega(0) = g(t_0, \omega_0)(t)$ is the state of the system (1.3) *at the current time t. $\omega_0(0)$ is the initial condition of the system* (1.3) *at the beginning $t = 0$. Equivalently,* (2.2) *can be written as*

$$ \begin{aligned} {}^{C}D_+^{\alpha}V(t,\omega(0),\omega) \ = \ & \limsup_{h\to 0^+} \frac{1}{h^{\alpha}} \Big\{ V(t,\omega(0)) + \sum_{l=1}^{[\frac{t-t_0}{h}]} (-1)^{l} \binom{\alpha}{l} V(t-lh, \omega(0) - h^{\alpha} f(t,\omega(0))) \Big\} \\ & - \frac{V(t_0,\omega_0(0))}{(t-t_0)^{\alpha}\Gamma(1-\alpha)}. \end{aligned} \tag{2.3} $$

Definition 2.4. *A function $G \in C[\mathbb{R}^n, \mathbb{R}^n]$ is considered quasi-monotone nondecreasing in x if, whenever $x \leq y$ and $x_i = y_i$ for $1 \leq i \leq n$, it follows that $G_i(x) \leq G_i(y)$ for all i.*

Definition 2.5. *[30] A function $a(r)$ is considered to be in the class $\mathcal{K}$ if a is a continuous function on $[0, \rho)$ with values in $\mathbb{R}_+$, $a(0) = 0$, and $a(r)$ is strictly increasing in r.*

Definition 2.6. *[33] The zero solution of* (1.3) *is considered*

(1) stable if, for every initial time $t_0 \in \mathbb{R}_+$ and any $\epsilon > 0$, there exists a $\delta = \delta(\epsilon, t_0) > 0$, continuous in t_0, such that for any initial function $\omega_0 \in \mathfrak{D}^N$ with $\|\omega_0\|_0 \leq \delta$, it follows that $\|g(t_0, \omega_0)(t)\| < \epsilon$ for $t \geq t_0$.

(2) asymptotically stable if, for every initial time $t_0 \in \mathbb{R}_+$ and any $\epsilon > 0$, there exists a $\delta = \delta(\epsilon, t_0) > 0$, continuous in t_0, such that for any initial function $\omega_0 \in \mathfrak{D}^N$ with $\|\omega_0\|_0 \leq \delta$, it follows that $\|g(t_0, \omega_0)(t)\| < \epsilon$ for $t \geq t_0$ and $\lim_{t\to\infty} \|g(t_0, \omega_0)(t)\| = 0$.

3. Main result

In this section, we present our findings on the stability and asymptotic stability of Caputo fractional time-dependent systems with delay. Our results are structured around lemmas and theorems that define the necessary conditions for stability and asymptotic stability.

Lemma 3.1. *Assume $p(t), r(t) \in C([t_0, T), \mathbb{R}^N)$ and suppose there exists $\tau_* \in (t_0, T]$ such that $p(\tau_*) = r(\tau_*)$ and $p(t) < r(t)$ for $t \in [t_0, \tau_*)$. The inequality ${}^C D_+^\alpha p(\tau_*) >{}^C D_+^\alpha r(\tau_*)$ holds if the Caputo fractional Dini derivative of p and r exists at $t = \tau_*$ for $\alpha \in (0, 1)$.*

Proof. Applying the definition of the Caputo Dini derivative in (2.3), we have

$$\begin{aligned}
&{}^C D_+^\alpha p(\tau_*) -{}^C D_+^\alpha r(\tau_*) \\
= \ & \limsup_{h\to 0^+} \frac{1}{h^\alpha}\Big\{p(\tau_*) + \sum_{l=1}^{[\frac{\tau-\tau_0}{h}]} (-1)^l \binom{\alpha}{l} p(\tau_* - lh)\Big\} - \frac{p(\tau_0)(\tau-\tau_0)^{-\alpha}}{\Gamma(1-\alpha)} \\
&-\Big(\limsup_{h\to 0^+} \frac{1}{h^\alpha}\Big\{r(\tau_*) + \sum_{l=1}^{[\frac{\tau-\tau_0}{h}]} (-1)^l \binom{\alpha}{l} r(\tau_* - lh)\Big\} - \frac{r(\tau_0)(\tau-\tau_0)^{-\alpha}}{\Gamma(1-\alpha)}\Big).
\end{aligned}$$

It is clear from the hypothesis of the lemma that for $\tau_* \in (\tau_0, T]$, $p(\tau_*) - r(\tau_*) = 0$ so that

$$\begin{aligned}
{}^C D_+^\alpha p(\tau_*) -{}^C D_+^\alpha r(\tau_*) \ = \ & \limsup_{h\to 0^+} \frac{1}{h^\alpha}\Big\{\sum_{l=1}^{[\frac{\tau-\tau_0}{h}]} (-1)^l \binom{\alpha}{l}\Big[p(\tau_* - lh) - r(\tau_* - lh)\Big]\Big\} \\
&-\frac{(p(\tau_0) - r(\tau_0))(\tau-\tau_0)^{-\alpha}}{\Gamma(1-\alpha)}.
\end{aligned}$$

Taking limit as $h \to 0^+$, we have

$${}^C D_+^\alpha p(\tau_*) -{}^C D_+^\alpha r(\tau_*) \ = \ -\frac{(p(\tau_0) - r(\tau_0))(\tau-\tau_0)^{-\alpha}}{\Gamma(1-\alpha)}.$$

Again by the hypothesis of the lemma, for $\tau = \tau_0$, $p(\tau_0) - r(\tau_0) < 0$ together with the fact that $\frac{(\tau-\tau_0)^{-\alpha}}{\Gamma(1-\alpha)} > 0$, leads to

$${}^C D_+^\alpha p(\tau_*) >{}^C D_+^\alpha r(\tau_*)$$

hence the result. □

Lemma 3.2. *Let $w, s : [t_0 - \gamma, T] \to \mathbb{R}^n$ be continuous on $[t_0, T]$, and let $\zeta \in C([t_0, T] \times \mathbb{R}^n \times C_q, \mathbb{R}^n)$ be quasi-monotone nondecreasing in w_t for each $(t, w) \in \mathbb{R}^n$. Additionally, for each t, we have*

(i) ${}^C D_+^\alpha w(t) \le \zeta(t, w, w_t)$,
(ii) ${}^C D_+^\alpha s(t) > \zeta(t, s, s_t)$, $t \in [t_0, T]$.

Then

$$w_{t_0} < s_{t_0}, \tag{3.1}$$

implies

$$w(t) < s(t),\ t \in [t_0, T]. \tag{3.2}$$

Proof. Assume that the conclusion (3.2) of the theorem is false. Then, there would be a $t_1 > t_0$ such that

$$w(t_1) = s(t_1) \text{ and } w(t) < s(t) \text{ for } t \in [t_0, t_1). \tag{3.3}$$

Applying Lemma 3.1, we obtain

$$^{C}D_{+}^{\alpha}w(t_1) >^{C} D_{+}^{\alpha}s(t_1). \tag{3.4}$$

Furthermore, from (3.1) and (3.3), we deduce that

$$w_{t_1} \leq s_{t_1}. \tag{3.5}$$

Combining condition (i), (3.4), condition (ii), (3.5), and the quasi-monotonicity of G, we have that at $t = t_1$

$$\zeta(t_1, w, w_{t_1}) \geq^{C} D_{+}^{\alpha}w(t_1) >^{C} D_{+}^{\alpha}s(t_1) \geq \zeta(t_1, s, s_{t_1}) \geq \zeta(t_1, w, w_{t_1}),$$

which is a contradiction, thus (3.5) is true. □

Theorem 3.1. *Let $\zeta \in C[R_c, \mathbb{R}^n]$, where $R_c \subset \mathbb{R}_+ \times \mathbb{R}^n \times C_q$ such that $R_c := \{(t, u, \xi) : t_0 \leq t \leq t_0 + a,\ \|u - \theta_0(0)\| \leq b,\ \|\xi - \theta_0\|_0 \leq b,\ u \in \mathbb{R}^n,\ \xi \in C_q := \{\xi \in \mathfrak{D}^n : \|\xi\| < q, q > 0\},\ \theta_0 \in \mathfrak{D}^n,\ a, b > 0\}$ and $\|\zeta(t, u, u_t)\| \leq H$ on R_c. Assume that $\zeta(t, u, u_t)$ is quasi-monotone nondecreasing in u_t for every $(t, u) \in \mathbb{R}_+ \times \mathbb{R}^n$. Then, the IVP (1.4) has a maximal solution $h(t, (t_0, \theta_0))$ defined on the interval $[t_0, t_0 + q]$, where $q = min\left\{a, \left(\frac{b\Gamma(\alpha+1)}{2H+b}\right)^{\frac{1}{\alpha}}\right\}$ and $\alpha \in (0, 1)$.*

Proof. Let $\eta \in \mathbb{R}^n_+$ be a small arbitrary vector, such that $\|\eta\| < \frac{b}{2}e$, where $e = (1, 1, ...1)^T$ with $\|e\| = 1$.

