



Research article

A symplectic approach to Schrödinger equations in the infinite-dimensional unbounded setting

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Abstract: By using the theory of analytic vectors and manifolds modeled on normed spaces, we provide a rigorous symplectic differential geometric approach to t -dependent Schrödinger equations on separable (possibly infinite-dimensional) Hilbert spaces determined by families of unbounded self-adjoint Hamiltonians admitting a common domain of analytic vectors. This allows one to cope with the lack of smoothness of structures appearing in quantum mechanical problems while using differential geometric techniques. Our techniques also allow for the analysis of problems related to unbounded operators that are not self-adjoint. As an application, the Marsden-Weinstein reduction procedure was employed to map the above-mentioned t -dependent Schrödinger equations onto their projective spaces. We also analyzed other physically and mathematically relevant applications, demonstrating the usefulness of our techniques.

Keywords: analytic vector; infinite-dimensional symplectic manifold; Marsden-Weinstein reduction; normed space; projective Schrödinger equation; unbounded operator

Mathematics Subject Classification: 17B66, 34A26, 34A34, 53Z05

Abbreviations

A, B, H, \dots : operators; $D(A)$: domain of the operator A ; E, E_1, E_2, \dots : normed spaces; $\mathcal{X}, \mathcal{Y}, \dots$: Banach spaces; \mathcal{H} : Hilbert space; $\mathcal{C}^0[0, 1]$: Banach space of continuous real-valued functions on $[0, 1]$ with the supremum norm; $\mathcal{C}^1[0, 1]$: Banach space of continuous and one-time differentiable real-valued functions on $[0, 1]$ with respect to its natural norm; $\mathfrak{D}_v f$: Gâteaux differential at $u \in U$ in the direction $v \in E$; $\mathcal{D}_u P$: operational tangent space at point u to the manifold P ; $T_u P$: (kinematic) tangent space at u to the manifold P ; $H_n^p(x)$: Hermite probabilistic polynomial of order n ;

$D^\infty(A) := \bigcap_{n \in \bar{\mathbb{N}}} D(A^n)$: space of smooth vectors of an operator B ; $D^a(A)$: space of analytic vectors of an operator A ; $\dot{\psi}_{\psi_0}$: directional derivative on a normed space in the direction ψ at the point ψ_0 ; $\nabla_{\dot{\psi}}$: partial derivative in the direction $\dot{\psi}$ on a normed space; $\mathcal{P}\mathcal{H}$: projective space of a Hilbert space \mathcal{H} ; $\bar{\mathbb{N}}$: set of natural numbers including zero

1. Introduction and state-of-the-art

The study of finite-dimensional symplectic manifolds has proven to be very fruitful in the description of mathematical and physical theories [1–3]. Among its applications, one finds the analysis of problems appearing in classical and quantum mechanics (see [1, 3–5] and references therein).

The analysis of symplectic manifolds modeled on (possibly infinite-dimensional) Banach spaces, the hereafter called *symplectic Banach manifolds* (SBMs), is an appealing topic of research. This is illustrated by the mathematical works [6–9], or, more specifically, because SBMs are a particular case of other geometric structures on infinite-dimensional manifolds [10, 11]. Moreover, SBMs are also interesting from a physical perspective [12–15], and we will focus primarily on works such as [16–18]. Furthermore, SBMs appear in relevant physical topics such as Schrödinger equations (SEs) [17–19], non-linear SEs [7, 12, 13, 15], and fluid mechanics [19–22].

The theory of SBMs emerges as a generalization of finite-dimensional symplectic geometry obtained by addressing appropriate topological and analytical issues due to the infinite-dimensional nature of general Banach spaces. SBMs differ significantly from their finite-dimensional counterparts, e.g., there exists no immediate analogue of the Darboux theorem (see [9] and references therein). Relevantly, quantum and classical problems related to infinite-dimensional SBMs frequently require the study of non-smooth mathematical structures [18, 19, 23].

Schrödinger quantum mechanics represents a relevant field of application of symplectic geometry [5, 14, 17, 24]. It is also worth noting that symplectic forms have been defined and used on manifolds modeled on different linear spaces: Banach [6, 22, 25], Hilbert [26, 27], Fréchet [15, 25, 28, 29], convenient spaces [19, 23, 25, 26, 30, 31], etc. Despite this, many researchers study rigorously the differential geometric properties of t -dependent SEs on infinite-dimensional Hilbert manifolds. In this regard, [19, 23] are probably the first geometric analyses, with other notable references including [26, 32]. Most geometric works in the literature focus only on t -dependent or t -independent Schrödinger equations in finite-dimensional Hilbert spaces [4, 17, 24, 27]. These works provide a clear geometric description of momentum maps in quantum mechanics, their relationships with other quantum mechanical formalisms, and more. However, few articles try to extend such results to an infinite-dimensional realm [17, 18, 28]. Let us review the literature on quantum mechanics approached through symplectic and other geometric methods in the infinite-dimensional context.

The first important pioneering works dealing with the problem of studying Hamiltonian symplectic systems on infinite-dimensional manifolds and quantum mechanics can be traced back to Marsden and his collaborators [19, 23, 26]. They developed methods to study Schrödinger equations on infinite-dimensional Hilbert spaces and other partial differential equations and problems mainly on manifolds modeled on normed spaces of different types. They also reviewed the differential calculus on locally convex topological vector spaces, introduced vector fields with domains, and other

techniques to deal with quantum mechanical systems and partial differential equations. Moreover, Marsden and his collaborators also presented a generalized Hamiltonian approach based on the use of distributions and applied it to analyze the Bohr-Dirac correspondence principle in [23].

Later, other authors continued the study of quantum mechanical problems in a geometric manner. Some skipped most technical explanations and mathematical details, dealing mostly with bounded operators, finite-dimensional manifolds and/or not studying differentiability problems or domains ([32] and references therein on the topic), or justifying results on physical grounds. It is worth stressing the relevance of [28], which is one of the fundamental references in the study of quantum mechanics via differential geometric methods. It should be noted that [28] uses, among other techniques, manifolds modeled on Fréchet spaces, which are quite difficult to deal with.

In the 1970s and 1980s, many authors were confident in the possibility of treating quantum mechanics on an infinite-dimensional manifold using differential geometry [12, 13, 33]. In particular, the authors in [33] treated the study of quantum mechanics in an infinite-dimensional projective space and analyzed the integrability of quantum mechanical systems. Moreover, [32] focuses on manifolds modeled on Hilbert spaces. It is worth stressing that [32] analyzes the literature on the geometric treatment of quantum mechanics available at that time. Other posterior works, like [29], presented certain quantum differentials and other approaches based on spaces of functions.

Furthermore, [18] provided a nice geometric approach to quantum mechanics modeled on Hilbert spaces, involving weak symplectic manifolds, and solving some problems in the literature. The work in [18] mostly focused on problems related to bounded operators, and partially skipped, in some interesting manners based on the use of relations, the study of the smoothness of geometric structures for self-adjoint operators. It is worth noting that [18] does not deal with t -dependent operators, which generally requires the study in detail of the common domain of several not necessarily self-adjoint operators on it, addressed in our paper. At moments, [18] assumes some domain of differentiability to apply its techniques; in others, relations are enough to prove their purposes.

Apart from [18], other researchers deal with potential approaches to the study of quantum mechanics via symplectic manifolds on special types of infinite-dimensional manifolds [15, 30, 31]. It is worth noting that there exists an analysis of Poisson manifolds and Hamiltonian systems in convenient spaces [30, 31, 34], which could be potentially used to study t -dependent Schrödinger equations, but it seems that no relevant advancements in this direction has been done so far. However, consider that convenient space methods are quite technical. Finally, there are several other works showing new potential approaches to t -dependent Schrödinger equations [35]. Of course, there are many works with applications in quantum mechanics using techniques from functional analysis [36–38] and many researchers who focus on the application of differential geometry in finite-dimensional Schrödinger equations or treat the problem at a physical level without dealing with technical problems. At this point, it is important to make some distinctions, as certain works can be very precise while omitting the study of specific aspects. For example, [18] provides a precise description by bypassing some smoothness considerations, for instance, by using an approach based on relations. The work [32] is supposed to be focusing on the infinite-dimensional case, but it does not analyze the mathematical challenges, such as smoothness, operator domains, etc. The work [17] examines key aspects of infinite-dimensional Schrödinger equations, like kinetic and operational tangent spaces, but does not thoroughly analyze operator domain definitions. Similar approaches are found in [4, 5, 39].

It should be stressed that several research groups may have their own conventions and assumptions in the study of symplectic geometry in infinite-dimensional manifolds. For instance, some researchers admit symplectic forms on infinite-dimensional manifolds to have a non-trivial kernel (see the work [11, Remark 2.1]), which goes against the classical definition, which we follow here. In a more standard manner, these structures are called presymplectic forms (see [40] for presymplectic forms on Banach spaces).

Summarizing, most researchers deal with the study via differential geometry of Schrödinger equations and other related equations on finite-dimensional manifolds [5, 24, 41, 42]. In the infinite-dimensional realm, one assumes smoothness of structures in different manners or studies problems one can deal with without treating differentiability and other technical issues [4, 17, 43]. Some standard assumptions are the use of bounded or self-adjoint operators [18]. In general, most researchers mix some of the previous assumptions at different parts of their analysis. This significantly restricts the field of application of differential geometry in quantum mechanics. Several problems, like those related to families of operators that can be integrated to a Lie group action on an infinite-dimensional space, are related with operators that are not self-adjoint in a common domain to be analyzed. Hence, the development of an infinite-dimensional differential geometric approach allowing for the extension of previous ideas to general t -dependent unbounded Hamiltonian operators is extremely important, and it will be the main topic of this work.

It is worth stressing that the lack of a fully general geometric infinite-dimensional approach to t -dependent Schrödinger equations is due, in part, to the lack of smoothness of the structures used to describe unbounded operators and other related structures; the problems to define differential geometric objects, like the commutator of operators [19], etc. Known methods to deal with these problems rely on using pure functional analysis techniques with no differential geometric methods or the justification of technical assumptions and results on physical grounds. Researchers using differential geometry techniques in quantum mechanical models in infinite-dimensional manifolds do not generally provide a detailed explanation of the differential geometric constructions they use. As far as we know, there is no rigorous differential geometric work dealing effectively with unbounded operators and their non-smoothness [37].

We apply infinite-dimensional geometric and functional analysis techniques to the rigorous geometrical study of t -dependent SEs determined by (possibly not bounded) families of self-adjoint operators on separable Hilbert spaces. It is worth noting that since we deal with several self-adjoint operators at the same time, our techniques have to deal with non self-adjoint operators. Indeed, our methods can also be applied in a natural manner to operators on domains where they are not self-adjoint. For instance, analytic vectors naturally appear in problems related to a Lie group acting on an infinite-dimensional Hilbert space. Separable Hilbert spaces occur in most physical problems, and separability ensures the existence of a countable set of coordinates [27]. As a first new contribution, we have applied the theory of normed spaces, which admits a relatively simple differential geometric analysis and extensions to the Banach case, and analytic vectors [44–46] to improve the standard geometric approach given in [19] to deal with unbounded Hamiltonian operators. In particular, we mostly restrict ourselves to t -dependent Hamiltonian operators taking values in a Lie algebra of operators with an appropriate common domain of analytic vectors. It is worth noting that our approach can be applied to more general cases provided an adequate space of analytic vectors is known. Although the problems we are dealing with are not generally smooth, we

will restrict ourselves to appropriate subspaces where problems are smooth enough to apply differential geometry. It is also worth noting that many of our practical applications restrict to the so-called strong SBMs, where a Darboux theorem exists [47] and other differential geometric techniques are applicable.

