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Research article

Structure of backward quantum Markov chains

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Abstract: This paper extended the framework of quantum Markovianity by introducing backward and inverse backward quantum Markov chains (QMCs). We established the existence of these models under general conditions, demonstrating their applicability to a wide range of quantum systems. Our findings revealed distinct structural properties within these models, providing new insights into their dynamics and relationships to finitely correlated states. These advancements contributed to a deeper understanding of quantum processes and have potential implications for various quantum applications, including hidden quantum Markov processes.

Keywords: quantum theory; Markov chain; C^*-algebras; transition expectation; dynamics
Mathematics Subject Classification: 46L35, 46L55

1. Introduction

The development of the concept of quantum Markovianity significantly advances our comprehension of quantum dynamics and serves as a critical link in connecting various related disciplines [3–5, 19]. This concept not only broadens the theoretical landscape of quantum physics but also catalyzes advancements in practical applications. By integrating rigorous mathematical frameworks with empirical investigations, quantum Markovianity facilitates novel insights into the behavior and control of quantum systems [7, 15, 16].

In the seminal works [1,2], a comprehensive framework for *quantum Markovianity* was introduced, extending the classical notion of Markov processes into the quantum domain. This advanced framework provides a nuanced understanding of the temporal evolution of quantum systems under Markovian conditions. Specifically, the authors constructed two pivotal classes of states to exemplify this notion: QMCs and *inverse QMCs*. These states are defined on infinite tensor products of finite-dimensional type I factors with a totally ordered index set, capturing different aspects of quantum temporal dynamics. QMCs represent systems where the evolution from one state to another follows a Markovian process, while inverse QMCs consider the reverse dynamics within the same framework. Building on these foundational contributions, subsequent research has explored various definitions and properties of QMCs, particularly in connection with quantum information theory.

Notable studies have explored the complex relationships between quantum Markovianity and informational concepts, providing insights into how these chains model information flow within quantum systems [8, 9, 13, 19]. Specifically, research in [9] has developed a potential theory for QMCs, examining their recurrence, transience, and irreducibility with applications to quantum random walks and entangled states. Additionally, the study of Markov chains has been approached from various perspectives, including quantum operations [13, 14] and quantum conditional mutual information [12].

An analysis of the proofs reveals that the core constructions are applicable to infinite tensor products of arbitrary C^*-algebras. Notably, in [2, 6], the distinction between forward and backward QMCs (BQMCs) was deliberately circumvented. This decision was due to an overemphasis on interpreting the index set as time, which could confine the framework. Instead, the term *inverse QMC* was adopted, providing a more neutral terminology that is applicable to both probabilistic (i.e., dynamical) and statistical mechanics interpretations. Despite this neutral approach, a more nuanced analysis shows that within each category, QMC and inverse QMC, a natural distinction emerges between forward (or inside) and backward (or outside) QMC. This differentiation highlights deeper structural aspects within each class, suggesting that the evolution and properties of quantum systems can exhibit different characteristics depending on the direction of time or perspective of the analysis.

In this paper, the framework of BQMCs and inverse BQMCs is thoroughly examined. It is shown that for any sequence of backward transition expectations $\mathcal{E}^{(n)} : \mathcal{B}_n \otimes \mathcal{B}_{n+1} \to \mathcal{B}_n$, there exists at least one BQMCs on $\mathcal{A}$ that aligns with this sequence. This result substantiates the feasibility of the backward QMC model, demonstrating that quantum systems adhering to the specified transition expectations can indeed be constructed.

Additionally, the study establishes that for any sequence of backward transition expectations $\mathcal{E}^{(n)} : \mathcal{B}_n \otimes \mathcal{B}_{-\infty} \to \mathcal{B}_{-\infty}$, there is at least one inverse BQMCs on $\mathcal{A}$ that accommodates these expectations.Moreover, concrete examples are provided to distinguish between the two Markovian structures under consideration. Additionally, the paper demonstrates the connection between the introduced QMCs and finitely correlated states, as discussed in [10, 11]. These examples and connections offer valuable insights into the distinctions and relationships among various quantum Markovian models.

This finding broadens the scope of quantum Markovianity by confirming the practicality of inverse BQMCs models, thus expanding the theoretical understanding of quantum systems within this framework. The obtained results extend QMCs into two-sided one-dimensional (1D) lattice.

Our work advances the study of quantum Markov processes by establishing the existence of these models and offering new perspectives on their dynamics. These advancements push the boundaries of both theoretical and applied quantum physics, offering new insights and methodologies that significantly impact our comprehension of quantum processes and their practical applications. Namely, the present work is promising in connection with the recent development on hidden quantum Markov processes [3, 4, 20].

Let us outline the organization of the paper. Following an introduction to the preliminary concepts in Section 2, Section 3 is dedicated to the study of BQMCs. Section 4 focuses on inverse BQMCs. In Section 5, the paper elucidates the distinct structural properties inherent to both inverse and backward Markov chains. Section 6 establishes a connection between the obtained QMCs and finitely correlated states, further elaborating on the relationships and implications of these models. Finally, Section 7 provide concluding remarks.

2. Tensor BQMCs on 1D lattice

In the following, all C^*–algebras are unital and separable unless otherwise stated.

Definition 1. Given two $*$–algebras $\mathcal{A}$, $\mathcal{B}$ and an integer $n \in \mathbb{N}$, a $*$-map P: $\mathcal{A} \to \mathcal{B}$ is called n**–positive** ($n \in \mathbb{N}^*$) if $\forall\, b_1, \ldots, b_n \in \mathcal{B}$, $\forall\, a_1, \ldots, a_n \in \mathcal{A}$

$$\sum_{j,k=1}^{n} b_j^* P(a_j^* a_k) b_k \geq 0 \tag{1}$$

1**–positive** maps are called positive; n**–positive** maps for each natural integer n are called **completely positive**.

Definition 2. A **quasi–conditional expectation** with respect to the triplet of $*$-algebras $C \subseteq \mathcal{B} \subseteq \mathcal{A}$ is a completely positive linear map $E : \mathcal{A} \to \mathcal{B}$, such that

$$E(ca) = cE(a) \quad , \quad \forall a \in \mathcal{A} \;\; \forall c \in C \tag{2}$$

If $E(1) = 1$, E is called a **normalized**-quasi–conditional expectation.