Consider the IVP for the following Caputo fractional time-dependent system of the form:

$$\begin{cases} ^{C}D^{\alpha}u_{\eta} = \zeta_{\eta}(t, u, u_t) + \eta, \\ u_{t_0} = \theta_0 + \eta, \end{cases} \tag{3.6}$$

where $\zeta_{\eta}(t, u, u_t) + \epsilon$ is continuous on R_{η} and is given by $R_{\eta} = \{R_{\eta} := \{(t, u, \xi) : t_0 \leq t \leq t_0 + a,\ \|u - (\theta_0(0) + \eta)\| \leq \frac{b}{2},\ \|\xi - (\theta_0 + \eta)\|_0 \leq \frac{b}{2},\ u \in \mathbb{R}^n,\ \xi \in C_q,\ \theta \in \mathfrak{D}^n,\ a, b > 0\}$ and $\|\zeta_{\eta}(t, u, u_t) + \eta\| \leq H$ on R_{η} with $R_{\eta} \subset R_c$.

Integrating (3.6) from t_0 to t in the Caputo sense, we obtain

$$u_{\eta}(t_0, \theta_0)(t) = \theta_0 + \eta + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - \xi)^{\alpha-1} \Big(\zeta_{\eta}(\xi, u(\xi), u_{\xi}) + \eta\Big) d\xi. \tag{3.7}$$

Now, consider the family of solutions $\{u_{\eta}(t_0, \theta_0)(t)\}$ on $[t_0, t_0 + q]$. Then from (3.7)

$$\begin{aligned} \|u_{\eta}(t_0, \theta_0)(t)\| &= \left\|\theta_0 + \eta + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - \xi)^{\alpha-1} \Big(\zeta_{\eta}(\xi, u(\xi), u_{\xi}) + \eta\Big)\right\| d\xi \\ &\leq \|\theta_0\|_0 + \|\eta\| + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - \xi)^{\alpha-1} \left(\left\|\zeta_{\eta}(\xi, u(\xi), u_{\xi})\right\| + \|\eta\|\right) d\xi \\ &\leq \|\theta_0\|_0 + \frac{b}{2} + \frac{1}{\Gamma(\alpha)} \Big(\frac{2H + b}{2}\Big) \frac{a^{\alpha}}{\alpha} = K. \end{aligned}$$

Therefore

$$\|u_{\eta}(t_0, \theta_0)(t)\| \leq K.$$

Thus, the set of solutions $\{u_\eta(t_0,\theta_0)(t)\}$ has a uniform bound with bound K. We take $t_1, t_2 \in [t_0, t_0+q]$, with $t_1 < t_2$, and produce the following estimate to demonstrate that the family of solutions $\{u_\eta(t_0,\theta_0)(t)\}$ is equi-continuous.

$$\begin{aligned}
&\|u_\eta(t_0,\theta_0)(t_2) - u_\eta(t_0,\theta_0)(t_1)\| = \Bigg\|\theta_0 + \eta + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_2}(t_2-\xi)^{\alpha-1}\Big(\zeta_\eta(\xi, u(\xi), u_\xi) + \eta\Big) \\
&-\Big(\theta_0 + \eta + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}(t_1-\xi)^{\alpha-1}\Big(\zeta_\eta(\xi, u(\xi), u_\xi) + \eta\Big)\Big)\Bigg\|\mathrm{d}\xi \\
= \;& \frac{1}{\Gamma(\alpha)}\Bigg\|\int_{t_0}^{t_2}(t_2-\xi)^{\alpha-1}\Big(\zeta_\eta(\xi, u(\xi), u_\xi) + \eta\Big) - \int_{t_0}^{t_1}(t_1-\xi)^{\alpha-1}\Big(\zeta_\eta(\xi, u(\xi), u_\xi) + \eta\Big)\Bigg\|\mathrm{d}\xi \\
= \;& \frac{1}{\Gamma(\alpha)}\Bigg\|\Big(\int_{t_0}^{t_1}(t_2-\xi)^{\alpha-1} - \int_{t_0}^{t_1}(t_1-\xi)^{\alpha-1}\Big)\Big(\zeta_\eta(\xi, u(\xi), u_\xi) + \eta\Big)\mathrm{d}\xi \\
&+ \int_{t_1}^{t_2}(t_2-\xi)^{\alpha-1}\Big(\zeta_\eta(\xi, u(\xi), u_\xi) + \eta\Big)\Bigg\|\mathrm{d}\xi \\
\leq \;& \frac{2H+b}{2\Gamma(\alpha)}\Bigg[\Bigg|\int_{t_0}^{t_1}(t_2-\xi)^{\alpha-1} - \int_{t_0}^{t_1}(t_1-\xi)^{\alpha-1}\Bigg|\mathrm{d}\xi + \Bigg|\int_{t_1}^{t_2}(t_2-\xi)^{\alpha-1}\Bigg|\mathrm{d}\xi\Bigg] \\
= \;& \frac{2H+b}{2\alpha\Gamma(\alpha)}\Big[(t_1-t_0)^\alpha + (t_2-t_1)^\alpha - (t_2-t_0)^\alpha + (t_2-t_1)^\alpha\Big] \\
\leq \;& \frac{2H+b}{\Gamma(\alpha+1)}(t_2-t_1)^\alpha < \epsilon,
\end{aligned}$$

provided $|t_2 - t_1| < \delta(\epsilon) = (\frac{\epsilon\Gamma(\alpha+1)}{2H+b})^{\frac{1}{\alpha}}$, hence the family $\{u_\eta(t_0,\theta_0)(t)\}$ is equi-continuous on $[t_0, t_0+q]$. Then, by the Arzela-Ascoli theorem, $\lim_{i\to\infty} u_{\eta_i}(t_0,\theta_0)(t) = h(t_0,\theta_0)(t)$ uniformly on $[t_0, t_0+q]$ for every decreasing sequence $\{\eta_i\}$, $\eta_i \to 0$ as $i \to \infty$.. The uniform continuity of ζ implies that $\zeta(t, u_t(t_0,\theta_0)) + \eta_i$ tends uniformly to $\zeta(t, h_t(t_0,\theta_0))$ as $\eta_i \to \infty$. Taking limit as $i \to \infty$ in (3.7) leads to

$$h(t_0,\theta_0)(t) = \theta_0 + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\xi)^{\alpha-1}\zeta(\xi, h(\xi), h_\xi)\mathrm{d}\xi,$$

which demonstrates that the limit $h(t_0,\theta_0)(t)$ is truly a solution of (1.4) on the interval $[t_0, t_0+q]$.

It is left to show that $h(t_0,\theta_0)(t)$ is the maximal solution of the comparison system (1.4). Let $u(t_0,\theta_0)(t)$ be any solution of the IVP (1.4) on $[t_0, t_0+q]$. Then in light of Lemma 3.2, we have that

$$\begin{aligned}
{}^{C}D_+^\alpha u(t_0,\theta_0)(t) &\leq \zeta(t, (t_0,\theta_0), u_t) \\
{}^{C}D_+^\alpha u_{\eta_i}(t_0,\theta_0)(t) + \eta_i &> \zeta(t, (t_0,\theta_0), u_t) + \eta_i.
\end{aligned}$$

Then $\theta_0 < \theta_0 + \eta$, $\eta > 0$ implies that $u(t_0,\theta_0)(t) < u_{\eta_i}(t_0,\theta_0)(t) + \eta_i$.