Our approach seems to be general enough to provide a rigorous theory while studying interesting physical systems, but it requires using some functional analysis techniques [19]. We have relied as much as possible on differential geometric techniques and simplifying technical assumptions. In particular, self-adjoint operators and their mean values are described as objects on manifolds modeled on normed spaces stemming from the manifold structure of Hilbert spaces. Due to the fact that functional analysis is not well-known for most researchers working on differential geometric methods in physics, we provide a brief introduction to the results on functional analysis and infinite-dimensional manifolds to be used.

Our work solves some technical difficulties to ensure the standard differential geometric approach of the examples under study and other similar ones. For instance, we provide several technical details in the theory of analytic vectors related to the metaplectic representation, angular momentum operators, and other physical examples, which seem to be not developed elsewhere. Once these details are given, the further application of differential geometric techniques is simpler and operational. In particular, our approach shows that the case of physical unbounded observables and other operators admitting a basis of eigenfunctions (without being necessarily symmetric), may be studied throughout our differential geometric techniques. It is worth noting that our approach is much simpler than the one in [19], which requires the use of topological vector spaces and deals with complicated issues.

As an application of our techniques, the above-mentioned t -dependent SEs are projected onto (generally infinite-dimensional) projective spaces using a Marsden-Weinstein (MW) reduction. Our methods represent an appropriate framework, for instance, for the generalization of the standard differential geometric works by G. Marmo and collaborators on quantum mechanics (see, e.g., [4, 5, 48]) or other authors [27, 49] to the analysis of quantum mechanical problems on infinite-dimensional Hilbert spaces not determined by bounded operators.

Let us detail the structure of this work. At the end of this introduction, we present a table summarizing the major symbols used throughout this work. Section 2 is devoted to the basic functional analysis techniques and notations to be used hereafter. Differential geometry modeled on normed spaces [19, 50] and Banach spaces [1, 16] are discussed in Section 3. In Section 4, the theory of Banach and normed symplectic manifolds is briefly addressed. After studying weak and strong SBMs, we survey their related Darboux theorems, the canonical one- and two-forms on cotangent bundles to Banach manifolds, and introduce the MW reduction theorem. In Section 5, we provide a symplectic approach to infinite-dimensional separable Hilbert spaces. In Section 6, we focus on vector fields induced by unbounded self-adjoint, observable, and diagonalizable operators, while we analyze Hamiltonian functions for quantum models in Section 7. In Section 8, we study a class of t -dependent Hamiltonian systems related to the t -dependent SEs appearing in this work. In Section 9, a MW reduction procedure is used to project the above-mentioned t -dependent SEs onto a, generally, infinite-dimensional projective space. We also consider t -dependent SEs related to unbounded t -dependent Hamiltonian operators as a special type of Hamiltonian systems on Hilbert manifolds. Finally, we summarize our results and survey prospective lines of research in Section 10.

2. Fundamentals on operators

Let us discuss several basic facts on functional analysis to be applied hereafter, e.g., in the mathematical formulation of Schrödinger quantum mechanics with unbounded operators (see [16,27,51] for details).

We call *operator* a linear map A defined on a dense subspace $D(A)$ of a normed space E_1 taking values on a normed space E_2 of the form

$$A : D(A) \subset E_1 \rightarrow E_2.$$

Operators are sometimes called *unbounded operators* to highlight that they do not need to be continuous. It is worth stressing that defining an operator amounts to giving both its domain and its value on it. If two operators

$$A_i : D(A_i) \subset E_1 \rightarrow E_2$$

with $i = 1, 2$, satisfy that $D(A_1) \subset D(A_2)$ while A_1 and A_2 take the same values in $D(A_1)$, then we say that A_2 is an *extension of* A_1 . Two operators are equal, and we write $A_1 = A_2$, when A_1 and A_2 have the same domain and they take the same values on it.

To define the sum and composition of operators with possibly different domains, we proceed as follows. Let

$$A_i : D(A_i) \subset E_1 \rightarrow E_2$$

with $i = 1, 2$, be two operators. If $D(A_1) \cap D(A_2)$ is dense in E_1 , then $A_1 + A_2$ is the operator on $D(A_1) \cap D(A_2)$ whose value at

$$\psi \in D(A_1) \cap D(A_2)$$

is $A_1\psi + A_2\psi$. Similarly, given an operator

$$B : D(B) \subset E_2 \rightarrow E_3$$

such that

$$D_{A_1} = A_1^{-1}(\text{Im}(A_1) \cap D(B))$$

is dense in E_1 , then we can define the composition

$$B \circ A_1 : D_{A_1} \subset E_1 \rightarrow E_3.$$

Let

$$H : D(H) \subset \mathcal{H} \rightarrow \mathcal{H}$$

be an operator on a Hilbert space \mathcal{H} . Consider the associated operator

$$H_\psi : \phi \in D(H) \subset \mathcal{H} \mapsto \langle \psi | H\phi \rangle \in \mathbb{C}.$$

Whenever H_ψ is continuous on $D(H)$, the Riesz representation theorem [27] ensures that there exists $\psi_0 \in \mathcal{H}$ such that

$$H_\psi(\phi) = \langle \psi_0 | \phi \rangle$$

for every $\phi \in D(H)$. If H_ψ is continuous for every ψ in a dense subset $D(H^\dagger)$ of \mathcal{H} , then one can define the *adjoint* of H , denoted by H^\dagger , as the operator

$$H^\dagger : \psi \in D(H^\dagger) \subset \mathcal{H} \mapsto \psi_0 \in \mathcal{H}.$$

In other words,

$$\langle \psi | H\phi \rangle = \langle H^\dagger \psi | \phi \rangle$$

for every $\psi \in D(H^\dagger)$ and $\phi \in D(H)$.

Let us now detail certain relevant types of operators.

Definition 2.1. An operator

$$H : D(H) \subset \mathcal{H} \rightarrow \mathcal{H}$$

is called:

- a) *Symmetric*: if $\langle \psi | H\phi \rangle = \langle H\psi | \phi \rangle$ for all $\phi, \psi \in D(H)$;
- b) *Skew-symmetric*: if $\langle \psi | H\phi \rangle = -\langle H\psi | \phi \rangle$ for all $\phi, \psi \in D(H)$;
- c) *Self-adjoint*: if $H^\dagger = H$;
- d) *Skew-self-adjoint*: if $H^\dagger = -H$;
- e) *Essentially self-adjoint*: if H admits a unique self-adjoint extension;
- f) *Observable*: if H is symmetric and admits a basis of eigenvectors.

Let us briefly comment several facts. If H is symmetric, then the equality

$$\langle \psi | H\phi \rangle = \langle H\psi | \phi \rangle$$

for $\psi, \phi \in D(H)$ ensures that H_ψ is continuous for every $\psi \in D(H)$ and

$$D(H^\dagger) \supset D(H).$$

Hence, $D(H^\dagger)$ is dense in \mathcal{H} and, consequently, H^\dagger is a well-defined operator. Moreover,

$$H^\dagger|_{D(H)} = H,$$

where $H^\dagger|_{D(H)}$ is the restriction of H^\dagger to $D(H)$, and H^\dagger is therefore an extension of H . On the other hand, if H is symmetric and

$$D(H^\dagger) = D(H),$$

then $H^\dagger = H$ and vice versa. An analogue discussion can be applied to the case of H being a skew-symmetric operator.

It is relevant to stress that any self-adjoint operator A on a separable Hilbert space is unitarily equivalent to a multiplication operator [27]. Moreover, by Stone's theorem on one-parameter unitary groups, self-adjoint operators correspond to infinitesimal generators of t -parametric unitary groups.

Definition 2.2. Let

$$A : D(A) \subset E_1 \rightarrow E_2$$

be an operator between normed spaces. We say that A is *open* if the image of every open set in E_1 by A is open. The operator A is said to be *closed* if its graph is closed in $E_1 \times E_2$ relative to the product topology.

Sometimes in the literature, a map whose graph is closed is said to be *graph-closed*, and the term *closed* is applied to operators so that $A(C)$ is closed whenever $C \subset D(A)$ is closed. However, we will stick to the terminology established in Definition 2.2.

Theorem 2.3. (*Closed graph theorem [27]*) Let

$$A : \mathcal{X} \rightarrow \mathcal{Y}$$

be an operator between Banach spaces. The operator A is continuous if and only if its graph is closed in $\mathcal{X} \times \mathcal{Y}$.

If the

$$A : \mathcal{X} \rightarrow \mathcal{Y}$$

in Theorem 2.3 is continuous, then the closed graph theorem yields that A is closed. Nevertheless, an operator defined on a non-closed domain may be non-continuous and closed simultaneously. For instance, the derivative

$$\partial : \mathcal{C}^1[0, 1] \subset \mathcal{C}^0[0, 1] \rightarrow \mathcal{C}^0[0, 1],$$

where $\mathcal{C}^1[0, 1]$ and $\mathcal{C}^0[0, 1]$ stand for the spaces of differentiable with continuous derivative and continuous functions on $[0, 1]$ with their standard norms, respectively, can be proved to be closed but not continuous. More generally, every self-adjoint operator is closed [27]. Then, if H is a symmetric operator such that $D(H) = \mathcal{H}$, then H is bounded.

Theorem 2.4. (*Open mapping theorem [51, Theorem 4.10]*) If

$$A : \mathcal{X} \rightarrow \mathcal{Y}$$

is a surjective continuous operator between Banach spaces, then A is an open map.

3. Geometry on normed and Banach manifolds

Our geometric study of quantum models will entail the analysis of classes of normed spaces [19]. Since the Banach fixed point theorem does not hold on normed spaces, first-order differential equations in normal form do not need to admit unique integral curves [25, 34]. There are no straightforward analogues of the Implicit function and Inverse function theorems either (see [25] for details). This section recalls the major results on differential calculus on normed spaces to be used hereafter. For details, we refer to [19, 25, 50].

We hereafter assume that every normed vector space E over a field \mathbb{K} admits an *unconditional Schauder basis*, namely a topological basis $\{e_i\}$ so that every $v \in E$ admits a unique decomposition

$$v = \sum_i \lambda^i e_i$$

for certain unique constants $\{\lambda^i\} \subset \mathbb{K}$ such that the sum does not depend on the order of summation of its terms. Hereafter, all basis are considered to be unconditional Schauder bases. The field \mathbb{K} is hereafter assumed to be the field of real or complex numbers.

Let E and F be normed spaces and let $B(E, F)$ denote the space of continuous linear maps from E to F . The space $B(E, F)$ admits an induced topology. It is worth noting that, if E, F are normed, the space $B(E, F)$ does not admit a topology with good properties (for details see [19]). Similarly, $B^r(E, F)$ stands for the space of continuous r -linear maps on E taking values in F .