Definition 3. Let $\mathcal{B}$, C be C^*–algebras. A completely positive, identity–preserving linear operator $\mathcal{E} : \mathcal{B} \to C$ is called a **Markov operator**. A Markov operator $\mathcal{E} : \mathcal{B} \otimes C \to \mathcal{B}$ is called a **backward transition expectation** from $\mathcal{B} \otimes C$ to $\mathcal{B}$.

It is notationally convenient to introduce the notations

$$\mathcal{E}_{2;b}(c) := \mathcal{E}(b \otimes c) =: \mathcal{E}_{1;c}(b) \quad ; \quad b \in \mathcal{B} \quad , \quad c \in C$$

Every backward transition expectation $\mathcal{E} : \mathcal{B} \otimes C \to \mathcal{B}$ defines two Markov operators: the $\mathcal{E}$**–forward Markov operator** (acting on the 1–st factor of the product):

$$\mathcal{E}_1 : b \in \mathcal{B} \to \mathcal{E}_1(b) := \mathcal{E}(b \otimes 1_C) = \mathcal{E}_{1;1_C}(b) = \mathcal{E}_{2;b}(1_C) \in \mathcal{B}$$

and the $\mathcal{E}$–**backward Markov operator** (acting on the 2-d factor of the product):

$$\mathcal{E}_2 : c \in C \to \mathcal{E}_2(c) := \mathcal{E}(1_{\mathcal{B}} \otimes c) = \mathcal{E}_{2;1_{\mathcal{B}}}(c) = \mathcal{E}_{1;c}(1_{\mathcal{B}}) \in \mathcal{B}$$

In the following $(\mathcal{B}_n)_{n\in\mathbb{Z}}$ will denote a sequence of C^*–algebras and

$$\mathcal{A} := \bigotimes_{n\in\mathbb{Z}} \mathcal{B}_n$$

$$j_n : \mathcal{B}_n \to j_n(\mathcal{B}_n) =: \mathcal{A}_n \subseteq \mathcal{A} \quad ; \quad n \in \mathbb{N}$$

with their infinite tensor product with respect to a fixed family of cross norms and the associated embeddings (see [18] Definition 1.23.11). For example, we can assume that each algebra $\mathcal{B}_n$ is realized on a Hilbert space $\mathcal{H}_n$ and that the tensor products are those induced by the tensor products of the corresponding spaces.

In the sequel, by $\mathcal{S}(\mathcal{A})$ we denote the set of states on $\mathcal{A}$, $1_{\mathcal{B}_n}$ the identity of $\mathcal{B}_n$, and $1_n := j_n(1_{\mathcal{B}_n}) = 1_{\mathcal{A}}$ the identity of $\mathcal{A}_n$.

Definition 4. *A state φ_{backw} on $\mathcal{A}$ is called a BQMC if there exist:*

(i) a sequence of backward transition expectations $\mathcal{E}^{(n)} : \mathcal{B}_n \otimes \mathcal{B}_{n+1} \to \mathcal{B}_n$ ($n \in \mathbb{Z}$),

*(ii) a sequence $\psi_n \in \mathcal{S}(\mathcal{B}_n)$ ($n \in \mathbb{Z}$), called a sequence of **boundary conditions for** φ_{backw}, such that for all $m \le n \in \mathbb{Z}$ and $b_h \in \mathcal{B}_h$, $h \in \{m, \dots, n\}$:*

$$\begin{aligned}\varphi_{backw}(j_m(b_m)\cdots j_n(b_n)) &:= \psi_m(\mathcal{E}^{(m)}(b_m \otimes \cdots \mathcal{E}^{(n-1)}(b_{n-1} \otimes \mathcal{E}^{(n)}(b_n \otimes 1_{\mathcal{B}_{n+1}})))\\ &= \psi_m(\mathcal{E}^{(m)}_{2;b_m} \circ \cdots \circ \mathcal{E}^{(n)}_{2;b_n}(1_{\mathcal{B}_{n+1}})) \qquad (3)\end{aligned}$$

3. Backward Markov Chains

Given a sequence of backward transition expectations $\mathcal{E}^{(n)} : \mathcal{B}_n \otimes \mathcal{B}_{n+1} \to \mathcal{B}_n$, define iteratively the vector spaces:

$$S^{backw}_{[n,n]} := S^{backw}_{[n]} := \mathcal{E}^{(n)}(\mathcal{B}_n \otimes 1_{\mathcal{B}_{n+1}}) = \mathcal{E}^{(n)}_1(\mathcal{B}_n) = \mathrm{Range}(\mathcal{E}^{(n)}_1) \subseteq \mathcal{B}_n$$

and, for $k \in \{1, \dots, n-m\}$,

$$S^{backw}_{[n-k,n]} := \mathcal{E}^{(n-k)}\left(\mathcal{B}_{n-k} \otimes S^{backw}_{[n-k+1,n]}\right) \subseteq \mathcal{B}_{n-k}$$

$$S^{backw}_{[m} := \bigvee_{n\ge m} S^{backw}_{[m,n]} \subseteq \mathcal{B}_m$$

where the righthand side denotes the linear sub–space of $\mathcal{B}_m$ generated by $S^{backw}_{[m,n]}$ for $n \ge m$. The above defined spaces are operator spaces in the sense of [17] with identity, so it makes sense to speak of states on them. The operator space $S^{backw}_{[m}$ is the domain of the boundary condition ψ_m and since

$$S^{backw}_{[m,n]} = \overline{\mathrm{lin.}\{\mathcal{E}^{(m+1)}_{2;b_m} \circ \cdots \circ \mathcal{E}^{(n)}_{2;b_n}(1_{\mathcal{B}_{n+1}}) \ : \ b_j \in \mathcal{B}_j\ ,\ j \in \{m, \dots, n\}\}} \qquad (4)$$

is a measure of the non–surjectivity of the sequence of transition expectations $\{\mathcal{E}^{(n)}\}_{n\ge m}$. In particular, if all the $\mathcal{E}^{(n)}_1$ and all the $\{\mathcal{E}^{(n)}\}_{n\ge m}$ are surjective, then $S^{backw}_{[m} = \mathcal{B}_m$.