Since $\lim_{i\to\infty} u_{\eta_i}(t_0,\theta_0)(t) = h(t_0,\theta_0)(t)$ uniformly on $[t_0, t_0+q]$, it follows by taking limits that $u(t_0,\theta_0)(t) < \lim_{i\to\infty}\{u_{\eta_i}(t_0,\theta_0)(t) + \eta_i\} = h(t_0,\theta_0)(t)$ and so the result follows. □

Theorem 3.2. *Assume that*

(1) $V \in C[(-\gamma,\infty) \times C_\rho, \mathbb{R}_+^N]$, where $V(t, g_t)$ is locally Lipschitz continuous with respect to the second argument.

(2) $\zeta \in C[\mathbb{R}_+ \times \mathbb{R}^n \times \mathfrak{D}_q, \mathbb{R}^n]$ *and* $\zeta(t, u, u_t)$ *is quasi-monotone nondecreasing with respect to* u_t.
(3) ${}^C D_+^\alpha V(t, \omega(0), \omega) \le \zeta(t, V(t, \omega(0)), V_t)$ *for all* $t \in \mathbb{R}_+$, *where* $V_t = V(t + s, \omega(s))$, $\xi \in J_0$.

If $h(t_0, \theta_0)(t)$ *is the maximal solution of* (1.4) *and* $g(t_0, \omega_0)(t)$ *is any solution of* (1.3) *defined in the future such that*

$$\sup_{\xi \in J_0} V(t_0, \omega_0)(\xi) \le \theta_0, \tag{3.8}$$

then the inequality

$$V(t, g(t_0, \omega_0)(t)) \le h(t_0, \theta_0)(t),\ t \ge t_0, \tag{3.9}$$

holds.

Proof. Let $g(t_0, \omega_0)(t)$ be any solution of (1.3) such that (3.8) holds.

For an arbitrary vector $\eta \in \mathbb{R}_+^n$ of sufficiently small magnitude, we examine the IVP associated with the Caputo fractional time-dependent system with delay as follows.

$$\begin{cases} {}^C D^\alpha u_\eta = \zeta_\eta(t, u, u_t) + \eta, \\ u_{t_0} = \theta_0 + \eta, \end{cases} \tag{3.10}$$

for $t \in \mathbb{R}_+$, where the solution $u_\eta(t_0, \theta_0)(t)$ exists as long as the maximal solution $h(t_0, \theta_0)(t)$ to the right of t_0 and satisfies the Volterra integral equation

$$u_\eta(t_0, \theta_0)(t) = \theta_0 + \eta + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - \xi)^{\alpha - 1} \Big(\zeta_\eta(\xi, u(\xi), u_\xi) + \eta \Big) d\xi,\ t \in \mathbb{R}_+. \tag{3.11}$$

Let $y(t) = V(t, g(t_0, \omega_0))$.

Since $\lim_{\eta \to 0} u_\eta(t_0, \theta_0)(t) = h(t_0, \theta_0)(t)$, it is sufficient to show that

$$y(t) < u_\eta(t_0, \theta_0)(t),\ \text{for}\ t \ge t_0. \tag{3.12}$$

In the event that the inequality (3.12) is false, there would be a point $\tau > t_0$ such that

$$y(\tau) = u_\eta(\tau, (t_0, \omega_0))\ \text{and}\ y(t) < u_\eta(t, (t_0, \omega_0))\ \text{for}\ t \in [t_0, \tau).$$

It follows from Lemma (3.1) that

$${}^C D_+^\alpha y(\tau) - {}^C D_+^\alpha u_\eta(\tau, (t_0, \omega_0)) > 0.$$

Thus,

$${}^C D_+^\alpha y(\tau) > {}^C D_+^\alpha u_\eta(\tau, (t_0, \omega_0)),$$

and using (3.10) we obtain

$${}^C D_+^\alpha y(\tau) > {}^C D_+^\alpha u_\eta(\tau, (t_0, \omega_0)) = \zeta_\eta(\tau, u(\tau), u_\tau) + \eta > \zeta(\tau, u(\tau), u_\tau).$$

Therefore,

$${}^C D_+^\alpha y(\tau) > \zeta(\tau, u(\tau), u_\tau). \tag{3.13}$$

Let $g(t) = g(t_0, \omega_0)(t)$ be any solution of (1.3) such that (3.8) holds. Using the Caputo fractional Dini derivative (1) for $y(t)$, we then obtain for $t \in [t_0, T]$ the following

$$
\begin{aligned}
{}^C D_+^\alpha y(t) &= \limsup_{h\to 0^+} \frac{1}{h^\alpha}\Big\{ y(t) - y(t_0) - \sum_{l=1}^{[\frac{t-t_0}{h}]} (-1)^{l+1}\binom{\alpha}{l}\Big[y(t-lh) - y(t_0)\Big]\Big\}\\
&= \limsup_{h\to 0^+} \frac{1}{h^\alpha}\Big\{ V(t, g(t)) - V(t_0, \omega_0(0)) - \sum_{l=1}^{[\frac{t-t_0}{h}]} (-1)^{l+1}\binom{\alpha}{l}\Big[V(t-lh, g(t-lh)) - V(t_0, \omega_0(0))\Big]\Big\}\\
&= \limsup_{h\to 0^+} \frac{1}{h^\alpha}\Big\{ V(t, g(t)) - V(t_0, \omega_0(0)) - \sum_{l=1}^{[\frac{t-t_0}{h}]} (-1)^{l+1}\binom{\alpha}{l}\Big[V(t-lh, \omega(0) - h^\alpha f(t, \omega(0), \omega)) - V(t_0, \omega_0(0))\Big]\\
&\quad + \sum_{l=1}^{[\frac{t-t_0}{h}]} (-1)^{l+1}\binom{\alpha}{l}\Big[V(t-lh, \omega(0) - h^\alpha f(t, \omega(0), \omega)) - V(t_0, \omega_0(0))\Big]\\
&\quad - \sum_{l=1}^{[\frac{t-t_0}{h}]} (-1)^{l+1}\binom{\alpha}{l} V(t-lh, g(t-lh)) - V(t_0, \omega_0(0))\Big\}\\
&= \limsup_{h\to 0^+} \frac{1}{h^\alpha}\Big\{ V(t, g(t)) - V(t_0, \omega_0(0)) - \sum_{l=1}^{[\frac{t-t_0}{h}]} (-1)^{l+1}\binom{\alpha}{l}\Big[V(t-lh, \omega(0) - h^\alpha f(t, \omega(0))) - V(t_0, \omega_0(0))\Big]\Big\}\\
&\quad - \limsup_{h\to 0^+} \frac{1}{h^\alpha} \sum_{l=1}^{[\frac{t-t_0}{h}]} (-1)^{l+1}\binom{\alpha}{l}\Big[V(t-lh, \omega(0) - h^\alpha f(t, \omega(0))) - V(t-lh, g(t-lh))\Big]\\
&= {}^C D_+^\alpha V(t, g(t)) - \limsup_{h\to 0^+} \frac{1}{h^\alpha} \sum_{l=1}^{[\frac{t-t_0}{h}]} (-1)^{l+1}\binom{\alpha}{l}\Big[V(t-lh, \omega(0) - h^\alpha f(t, \omega(0))) - V(t-lh, g(t-lh))\Big].
\end{aligned}
$$