Partial derivatives on normed vector spaces can be defined using the *Gâteaux differential*. In particular, a function

$$f : U \subset E \rightarrow \mathbb{R}$$

on an open U of the normed space E admits a *Gâteaux differential* at $u \in U$ in the direction $v \in E$ if the limit

$$(\mathfrak{D}_v f)(u) := \lim_{t \rightarrow 0} \frac{f(u + tv) - f(u)}{t}$$

exists. We call the function $u \mapsto (\mathfrak{D}_v f)(u)$ the *partial derivative* of f in the direction $v \in E$. If $\mathfrak{D}_v f(u)$ exists for every $v \in E$ at $u \in U$, we say that f is *Gâteaux differentiable* at u and we define

$$(df)_u : v \in E \mapsto \mathfrak{D}_v f(u) \in \mathbb{R}.$$

This function satisfies that

$$(df)_u(\lambda v) = \lambda(df)_u(v)$$

for every $v \in E$ and $\lambda \in \mathbb{K}$. Nevertheless, $(df)_u$ does not need to be linear and, even if it is linear, it does not need to be continuous. If $(df)_u$ is linear and continuous, then we say that f is *Fréchet differentiable* at u .

The function

$$f : U \subset E \rightarrow \mathbb{R}$$

is said to be differentiable if it is Fréchet differentiable at every point of U and the mapping $u \mapsto df_u$ is continuous [50, p.38]. The function

$$f : U \subset E \rightarrow \mathbb{R}$$

is said to be differentiable of order $r + 1$ at $a \in U$ if it is differentiable of order r and the differential of order r of f , let us say $d^r f$, exists at a [50, Section 4.4] and [52, Chapter 9]. We say that f is *smooth* or of class \mathcal{C}^∞ if it is differentiable for every r .

The composite mapping theorem and the Leibniz rule hold on sufficiently differentiable functions on normed spaces [19]. Although there is no *mean value theorem* in topological vector spaces, a mean value theorem holds on normed spaces. In any case, if

$$(df)_u = 0$$

for every $u \in U$ and U is connected, then f is constant on U (see [19, 50]). This result will be employed for studying the existence of Hamiltonian functions related to self-adjoint operators in quantum mechanics (see Section 6).

Once a notion of differentiability has been stated on normed vector spaces, the definition of a manifold modeled on a normed vector space is similar to the one on a standard manifold modeled on a finite-dimensional vector space (see [19, 50, 52]).

If not otherwise stated, manifolds are assumed to be smooth and modeled on a separable Banach space \mathcal{X} , i.e., the manifold P admits a smooth differentiable structure whose charts take values in \mathcal{X} . Exceptions to this general rule appear in our description of quantum mechanical problems related to unbounded operators, which require the use of smooth manifolds modeled on normed spaces that do not need to be complete. It is worth noting that avoiding spaces that are not normed is a key to obtain a description that is technically appropriate for physical problems without adding intricate technical problems.

It can be proved that there exists a natural one-to-one correspondence between the space $T_u P$ of tangent vectors at $u \in P$, which are understood as equivalence classes of curves

$$\gamma : t \in \mathbb{R} \mapsto \gamma(t) \in P$$

passing through u at $t = 0$ and having the same, well-defined, Gâteaux differential at $t = 0$ in the direction $1 \in \mathbb{R}$ [16, 17, 19], and the space $\mathcal{D}_u P$ of derivations

$$\mathcal{D}_u : \mathcal{C}_u^\infty(P) \rightarrow \mathbb{R},$$

where $\mathcal{C}_u^\infty(P)$ is the space of germs at $u \in P$ of functions on P , of the form

$$\mathcal{D}_u f = \left. \frac{d}{dt} \right|_{t=0} (f \circ \gamma)$$

for a curve γ with $\gamma(0) = u$ and any $f \in \mathcal{C}_u^\infty(P)$. Let us prove the analogue of this result on normed spaces under our given general assumptions. The case for general manifolds modeled on normed spaces follows then by using local charts and their compatibility relations.

Let $(E, \|\cdot\|)$ be a real normed space. By our general assumption on normed spaces, E admits an unconditional Schauder basis $\{\psi_n\}$. Let $\{f^n\}$ be the corresponding coordinate functions in E^* , which are continuous. Thus, if $\psi \in E$, then there exist unique scalars $c^n = f^n(\psi)$ such that

$$\psi = \sum_n c^n \psi_n.$$

The elements of $\{f^n\}$ do not need to be a basis of the dual space E^* in the sense that certain elements of E^* may not be the limit of a sequence of finite linear combinations of $\{f^n\}$ in the strong topology of E^* (consider for instance ℓ^1 with the basis $e_n = (0, \dots, 1(n\text{-th position}), 0, \dots)$ with $n \in \mathbb{N}$, and its dual $(\ell^1)^* = \ell^\infty$ with the elements $\sum_{n \in \mathbb{N}} f^n \in \ell^\infty$). If the functions $\{f^n\}$ are a basis of E^* , then $\{\psi_n\}$ is called a shrinking basis. Anyhow, the value of any element of E^* at an element $v \in E$ can be obtained as a linear combination $\sum_n \lambda_n f^n(v)$, where the sum is understood as a limit in the field \mathbb{K} . In general, the $\{\lambda_n\}$ satisfy relations ensuring that the sum exists and is continuous. These relations can be different from the ones satisfied by the $\{c^n\}$, e.g., the space ℓ^p of sequences $\mathbf{c} = (c^1, c^2, \dots)$ of numbers with

$$\sum_{n \in \mathbb{N}} |c^n|^p < \infty$$

with respect to the norm

$$\|\mathbf{c}\| = \left(\sum_{n \in \mathbb{N}} |c^n|^p \right)^{1/p}$$

satisfies that $\ell^{p^*} \simeq \ell^q$ for $1/p + 1/q = 1$.

Every equivalence class of curves passing through $\psi_0 \in E$ is determined by the Gâteaux differential of the curves at $t = 0$ in the direction $1 \in \mathbb{R}$ and vice versa. Then, there is a one-to-one correspondence between $T_{\psi_0}E$ and E , and we can write $T_{\psi_0}E \simeq E$.

In turn, $\psi \in E$ determines a derivation

$$\dot{\psi}_{\psi_0} f = \left. \frac{d}{dt} \right|_{t=0} f(\psi_0 + t\psi), \quad \forall f \in \mathcal{C}^\infty(E). \quad (3.1)$$

Obviously $\dot{\psi}_{\psi_0} \in \mathcal{D}_{\psi_0}E$. The subindex ψ_0 will be omitted if it is clear from context, or irrelevant, so as to simplify the notation.

If

$$\dot{\psi}_{\psi_0} = \dot{\phi}_{\psi_0},$$

then it follows from (3.1) that

$$f^n(\psi) = f^n(\phi)$$

for each n and $\psi = \phi$. Then, every element of $\mathcal{D}_{\psi_0}E$ determines a unique element of E , which defines a unique tangent vector at ψ_0 . Then, we can identify $T_{\psi_0}E$ and $\mathcal{D}_{\psi_0}E$ and write

$$T_{\psi_0}E := \{\dot{\psi}_{\psi_0} \mid \psi \in E\}.$$

Moreover, each $\psi \in E$ is associated with a derivation $\dot{\psi} \in T_{\psi_0}E$ via the isomorphism

$$\lambda_{\psi_0} : \psi \in E \mapsto \dot{\psi}_{\psi_0} \in T_{\psi_0}E.$$

The previous construction allows for building the tangent and cotangent bundles of a normed manifold P along with other geometric structures, e.g., generalized tangent bundles, following the same ideas as in the finite-dimensional case, provided the infinite-dimensional nature of these structures is taken into account. In particular, a *vector field* is a section of the tangent bundle TP , more precisely, it is a map

$$X : P \rightarrow TP$$

such that

$$\tau \circ X = \text{Id}_P,$$

where

$$\tau : TP \rightarrow P$$

is the projection onto the base space P and Id_P is the identity map on P (see [19]). Similarly, a *k-form* is a section of the bundle

$$\bigwedge^k T^*P \rightarrow P,$$

where $(\bigwedge^k T^*P)_u$ is the space of continuous \mathbb{R} -valued k -linear skew-symmetric mappings on T_uP . We hereafter write $\Omega^k(P)$ for the space of k -forms on P and we denote by $\mathcal{C}^\infty(P)$ the space of smooth functions on P . Similarly to the finite-dimensional manifold case, one can define the *exterior derivative*

$$d : \Omega^k(P) \rightarrow \Omega^{k+1}(P),$$

which exists on an appropriate subset of $\Omega^k(P)$ of, differentiable enough, k -forms.

If P is not modeled on a Banach space, then a vector field on P may not have certain integral curves [34]. Hence, additional conditions must be required to ensure the existence of integral curves. Moreover, our study of quantum mechanical problems will demand the definition of vector fields with a domain (see [19, p.366]) for details).

Definition 3.1. A vector field on a domain $D \subset P$ is a map

$$X : D \subset P \rightarrow TP$$

such that D is a manifold modeled on a normed space that, as a subset of P , is dense in P , the inclusion $\iota : D \hookrightarrow P$ is smooth, and $\tau \circ X = \text{Id}_D$. Moreover, X is called *smooth* if

$$X : D \rightarrow TP$$

is smooth.

Example 3.2. Let us provide an example of a vector field on a domain that will be generalized and studied in detail in Section 6. Consider the complex Hilbert space $L^2(\mathbb{R})$ of equivalence classes of square integrable complex functions on \mathbb{R} that coincide almost everywhere. This space is naturally a real manifold modeled on a real Hilbert space given by the differential structure induced by the global map

$$J : L^2(\mathbb{R}) \rightarrow L_{\mathbb{R}}^2(\mathbb{R}),$$

where $L_{\mathbb{R}}^2(\mathbb{R})$ is $L^2(\mathbb{R})$ considered, in the natural way, as a real Hilbert space. The isomorphisms

$$\lambda_{\psi} : L^2(\mathbb{R}) \rightarrow T_{\psi}L^2(\mathbb{R}),$$

for every $\psi \in L^2(\mathbb{R})$, allow us to write

$$TL^2(\mathbb{R}) \simeq L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$$

and to consider every operator

$$A : D(A) \subset L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}),$$

as a map

$$X_A : \psi \in D(A) \subset L^2(\mathbb{R}) \mapsto (\psi, A\psi) \in T_{\psi}L^2(\mathbb{R}). \quad (3.2)$$

In particular, consider

$$\partial_x : D(\partial_x) \subset L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$$

such that $D(\partial_x)$ is spanned by linear combinations of a finite family of elements of the basis of functions

$$H_n(x)e^{-x^2/2}/(\pi^{1/4}2^{n/2}\sqrt{n!})$$

with

$$n \in \tilde{\mathbb{N}} = \mathbb{N} \cup \{0\}$$

and $H_n(x)$ being the Hermite polynomial of order n . Then, $D(\partial_x)$ is dense in $L^2(\mathbb{R})$. Moreover, $D(\partial_x)$ is a normed vector space relative to the restriction of the topology of $L^2(\mathbb{R})$ to $D(\partial_x)$, but it is not complete.