Definition 5.
A sequence $(\varphi_{[0,n]})$ of states on $\mathcal{A}$ is called **convergent in the strongly finite sense** if, for any $a \in \mathcal{A}$, there exists $n_a \in \mathbb{N}$ such that for any $n \geq n_a$,

$$\varphi_{[0,n]}(a) = \varphi_{[0,n_a]}(a)$$

Lemma 1. *For a sequence of backward transition expectations* $\mathcal{E}^{(n)} : \mathcal{B}_n \otimes \mathcal{B}_{n+1} \to \mathcal{B}_n$*, the following statements are true.*
(B1) For each $n < N \in \mathbb{Z}$*, the map*

$$E_{N],N-1]} := id_{\mathcal{A}_{N-2]}} \otimes \left(j_{N-1} \circ \mathcal{E}^{(N-1)} \circ (j_{N-1} \otimes j_N)^{-1} \right) : \mathcal{A}_{N]} \to \mathcal{A}_{N-1]}$$

is a Markov quasi–conditional expectation with respect to the Markov localization

$$\{\mathcal{A}_{N-2]} \subseteq \mathcal{A}_{N-1]} \subseteq \mathcal{A}_{N]} \ , \ \mathcal{A}_{[N} \ , \ \mathcal{A}_N\}$$

(B2) For each $n < N \in \mathbb{Z}$*, the limit*

$$E_{n]} := \lim_{N\to+\infty} E_{n+1],n]} \circ E_{n+2],n+1]} \cdots \circ E_{N],N-1]} : \mathcal{A}_{N]} \to \mathcal{A}_{n]}$$

exists point-wise in the strongly finite sense on $\mathcal{A}_{loc}$ *and defines a Markov quasi–conditional expectation with respect to the Markov localization*

$$\{\mathcal{A}_{n-1]} \subseteq \mathcal{A}_{n]} \subseteq \mathcal{A} \ , \ \mathcal{A}_{[n} \ , \ \mathcal{A}_n\}$$

Moreover,

$$E_{n]}(\mathcal{A}_{[n}) = S^{backw}_{[n}$$

Proof. (B1) and (B2) follow from standard arguments on QMC. □

Lemma 2. *Let* φ_{backw} *be backward QMC on* $\mathcal{A}$ *with sequence of transition expectations* $\{\mathcal{E}^{(n)}\}_{n\in\mathbb{Z}}$ *and boundary conditions* (ψ_n)*. Then:*

(B3) The sequence (ψ_n) *satisfies the compatibility condition*

$$\psi_m \circ \mathcal{E}^{(m)}_2(b) = \psi_{m+1}(b) \quad , \quad \forall b \in S^{backw}_{[m} \quad , \quad \forall m \in \mathbb{Z} \tag{5}$$

In particular, if all the $\{\mathcal{E}^{(n)}_1\}_{n\geq m}$ *and all the transition expectations* $\{\mathcal{E}^{(n)}\}_{n\geq m}$ *are surjective, then* $\psi_m \circ \mathcal{E}^{(m)}_2 = \psi_{m+1}$*.*
(B4) The marginal distributions of φ_{backw} *are*

$$\varphi_n := \varphi_{backw} \circ j_n = \psi_{n-1} \circ \mathcal{E}^{(n)}_1 = \psi_n \circ \mathcal{E}^{(n)}((\,\cdot\,) \otimes 1_{\mathcal{B}_{n+1}}) \in \mathcal{S}(\mathcal{B}_n) \ ; \ n \in \mathbb{Z} \tag{6}$$

(B5) For any $m_0 \in \mathbb{Z}$ *choosing arbitrarily* ψ_{m_0}*, the state on* $\mathcal{A}_{[m_0}$ *defined by*

$$\varphi_{backw;[m_0} := \psi_{m_0} \circ E_{m_0]}$$

where $E_{m_0]}$ *is defined by Lemma 1, is a one–sided backward QMC with the sequence of boundary conditions* ψ_m $(m > m_0)$ *defined by* (5).

Proof. (B3) follows from (4) and:

$$\varphi_{backw}(j_m(1_{\mathcal{B}_m})j_{m+1}(b_{m+1})\cdots j_n(b_n)) = \psi_m(\mathcal{E}_2^{(m)}\circ\mathcal{E}_{2;b_{m+1}}^{(m+1)}\circ\cdots\circ\mathcal{E}_{2;b_n}^{(n)}(1_{\mathcal{B}_{n+1}}))$$

$$= \psi_{m+1}(\mathcal{E}_{2;b_{m+1}}^{(m+1)}\circ\cdots\circ\mathcal{E}_{2;b_n}^{(n)}(1_{\mathcal{B}_{n+1}}))$$

(B4) follows from

$$\varphi_n(b_n) := \varphi(j_n(b_n)) = \psi_n(\mathcal{E}_{2;b_n}^{(n)}(1_{n+1})) = \psi_{n-1}(\mathcal{E}_2^{(n-1)}\circ\mathcal{E}_{2;b_n}^{(n)}(1_{n+1})) \quad ; \quad \forall b_n\in\mathcal{B}_n$$

(B5) follows from standard arguments on QMC. □

Theorem 1. *Given a sequence of backward transition expectations* $\mathcal{E}^{(n)} : \mathcal{B}_n\otimes\mathcal{B}_{n+1}\to\mathcal{B}_n$, *the set of backward QMC on* $\mathcal{A}$ *admitting* $(\mathcal{E}^{(n)})$ *as sequence transition expectations is not empty.*

Proof. Let $\psi_n\in\mathcal{S}(\mathcal{B}_n)$ and $\chi_{n)}\in\mathcal{S}(\mathcal{A}_{n)})$ $(n\in\mathbb{Z})$ be two arbitrary sequences of states. Then, because of the Markov property, for any $n\in\mathbb{Z}$:

$$\varphi^{(n)} := \left(\chi_{n)}\Big|_{\mathcal{A}_{n)}}\right)\otimes\left(\psi_n\circ j_n^{-1}\circ E_{n]}\Big|_{\mathcal{A}_{[n}}\right)\in\mathcal{S}(\mathcal{A})\equiv\mathcal{S}(\mathcal{A}_{n)}\vee\mathcal{A}_{[n)}) \tag{7}$$