Given that V is locally Lipschitz in the second variable with a Lipschitz constant $L > 0$, we derive

$$
\begin{aligned}
{}^C D_+^\alpha y(t) &\le {}^C D^\alpha V(t, \omega(0), \omega) - L \limsup_{h\to 0^+} \frac{1}{h^\alpha}\Big| \sum_{l=1}^{[\frac{t-t_0}{h}]} (-1)^{l+1}\binom{\alpha}{l}\Big| \big\|\omega(0) - h^\alpha f(t, \omega(0)) - g(t-lh)\big\|\\
&\le {}^C D^\alpha V(t, \omega(0), \omega) - L \limsup_{h\to 0^+} \frac{1}{h^\alpha}\Big| \sum_{l=1}^{[\frac{t-t_0}{h}]} (-1)^{l+1}\binom{\alpha}{l}\Big| \big\|\omega(0)\big\| + h^\alpha\big\|f(t, \omega(0))\big\| + \big\|g(t)\big\|\\
&= {}^C D^\alpha V(t, \omega(0), \omega) - L\Big| \sum_{l=1}^{[\frac{t-t_0}{h}]} (-1)^{l+1}\binom{\alpha}{l}\Big| \big\|\omega(0)\big\| + \big\|f(t, \omega(0))\big\| + \big\|g(t)\big\|.
\end{aligned}
$$

Let

$$
M = \Big| \sum_{l=1}^{[\frac{t-t_0}{h}]} (-1)^{l+1}\binom{\alpha}{l}\Big| \big\|\omega(0)\big\| + \big\|f(t, \omega(0))\big\| + \big\|g(t)\big\| > 0,
$$

so that

$$
{}^C D_+^\alpha y(t) \le {}^C D^\alpha V(t, \omega(0), \omega) - LM
$$

$$\leq \ {}^{C}D^{\alpha}V(t,\omega(0),\omega).$$

Therefore, using condition 3 of the theorem, we have that

$$ {}^{C}D^{\alpha}_{+}y(t) \leq^{C} D^{\alpha}V(t,\omega(0),\omega) \leq \zeta(t, V(t,\omega(0),V_t)) = \zeta(t, y(t)). \tag{3.14}$$

Now (3.14) with $t = \tau$ gives

$$ {}^{C}D^{\alpha}_{+}y(\tau) \leq \zeta(\tau, y(\tau)), \tag{3.15}$$

which contradicts (3.13), and hence (3.12) is true.

From the proof of Theorem 3.1, it can be concluded that the set of solutions $\{u_{\eta}(t_0,\theta_0)(t)\}$ is uniformly bounded and equi-continuous on the interval $[t_0, T]$. Therefore, according to the Arzelà-Ascoli theorem, there exist a decreasing subsequence $\{u_{\eta_k}(t_0,\theta_0)(t)\}$ and a continuous function $p(t_0,\theta_0)(t)$ that serves as the uniform limit of $u_{\eta_k}(t_0,\theta_0)(t)$ on the interval $[t_0, T]$. From (3.11) we have

$$u_{\eta_k}(t_0,\theta_0)(t) = \theta_0 + \eta_k + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\xi)^{\alpha-1}\Big(\zeta_{\eta_k}(\xi, u(\xi), u_{\xi}) + \eta_k\Big)d\xi,\ t \in \mathbb{R}_+. \tag{3.16}$$

Taking the limit as $k \to \infty$ in (3.16) leads to

$$p(t_0,\theta_0)(t) = \theta_0 + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\xi)^{\alpha-1}\zeta(\xi, p(\xi), p_{\xi})d\xi, \tag{3.17}$$

which demonstrates that $p(t_0,\theta_0)(t)$ serves as a solution to (1.4) over the interval $[t_0, T]$. We claim that $p(t_0,\theta_0)(t)$ converges to the maximal solution $h(t_0,\theta_0)(t)$ on $[t_0, T]$. In order to demonstrate this, we take the limit in (3.12) for $\eta = \eta_k$ as $k \to \infty$. From there, we get $V(t,(t_0,\omega_0)(t)) \leq h(t_0,\theta_0)(t)$. □

Theorem 3.3. *Assume that*

(1) $\zeta \in C(\mathbb{R}_+ \times \mathbb{R}^n \times \mathfrak{D}^n, \mathbb{R}^n)$*, and* $\zeta(t,u,u_t)$ *is quasi-monotone nondecreasing in* u_t *with* $\zeta(t,0,0) = 0$*.*
(2) $V \in C[(-\gamma,\infty) \times C_{\rho}, \mathbb{R}^N_+]$, $V(t,0) = 0$*, and* $V(t,g_t)$ *is locally Lipschitz continuous in* g_t *such that*

$$ {}^{C}D^{\alpha}_{+}V(t,\omega(0),\omega) \leq \zeta(t,\omega(0),V_t), \tag{3.18}$$

holds for all $(t,g) \in \mathbb{R}_+ \times B_{\rho}$*.*
(3) $a(\|g\|) \leq V_0(t,g_t)$*, where* $a \in \mathcal{K}$ *and* $V_0(t,g_t) = \sum_{i=1}^{N} V_i(t,g_t)$*.*
The stability of the trivial solution $g = 0$ *of the system* (1.3) *is therefore implied by the stability of the trivial solution* $u = 0$ *of the system* (1.4)*.*

Proof. Given $\epsilon \in (0,\rho]$ and $t_0 \in \mathbb{R}_+$, the stability of the trivial solution $u = 0$ of (1.4) indicates that for any $a(\epsilon) > 0$, $t_0 \in \mathbb{R}_+$, and initial function $\theta_0 \in \mathfrak{D}^n$, there exists a $\delta = \delta(t_0,\epsilon) > 0$ which is continuous in t_0 such that

$$\theta_0 = \Big\|\sum_{i=1}^{n}\theta_{i0}\Big\|_0 < \delta \,\text{implies}\, \sum_{i=1}^{n}u_i(t_0,\theta_0)(t) \leq a(\epsilon), t \geq t_0, \tag{3.19}$$

where $u(t_0,\theta_0)(t)$ is any solution of (1.4). With $V(t,0) = 0$ and the continuity of $V(t_0,\theta_0(0))$, it is ensured that there exists a $\delta_1 = \delta_1(t_0,\delta) > 0$ such that

$$\|\theta_0\|_0 < \delta_1 \text{ implies } V_0(t_0,\theta_0(0))(t) < \delta. \tag{3.20}$$

Let $g(t_0, \omega_0)(t)$ be any solution of (1.3), with $\|\omega_0\|_0 < \delta_1$.
Claim:

$$\|g(t_0, \omega_0)(t)\|_0 < \epsilon,\ t \geq t_0. \tag{3.21}$$

Assuming (3.21) does not hold, there exists a $\tau > t_0$ such that $\|g(t_0, \omega_0(0))(\tau)\|_0 = \epsilon$ and $\|g(t_0, \omega_0(0))(t)\|_0 < \epsilon$ for $t \in [t_0, \tau)$.

Let $\theta_0 = V_0(t_0, \omega_0)$. Then, from (3.19), we have $V_0(t_0, \omega_0) < \delta < \epsilon$.

Let $h_m(t_0, \theta_0)(t) = \sum_{i=1}^{n} h_i(t_0, \theta_0)(t)$ with $h_0(t_0, \theta_0) < \delta$ be the maximal solution of (1.4) such that

$$V_0(t_0, \omega_0(0))(t) \leq h_m(t_0, \theta_0)(t). \tag{3.22}$$

Therefore at $t = \tau$, we have that $\|g(t_0, \omega_0(0))(\tau)\|_0 = \epsilon$. Combining condition 3 of the theorem, (3.19) and (3.22) we obtain

$$a(\|g(\tau_0, \omega_0(0))(\tau)) \leq V_0(\tau_0, \omega_0(0))(\tau) \leq h_m(\tau_0, \theta_0)(\tau) < a(\epsilon).$$

This yields

$$a(\epsilon) \leq V_0(t_0, \omega_0(0))(\tau) \leq h_m(t_0, \theta_0)(\tau) < a(\epsilon),$$

which is a contradiction. Thus, (3.21) holds, leading us to conclude that the trivial solution $g = 0$ of (1.3) is stable. □

Theorem 3.4. *Assume that*

(1) $\zeta \in C(\mathbb{R}_+ \times \mathbb{R}^n \times \mathfrak{D}^n, \mathbb{R}^n)$*, and* $\zeta(t, u, u_t)$ *is quasi-monotone nondecreasing in* u_t *with* $\zeta(t, 0, 0) = 0$.
(2) $V \in C[(-\gamma, \infty) \times C_\rho, \mathbb{R}_+^N]$, $V(t, 0) = 0$*, and* $V(t, g_t)$ *is locally Lipschitzian in* g_t *such that*

$$^{C}D_+^{\alpha} V(t, \omega(0), \omega) \leq -cV(t, \omega(0)), \tag{3.23}$$

holds for all $(t, g) \in \mathbb{R}_+ \times B_\rho$.
(3) $a(\|g\|) \leq V_0(t, g_t)$*, where* $a \in \mathcal{K}$ *and* $V_0(t, g_t) = \sum_{i=1}^{N} V_i(t, g_t)$.
Consequently, the asymptotic stability of the trivial solution $g = 0$ *of the system* (1.3) *is implied by the asymptotic stability of the trivial solution* $u = 0$ *of the system* (1.4).