The restriction of the global atlas of $L^2(\mathbb{R})$ to $D(\partial_x)$ induces a second one on $D(\partial_x)$ that is modeled only on a normed, not complete, space because $L^2(\mathbb{R})$ is the topological closure of $D(\partial_x)$, which is therefore different from $D(\partial_x)$. Note that

$$D(\partial_x) \simeq \bigoplus_{n \in \mathbb{N}} \mathbb{R},$$

where one assigns every element of $D(\partial_x)$ its coordinates in the given basis. Then, the inclusion of $D(\partial_x)$ into $L^2(\mathbb{R})$ reads in these coordinates as

$$D(\partial_x) \ni (x_1, x_2, x_3, \dots) \longmapsto (x_1, x_2, \dots) \in L^2(\mathbb{R}).$$

It is therefore a linear bounded mapping and it becomes smooth. Moreover, the coordinates of an element in $D(\partial_x)$ may be any finite sequence of non-zero constants, while any element of $L^2(\mathbb{R})$ is a square summable sequence. Evidently, every smooth function on $L^2(\mathbb{R})$ is smooth on $D(\partial_x)$. Since

$$\partial_x(D(\partial_x)) \subset D(\partial_x),$$

we can write that

$$X_{\partial_x} : D(\partial_x) \rightarrow TD(\partial_x).$$

In coordinates adapted to the basis $H_n^p(x)e^{-x^2/2}$ of $D(\partial_x)$ and $L^2(\mathbb{R})$, where $H_n^p(x)$ is the Hermite probabilistic polynomial of order n , and using that

$$\partial_x H_n^p(x) - x H_n^p(x) = -H_{n+1}^p(x)$$

for any non-negative integer n , one has

$$X_{\partial_x}(x_1, x_2, x_3, \dots) = (x_1, x_2, x_3, \dots; 0, -x_1, -x_2, -x_3, \dots) \in TD(\partial_x).$$

Hence, X_{∂_x} becomes a vector field on a domain.

Example 3.2 can be generalized to vector fields on domains induced by observables, which will be the standard case in our study of quantum mechanics. This case will also turn out to be very frequent in the literature [19].

Similarly to vector fields on a domain, one can define differential k -forms on a domain. We write $\Omega^k(D)$ for the space of differential k -forms on a domain D . Moreover, it is straightforward to define t -dependent vector fields on a domain [19].

Definition 3.3. A (local) flow for

$$X : D \subset P \rightarrow TP$$

is a map of the form

$$F : (t, u) \in (-\varepsilon, \varepsilon) \times U \mapsto F_t(u) := F(t, u) \in P,$$

where $\varepsilon > 0$ and $U \subset P$ is open and contains D , such that

$$X(F_t(u)) = \left. \frac{d}{ds} \right|_{s=t} F_s(u),$$

for all $u \in D$ and $t \in (-\varepsilon, \varepsilon)$.

At first glance, it may seem strange that the flow of a vector field X on a domain D may be defined on an open set U strictly containing D , but this will be useful and natural in this and other works (see [19]). More precisely, skew-self-adjoint operators on a Hilbert space will give rise to vector fields on domains whose flows are defined on the full $\mathbb{R} \times \mathcal{H}$ but they are differentiable only on a subspace of it. It is worth noting that an extension of the above for the case of t -dependent vector fields can be accomplished quite straightforwardly [19].

Example 3.4. Recall the vector field X_{∂_x} from Example 3.2. For an element $\psi \in D(\partial_x)$, since $\partial_x \psi \in D(\partial_x)$ and ψ is an analytic function, one has that

$$[\exp(t\partial_x)\psi](y) = \sum_{n=0}^{\infty} \frac{t^n \partial_x^n \psi}{n!}(y) = \psi(y + t)$$

and $\exp(t\partial_x)\psi$ may not belong to $D(\partial_x)$ [27]. Since $i\partial_x$ admits a self-adjoint extension, the Stone-von Neumann theorem ensures that $\exp(t\partial_x)$ can be defined as a unitary map on $L^2(\mathbb{R})$ and X_{∂_x} admits a flow given by

$$F : (t; \psi) \in \mathbb{R} \times L^2(\mathbb{R}) \mapsto \exp(t\partial_x)\psi \in L^2(\mathbb{R}).$$

For elements $\psi \in D(\partial_x)$, one observes that

$$\frac{d}{dt}F(t, \psi) = \partial_x \psi,$$

but F may not be differentiable for $\psi \notin D(\partial_x)$.

Note that $D(\partial_x)$ can be chosen so as to give rise to a flow F of X_{∂_x} on a domain that is invariant under the maps F_t for $t \in \mathbb{R}$. For instance, consider ∂_x to be defined on the Schwartz space $\mathcal{S}(\mathbb{R})$. It can be proved that

$$(\exp(t\partial_x)\psi)(x) = \psi(x + t)$$

for every $\psi \in \mathcal{S}(\mathbb{R})$ and $x \in \mathbb{R}$. Hence, the new $D(\partial_x)$ is invariant relative to $\exp(t\partial_x)$. Moreover,

$$\frac{d}{dt}F(t, \psi) = \partial_x \psi$$

on elements of $\mathcal{S}(\mathbb{R})$. Note that we are using the Schwarz space with the norm induced by $L^2(\mathbb{R})$.

Since the sum and composition of vector fields related to operators with different domains do not need to be defined at a new common domain, the Lie bracket of such vector fields may not be defined. In this work, conditions will be given to ensure that the Lie bracket of studied vector fields is defined on a dense subspace of P . This happens, for instance, for observables having a common basis of eigenvalues on the space they are defined on. Then, one can consider the direct sum of the subspaces generated by the elements of such a basis. This is a normed, but generally not complete, dense subspace.

It is interesting to our purposes to consider the case when a subset D of a manifold P modeled on a Banach space is a manifold modeled on a normed space. In this case, the exterior differential operator d can be similarly defined on differential forms on D . Moreover, $T_p D \subset T_p P$ and one can consider whether a function $f: D \rightarrow \mathbb{R}$ is such that its differential, namely

$$df_u : T_u D \rightarrow \mathbb{R}$$

can be extended at points $u \in D$ to elements of $(df)_u \in T_p^*P$. This will be important in our quantum mechanical applications and it becomes the main reason to provide the following definition.

Definition 3.5. Let D be a subset of a manifold P that is, again, a manifold modeled on a normed space. We say that $f: D \rightarrow \mathbb{R}$ is *admissible* when

$$(df)_u : T_uD \rightarrow \mathbb{R}$$

can be extended continuously to T_uP .

For a vector field X on a domain $D_X \subset P$, which is assumed to be a manifold modeled on a normed space such that $X(D_X) \subset TD_X$, the *inner contraction*, ι_X , on differential forms with $D_X \supset D_\alpha$ is defined by restricting the contraction with X of differential forms at points of P and its evaluation on vector fields on D_X .

Similarly, the *Lie derivative*

$$\mathcal{L}_X : \alpha \in \Omega^k(P) \mapsto \mathcal{L}_X\alpha \in \Omega^k(D_X)$$

is defined using Cartan's formula

$$(\mathcal{L}_X\alpha)(u) = (\iota_X d\alpha + d\iota_X\alpha)(u), \quad \forall u \in D_X.$$

Theorem 3.6. Let X be a vector field on a domain $D_X \subset P$ and let

$$F : (-\epsilon, \epsilon) \times D_X \rightarrow D_X$$

be the flow of X . Then,

$$[F_t^*(\mathcal{L}_X\alpha)]_u = \left. \frac{d}{ds} \right|_{s=t} (F_s^*\alpha)_u$$

for any $\alpha \in \Omega^k(P)$, values $t, s \in (-\epsilon, \epsilon)$, and arbitrary $u \in D_X$.

For a diffeomorphism $\phi: P_1 \rightarrow P_2$ and a vector field X on a domain $D_X \subseteq P_1$, the *push-forward* of X by ϕ , if it exists, has domain $\phi(D_X) \subseteq P_2$ and is defined by

$$(\phi_*X)_u = T_u\phi[X_{\phi^{-1}(u)}].$$

If the flow of X is given by F_t , then the flow of ϕ_*X is

$$\hat{F}_t := \phi \circ F_t \circ \phi^{-1}$$

for every $t \in (-\epsilon, \epsilon)$. In particular, one has to assume that $\phi(D_X)$ is dense in P_2 for the vector field ϕ_*X to exist.

Definition 3.7. A diffeomorphism $\phi: D \rightarrow D$ of a domain $D \subset P$ is said to be *admissible* if and only if for every $u \in D$, the map

$$T_u\phi : T_uD \rightarrow T_{\phi(u)}D$$

extends to a continuous linear map

$$T_uP \rightarrow T_{\phi(u)}P.$$

The following results are the generalization to the infinite-dimensional context of the quotient manifold theorem [16].

Theorem 3.8. *Let*

$$\Phi : G \times P \rightarrow P$$

be a free and proper action on a Banach manifold. Then, P/G is a manifold and

$$\pi : P \rightarrow P/G$$

is a submersion.

Proposition 3.9. *If G is a compact Lie group, then every Lie group action*

$$\Phi : G \times P \rightarrow P$$

is proper.

4. Symplectic geometry in the infinite-dimensional context

In this section, we introduce and study symplectic and related structures on manifolds modeled on normed and Banach spaces. As shown next, the fact that a symplectic form is defined on such, generally infinite-dimensional, manifolds yields several relevant differences with respect to the case of symplectic forms on finite-dimensional manifolds. It is worth noting that our presentation is focused on Banach manifolds and related cases, which are enough for our purposes. We skip the theory of symplectic forms on convenient spaces, given for instance in [30,31,34], which is much more technical.

Definition 4.1. A *symplectic form* is a closed differential two-form ω on a manifold P modeled on a normed space that is *non-degenerate*, namely

$$\omega_u^b : v_u \in T_u P \mapsto \omega_u(v_u, \cdot) \in T_u^* P$$

is injective for every $u \in P$. If ω_u^b is an isomorphism for every $u \in P$, then we call ω a *strong symplectic form*. A (*resp. strong*) *symplectic manifold* is a pair (P, ω) , where ω is a (*resp. strong*) symplectic form on P .

Every strong symplectic form is always a symplectic form, whereas a symplectic form ω on P is strong provided ω_u^b is also surjective for every $u \in P$. The latter can be warranted, for instance, when ω is a symplectic form and P is finite-dimensional. If the symplectic form ω of a symplectic manifold (P, ω) is known from context, then one will simply say that P is a symplectic manifold.

The Darboux theorem is a well-known result from standard finite-dimensional symplectic geometry that states that every symplectic manifold admits a local coordinate system in which the associated symplectic form takes a canonical form [1]. This result cannot be extended to symplectic forms on manifolds modeled on general Banach spaces, as shown by Marsden for the case of SBMs [19], which in turn led to different generalizations of the Darboux theorem for SBMs [1, 49].

The extension of the standard Darboux theorem to SBMs is accomplished through *constant differential forms*. Let us describe this notion. Every skew-symmetric continuous bilinear form on a normed space E , let us say

$$B : E \times E \rightarrow \mathbb{R}$$

gives rise to a unique differential two-form ω_0 on E by considering the natural isomorphism

$$\lambda_u : E \simeq T_u E$$

and defining ω_0 to be the only differential form satisfying that

$$\lambda_u^*(\omega_0)_u = B$$

for every $u \in E$. Such a type of differential two-form on a normed space E defined from of a skew-symmetric bilinear form on E is called a *constant* differential two-form.

Theorem 4.2. (*Strong Darboux theorem [1]*) *If (P, ω) is a strong Banach symplectic manifold, then there exists an open neighbourhood around every $u \in P$ where ω is the pull-back of a constant differential two-form.*

The cotangent bundle of any finite-dimensional manifold admits a so-called *tautological differential one-form* whose differential gives rise to a strong symplectic form [16]. Similarly, the cotangent bundle of a Banach manifold P is endowed with a canonical differential one-form and a related symplectic form, which is not strong in general [19, Theorem 2.4].

Definition 4.3. Let P be a manifold modeled on a Banach space \mathcal{X} and let

$$\pi : T^*P \rightarrow P$$

be the cotangent bundle projection. The *tautological differential one-form* on T^*P is the differential one-form θ on T^*P of the form

$$\theta_{\alpha_u}(w_{\alpha_u}) := \alpha_u(T\pi(w_{\alpha_u})), \quad \forall \alpha_u \in T_u^*P, \quad \forall w_{\alpha_u} \in T_{\alpha_u}(T^*P), \quad \forall u \in P,$$

and the *canonical two-form* ω on P is defined as $\omega = -d\theta$.