Since $\mathcal{A}$ is a C^*–algebra with unit, its state space is compact. Therefore the set of limit points of the sequence (7) is nonempty.
Let $(\varphi^{(n_k)})$ be a sub–sequence of (7) and $\varphi\in\mathcal{S}(\mathcal{A})$ be such that

$$\varphi = \lim_{k\to\infty}\varphi^{(n_k)} = \lim_{n_k\to-\infty}\varphi^{(n_k)}$$

point-wise on $\mathcal{A}$. Let $a\in\mathcal{A}_{loc}$, then there exists $n_0 < N_0\in\mathbb{N}$ such that $a\in\mathcal{A}_{[n_0,N_0]}$. Therefore, for $n_k < n_0$,

$$\varphi(a) := \lim_{n_k\to-\infty}\chi_{n_k)}(1_{n_k)})(\psi_{n_k}\circ j_{n_k}^{-1})\left(E_{n_k]}(a)\right) = \lim_{n_k\to-\infty}(\psi_{n_k}\circ j_{n_k}^{-1})\left(E_{n_k]}(a)\right)$$

$$= \lim_{n_k\to-\infty}\psi_{n_k}\left(\mathcal{E}_2^{(n_k)}\cdots\mathcal{E}_2^{(n_0-2)}\mathcal{E}_2^{(n_0-1)}j_{n_0}^{-1}E_{n_0]}(a)\right)$$

and since N_0 is arbitrary, this identity holds for all $a\in\mathcal{A}_{[n_0}$. This means that the sequence of states on $S_{[n_0}^{backw}\subseteq\mathcal{B}_{n_0}$ given by

$$\psi_{n_k}\circ\mathcal{E}_2^{(n_k)}\cdots\mathcal{E}_2^{(n_0-2)}\mathcal{E}_2^{(n_0-1)}$$

converges to a linear functional ϕ_{n_0}. Moreover, the identity

$$\varphi(a) = \phi_{n_0}\left(j_{n_0}^{-1}E_{n_0]}(a)\right) \quad ; \quad \forall a\in\mathcal{A}_{[n_0} \tag{8}$$

shows that ϕ_{n_0} is a state on $S_{[n_0}^{backw}$. By definition of $E_{n_0]}$, (14) implies that the identity (3) holds with the replacements

$$\varphi_{backw}\to\varphi \quad ; \quad n\to n_0 \quad ; \quad n\geq n_0$$

Since n_0 is arbitrary, it follows that φ is a BQMC with transitions expectations $(\mathcal{E}^{(n)})$ and sequence of boundary conditions (ϕ_n). □

4. Inverse BQMCs

In this section, we are going to define and investigate inverse BQMCs and prove an existence result.

Definition 6.
A state φ on $\mathcal{A}$ is called an **inverse BQMC** if there exist:
(j1) an algebra $\mathcal{B}_{-\infty}$;
(j2) a sequence of backward transition expectations $\mathcal{E}^{(n)} : \mathcal{B}_{-\infty} \otimes \mathcal{B}_n \to \mathcal{B}_{-\infty}$, $n \in \mathbb{Z}$;
(j3) a state $\psi_{-\infty} \in \mathcal{S}(\mathcal{B}_{-\infty})$, such that for all $m \leq n \in \mathbb{N}$ and $b_h \in M_{d_h}(\mathbb{C})$, $h \in \{m, \ldots, n\}$.

$$\begin{aligned}\varphi(j_m(b_m) \cdots j_n(b_n)) &:= \psi_{-\infty}(\mathcal{E}^{(n)}((\cdots \mathcal{E}^{(m)}(1_{\mathcal{B}_{-\infty}} \otimes b_m) \cdots) \otimes b_n)) \\ &:= \psi_{-\infty}(\mathcal{E}_{2;b_n} \circ \cdots \circ \mathcal{E}_{2;b_{m+1}} \circ \mathcal{E}_{2;1_{\mathcal{B}_{-\infty}}}(b_m)). \end{aligned} \tag{9}$$

Given a sequence of backward transition expectations $\mathcal{E}^{(n)} : \mathcal{B}_{-\infty} \otimes \mathcal{B}_n \to \mathcal{B}_{-\infty}$, define iteratively the operator spaces:

$$S^{backw}_{(n,n)} := S^{backw}_{(n)} := \mathcal{E}^{(n)}\left(1_{\mathcal{B}_{-\infty}} \otimes \mathcal{B}_n\right) = \mathcal{E}^{(n)}_2(\mathcal{B}_n) = \mathrm{Range}(\mathcal{E}^{(n)}_2) \subseteq \mathcal{B}_{-\infty}$$

and, for $k \in \{1, \ldots, n-m\}$,

$$S^{backw}_{(m,m+k)} := \mathcal{E}^{(m+k)}\left(S^{backw}_{(m,m+k-1)} \otimes \mathcal{B}_{m+k}\right) \subseteq \mathcal{B}_{-\infty}$$

$$S^{backw}_{(m} := \bigvee_{n \geq m} S^{backw}_{(m,n)} \subseteq \mathcal{B}_{-\infty}$$

where the right hand side denotes the linear sub–space of $\mathcal{B}_{-\infty}$ generated by $S^{backw}_{(m,n)}$ for $n \geq m$. $S^{backw}_{(m}$ is the domain of the boundary condition $\psi_{-\infty}$ and since

$$S^{backw}_{(m,n)} = \mathrm{lin.}\{\mathcal{E}^{(n)}_{2;b_n} \circ \cdots \circ \mathcal{E}^{(m+1)}_{2;b_{m+1}} \circ \mathcal{E}^{(m)}_{2;1_{\mathcal{B}_{-\infty}}}(b_m) \;:\; b_j \in \mathcal{B}_j\,,\; j \in \{m, \ldots, n\}\} \tag{10}$$

is a measure of the non–surjectivity of the sequence of transition expectations $\{\mathcal{E}^{(n)}\}_{n \geq m}$. In particular, if all the $\mathcal{E}^{(n)}_1$ and all the $\{\mathcal{E}^{(n)}\}_{n \geq m}$ are surjective, then $S^{backw}_{(m} = \mathcal{B}_{-\infty}$.