Proof. According to Theorem (1.3), the trivial solution of (1.4) is stable. Condition (ii) of the theorem ensures that $V(t, \omega(0))$ is monotonically decreasing, and condition (iii) further ensures that it is bounded below by zero. Therefore, there exists a limit

$$\lim_{t \to \infty} V(t, \omega(0), \omega) = G_0(say). \tag{3.24}$$

Claim: $G_0 = 0$

Assume that the claim is false. In other words, if we assume $G_0 \neq 0$, then $c(G_0) \neq 0$ because $c \in \mathcal{K}$. $V(t, \omega(0))$ being monotonically decreasing combined with (3.24) guarantees that $V(t, \omega(0)) > G_0$. Given that $c(r)$ is a monotonically increasing function of r, we can state that

$$c(V(t, \omega(0))) > c(G_0),$$

so that

$$-c(V(t, \omega(0))) < -c(G_0).$$

In terms of (3.23) we have

$$ {}^{C}D_{+}^{\alpha}V(t,\omega(0)) \leq -c(G_0). \tag{3.25}$$

Integrating (3.25) from t_0 to t we have

$$V(t,\omega(0),\omega) \leq V(t_0,\omega(0),\omega) - \frac{c(G_0)}{\Gamma(\alpha)}\Big(\int_{t_0}^{t}(t-\xi)^{\alpha-1}d\xi\Big)\mathbb{I}_N,$$

where $\mathbb{I}_N$ denotes an identity matrix of order N.

This implies that

$$V(t_0,\omega(0),\omega) \leq V(t_0,\omega(0),\omega) - \frac{c(G_0)}{\alpha\Gamma(\alpha)}\Big((t-t_0)^{\alpha}\Big)\mathbb{I}_N, \tag{3.26}$$

so that as $t \to \infty$ in (3.26), we have that $\frac{c(G_0)}{\alpha\Gamma(q)}\Big((t-t_0)^{\alpha}\Big)\mathbb{I}_N \to \infty$ so that $V(t,\omega(0),\omega) \to -\infty$. This contradicts condition (3) of the theorem and so our claim that $V_0 = 0$ is true, that is $\lim_{t\to\infty} V(t,\omega(0),\omega) = 0$. This demonstrates that the zero solution $u = 0$ of (1.4) is asymptotically stable. □

4. Example

We demonstrate the benefit of employing the vector Lyapunov function over the scalar Lyapunov function with this example.

Consider the system of retarded nonlinear Caputo fractional differential equations

$$\begin{aligned} {}^{C}D^{\alpha}g_1(t) =&\ 8g_1(t-2)\cos g_2(t-2) + g_2(t-2)\sin^2 g_1(t-2),\\ {}^{C}D^{\alpha}g_2(t) =&\ -4g_2(t-2)\sin^2 g_1(t-2) + 2g_1(t-2)\cos^2 g_2(t-2), \end{aligned} \tag{4.1}$$

for $t \geq t_0$, with initial functions

$$g_1(s) = \omega_1(s),\ g_2(s) = \omega_2(s),\ for\, s \in [-2,0],$$

where $\omega_1(s)$ and $\omega_2(s)$ are the initial functions defined on $-2 \leq s \leq 0$.We recall that the initial function $\omega_1(s)$ and $\omega_2(s)$ captures the state of the system at time $t+s$. In this example, $g_1(t) = \omega_1(s) = g_1(t+s)$, so that at $s = -2$ we have $g_1(t) = \omega_1(-2) = g_1(t-2)$. Similarly, $g_2(t) = \omega_2(-2) = g_2(t-2)$. With these, the system (4.1) can therefore be written as

$$\begin{aligned} {}^{C}D^{\alpha}g_1(t) =&\ 8\omega_1(-2)\cos\omega_2(-2) + \omega_2(-2)\sin^2\omega_1(-2),\\ {}^{C}D^{\alpha}g_2(t) =&\ -4\omega_2(-2)\sin^2\omega_1(-2) + 2\omega_1(-2)\cos^2\omega_2(-2). \end{aligned} \tag{4.2}$$

Now we consider a scalar Lyapunov function for (4.1) given by

$$V(t,\omega) = |\omega_1(-2)| + |\omega_2(-2)|.$$

Then according to (2.3) we obtain

$$ {}^{C}D_{+}^{\alpha}V = \limsup_{h\to 0^+}\frac{1}{h^{\alpha}}\Big\{|\omega_1(-2)| + |\omega_2(-2)| + \sum_{l=1}^{[\frac{t-t_0}{h}]}(-1)^{l}\binom{\alpha}{l}\Big[|\omega_1(-2) - h^{\alpha}f_1(t,\omega_1(0))|$$

$$
\begin{aligned}
&+|\omega_2(-2) - h^\alpha f_2(t, \omega_2(0))|\Big]\Big\} - \frac{\Big[|\omega_{01}(-2)| + |\omega_{02}(-2)|\Big]}{t^\alpha \Gamma(1-\alpha)} \\
\leq\ & \limsup_{h\to 0^+} \frac{1}{h^\alpha}\Big\{|\omega_1(-2)| + |\omega_2(-2)| + \sum_{l=1}^{[\frac{t-t_0}{h}]}(-1)^l\binom{\alpha}{l}|\omega_1(-2)| + \sum_{l=1}^{[\frac{t-t_0}{h}]}(-1)^l\binom{\alpha}{l} \\
&\times h^\alpha|f_1(t, \omega_1(0))| + \sum_{l=1}^{[\frac{t-t_0}{h}]}(-1)^l\binom{\alpha}{l}|\omega_2(-2)| + \sum_{l=1}^{[\frac{t-t_0}{h}]}(-1)^l\binom{\alpha}{l}h^\alpha|f_2(t, \omega_2(0))|\Big\} \\
&-\frac{\Big[|\omega_{01}(-2)| + |\omega_{02}(-2)|\Big]}{t^\alpha \Gamma(1-\alpha)} \\
=\ & \limsup_{h\to 0^+} \frac{1}{h^\alpha}\Big\{ \sum_{l=0}^{[\frac{t-t_0}{h}]}(-1)^l\binom{\alpha}{l}|\omega_1(-2)| + \sum_{l=0}^{[\frac{t-t_0}{h}]}(-1)^l\binom{\alpha}{l}|\omega_2(-2)| + \sum_{l=1}^{[\frac{t-t_0}{h}]}(-1)^l\binom{\alpha}{l} \\
&\times h^\alpha\Big[|f_1(t, \omega_1(0))| + |f_2(t, \omega_2(0))|\Big]\Big\} - \frac{\Big[|\omega_{01}(-2)| + |\omega_{02}(-2)|\Big]}{t^\alpha \Gamma(1-\alpha)} \\
=\ & \limsup_{h\to 0^+} \frac{1}{h^\alpha} \sum_{l=0}^{[\frac{t-t_0}{h}]}(-1)^l\binom{\alpha}{l}|\omega_1(-2)| + \limsup_{h\to 0^+} \frac{1}{h^\alpha} \sum_{l=0}^{[\frac{t-t_0}{h}]}(-1)^l\binom{\alpha}{l}|\omega_2(-2)| \\
&+\Big[|f_1(t, \omega_1(0))| + |f_2(t, \omega_2(0))|\Big] \limsup_{h\to 0^+} \sum_{l=1}^{[\frac{t-t_0}{h}]}(-1)^l\binom{\alpha}{l} - \frac{\Big[|\omega_{01}(-2)| + |\omega_{02}(-2)|\Big]}{t^\alpha \Gamma(1-\alpha)}.
\end{aligned}
$$