Let us give a coordinate expression for θ and ω on a separable Banach manifold P modeled on a Banach space \mathcal{X} with an unconditional Schauder basis. It is worth recalling that the separability does not imply the existence of an unconditional Schauder basis, as follows from the very famous counterexample found by Enflo. On a sufficiently small open set $U \subset P$, one can consider a coordinate system $\{x^j\}$. The coordinate system $\{x^j, p_j\}$ adapted to T^*P allows us to establish a local diffeomorphism

$$T^*U \simeq U \times \mathcal{X}^*.$$

In turn, the coordinate system $\{x^j, p_j\}$ induces another adapted coordinate system to TT^*U , namely $\{x^j, p_j, \dot{x}^j, \dot{p}_j\}$, allowing us to define a diffeomorphism

$$TT^*U \simeq (U \times \mathcal{X}^*) \times (\mathcal{X} \times \mathcal{X}^*).$$

In these coordinates,

$$\sum_j \theta_{(x,p)}(\dot{x}^j(x,p)\partial_{x^j} + \dot{p}_j(x,p)\partial_{p_j}) = \sum_j p_j \dot{x}^j \implies \theta = \sum_j p_j dx^j.$$

Then,

$$\omega = -d\theta = \sum_j dx_j \wedge dp^j.$$

More generally, one has the following result for Banach spaces (see [19, Proof of Theorem 2.4] for details). In essence, the reflexivity of \mathcal{X} amounts to the fact that ω gives rise to an isomorphism between TP and T^*P .

Theorem 4.4. *The canonical two-form ω on T^*P is a symplectic form. In addition, ω is a strong symplectic form if and only if P is modeled on a reflexive Banach space.*

Definition 4.5. Let P be a Banach manifold and let

$$\{\cdot, \cdot\} : \mathcal{C}^\infty(P) \times \mathcal{C}^\infty(P) \rightarrow \mathcal{C}^\infty(P)$$

denote a bilinear map. If $(\mathcal{C}^\infty(P), \{\cdot, \cdot\})$ is a Lie algebra and $\{\cdot, \cdot\}$ satisfies the *Leibniz rule*, namely

$$\{fg, h\} = \{f, h\}g + f\{g, h\}$$

for all $f, g, h \in \mathcal{C}^\infty(P)$, then $\{\cdot, \cdot\}$ is called a *Poisson bracket* or a *Poisson structure* on P , and in this case $(P, \{\cdot, \cdot\})$ is said to be a *Poisson manifold*.

Note that the Poincare Lemma works also for infinite-dimensional manifolds (see [53]).

Definition 4.6. Let (P, ω) be a symplectic manifold. A vector field $X: P \rightarrow TP$ is called *Hamiltonian* if there exists a function $h \in \mathcal{C}^\infty(P)$ such that

$$\iota_X \omega = dh.$$

The function h is called a *Hamiltonian function* of X . Whenever $\iota_X \omega$ is closed, and thus locally exact (see [19, 53]), we say that X is a *locally Hamiltonian vector field*.

The set of (locally) Hamiltonian vector fields on (P, ω) will be denoted by (resp. $\text{Ham}_{\text{loc}}(P, \omega)$) $\text{Ham}(P, \omega)$ or (resp. $\text{Ham}_{\text{loc}}(P)$) $\text{Ham}(P)$ when ω is understood from context.

If ω is a strong symplectic form, then every $h \in \mathcal{C}^\infty(P)$ is the Hamiltonian function of a Hamiltonian vector field as in the case of symplectic forms on finite-dimensional manifolds. For general SBMs, each ω_u^b , with $u \in P$, is only in general injective and some $h \in \mathcal{C}^\infty(P)$ may not give rise to any Hamiltonian vector field. If a function $h \in \mathcal{C}^\infty(P)$ has a Hamiltonian vector field, it is unique. This motivates the following definition.

Definition 4.7. Let (P, ω) be a Banach symplectic manifold. We call a function $h \in \mathcal{C}^\infty(P)$ *admissible* if it is the Hamiltonian function of a Hamiltonian vector field.

We will denote by X_h the unique Hamiltonian vector field associated with an admissible function h . The following result is immediate.

Proposition 4.8. *Let (P, ω) be a symplectic manifold. The set $\text{Ham}_{\text{loc}}(P)$ is a Lie algebra, while $\text{Ham}(P)$ is an ideal of $\text{Ham}_{\text{loc}}(P)$ and*

$$[\text{Ham}_{\text{loc}}(P), \text{Ham}_{\text{loc}}(P)] \subset \text{Ham}(P).$$

In particular, if $X_1, X_2 \in \text{Ham}_{\text{loc}}(P)$, then

$$[X_1, X_2] = -X_{\omega(X_1, X_2)}.$$

Every symplectic form on a finite-dimensional manifold gives rise to a Poisson structure. This is no longer true for a general symplectic form, as not every function is associated with a Hamiltonian vector field. A way to ensure this is given by the following proposition.

Proposition 4.9. *A strong symplectic manifold (P, ω) induces a Poisson structure on P given by*

$$\{f, g\}_\omega := \omega(X_f, X_g), \quad \forall f, g \in \mathcal{C}^\infty(P).$$

Definition 4.10. Let (P_1, ω_1) and (P_2, ω_2) be two symplectic manifolds. A diffeomorphism $\phi: P_1 \rightarrow P_2$ is called a *symplectic map*, or a *symplectomorphism*, if $\phi^*\omega_2 = \omega_1$.

We hereafter focus on physical systems determined by the so-called Hamiltonian systems.

Definition 4.11. A *Hamiltonian system* is a triple (P, ω, h) , where (P, ω) is a symplectic manifold and

$$h : (t, x) \in \mathbb{R} \times P \mapsto h_t(x) = h(t, x) \in \mathbb{R}$$

is a t -dependent function on P such that for every $t \in \mathbb{R}$, the function $h_t \in \mathcal{C}^\infty(P)$ is an admissible function for (P, ω) . We call h the *Hamiltonian function* of the system.

Every Hamiltonian system (P, ω, h) gives rise to a unique t -dependent vector field X , called the *Hamiltonian vector field* of the system, satisfying that

$$\iota_{X_t}\omega = dh_t.$$

The system of differential equations describing the integral curves

$$c : t \in \mathbb{R} \mapsto c(t) \in P$$

of X reads

$$\frac{dc(t)}{dt} = (X_t \circ c)(t).$$

This system of differential equations is called the *Hamilton's equations* for the system (P, ω, h) . To emphasise the role of the Hamiltonian function h to determine its associated vector field, we will sometimes write $X_h = X$. In the MW reduction procedure, one is interested in a special family of Hamiltonian systems admitting a particular type of symmetries described next.

Definition 4.12. A *G -invariant Hamiltonian system* relative to the Lie group action

$$\Phi : G \times P \rightarrow P$$

is a Hamiltonian system (P, ω, h) such that

$$\Phi_g^*h_t = h_t, \quad \Phi_g^*\omega = \omega, \quad \forall g \in G, \quad \forall t \in \mathbb{R}.$$

If

$$\Phi : G \times P \rightarrow P$$

is understood from context, we will simply talk about a G -invariant Hamiltonian systems, omitting the Lie group action.

Definition 4.13. A Lie group action

$$\Phi : G \times P \rightarrow P$$

on a symplectic manifold (P, ω) is called *symplectic* if, for every $g \in G$, the map

$$\Phi_g : P \rightarrow P$$

is a symplectomorphism. Moreover, Φ is called *Hamiltonian* if the fundamental vector fields of Φ are Hamiltonian relative to the symplectic form ω .

Example 4.14. Let \mathcal{H} be a complex Hilbert space and consider the constant symplectic form ω_B on \mathcal{H} induced by the continuous skew-symmetric bilinear form

$$B : (\psi_1, \psi_2) \in \mathcal{H} \times \mathcal{H} \mapsto \Im \langle \psi_1 | \psi_2 \rangle \in \mathbb{R}.$$

Given the skew-adjoint operator $i\text{Id}_{\mathcal{H}}$, the Stone-von Neumann theorem [27] states that each $\exp(it\text{Id}_{\mathcal{H}})$ is a continuous mapping on \mathcal{H} and, since it is linear, it is also smooth. Since $ti\text{Id}_{\mathcal{H}}$ is continuous

$$\exp(it\text{Id}_{\mathcal{H}}) = e^{it}\text{Id}_{\mathcal{H}},$$

and the operators $\exp(it\text{Id}_{\mathcal{H}})$, for $t \in \mathbb{R}$, span a group isomorphic to $U(1)$, namely the unitary group of transformations on a one-dimensional Hilbert space. Define the Lie group action

$$\Phi : (\exp(it\text{Id}_{\mathcal{H}}), \psi) \in U(1) \times \mathcal{H} \mapsto e^{it}\psi \in \mathcal{H}.$$

Then,

$$B(\Phi_g\psi_1, \Phi_g\psi_2) = B(\psi_1, \psi_2)$$

for every $\psi_1, \psi_2 \in \mathcal{H}$ and $g \in U(1)$. Consequently,

$$\Phi_g^* \omega_B = \omega_B$$

for every $g \in G$ and Φ is a symplectic action.

Definition 4.15. Let (P, ω) be a symplectic manifold. A *momentum map* for a Lie group action

$$\Phi : G \times P \rightarrow P$$

is a mapping $J : P \rightarrow \mathfrak{g}^*$ such that

$$\iota_{\xi_P} \omega = d\langle J, \xi \rangle, \quad \forall \xi \in \mathfrak{g}, \quad \forall u \in P,$$

where ξ_P is the fundamental vector field of Φ related to $\xi \in \mathfrak{g}$, namely

$$\xi_P(u) := \left. \frac{d}{dt} \right|_{t=0} \Phi(\exp(t\xi), u), \quad \forall u \in P.$$

Thus,

$$J_\xi : u \in P \mapsto \langle J(u), \xi \rangle \in \mathbb{R}$$

is a Hamiltonian function of ξ_P for each $\xi \in \mathfrak{g}$.

Theorem 4.16. (Hamiltonian Noether's theorem [54]) Consider a Hamiltonian system (P, ω, h) . Let

$$\Phi : G \times P \rightarrow P$$

be a Lie group action on a symplectic manifold (P, ω) with a momentum map

$$J : P \rightarrow \mathfrak{g}^*.$$

If the t -dependent function h on P is G -invariant, namely

$$h_t \circ \Phi_g = h_t$$

for all $g \in G$ and $t \in \mathbb{R}$, then J is conserved on the trajectories of X_h , that is

$$J \circ F_t = J$$

for all $t \in \mathbb{R}$, where F is the flow of X_h .

To introduce the symplectic MW reduction, one may consider a quite general property of momentum maps given by the following definition [54], which makes use of the coadjoint action

$$\text{Ad} : G \times \mathfrak{g}^* \rightarrow \mathfrak{g}^*$$

on the dual to the Lie algebra \mathfrak{g}^* of G .