Lemma 3. *For a sequence of backward transition expectations $\mathcal{E}^{(n)} : \mathcal{B}_{-\infty} \otimes \mathcal{B}_n \to \mathcal{B}_{-\infty}$, the following statements are true.*
(IB1) For each $m \in \mathbb{Z}$, the map

$$E_{[m,[(m+1)} := \left(j_{-\infty} \circ \mathcal{E}^{(m)} \circ (j_{-\infty} \otimes j_m)^{-1}\right) \otimes id_{\mathcal{A}_{[(m+1)}} : \mathcal{A}_{-\infty} \vee \mathcal{A}_{[m} \to \mathcal{A}_{-\infty} \vee \mathcal{A}_{[(m+1)}$$

is a Markov quasi–conditional expectation with respect to the Markov localization

$$\{\mathcal{A}_{[(m+1)} \subseteq \mathcal{A}_{-\infty} \vee \mathcal{A}_{[(m+1)} \subseteq \mathcal{A}_{-\infty} \vee \mathcal{A}_{[m}\,,\ \mathcal{A}_{-\infty} \vee \mathcal{A}_m\,,\ \mathcal{A}_{-\infty}\}$$

(IB2) For each $n \in \mathbb{Z}$, the limit

$$E_{[n} := \lim_{m \to -\infty} E_{[n-1,[n} \circ E_{[n-2,[n-1} \cdots \circ E_{[m,[m+1} : \mathcal{A}_{-\infty} \vee \mathcal{A}_{[m} \to \mathcal{A}_{-\infty} \vee \mathcal{A}_{[n}$$

exists point-wise in the strongly finite sense on $\mathcal{A}_{loc}$ and defines a Markov quasi–conditional expectation with respect to the Markov localization

$$\{\mathcal{A}_{[(n+1)} \subseteq \mathcal{A}_{-\infty} \vee \mathcal{A}_{[n} \subseteq \mathcal{A}_{-\infty} \vee \mathcal{A}\,,\ \mathcal{A}_{-\infty} \vee \mathcal{A}_n\,,\ \mathcal{A}_{-\infty}\}$$

Moreover,

$$E_{(n}(\mathcal{A}_{[n}) = S^{backw}_{[n}$$

Proof. (IB1) holds because by definition the map $E_{[m,[(m+1)}$ satisfies

$$E_{[m,[(m+1)}\left(a_{\{-\infty\}\cup[m,+\infty)}a_{[(m+1)}\right) = E_{[m,[(m+1)}\left(a_{\{-\infty\}\cup[m,+\infty)}\right)a_{[(m+1)}$$

$$E_{[m,[(m+1)}\left(j_{-\infty}(b_{-\infty})j_{-\infty}(b_m)\right) = j_{-\infty}\left(\mathcal{E}^{(m)}(b_{-\infty}\otimes b_m)\right) \in \mathcal{A}_{-\infty}$$
$$\iff E_{[m,[(m+1)}\left(\mathcal{A}_{-\infty}\vee\mathcal{A}_n\right) \subseteq \mathcal{A}_{-\infty}$$

(IB2) follows from standard arguments on QMC together with the fact that, as $m \to -\infty$, $\mathcal{A}_{[m} \uparrow \mathcal{A}_{(-\infty} = \mathcal{A}$. □

Lemma 4. *Let φ_{inv} be an inverse BQMC on $\mathcal{A}$ with sequence of transition expectations $\{\mathcal{E}^{(n)}\}_{n\in\mathbb{Z}}$ and boundary condition $\psi_{-\infty}$. Then:*
(IB3) $\psi_{-\infty}$ satisfies the compatibility condition

$$\psi_{-\infty}\circ\mathcal{E}_2^{(m)}(b) = \psi_{-\infty}(b) \quad , \quad \forall b \in S_{[m}^{backw} \subseteq \mathcal{B}_{-\infty} \quad , \quad \forall m\in\mathbb{Z} \tag{11}$$

In particular, if all the $\{\mathcal{E}_1^{(n)}\}_{n\geq m}$ and all the transition expectations $\{\mathcal{E}^{(n)}\}_{n\geq m}$ are surjective, then $\psi_{-\infty}\circ\mathcal{E}_2^{(m)} = \psi_{-\infty}$.
(IB4) The marginal distributions of φ_{inv} are

$$\varphi_n := \varphi_{inv}\circ j_n = \psi_{-\infty}\circ\mathcal{E}_1^{(n)} = \psi_{-\infty}\circ\mathcal{E}^{(n)}(1_{\mathcal{B}_{-\infty}}\otimes(\,\cdot\,)) \in \mathcal{S}(\mathcal{B}_n)\ ;\ n\in\mathbb{Z} \tag{12}$$

(IB5) For any $m_0\in\mathbb{Z}$, choosing arbitrarily $\psi_{-\infty}$, the state on $\mathcal{A}_{[m_0}$ defined by

$$\varphi_{forw;[m_0} := \psi_{-\infty}\circ E_{m_0]}$$

where $E_{m_0]}$ is defined by Lemma 3. is a one–sided BQMC with boundary condition $\psi_{-\infty}$ defined by (11).

Proof. (IB3) follows from (10) and the fact that for every n and $b_j\in\mathcal{B}_j$, $j\in\{m,\ldots,n\}$:

$$\begin{aligned}
& \varphi_{inv}(j_m(b_m)j_{m+1}(b_{m+1})\cdots j_{n-1}(b_{n-1})) \\
= \ & \psi_{-\infty}(\mathcal{E}_{2;b_{n-1}}^{(n-1)}\circ\cdots\circ\mathcal{E}_{2;b_{m+1}}^{(m+1)}\circ\mathcal{E}_{2;1_{\mathcal{B}_{-\infty}}}^{(m)}(b_m)) \\
= \ & \varphi_{inv}(j_m(b_m)j_{m+1}(b_{m+1})\cdots j_{n-1}(b_{n-1})j_n(1_{\mathcal{B}_n})) \\
= \ & \psi_{-\infty}(\mathcal{E}_{2;1_{\mathcal{B}_n}}^{(n)}\circ\mathcal{E}_{2;b_{n-1}}^{(n-1)}\circ\cdots\circ\mathcal{E}_{2;b_{m+1}}^{(m+1)}\circ\mathcal{E}_{2;1_{\mathcal{B}_{-\infty}}}^{(m)}(b_m))
\end{aligned}$$

$$= \psi_{-\infty}(\mathcal{E}_2^{(n)}\circ\mathcal{E}_{2;b_{n-1}}^{(n-1)}\circ\cdots\circ\mathcal{E}_{2;b_{m+1}}^{(m+1)}\circ\mathcal{E}_{2;1_{\mathcal{B}_{-\infty}}}^{(m)}(b_m))$$

(IB4) follows from

$$\varphi_n(b_n) := \varphi(j_n(b_n)) = \psi_{-\infty}(\mathcal{E}_{2;b_n}^{(n)}(1_{\mathcal{B}_{-\infty}})) = \psi_{-\infty}(\mathcal{E}_1^{(n)}(b_n)) \quad ; \quad \forall b_n\in\mathcal{B}_n$$

(IB5) follows from standard arguments on QMC. □

Theorem 2. *Given a sequence of backward transition expectations*
$\mathcal{E}^{(n)} : \mathcal{B}_n\otimes\mathcal{B}_{-\infty}\to\mathcal{B}_{-\infty}$, the set of BQMC on $\mathcal{A}$ admitting $(\mathcal{E}^{(n)})$ as sequence transition expectations is not empty.