Applying Eqs (3.7) and (3.8) in [26], we obtain

$$
\begin{aligned}
&{}^C D_+^\alpha V \leq \frac{|\omega_1(-2)|}{t^\alpha \Gamma(1-\alpha)} + \frac{|\omega_2(-2)|}{t^\alpha \Gamma(1-\alpha)} - \frac{\Big[|\omega_{01}(-2)| + |\omega_{02}(-2)|\Big]}{t^\alpha \Gamma(1-\alpha)} - \Big[|f_1(t, \omega_1(0))| + |f_2(t, \omega_2(0)|\Big] \\
\leq\ & \frac{|\omega_1(-2)|}{t^\alpha \Gamma(1-\alpha)} + \frac{|\omega_2(-2)|}{t^\alpha \Gamma(1-\alpha)} - \Big[|f_1(t, \omega_1(0))| + |f_2(t, \omega_2(0))|\Big] \\
\leq\ & \frac{|\omega_1(-2)|}{t^\alpha \Gamma(1-\alpha)} + \frac{|\omega_2(-2)|}{t^\alpha \Gamma(1-\alpha)} + \Big[|f_1(t, \omega_1(0))| + |f_2(t, \omega_2(0))|\Big].
\end{aligned}
$$

As $t \to \infty$, the first two terms tend to zero, and using (4.2) we have

$$
\begin{aligned}
{}^C D_+^\alpha V \ \leq\ & \Big[|f_1(t, \omega_1(0))| + |f_2(t, \omega_2(0))|\Big] \\
=\ & \Big[|8\omega_1(-2)\cos\omega_2(-2) + \omega_2(-2)\sin^2\omega_1(-2)| + |-4\omega_2(-2)\sin^2\omega_1(-2) \\
&+2\omega_1(-2)\cos^2\omega_2(-2)|\Big] \\
\leq\ & \Big[8|\omega_1(-2)||\cos\omega_2(-2)| + |\omega_2(-2)||\sin^2\omega_1(-2)| + 4|\omega_2(-2)||\sin^2\omega_1(-2)| \\
&+2|\omega_1(-2)||\cos^2\omega_2(-2)|\Big]
\end{aligned}
$$

$$
\begin{aligned}
&\leq \Big[8|\omega_1(-2)| + |\omega_2(-2)| + 4|\omega_2(-2)| + 2|\omega_1(-2)|\Big] \\
&= \Big[10|\omega_1 - (2)| + 5|\omega_2(-2)|\Big] = 10|\omega_1(-2)| + 5|\omega_2(-2)| \leq 10|\omega_1(-2)| + 10|\omega_2(-2)| \\
&= 10(|\omega_1(-2)| + |\omega_2(-2)|) = 10V(t,\omega).
\end{aligned}
$$

Therefore, we have

$$
{}^{C}D_{+}^{\alpha}V \leq 10V(t,\omega) = \zeta(t, V(t,\omega)). \tag{4.3}
$$

Now consider the scalar comparison equation

$$
\begin{aligned}
{}^{C}D^{\alpha}u = \zeta(t, u(t), u(t-2)) = 10u(t-2), \\
u(s) = \theta(s) = \theta_0,\ for\, s \in [-2, 0],
\end{aligned} \tag{4.4}
$$

where $\theta_0 = 2$ remains constant throughout the given interval. Solving (4.4) by the Laplace transform method and noting that $u(t-2)$ is a Heaviside step function, we obtain the following:

$$
\mathcal{L}({}^{C}D^{\alpha}u) = 10\mathcal{L}(u(t-2)).
$$

This implies that

$$
s^{\alpha}U(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1}U^{k}(0) = 10\frac{e^{-2s}}{s},
$$

so that

$$
\begin{aligned}
s^{\alpha}U(s) - 2s^{\alpha-1} &= 10\frac{e^{-2s}}{s}, \\
s^{\alpha}U(s) &= 2s^{\alpha-1} + 10\frac{e^{-2s}}{s}, \\
U(s) &= \frac{2}{s} + 10\frac{e^{-2s}}{s^{\alpha+1}}.
\end{aligned}
$$

Taking the inverse Laplace transforms we obtain

$$
\mathcal{L}^{-1}U(s) = \mathcal{L}^{-1}\Big(\frac{2}{s}\Big) + 10\mathcal{L}^{-1}\Big(\frac{e^{-2s}}{s^{\alpha+1}}\Big),
$$

so that

$$
u(t) = 2 + 10\mathcal{L}^{-1}\Big(\frac{e^{-2s}}{s^{\alpha+1}}\Big).
$$

Using the fact that $\mathcal{L}(t^{\alpha}) = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$, we have

$$
u(t) \;=\; 2 + 10(t-2)^{\frac{1}{\Gamma(\alpha)}}u(t-2). \tag{4.5}
$$

We observe that

$$
|u(t)| = |2 + 10(t-2)^{\frac{1}{\Gamma(\alpha)}}u(t-2)| \text{ with } |\theta_0| = 2.
$$

As t increases, the term $10(t-2)^{\frac{1}{\Gamma(\alpha)}}$ grows causing $u(t)$ to be unbounded. This indicates that for any nonzero initial condition θ_0, $u(t)$ will eventually grow without bound as t increases. Hence for any small $\delta > 0$ such that $\|\theta_0\|_0 < \delta$, there exist some $t > 0$ at which $u(t)$ becomes unbounded. This means that no matter how small we choose δ, $u(t)$ will eventually exceed any prescribed ϵ.

All the conditions of Theorem 3.3 are satisfied, except that the trivial solution $u = 0$ of (4.5) is not stable (also see Figure 1). Therefore Theorem 3.3 cannot yield any stability information for the zero solution of (4.4).

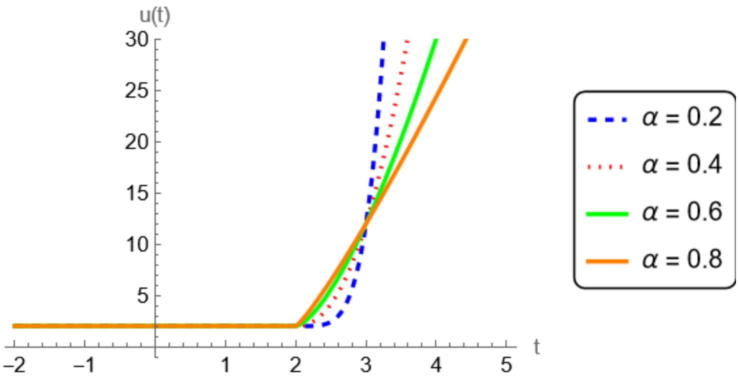

Figure 1. Plot of $u(t) = 2 + 10(t-2)^{\frac{1}{\Gamma(\alpha)}} u(t-2)$, $\alpha = 0.2, 0.4, 0.6$ and 0.8.

We now examine a vector Lyapunov function with the following form:

$$V(t, \omega(0)) = (V_1, V_2)^T = \Big(|\omega_1(-2)|, |\omega_2(-2)|\Big)^T, \tag{4.6}$$

where $V_1 = |\omega_1(-2)|$ and $V_2 = |\omega_2(-2)|$, with $\omega = (\omega_1, \omega_2) \in \mathbb{R}^2$, so that the associated norm $\|\omega\| = \sqrt{\omega_1^2 + \omega_2^2}$.