Definition 4.17. A momentum map

$$J : P \rightarrow \mathfrak{g}^*$$

of a Lie group action

$$\Phi : G \times P \rightarrow P$$

is said to be Ad^* -equivariant when

$$J \circ \Phi_g = (\text{Ad}^*)_g \circ J$$

for all $g \in G$, where Ad^* is the coadjoint action of G , namely

$$\text{Ad}^* : (g, \theta) \in G \times \mathfrak{g}^* \mapsto (\text{Ad}^*)_g := \theta \circ \text{Ad}_{g^{-1}} \in \mathfrak{g}^*$$

and $(\text{Ad}^*)_g$ is the transpose of the morphism

$$\text{Ad}_{g^{-1}} : \mathfrak{g} \rightarrow \mathfrak{g}.$$

In other words, J is Ad^* -equivariant if the diagram aside commutes for every $g \in G$

$$\begin{array}{ccc} P & \xrightarrow{J} & \mathfrak{g}^* \\ \Phi_g \downarrow & & \downarrow (\text{Ad}^*)_g \\ P & \xrightarrow{J} & \mathfrak{g}^* \end{array}.$$

The Ad^* -equivariance of momentum maps is not really necessary in what follows: It is only commonly used. Instead, one can use that a momentum map related to a Lie group action

$$\Phi : G \times P \rightarrow P$$

is always equivariant related to a new Lie group action

$$\Delta : G \times \mathfrak{g}^* \rightarrow \mathfrak{g}^*$$

defined by

$$\Delta(g, \vartheta) = (\text{Ad}^*)_g + \sigma(g)$$

for every $g \in G$ and $\vartheta \in \mathfrak{g}^*$. Moreover,

$$\sigma : G \rightarrow \mathfrak{g}^*$$

is defined to be the well defined constant element in \mathfrak{g}^* of the form

$$\sigma(g) = J \circ \Phi_g - (\text{Ad}^*)_g \circ J, \quad \forall g \in G.$$

The Lie group action Δ is frequently called the affine Lie group action related to J and Φ . We hereafter write G_μ for the isotropy subgroup of G leaving invariant an element $\mu \in \mathfrak{g}^*$ relative to its associated affine Lie group action.

When (P, ω, h) is a G -invariant Hamiltonian system, one expects that the G -invariance of the system will enable the simplification of the system, e.g., giving rise to a new symplectic manifold of “smaller” dimension. This is accomplished through the MW reduction theorem enabling one to go from a symplectic manifold having a Hamiltonian Lie group action of symmetries of h to another “reduced” symplectic manifold. Then, the initial Hamiltonian function h will also give rise to a new t -dependent function on the ‘reduced’ manifold, which finishes the reduction procedure. To do so, we first need to give the following definition.

Definition 4.18. Given a map

$$f : P \rightarrow N$$

between two manifolds, we say that $n \in N$ is a *weakly regular value* of f if $f^{-1}(n) \subset P$ is a submanifold,

$$T_u f^{-1}(n) = \ker T_u f$$

for every $u \in f^{-1}(n)$, and $\ker T_u f$ admits a closed supplementary in $T_u P$.

Note that the condition on the closed supplementary in $T_u P$ is immediately satisfied when P is a finite-dimensional manifold. Nevertheless, this property is not always satisfied in the infinite-dimensional case. In the infinite-dimensional realm, immersions, submersions and other types of mappings are required to satisfy similar conditions to extend to the infinite-dimensional case the properties they have on finite-dimensional manifolds.

Theorem 4.19. (*MW reduction theorem*) Consider a symplectic manifold (P, ω) . Let G be a Lie group acting on P and let

$$J : P \rightarrow \mathfrak{g}^*$$

be an associated momentum map. Let μ be a weakly regular value of J and suppose that G_μ acts freely and properly on the submanifold $J^{-1}(\mu)$. If

$$J_\mu : J^{-1}(\mu) \hookrightarrow P$$

is the natural embedding of $J^{-1}(\mu)$ within P and

$$\pi_\mu : J^{-1}(\mu) \rightarrow P_\mu$$

is the canonical projection, then there exists a unique symplectic form ω_μ on

$$P_\mu := J^{-1}(\mu)/G_\mu$$

such that

$$\pi_\mu^* \omega_\mu = J_\mu^* \omega.$$

It is important to point out that Theorem 4.19 does not require the Banach symplectic manifold P to be strongly symplectic [54]. The following theorem uses the MW reduction theorem to reduce the dynamics.

Theorem 4.20. *Let the conditions of Theorem 4.19 hold. Let (P, ω, h_t) be a G -invariant Hamiltonian system relative to the Lie group action*

$$\Phi : G \times P \rightarrow P.$$

Then,

- The flow of X_h , namely F , leaves $J^{-1}(\mu)$ invariant;
- The diffeomorphisms F_t commute with every Φ_g , for $g \in G_\mu$, giving rise to a flow K on P_μ such that $\pi_\mu \circ F_t = K_t \circ \pi_\mu$;
- The flow K corresponds to a vector field with a Hamiltonian function $h_\mu \in \mathcal{C}^\infty(P_\mu)$ determined by $h_\mu \circ \pi_\mu = h \circ J_\mu$.

It is remarkable that the proper description of the theorem in the infinite-dimensional case requires Theorem 3.8 so as to ensure that $J^{-1}(\mu)/G_\mu$ is a submanifold.

5. Symplectic forms on separable Hilbert spaces

Most quantum mechanical theories assume Hilbert spaces to be separable, i.e., to admit a countable dense subset of vectors [55], and so is done hereafter. Let us now endow a separable Hilbert space with a strong symplectic form. This will provide a rigorous mathematical extension to the infinite-dimensional setting of the standard approach to finite-dimensional Hilbert spaces [4, 17, 39, 48]. We provide certain details that are frequently absent in the literature [32] and focus on problems on infinite-dimensional Hilbert spaces, but the finite-dimensional case is immediate from our considerations.

A complex Hilbert space $(\mathcal{H}, \langle \cdot | \cdot \rangle)$ is separable if and only if it admits a countable orthonormal basis $\{\psi_j\}_{j \in \mathbb{N}}$ [27]. This is quite a ubiquitous condition in quantum problems, but it may fail to hold, for instance, in scattering processes [56]. When \mathcal{H} is separable, it admits an orthonormal countable basis that enables us to endow \mathcal{H} with a real differentiable structure modeled on a Hilbert space.

Choose real-valued coordinates $\{q^j, p_j\}_{j \in \mathbb{N}}$ on \mathcal{H} given by

$$q^j(\psi) := \Re \langle \psi_j | \psi \rangle, \quad p_j(\psi) := \Im \langle \psi_j | \psi \rangle, \quad \forall j \in \mathbb{N}, \quad \forall \psi \in \mathcal{H}.$$

Let ℓ_2 be the Hilbert space of complex square-summable series considered in the natural way as a real vector space. One can then construct a global chart

$$\varphi : \psi \in \mathcal{H} \mapsto (q^j(\psi), p_j(\psi))_{j \in \mathbb{N}} \in \ell^2.$$

In fact, every $\psi \in \mathcal{H}$ admits a unique family of complex constants $c_j \in \mathbb{C}$ such that

$$\psi = \sum_{j \in \mathbb{N}} c_j \psi_j.$$

It is worth recalling that since \mathcal{H} is a Hilbert space

$$\lim_{n \rightarrow \infty} \sum_{j=1}^n c_j \psi_j = \psi,$$

and this holds independently of the order of the elements of the basis. This is not true for a general Schauder basis of a Banach space. Consequently,

$$\|\psi\|_{\mathcal{H}} := \sqrt{\langle \psi | \psi \rangle} = \sum_{j \in \mathbb{N}} |c_j|^2 < \infty$$

and

$$\sum_{j \in \mathbb{N}} |c_j|^2 = \sum_{j \in \mathbb{N}} (\Re(c_j)^2 + \Im(c_j)^2) = \sum_{j \in \mathbb{N}} (q^j(\psi)^2 + p_j(\psi)^2) < \infty. \quad (5.1)$$

Thus,

$$\varphi(\psi) = (q^j(\psi), p_j(\psi))_{j \in \mathbb{N}} \in \ell^2.$$

Conversely, every sequence

$$(x_0^j, p_{0j})_{j \in \mathbb{N}} \in \ell^2$$

is the image of a unique element

$$\sum_{j \in \mathbb{N}} (x_0^j + i p_{0j}) \psi_j \in \mathcal{H}.$$

In view of (5.1), one has that

$$\|\varphi(\psi)\|_{\ell^2} = \|\psi\|_{\mathcal{H}}$$

and φ is an isometry. Since φ is an isometry and it has an inverse, its inverse is also an isometry and it is therefore continuous. Consequently, \mathcal{H} and ℓ^2 are homeomorphic. This gives rise to a real differentiable structure on \mathcal{H} .

Recall that the tangent space to \mathcal{H} at ϕ is the space of equivalence classes of curves

$$\gamma_\psi : \mathbb{R} \rightarrow \mathcal{H}$$

such that

$$\gamma_\psi(0) = \phi \quad \text{and} \quad d\gamma_\psi/dt(0) = \psi.$$

We will denote by $\dot{\psi}_\phi$ the equivalence class of the curve γ_ψ . Moreover,

$$T_\phi \mathcal{H} = \{\dot{\psi}_\phi : \psi \in \mathcal{H}\}$$

and there exists an isomorphism

$$\lambda_\phi : \psi \in \mathcal{H} \mapsto \dot{\psi}_\phi \in T_\phi \mathcal{H}.$$

Moreover, we represent $\lambda_\phi(\psi_j)$ and $\lambda_\phi(i\psi_j)$ by $(\partial_{q^j})_\phi$ and $(\partial_{p_j})_\phi$, respectively. Hence, $\{(\partial_{q^j})_\phi, (\partial_{p_j})_\phi\}_{j \in \mathbb{N}}$ becomes a basis of $T_\phi \mathcal{H}$ and

$$\lambda_\phi(\psi) = \sum_{j \in \mathbb{N}} \left(q^j(\psi) (\partial_{q^j})_\phi + p_j(\psi) (\partial_{p_j})_\phi \right).$$

It is worth noting that each $T_\phi \mathcal{H}$ is then isomorphic to ℓ^2 and the chosen basis is shrinkable.

The cotangent space of \mathcal{H} at ϕ , i.e.,

$$T_\phi^* \mathcal{H} := (T_\phi \mathcal{H})^*,$$

admits a basis $\{(dq^j)_\phi, (dp_j)_\phi\}_{j \in \mathbb{N}}$ dual to $\{(\partial_{q^i})_\phi, (\partial_{p_j})_\phi\}_{j \in \mathbb{N}}$. Then,

$$(dq^j)_\phi(\lambda_\phi(\psi)) = q^j(\psi) = \Re \langle \psi_j | \psi \rangle, \quad (dp_j)_\phi(\lambda_\phi(\psi)) = p_j(\psi) = \Im \langle \psi_j | \psi \rangle.$$

At each ϕ , one has then that

$$T_\phi^* \mathcal{H} \simeq (\ell^2)^* \simeq \ell^2.$$

In other words, the elements of $T_\phi^* \mathcal{H}$ are those of the form $\lambda_j(dq^j)_\phi + \mu^j(dp_j)_\phi$ with $(\lambda_j, \mu^j)_{j \in \mathbb{N}}$ forming a sequence in ℓ^2 .