Proof. Let $\psi_{-\infty} \in \mathcal{S}(\mathcal{B}_{-\infty})$ and $\chi_{(n} \in \mathcal{S}(\mathcal{A}_{(n})$ $(n \in \mathbb{Z})$ be two arbitrary sequences of states. Then, because of the Markov property, for any $n \in \mathbb{Z}$:

$$\varphi^{(n)} := \left(\psi_{-\infty} \circ j_{-\infty}^{-1} \circ E_{[n}\Big|_{\mathcal{A}_{n]}}\right) \otimes \left(\chi_{(n}\Big|_{\mathcal{A}_{(n}}\right) \in \mathcal{S}(\mathcal{A}) \equiv \mathcal{S}(\mathcal{A}_{n]} \vee \mathcal{A}_{(n}) \tag{13}$$

Since $\mathcal{A}$ is a C^*–algebra with unit, its state space is compact. Therefore, the set of limit points of the sequence (13) is nonempty.
Let $(\varphi^{(n_k)})$ be a sub–sequence of (7) and $\varphi \in \mathcal{S}(\mathcal{A})$ be such that

$$\varphi = \lim_{k\to\infty} \varphi^{(n_k)} = \lim_{n_k\to\infty} \varphi^{(n_k)}$$

point-wise on $\mathcal{A}$. Let $a \in \mathcal{A}_{loc}$, then there exists $n_0 < N_0 \in \mathbb{N}$ such that $a \in \mathcal{A}_{[n_0,N_0]}$. Therefore, for $n_k > n_0$,

$$\begin{aligned}
\varphi(a) &:= \lim_{n_k\to+\infty} (\psi_{-\infty} \circ j_{-\infty}^{-1}) \left(E_{[n_k}(a)\right) \chi_{n_k)}(1_{n_k)}) \\
&= \lim_{n_k\to+\infty} (\psi_{-\infty} \circ j_{-\infty}^{-1}) \left(E_{[n_k}(a)\right) \\
&= \lim_{n_k\to+\infty} \psi_{-\infty} \circ \mathcal{E}_2^{(n_k)} \circ \cdots \circ \mathcal{E}_2^{(n_0+2)} \circ \mathcal{E}_2^{(n_0+1)} j_{-\infty}^{-1} E_{[n_0}(a)
\end{aligned}$$

and since N_0 is arbitrary, this identity holds for all $a \in \mathcal{A}_{[n_0}$. This means that the sequence of states on $S_{[n_0}^{backw} \subseteq \mathcal{B}_{-\infty}$ given by

$$\psi_{-\infty} \circ \mathcal{E}_2^{(n_k)} \cdots \mathcal{E}_2^{(n_0+2)} \circ \mathcal{E}_2^{(n_0+1)}$$

converges to a linear functional $\phi_{-\infty}$. Moreover, the identity

$$\varphi(a) = \phi_{-\infty}\left(j_{n_0}^{-1} E_{[n_0}(a)\right) \quad ; \quad \forall a \in \mathcal{A}_{[n_0} \tag{14}$$

shows that $\phi_{-\infty}$ is a state on $S_{[n_0}^{backw}$. By definition of $E_{[n_0}$, (14) implies that the identity (3) holds with the replacements

$$\varphi_{inv} \to \varphi \quad ; \quad n \to n_0 \quad ; \quad n \geq n_0$$

Since n_0 is arbitrary, it follows that φ is an inverse BQMC with transitions expectations $(\mathcal{E}^{(n)})$ and boundary conditions $\phi_{-\infty}$. □

5. Example of inverse and BQMCs

Within this section, we aim to elucidate the distinct structural properties inherent in inverse and backward Markov chains, employing a concrete example as an illustrative tool.
Let $\mathcal{M}_2 := M_2(\mathbb{C})$ be our starting C^*-algebra. By σ_x, σ_y, σ_z we denote the Pauli spin operators, i.e.,

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $E = \frac{1}{\sqrt{2}}(1, 1)$ and define $\mathcal{E} : \mathcal{M}_2 \otimes \mathcal{M}_2 \to \mathcal{M}_2$, for all $a = (a_{ij})$, $b = (b_{ij})$ by

$$\mathcal{E}(a \otimes b) = \sum_{i=0}^{3} M_i a \otimes b M_i^* \tag{15}$$
$$= \frac{(1-p)}{2}\sum_{i,j} a_{ij} b + \frac{p}{6}\sum_{i,j} a_{ij}\sigma_x b\sigma_x + \frac{p}{6}\sum_{i,j} a_{ij}\sigma_y b\sigma_y + \frac{p}{6}\sum_{i,j} a_{ij}\sigma_z b\sigma_z$$

where

$$M_0 = \sqrt{1-p}E \otimes I; \quad M_1 = \sqrt{\frac{p}{3}}E \otimes \sigma_x; \quad M_2 = \sqrt{\frac{p}{3}}E \otimes \sigma_y; \quad M_3 = \sqrt{\frac{p}{3}}E \otimes \sigma_z$$

Lemma 5. *$\mathcal{E}$ defined by* (15) *is a backward Markov transition expectation, i.e., a completely positive identity preserving linear map.*

Proof. $\mathcal{E}$ is identity preserving, in fact, from (15),

$$\mathcal{E}(I \otimes I) \;=\; (1-p)I + \frac{p}{3}I + \frac{p}{3}I + \frac{p}{3}I = I$$