Now,

$$V_0 = \sum_{i=1}^{2} V_i = |\omega_1(-2)| + |\omega_2(-2)|,$$

and so $a(\|\omega\|) \leq V_0(t, x_t)$ with $a(r) = r$, implying that $a \in \mathcal{K}$. We compute the Caputo fractional Dini derivative for $V_1 = |\omega_1(-2)|$ using (2.3) as follows:

$$\begin{aligned}
{}^C D_+^\alpha V_1 &= \limsup_{h\to 0^+} \frac{1}{h^\alpha}\Big\{|\omega_1(-2)| + \sum_{l=1}^{[\frac{t-t_0}{h}]} (-1)^l \binom{\alpha}{l} \Big|\omega_1(-2) - h^\alpha f_1(t, \omega_1(0))\Big|\Big\} - \frac{|\omega_{01}(-2)|}{t^\alpha \Gamma(1-\alpha)} \\
&\leq \limsup_{h\to 0^+} \frac{1}{h^\alpha}\Big\{|\omega_1(-2)| + \sum_{l=1}^{[\frac{t-t_0}{h}]} (-1)^l \binom{\alpha}{l} \Big[|\omega_1(-2)| + |h^\alpha f_1(t, \omega_1(0))|\Big]\Big\} - \frac{|\omega_{01}(-2)|}{t^\alpha \Gamma(1-\alpha)} \\
&= \limsup_{h\to 0^+} \frac{1}{h^\alpha}\Big\{|\omega_1(-2)| + \sum_{l=1}^{[\frac{t-t_0}{h}]} (-1)^l \binom{\alpha}{l} |\omega_1(-2)| + \sum_{l=1}^{[\frac{t-t_0}{h}]} (-1)^l \binom{\alpha}{l} |h^\alpha f_1(t, \omega_1(0))|\Big\} - \frac{|\omega_{01}(-2)|}{t^\alpha \Gamma(1-\alpha)} \\
&= |\omega_1(-2)| \limsup_{h\to 0^+} \frac{1}{h^\alpha} \sum_{l=0}^{[\frac{t-t_0}{h}]} (-1)^l \binom{\alpha}{l} + |f_1(t, \omega_1(0))| \limsup_{h\to 0^+} \sum_{l=1}^{[\frac{t-t_0}{h}]} (-1)^l \binom{\alpha}{l} - \frac{|\omega_{01}(-2)|}{t^\alpha \Gamma(1-\alpha)}.
\end{aligned}$$

Applying Eqs (3.7) and (3.8) in [26], we obtain

$$ {}^{C}D_{+}^{\alpha}V_1 \leq \frac{\Big(|\omega_1(-2)| - |\omega_{01}(-2)|\Big)}{t^{\alpha}\Gamma(1-\alpha)} - |f_1(t,\omega_1(0))|. $$

As $t \to \infty$, the first term tends to zero, and using (4.2) we obtain

$$ \begin{aligned} {}^{C}D_{+}^{\alpha}V_1 &\leq -|8\omega_1(-2)\cos\omega_2(-2) + \omega_2(-2)\sin^2\omega_1(-2)| \\ &\leq -\Big(|8\omega_1(-2)||\cos\omega_2(-2)| + |\omega_2(-2)||\sin^2\omega_1(-2)|\Big) \\ &\leq -\Big(8|\omega_1(-2)| + |\omega_2(-2)|\Big) = -8|\omega_1(-2)| - |\omega_2(-2)| = -8V_1 - V_2 \leq -8V_1 + V_2. \end{aligned} $$

Therefore

$$ {}^{C}D_{+}^{q}V_1 \leq -8V_1 + V_2. \tag{4.7} $$

Similarly, we compute the Caputo fractional Dini derivative for $V_2 = |\omega_2(-2)|$ using (2.3) as follows:

$$ \begin{aligned} {}^{C}D_{+}^{\alpha}V_2 &= \limsup_{h\to 0^+}\frac{1}{h^{\alpha}}\Big\{|\omega_2(-2)| + \sum_{l=1}^{[\frac{t-t_0}{h}]}(-1)^l\binom{\alpha}{l}\Big|\omega_2(-2) - h^{\alpha}f_2(t,\omega_2(0))\Big|\Big\} - \frac{|\omega_{20}(-2)|}{t^{\alpha}\Gamma(1-\alpha)} \\ &\leq \limsup_{h\to 0^+}\frac{1}{h^{\alpha}}\Big\{|\omega_2(-2)| + \sum_{l=1}^{[\frac{t-t_0}{h}]}(-1)^l\binom{\alpha}{l}\Big[|\omega_2(-2)| + |h^{\alpha}f_2(t,\omega_2(0))|\Big]\Big\} - \frac{|\omega_{20}(-2)|}{t^{\alpha}\Gamma(1-\alpha)} \\ &= \limsup_{h\to 0^+}\frac{1}{h^{\alpha}}\Big\{|\omega_2(-2)| + \sum_{l=1}^{[\frac{t-t_0}{h}]}(-1)^l\binom{\alpha}{l}|\omega_2(-2)| + \sum_{l=1}^{[\frac{t-t_0}{h}]}(-1)^l(\binom{\alpha}{l})|h^{\alpha}f_2(t,\omega_2(0))| - \frac{|\omega_{20}(-2)|}{t^{\alpha}\Gamma(1-\alpha)}\Big\} \\ &= |\omega_2(-2)|\limsup_{h\to 0^+}\frac{1}{h^{\alpha}}\sum_{l=0}^{[\frac{t-t_0}{h}]}(-1)^l\binom{\alpha}{l} + |f_2(t,\omega_2(0))|\limsup_{h\to 0^+}\sum_{l=1}^{[\frac{t-t_0}{h}]}(-1)^l\binom{\alpha}{l} - \frac{|\omega_{20}(-2)|}{t^{\alpha}\Gamma(1-\alpha)}. \end{aligned} $$

Applying Eqs (3.7) and (3.8) in [26], we obtain

$$ {}^{C}D_{+}^{\alpha}V_2 \leq \frac{\Big(|\omega_2(-2)| - |\omega_{20}(-2)|\Big)}{t^{\alpha}\Gamma(1-\alpha)} - |f_2(t,\omega_2(0))|. $$

As $t \to \infty$, the first term tends to zero, and using (4.2) we obtain

$$ \begin{aligned} {}^{C}D_{+}^{\alpha}V_2 &\leq -|-4\omega_2(-2)\sin^2\omega_1(-2) + 2\omega_1(-2)\cos^2\omega_2(-2)| \\ &\leq -\Big(|-4\omega_2(-2)\sin^2\omega_1(-2)| + |2\omega_1(-2)\cos^2\omega_2(-2)|\Big) \\ &\leq -\Big(4|\omega_2(-2)||\sin^2\omega_1(-2)| + 2|\omega_1(-2)||\cos^2\omega_2(-2)|\Big) \\ &\leq -\Big(4|\omega_2(-2)| + 2|\omega_1(-2)|\Big) = -4|\omega_2(-2)| - 2|\omega_1(-2)| = -2V_1 - 4V_2 \leq 2V_1 - 4V_2. \end{aligned} $$

Therefore,

$$ {}^{C}D_{+}^{q}V_1 \leq 2V_1 - 4V_2. \tag{4.8} $$

Combining (4.7) and (4.8), we obtain

$$ {}^{C}D_{+}^{\alpha}V \leq \begin{pmatrix} -8 & 1 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \zeta(t, V(t, \omega)). \tag{4.9} $$

Now consider the comparison system

$$ {}^{C}D^{\alpha}u = \zeta(t, u(t-2)) = Au(t-2), \tag{4.10} $$

where $A = \begin{pmatrix} -8 & 1 \\ 2 & -4 \end{pmatrix}$, $u(\xi) = \theta_0$ for $\xi \in [-2, 0]$, with $\theta_0 = (2, 2)^T$ being a constant function defined over the interval.

The vector inequality (4.9), along with the required conditions for using vector Lyapunov functions as outlined in Theorem 3.3, is fulfilled by (4.6). In fact, the eigenvalues of A have negative real components. Consequently, by Theorem 3.3, we can conclude that the zero solution $g = 0$ of the system (4.1) is not only stable but also asymptotically stable.