Observe that

$$\mathcal{B} : (\phi, \psi) \in \mathcal{H} \times \mathcal{H} \mapsto \Im \langle \phi | \psi \rangle \in \mathbb{R}$$

is a non-degenerate bilinear (over \mathbb{R}) form. Indeed, \mathcal{B} is well-defined because if $\mathbf{c}_1, \mathbf{c}_2 \in \ell_2$, then

$$\sum_{n \in \mathbb{N}} c_{1n} c_{2n} < \infty,$$

and the map \mathcal{B} is bilinear and continuous. Since

$$\mathcal{B}(\psi_1, \psi_2) = \Im \langle \psi_1 | \psi_2 \rangle = \Im \overline{\langle \psi_2 | \psi_1 \rangle} = -\Im \langle \psi_2 | \psi_1 \rangle = -\mathcal{B}(\psi_2, \psi_1), \quad \forall \psi_1, \psi_2 \in \mathcal{H},$$

we obtain that \mathcal{B} is skew-symmetric. Then, \mathcal{B} allows us to endow \mathcal{H} with a unique constant differential two-form ω such that

$$\mathcal{B} = \lambda_\phi^* \omega_\phi$$

on every $\phi \in \mathcal{H}$. By using the isomorphism λ_ϕ , one has that

$$\omega_\phi : T_\phi \mathcal{H} \times T_\phi \mathcal{H} \rightarrow \mathbb{R}$$

takes the local form

$$\omega_\phi(\dot{\psi}_1, \dot{\psi}_2) := \Im \langle \psi_1 | \psi_2 \rangle = \sum_{j \in \mathbb{N}} (q^j(\psi_1) p_j(\psi_2) - q^j(\psi_2) p_j(\psi_1)) = \sum_{j \in \mathbb{N}} (dq^j \wedge dp_j)(\dot{\psi}_1, \dot{\psi}_2). \quad (5.2)$$

Then, $d\omega = 0$ and ω is a non-degenerate form.

To see that ω is a strong symplectic form, note that \mathcal{H} admits a submanifold Q given by the zeros of the functions $\{p^j\}_{j \in \mathbb{N}}$. In fact, the $\{q^j\}_{j \in \mathbb{N}}$ form a coordinate system on Q taking values in the real vector space of square-summable sequences $(q_j)_{j \in \mathbb{N}}$, making Q into a manifold modeled on a Hilbert space. Then, the variables $\{q^j, p_j\}_{j \in \mathbb{N}}$ can be understood as the adapted variables to

$$T^*Q = \mathcal{H}.$$

In view of (5.2), we get that ω is the canonical two-form of T^*Q . Since T^*Q is modeled on a Hilbert space, which is always reflexive [16], Theorem 4.4 shows that ω is a strong symplectic form.

6. Vector fields related to operators

In geometric finite-dimensional quantum mechanics, operators can easily be described via the so-called *linear vector fields*, which are smooth and even analytic functions [17, 19]. Let us extend this idea to bounded, first, and to general (possibly unbounded) operators on infinite-dimensional Hilbert spaces. This will involve addressing several issues concerning the lack of differentiability of the linear vector fields related to unbounded operators.

We showed in Section 5 that \mathcal{H} is a smooth manifold modeled on a real Hilbert space through a global chart given by the coordinates $\{q^j, p_j\}_{j \in \mathbb{N}}$. This gives rise to a global coordinate system $\{q^j, p_j, \dot{q}^j, \dot{p}_j\}_{j \in \mathbb{N}}$ adapted to $T\mathcal{H}$. This coordinate system gives rise to a diffeomorphism between $T\mathcal{H}$ and $\mathcal{H} \oplus \mathcal{H}$. Then, every bounded operator

$$B : \mathcal{H} \rightarrow \mathcal{H}$$

gives rise to a vector field

$$X_B : \phi \in \mathcal{H} \mapsto (\phi, B\phi) \in T_\phi \mathcal{H} \subset T\mathcal{H}.$$

Since B is bounded, it is continuous and

$$B : \mathcal{H} \rightarrow \mathcal{H}$$

is also a smooth mapping. It follows immediately that X_B , which amounts to a bounded operator

$$\text{Id}_{\mathcal{H}} \times B : \phi \in \mathcal{H} \mapsto (\phi, B\phi) \in \mathcal{H} \times \mathcal{H},$$

where $\mathcal{H} \times \mathcal{H}$ is endowed with its natural Hilbert structure induced by the one on \mathcal{H} , is a smooth vector field.

In infinite-dimensional quantum mechanics, operators are standardly unbounded, they have a domain, and it is non-trivial to provide a procedure to ensure that they can be described somehow through linear vector fields on a domain that will be smooth enough for their differential geometric treatment. In particular, one knows that an unbounded operator

$$B : D(B) \subset \mathcal{H} \rightarrow \mathcal{H}$$

on a Hilbert space \mathcal{H} is related to a vector field with a domain (3.2). However, it is important to discover what else can be said. Let us study this process by means of the theory of analytic vectors (see [27, 44, 46, 57] for details).

Definition 6.1. An analytic vector for an operator

$$B : D(B) \subset \mathcal{H} \rightarrow \mathcal{H}$$

is an element $\phi \in \mathcal{H}$ such that

$$\phi \in D^\infty(B) := \bigcap_{n \in \bar{\mathbb{N}}} D(B^n)$$

and

$$\|B^n \phi\| \leq C_\phi^n n!$$

for some $C_\phi > 0$ and every $n \in \bar{\mathbb{N}}^*$. The elements of $D^\infty(B)$ are called *smooth* vectors (of B) [58]. We write $D^a(B)$ for the space of analytic vectors of B .

Both smooth and analytic vectors for a given operator B are vector spaces. The fact that $D^\infty(B)$ is a vector space is clear from the linearity of B . Moreover if $\phi, \psi \in D^a(B)$, then a simple calculation, namely

$$\|B^n(\phi + \psi)\| \leq \|B^n \phi\| + \|B^n \psi\| \leq C_\phi^n n! + C_\psi^n n! \leq (C_\phi + C_\psi)^n n! = C_{\phi+\psi}^n n!, \quad \forall n \in \bar{\mathbb{N}}$$

shows that $D^a(B)$ is a vector space.

The term “smooth” vector comes from the fact that, if B is skew-self-adjoint, the mapping

$$\gamma : t \in \mathbb{R} \mapsto \exp(tB)\psi \in \mathcal{H}$$

is smooth if and only if $\psi \in D^\infty(B)$ (see [57, p. 145]). For a general operator B , one has that

$$D^a(B) \subset D^\infty(B).$$

The existence of analytic vectors for operators can be illustrated by *Nelson’s analytic vectors theorem*, which states that a symmetric operator A on a Hilbert space \mathcal{H} is essentially self-adjoint if and only if A admits a dense domain $D^a(A)$ of analytic vectors [46]. We hereafter assume B to be skew-self adjoint on some domain, if not otherwise stated. Analytic vectors allow for many other interesting simplifications in the theory of skew-adjoint operators. They are also useful in the differential geometric study of quantum mechanical problems. Let us also consider the following interesting result.

Proposition 6.2. ([57, Proposition 7.8]) Let A be a self-adjoint operator and consider an analytic vector $\psi \in D^a(A)$. Let $C_\psi \in \mathbb{C}$ be such that

$$\|A^n \psi\| \leq |C_\psi|^n n!$$

for every $n \in \bar{\mathbb{N}}$. If $z \in \mathbb{C}$ and $|z| < C_\psi^{-1}$, then $\psi \in D(e^{zA})$ and

$$e^{zA} \psi = \lim_{n \rightarrow +\infty} \sum_{k=0}^n \frac{z^k}{k!} A^k \psi.$$

*Note that there is a typo in [57, Definition 7.1].

Let A be a skew-self-adjoint operator. Since e^{tA} is a continuous operator defined on \mathcal{H} , it stems from the definitions of smooth and analytic vectors for A that

$$\exp(tA)D^\infty(A) \subset D^\infty(A)$$

for every $t \in \mathbb{R}$ [57, p.149]. In fact, if $\psi \in D^\infty(A)$, then $s \mapsto \exp(sA)\psi$ is smooth and

$$s \mapsto \exp((s-t)A)\exp(tA)\psi$$

is again smooth.

Assume that V is a finite-dimensional Lie algebra of skew-self-adjoint operators on \mathcal{H} . One may wonder when the exponential of the elements of V , which are continuous automorphisms on \mathcal{H} by the Stone-von Neumann Theorem [27], can be understood as automorphisms on \mathcal{H} related to a continuous Lie group representation on \mathcal{H} . The following definition, based on the one given in [7] establishes the family of Lie algebras V admitting such a property and defines rigorously what we mean.

Definition 6.3. A finite-dimensional Lie algebra V of operators with a common domain \mathcal{D} within a Hilbert space \mathcal{H} is *integrable* if there exists an injective Lie algebra representation

$$\rho : \mathfrak{g} \rightarrow V$$

and a continuous Lie group action

$$\Phi : G \times \mathcal{H} \rightarrow \mathcal{H}$$

(relative to the natural topologies on $G \times \mathcal{H}$ and \mathcal{H}), where G is the connected and simply connected Lie group of \mathfrak{g} , such that

$$\left. \frac{d}{dt} \right|_{t=0} \Phi(\exp(tv), \psi) = \rho(v)\psi, \quad \forall \psi \in \mathcal{D}, \quad \forall v \in \mathfrak{g}.$$

We call ρ the *infinitesimal Lie group action* associated with Φ .

Geometrically, the above definition tells us that the fundamental vector fields of the Lie group action Φ of G coincide on \mathcal{D} with the vector fields associated with the elements of V . Moreover, the fundamental vector fields of this action satisfy that

$$[X_{\rho(v_1)}, X_{\rho(v_2)}]\psi = \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} \exp(-t\rho(v_2)) \exp(-t\rho(v_1)) \exp(t\rho(v_2)) \exp(t\rho(v_1))\psi, \quad \forall \psi \in \mathcal{D}, \quad (6.1)$$

where \mathcal{D} is an invariant common domain of smooth functions where the operators exponentials of the elements of V are well defined and smooth enough. In other words, despite the lack of differentiability of the vector fields related to operators, the geometric expression of the Lie derivative of vector fields still holds.

Example 6.4. Consider the operators $i\hat{p}$, $i\hat{x}$ and $i\widehat{\text{Id}}$ on $L^2(\mathbb{R})$ spanning a three-dimensional Lie algebra of skew-self-adjoint operators. Consider the subspace of analytic functions of the Schwarz space $\mathcal{S}(\mathbb{R})$. This Lie algebra of operators admit a dense subspace of analytic vectors given by the

functions $H_n(x)e^{-x^2/2}$ and it can be integrated. Moreover, these functions belong to $\mathcal{S}(\mathbb{R})$, which is invariant relative to the action of $\exp(\lambda i\widehat{p})$, $\exp(\lambda i\widehat{x})$ and $\exp(\lambda i\widehat{\text{Id}})$ for $\lambda \in \mathbb{R}$. For instance,

$$(\exp(it\widehat{p})\psi)(x) = \psi(x - t), \quad (\exp(it\widehat{x})\psi)(x) = e^{ix}\psi(x), \quad \forall x \in \mathbb{R}.$$

Hence, expression (6.1) gives

$$\begin{aligned} ([X_{i\widehat{p}}, X_{i\widehat{x}}]\psi)(x) &= \frac{1}{2} \frac{\partial^2}{\partial t^2} \Big|_{t=0} \exp(-it\widehat{x}) \exp(-it\widehat{p}) e^{ix} \psi(x - t) \\ &= \frac{1}{2} \frac{\partial^2}{\partial t^2} \Big|_{t=0} \exp(it^2) \psi(x) \\ &= i\psi(x) \\ &= X_{i\widehat{\text{Id}}}. \end{aligned}$$

It is a consequence of an argument given by Goodman [45] and Nelson [46] that if a finite-dimensional real Lie algebra of skew-symmetric operators is integrable, then the next criterium holds [44].