Additionally, $\mathcal{E}$ is expressed in the Kraus representation, indicating its property of being completely positive. □

Lemma 6. *For all $n \in \mathbb{Z}$, let $\mathcal{E}_n := \mathcal{E}$ be defined by* (15) *and ψ_n be a state on $\mathcal{B}_n \equiv \mathcal{M}_2$. Then for all $m \leq n \in \mathbb{Z}$ and $b_h = \begin{pmatrix} b_{h,11} & b_{h,12} \\ b_{h,21} & b_{h,22} \end{pmatrix} \in \mathcal{B}_h$, let $h \in \{m, \ldots, n\}$*

$$\psi_m(\mathcal{E}(b_m \otimes \cdots \mathcal{E}(b_{n-1} \otimes \mathcal{E}(b_n \otimes 1_{\mathcal{B}_{n+1}}))) = \sum_{i_m, j_m \cdots i_n, j_n} b_{n,i_n j_n} b_{n-1,i_{n-1} j_{n-1}} \cdots b_{m,i_m j_m} \tag{16}$$

Proof By observing (15), it becomes apparent that

$$\mathcal{E}(b_n \otimes 1_{\mathcal{B}_{n+1}}) = \sum_{i,j} b_{n,ij} \tag{17}$$

Hence, (18) can be derived through iterative processes. □

Lemma 7. *Let $(\mathcal{E}^{(n)} = \mathcal{E})_{n\in\mathbb{Z}}$ denote a sequence of backward transition expectations, defined by* (15)*. Consider $\psi_{-\infty}$ as a state on $\mathcal{B}_{-\infty} \equiv \mathcal{M}_2$. Then, for all $m \leq n \in \mathbb{Z}$ and $b_h = \begin{pmatrix} b_{h,11} & b_{h,12} \\ b_{h,21} & b_{h,22} \end{pmatrix} \in \mathcal{B}_h$, where $h \in \{m, \ldots, n\}$, we have the following expression:*

$$\psi_{-\infty}(\mathcal{E}^{(n)}((\cdots \mathcal{E}^{(m+1)}(\mathcal{E}^{(m)}(1_{\mathcal{B}-\infty} \otimes bm) \otimes b_{m+1}) \cdots) \otimes b_n)) \tag{18}$$
$$= \frac{1}{2^{n-m}}\left(\sum_{i,j} b_{m,ij} - \frac{4p}{3}\sum_{i\neq j} b_{m,ij}\right)\cdots\left(\sum_{i,j} b_{n-1,ij} - \frac{4p}{3}\sum_{i\neq j} b_{n-1,ij}\right)$$
$$\times\psi_{-\infty}\left[\frac{(1-p)}{2}b_n + \frac{p}{6}\left(\sigma_x b_n \sigma_x + \sigma_y b_n \sigma_y + \sigma_z b_n \sigma_z\right)\right]$$

Proof By observing (15), it becomes apparent that

$$\mathcal{E}(1_{\mathcal{B}_{-\infty}} \otimes b_m) = \sum_{i=0}^{3} M_i 1_{\mathcal{B}_{-\infty}} \otimes b_m M_i^* \tag{19}$$

then

$$\mathcal{E}(1_{\mathcal{B}_{-\infty}} \otimes b_m) = (1-p)b_m + \frac{p}{3}\sigma_x b_m \sigma_x + \frac{p}{3}\sigma_y b_m \sigma_y + \frac{p}{3}\sigma_z b_m \sigma_z$$

One can see that $\mathcal{E}(\sigma_z b_m \sigma_z \otimes b_{m+1}) = \mathcal{E}(\sigma_y b_m \sigma_y \otimes b_{m+1})$. Consequently, we obtain:

$$\begin{aligned}
\mathcal{E}(\mathcal{E}(1_{\mathcal{B}_{-\infty}} \otimes b_m) \otimes b_{m+1}) &= (1-p)\mathcal{E}(b_m \otimes b_{m+1}) + \frac{p}{3}\mathcal{E}(\sigma_x b_m \sigma_x \otimes b_{m+1}) \\
&\quad + \frac{2p}{3}\mathcal{E}(\sigma_y b_m \sigma_y \otimes b_{m+1}) \\
&= (1-p)\sum_{i,j} b_{m,ij}\left[\frac{(1-p)}{2}b_{m+1} + \frac{p}{6}\left(\sigma_x b_{m+1}\sigma_x + \sigma_y b_{m+1}\sigma_y + \sigma_z b_{m+1}\sigma_z\right)\right] \\
&\quad + \frac{p}{3}\sum_{i,j} b_{m,ij}\left[\frac{(1-p)}{2}b_{m+1} + \frac{p}{6}\left(\sigma_x b_{m+1}\sigma_x + \sigma_y b_{m+1}\sigma_y + \sigma_z b_{m+1}\sigma_z\right)\right] \\
&\quad + \frac{2p}{3}\sum_{i} b_{m,ii}\left[\frac{(1-p)}{2}b_{m+1} + \frac{p}{6}\left(\sigma_x b_{m+1}\sigma_x + \sigma_y b_{m+1}\sigma_y + \sigma_z b_{m+1}\sigma_z\right)\right] \\
&\quad - \frac{2p}{3}\sum_{i\neq j} b_{m,ij}\left[\frac{(1-p)}{2}b_{m+1} + \frac{p}{6}\left(\sigma_x b_{m+1}\sigma_x + \sigma_y b_{m+1}\sigma_y + \sigma_z b_{m+1}\sigma_z\right)\right] \\
&= \left(\sum_{i,j} b_{m,ij} - \frac{4p}{3}\sum_{i\neq j} b_{m,ij}\right)\left[\frac{(1-p)}{2}b_{m+1} + \frac{p}{6}\left(\sigma_x b_{m+1}\sigma_x + \sigma_y b_{m+1}\sigma_y + \sigma_z b_{m+1}\sigma_z\right)\right]
\end{aligned}$$