5. Conclusions

Due to the increasing scholarly interest in fractional time-dependent systems with delay, known for their improved accuracy in modeling problems with hereditary and memory behaviors, this paper examines the stability and asymptotic stability dynamics of Caputo fractional time-dependent systems with delays using vector Lyapunov functions. By applying the Caputo fractional Dini derivative and introducing a new comparison theorem, we have established robust stability and asymptotic stability conditions for these systems. Our method advances beyond traditional scalar Lyapunov function approaches and enhances existing stability results. The provided example highlights the practical benefits and improved accuracy of our approach, representing a significant advancement in the field.

Author contributions

Jonas Achuobi: Conceptualization, methodology and writing of the original draft of the manuscript. Edet Akpan: Conceptualization, methodology and supervision. Reny George: Conceptualization, review, editing, validation and funding acquisition. Autine Ofem: Validation. All authors have read and agreed to the published version of the manuscript.

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Conflict of interest

The authors declare that they have no conflict of interest.

References

1. J. B. Hu, G. P. Lu, S. B. Zhang, L. D. Zhao, Lyapunov stability theorem about fractional system without and with delay, *Commun. Nonlinear Sci. Numer. Simul.*, **20** (2015), 905–913. https://doi.org/10.1016/j.cnsns.2014.05.013
2. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, 2006.
3. K. S. Miller, B. Ross, *An introduction to the fractional calculus and fractional differential equations*, New York: Wiley, 1993.
4. I. Podlubny, *Fractional differential equations: An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, San Diego: Academic Press, 1999.
5. R. P. Agarwal, Y. Zhou, Y. He, Existence of fractional neutral functional differential equations, *Comput. Math. Appl.*, **59** (2010), 1095–1100. https://doi.org/10.1016/j.camwa.2009.05.010
6. M. Benchohra, J. Henderson, S. K. Ntouyas, A. Ouahab, Existence results for fractional order functional differential equations with infinite delay, *J. Math. Anal. Appl.*, **338** (2008), 1340–1350. https://doi.org/10.1016/j.jmaa.2007.06.021
7. A. Chauhan, J. Dabas, Local and global existence of mild solution to an impulsive fractional functional integro-differential equation with nonlocal condition, *Commun. Nonlinear Sci. Numer. Simul.*, **19** (2014), 821–829. https://doi.org/10.1016/j.cnsns.2013.07.025
8. T. Jankowski, Existence results to delay fractional differential equations with nonlinear boundary conditions, *Appl. Math. Comput.*, **219** (2013), 9155–9164. https://doi.org/10.1016/j.amc.2013.03.045
9. W. Deng, C. Li, J. Lü, Stability analysis of linear fractional differential system with multiple time delays, *Nonlinear Dyn.*, **48** (2007), 409–416. https://doi.org/10.1007/s11071-006-9094-0
10. J. Čermák, Z. Došlá, T. Kisela, Fractional differential equations with a constant delay: Stability and asymptotics of solutions, *Appl. Math. Comput.*, **298** (2017), 336–350. https://doi.org/10.1016/j.amc.2016.11.016
11. H. T. Tuan, H. Trinh, A linearized stability theorem for nonlinear delay fractional differential equations, *IEEE Trans. Automat. Control*, **63** (2018), 3180–3186. https://doi.org/10.1109/TAC.2018.2791485
12. M. Li, J. Wang, Exploring delayed Mittag-Leffler type matrix functions to study finite time stability of fractional delay differential equations, *Appl. Math. Comput.*, **324** (2018), 254–265. https://doi.org/10.1016/j.amc.2017.11.063
13. N. T. Thanh, V. N. Phat, P. Niamsup, New finite-time stability analysis of singular fractional differential equations with time-varying delay, *Fract. Calc. Appl. Anal.*, **23** (2020), 504–519. https://doi.org/10.1515/fca-2020-0024
14. Y. Chen, K. L. Moore, Analytical stability bound for a class of delayed fractional-order dynamic systems, *Nonlinear Dyn.*, **29** (2002), 191–200. https://doi.org/10.1023/A:1016591006562

15. E. Kaslik, S. Sivasundaram, Analytical and numerical methods for the stability analysis of linear fractional delay differential equation, *J. Comput. Appl. Math.*, **236** (2012), 4027–4041. https://doi.org/10.1016/j.cam.2012.03.010

16. M. P. Lazarevic, Finite time stability analysis of PDα fractional control of robotic time-delay systems, *Mech. Res. Commun.*, **33** (2006), 269–279. https://doi.org/10.1016/j.mechrescom.2005.08.010

17. A. Mesbahi, M. Haeri, Stability of linear time invariant fractional delay systems of retardee type in the space of delay parameters, *Automatica*, **49** (2013), 1287–1294. https://doi.org/10.1016/j.automatica.2013.01.041

18. S. B. Bhalekar, Stability analysis of a class of fractional delay differential equations, *Pramana J. Phys.*, **81** (2013), 215–224. https://doi.org/10.1007/s12043-013-0569-5

19. V. Lakshmikantham, Theory of fractional functional differential equations, *Nonlinear Anal. Theory Methods Appl.*, **69** (2008), 3337–3343. https://doi.org/10.1016/j.na.2007.09.025

20. I. Stamova, Global Mittag-Leffler stability and synchronization of impulsive fractional-order neural networks with time-varying delays, *Nonlinear Dyn.*, **77** (2014), 1251–1260. https://doi.org/10.1007/s11071-014-1375-4

21. S. J. Sadati, R. Ghaderi, N. Ranjbar, Some fractional comparison results and stability theorem for fractional time delay systems, *Rom. Rep. Phys.*, **65** (2013), 94–102.

22. R. Agarwal, R. Almeida, S. Hristova, D. O'Regan, Caputo fractional differential equation with state dependent delay and practical stability, *Dyn. Syst. Appl.*, **28** (2019), 715–742.

23. Y. Li, Y. Chen, I. Podlubny, Stability of fractional-order nonlinear dynamic systems: Lyapunov direct method and generalized Mittag-Leffler stability, *Comput. Math. Appl.*, **59** (2010), 1810–1821. https://doi.org/10.1016/j.camwa.2009.08.019

24. R. Agarwal, S. Hristova, D. O'Regan, Lyapunov functions and strict stability of Caputo fractional differential equations, *Adv. Differ. Equ.*, **2015** (2015), 346. https://doi.org/10.1186/s13662-015-0674-5

25. R. Agarwal, S. Hristova, D. O'regan, Lyapunov functions and stability of Caputo fractional differential equations with delays, *Differ. Equ. Dyn. Syst.*, **30** (2022), 513–534. https://doi.org/10.1007/s12591-018-0434-6

26. R. Agarwal, D. O'Regan, S. Hristova, Stability of Caputo fractional differential equations by Lyapunov functions, *Appl. Math.*, **60** (2015), 653–676. https://doi.org/10.1007/s10492-015-0116-4

27. R. Agarwal, S. Hristova, D. O'Regan, A survey of Lyapunov functions, stability and impulsive Caputo fractional differential equations, *Fract. Calc. Appl. Anal.*, **19** (2016), 290–318. https://doi.org/10.1515/fca-2016-0017

28. R. Agarwal, D. O'Regan, S. Hristova, M. Cicek, Practical stability with respect to initial time difference for Caputo fractional differential equations, *Commun. Nonlinear Sci. Numer. Simul.*, **42** (2017), 106–120. https://doi.org/10.1016/j.cnsns.2016.05.005

29. W. M. Haddad, V. Chellaboina, *Nonlinear dynamical systems and control: A Lyapunov-based approach*, Princeton University Press, 2008.

30. H. K. Khalil, *Control of nonlinear systems*, New York: Prentice Hall, 2002.

31. K. S. Narendra, J. Balakrishnan, A common Lyapunov function for stable LTI systems with commuting A-matrices, *IEEE Trans. Automat. Control*, **39** (1994), 2469–2471. https://doi.org/10.1109/9.362846

32. S. Raghavan, J. K. Hedrick, Observer design for a class of nonlinear systems, *Int. J. Control*, **59** (1994), 515–528. https://doi.org/10.1080/00207179408923090

33. E. P. Akpan, On the ϕ_0-stability of functional differential equations, *Aequationes Math.*, **52** (1996), 81–104. https://doi.org/10.1007/BF01818328

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