One more step,

$$\begin{aligned}
\mathcal{E}(\mathcal{E}(\mathcal{E}(1_{\mathcal{B}_{-\infty}} \otimes b_m) \otimes b_{m+1}) \otimes b_{m+2}) &= \frac{1}{2}\left(\sum_{i,j} b_{m,ij} - \frac{4p}{3}\sum_{i\neq j} b_{m,ij}\right)(1-p)\mathcal{E}(b_{m+1} \otimes b_{m+2}) \\
&\quad + \frac{1}{2}\left(\sum_{i,j} b_{m,ij} - \frac{4p}{3}\sum_{i\neq j} b_{m,ij}\right)\frac{p}{3}\left(\mathcal{E}(\sigma_x b_{m+1}\sigma_x \otimes b_{m+2}) + 2\mathcal{E}(\sigma_y b_{m+1}\sigma_y \otimes b_{m+2})\right) \\
&= \frac{1}{2}\left(\sum_{i,j} b_{m,ij} - \frac{4p}{3}\sum_{i\neq j} b_{m,ij}\right)\left(\sum_{i,j} b_{m+1,ij} - \frac{4p}{3}\sum_{i\neq j} b_{m+1,ij}\right) \\
&\quad \times\left[\frac{(1-p)}{2}b_{m+2} + \frac{p}{6}\left(\sigma_x b_{m+2}\sigma_x + \sigma_y b_{m+2}\sigma_y + \sigma_z b_{m+2}\sigma_z\right)\right]
\end{aligned}$$

Through the process of iteration, we find that

$$\mathcal{E}^{(n)}((\cdots\mathcal{E}^{(m+1)}(\mathcal{E}^{(m)}(1_{\mathcal{B}-\infty} \otimes b_m) \otimes b_{m+1})\cdots) \otimes b_n)$$

$$= \frac{1}{2^{n-m}} \left(\sum_{i,j} b_{m,ij} - \frac{4p}{3} \sum_{i \neq j} b_{m,ij} \right) \cdots \left(\sum_{i,j} b_{n-1,ij} - \frac{4p}{3} \sum_{i \neq j} b_{n-1,ij} \right)$$
$$\times \left[\frac{(1-p)}{2} b_n + \frac{p}{6} \left(\sigma_x b_n \sigma_x + \sigma_y b_n \sigma_y + \sigma_z b_n \sigma_z \right) \right]$$

□

Remark 1. *The above example illustrates the difference between the structure of the backward Markov chain and the inverse Markov chains.*

6. Recovering finitely correlated states

In this section, we will check whether, BQMCs and the inverse QMCs have finitely correlated states structure. Recall that, a finitely correlated state (FCS) [10, 11] is a translation invariant state φ on the quasi-local algebra $\mathcal{A} = \bigotimes_{n\in\mathbb{Z}} \mathcal{B}_n$, where each $\mathcal{B}_n$ is a copy of a C*-algebra $\mathcal{B}$ with unit 1, characterized by the existence of a finite dimensional linear space $\mathcal{V}$, a linear map $\mathbb{E} : A \in \mathcal{B} \mapsto \mathcal{L}(\mathcal{V})$ ($\mathcal{L}(\mathcal{V})$ being the set of linear maps from $\mathcal{V}$ into itself), an element $e \in \mathcal{V}$, and a linear functional $\rho \in \mathcal{V}^*$ such that

$$\mathbb{E}_1(e) = e, \quad \rho \circ \mathbb{E}_1 = \rho$$

and for $n \in \mathbb{Z}, m \in \mathbb{N}$, and $a_j \in \mathcal{B}_j$:

$$\varphi(a_n \otimes \cdots a_{n+m}) = \frac{1}{\rho(e)} \rho \circ \mathbb{E}_{a_n} \circ \mathbb{E}_{a_{n+1}} \circ \cdots \circ \mathbb{E}_{a_{n+m}}(e) \tag{20}$$

Let $\varphi_{backw} \equiv ((\mathcal{B}_n)_n, (\mathcal{E}^{(n)})_n, (\psi_n)_n)$ be a BQMC whose correlations are given by (3). In the homogenous, for which all the $\mathcal{B}_n$ are copies of finite dimensional C*-algebra, all the transition expectations $\mathcal{E}_n$ are copies of a transition expectation $\mathcal{E} : \mathcal{B} \otimes \mathcal{B} \to \mathcal{B}$, and the functional ψ_n is a copy of a state $\psi \in \mathcal{S}(\mathcal{B})$. By taking $\mathcal{V} = \mathcal{B}, \mathbb{E} = \mathcal{E}, \rho = \psi$ and $e = \mathbf{1}$, one can see that φ_{backw} defines an FCS on $\mathcal{A}$.

Similarly, let $\varphi \equiv ((\mathcal{B}_n)_n, \mathcal{B}_\infty, (\mathcal{E}_n), \psi_\infty)$ be an inverse backward Markov chain. Assume that $\mathcal{B}_n$ are copies of a C*-algebra $\mathcal{B}$ and $\mathcal{E}^{(n)}$ are copies of a transition expectation $\mathcal{E} : \mathcal{B}_{-\infty} \otimes \mathcal{B} \to \mathcal{B}_{-\infty}$. By taking $\mathcal{V} = \mathcal{B}_{-\infty}, \rho = \psi_{-\infty}$ and $e = 1$, then the state φ generates an FCS. Contrary, to the FCS generated by BQMCs, the linear space $\mathcal{V} = \mathcal{B}_{-\infty}$ is not assumed to coincide with the algebra $\mathcal{B}$.

7. Conclusions

In conclusion, this paper significantly advances the field of quantum Markovianity by thoroughly examining BQMCs and inverse BQMCs. The results demonstrate that for any given sequence of backward transition expectations, both backward QMCs and inverse BQMCs can be constructed, thus affirming the feasibility and applicability of these models. This work not only clarifies the distinctions between different Markovian structures but also establishes important connections between QMCs and finitely correlated states. By providing concrete examples and elucidating the structural differences, this study enhances our understanding of quantum systems dynamics. The findings contribute to the broader theoretical framework and offer new perspectives that could impact practical applications in quantum information science. The insights gained from this research are expected to influence ongoing studies, including those related to hidden quantum Markov processes, and pave the way for further exploration in quantum dynamics and information theory.

Author contributions

Luigi Accardi: Conceptualization, Methodology, Writing – original draft; El Gheteb Soueidi: Methodology, Writing – original draft; Abdessatar Souissi: Writing – original draft, Writing – review & editing; Mohamed Rhaima: Validation, Funding acquisition; Farrukh Mukhamedov: Conceptualization, Methodology; Farzona Mukhamedova: Visualization, Validation.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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