Theorem 6.5. (*FS³ criterium [44]*) *Every finite-dimensional Lie algebra of skew-symmetric operators on a Hilbert space \mathcal{H} is integrable if it admits a basis with a common dense invariant subspace of analytic vectors.*

There are other criteria of integrability of finite-dimensional Lie algebras of skew-symmetric operators on Hilbert spaces [46, 58], but they are, in general, more complicated to use. On the other hand, the FS³ criterion can be relatively easily proved to hold in many physically relevant problems appearing in physics [17, 59]. In particular, if A_1, \dots, A_n are observables admitting a common basis of eigenvectors $\{e_n\}_{n \in \mathbb{N}}$, then iA_1, \dots, iA_n admit a basis in \mathcal{H} of common eigenvectors $\{e_n\}_{n \in \mathbb{N}}$ and all their elements are analytic relative to iA_1, \dots, iA_n . Moreover, the direct sum of subspaces

$$\bigoplus_{n \in \mathbb{N}} \langle e_n \rangle \simeq \mathbb{R}^{\mathbb{N}}$$

is a normed not complete space (in general) of analytic vectors relative to iA_1, \dots, iA_n that is invariant relative to the operators $\exp(i\lambda_\alpha A_\alpha)$ with $\lambda_\alpha \in \mathbb{R}$ and $\alpha = 1, \dots, r$.

Assume then that the skew-symmetric operators of the Lie algebra V have a common invariant domain D_V of smooth vectors on \mathcal{H} . In practical cases, each element of V is skew-self-adjoint in an appropriate domain and its exponential exists on \mathcal{H} . Note that D_V can be extended by the exponentials of the elements of V to give a domain that is invariant relative to them. The related continuous unitary action

$$\Phi : G \times \mathcal{H} \rightarrow \mathcal{H}$$

satisfies that

$$\Phi_\psi : g \in G \mapsto \Phi(g, \psi) \in \mathcal{H}$$

is smooth on D_V^∞ [60].

Let us now prove the following useful result:

Theorem 6.6. Every observable B on a Hilbert space \mathcal{H} gives rise to an integrable vector field on a domain

$$X_B : \psi \in D^\infty(B) \subset \mathcal{H} \mapsto (\psi, B\psi) \in \mathcal{H} \oplus \mathcal{H} \simeq T\mathcal{H}$$

and

$$X_B(\psi) \in T_\psi D^\infty(B).$$

Proof. We know that $T\mathcal{H}$ is diffeomorphic to $\mathcal{H} \oplus \mathcal{H}$. Hence, B can be associated with the vector field on a domain given by X_B . Note that the direct sum of the subspaces generated by the eigenvalues of B gives a dense subspace

$$E_B \simeq \mathbb{R}^{\mathbb{N}}$$

of smooth vectors for B in \mathcal{H} . Hence, $D^\infty(B)$ is not empty and becomes dense in \mathcal{H} . Moreover, $D^a(B) \supset E_B$ is also dense in \mathcal{H} and the Nelson's analytic vectors theorem shows that B is integrable. The norm of \mathcal{H} can be restricted to $D^\infty(B)$, which becomes a normed space. The space E_B is dense in $D^\infty(B)$ and we obtain a basis for $D^\infty(B)$. The maps of the global atlas on \mathcal{H} can be restricted to $D^\infty(B)$, which makes $D^\infty(B)$ into a manifold modeled on a normed space. The inclusion

$$J : D^\infty(B) \hookrightarrow \mathcal{H}$$

is smooth relative to the previous differential structures because every smooth function on \mathcal{H} remains smooth when restricted to $D^\infty(B)$. In fact, the variations of a smooth function on \mathcal{H} remain smooth the directions tangent to $D^\infty(B)$ at points of $D^\infty(B)$.

On an element $\psi \in D^\infty(B)$, one has that F_ψ is smooth. In view of this, one can see that X_B is a vector field on the domain $D^\infty(B)$ and

$$B(D^\infty(B)) \subset D^\infty(B),$$

see [60] for further details. Hence, X_B is an integrable vector field on the domain $D^\infty(B)$. Since

$$B(D^\infty(B)) \subset D^\infty(B),$$

one has that X_B can be considered also as a vector field on a domain $D^\infty(B)$ taking values in $TD^\infty(B)$. \square

It is worth studying non-symmetric operators for which a vector field approach is possible. Let us give an example.

Example 6.7. Consider the space of square integrable functions on $[-l, l] \subset \mathbb{R}$ for $l > 0$, which is a Hilbert space, and the operator

$$-i\widehat{p}^2 : D(-i\widehat{p}^2) \subset \mathcal{H} \rightarrow \mathcal{H}.$$

In particular, consider $D(-i\widehat{p}^2)$ to be the space spanned by finite linear combinations of functions

$$\frac{1}{\sqrt{l}} \sin\left(\frac{\pi(k+1)x}{l}\right), \quad \frac{1}{\sqrt{l}} \cos\left(\frac{\pi kx}{l}\right), \quad k = 0, 1, 2, 3, \dots$$

Hence, $D(-i\widehat{p}^2)$ becomes a normed space relative to the norm induced by $L^2([-l, l])$ that is isomorphic to $\mathbb{R}^{\mathbb{N}}$ and dense in $L^2([-l, l])$. Then, one obtains an operator

$$X_{-i\widehat{p}^2} : \psi \in D(-i\widehat{p}^2) \subset \mathcal{H} \mapsto (\psi, -i\widehat{p}^2\psi) \in T\mathcal{H}.$$

It is worth stressing that $X_{-i\widehat{p}^2}$ is not smooth and $-i\widehat{p}^2$ is not symmetric. However, one may define a flow

$$\varphi : (t, \psi) \in \mathbb{R} \times D(-i\widehat{p}^2) \mapsto \exp(-it\widehat{p}^2)\psi \in \mathcal{H}.$$

Note that this map is well defined as

$$\exp(-it\widehat{p}^2) \sin(\pi kx/L) = \exp(it\pi^2 k^2/L^2) \sin(\pi kx/L),$$

$$\exp(-it\widehat{p}^2) \cos(\pi kx/L) = \exp(it\pi^2 k^2/L^2) \cos(\pi kx/L).$$

Moreover, $D(-i\widehat{p}^2)$ consists of analytic vectors for $-i\widehat{p}^2$ and $\exp(-it\widehat{p}^2)$ is continuous on $D(-i\widehat{p}^2)$. Indeed, $D(-i\widehat{p}^2)$ is invariant relative to $\exp(-it\widehat{p}^2)$. Moreover,

7. Hamiltonian functions induced by self-adjoint operators

In the previous section, we showed that vector fields generated by skew-self adjoint operators may be well-defined on an appropriate domain. We will show that these vector fields can be considered as Hamiltonian vector fields in a domain using functions on domains. For instance, we have to deal with functions on Hilbert spaces of the type

$$f_H : \psi \in D(H) \subset \mathcal{H} \mapsto \langle \psi | H \psi \rangle \in \mathbb{R},$$

where H is a symmetric operator. The function f_H is called the *average value* of the operator H [27] and it is defined only on the domain of H , which is dense in \mathcal{H} .

In general, the whole symplectic formalism can be modified to deal with quantum mechanical systems by considering that structures are defined on a dense subset of \mathcal{H} having the structure of a normed space and using the fact that the operators of the theory are self- or skew-self-adjoint. Most structures to be used are not smooth, but they are differentiable enough to apply differential geometric techniques. Many results can be extended to symmetric operators and other more general setups.

Let us start by defining how to write in terms of quantum operators the differentials of the real and imaginary parts of certain average values of operators. This is done in the following lemma:

Lemma 7.1. *If f_A is the average function of a symmetric operator*

$$A : D(A) \subset \mathcal{H} \rightarrow \mathcal{H},$$

then

$$\iota_{\dot{\phi}}(\mathrm{d}f_A)_\psi = 2\Re\langle \phi | A \psi \rangle, \quad \forall \dot{\phi} \in T_\psi D(A).$$

Therefore, $(\mathrm{d}f_A)_\psi$ can be extended to a unique element of $T_\psi^* \mathcal{H}$.

Proof. By definition of the contraction of the differential of a function with a vector field

$$\iota_{\dot{\phi}}(\mathrm{d}f_A)_\psi = \lim_{t \rightarrow 0} \frac{f_A(\psi + t\phi) - f_A(\psi)}{t} = \langle \psi | A \phi \rangle + \langle \phi | A \psi \rangle = 2\Re\langle \phi | A \psi \rangle,$$

for all $\phi, \psi \in D(A)$. Now, observe that

$$(\mathrm{d}f_A)_\psi(\dot{\phi}) = 2\Re\langle A \psi | \phi \rangle \leq 2\|A \psi\| \|\phi\|,$$

so that $(df_H)_\psi$ is bounded on $D(A)$. Since $D(A)$ is a dense subspace of \mathcal{H} and $(df_A)_\psi(\dot{\phi})$ is bounded on $D(A)$, then, in view of the Hahn-Banach theorem, $(df_A)_\psi$ may be continuously extended to a linear continuous form on $T_\psi\mathcal{H}$. Since $D(A)$ is dense in \mathcal{H} , this extension is unique. \square

The above lemma shows that f_A is Fréchet differentiable at every point of $D(A)$ and it allows us to introduce the following definition:

Definition 7.2. Let f_H be the average value of a symmetric operator

$$H : D(H) \subset \mathcal{H} \rightarrow \mathcal{H}.$$

Its *differential* is the mapping

$$df_H : \psi \in D(H) \subset \mathcal{H} \mapsto (df_H)_\psi \in T_\psi^*\mathcal{H} \subset T^*\mathcal{H},$$

where

$$(df_H)_\psi(\dot{\phi}) = 2\Re\langle\dot{\phi}|H\psi\rangle.$$

Definitions from symplectic geometry can be adapted to the context of quantum mechanics by assuming that they must be restricted appropriately to domains.

Definition 7.3. We say that a vector field on a domain

$$X : D \subset \mathcal{H} \rightarrow T\mathcal{H}$$

is *Hamiltonian* relative to a symplectic manifold (\mathcal{H}, ω) if

$$\iota_X\omega = dh,$$

where the right-hand side is understood as the Gâteaux differential of a Fréchet differentiable function

$$h : D \subset \mathcal{H} \rightarrow \mathbb{R},$$

which is defined at points of D .

It is worth noting that the mapping dh_ψ does not need to exist for $\psi \notin D(H)$, while the contraction is in points of the domain of X .

Proposition 7.4. *The vector field with a domain X_{-iH} on \mathcal{H} induced by a symmetric operator*

$$H : D(H) \subset \mathcal{H} \rightarrow \mathcal{H},$$

where $D(H)$ is a domain of H , is *Hamiltonian* relative to the Hamiltonian function

$$f_H : \psi \in D(H) \mapsto \frac{1}{2}\langle\psi|H\psi\rangle \in \mathbb{R}.$$

Proof. Using the expression for the canonical symplectic form on \mathcal{H} , we obtain, in view of Lemma 7.1, that

$$(\iota_{X_{-iH}}\omega)_\psi(\dot{\phi}) = \Im\langle -iH\psi|\dot{\phi}\rangle = \Re\langle H\psi|\dot{\phi}\rangle = \frac{1}{2}\iota_{\dot{\phi}}d\langle\psi|H\psi\rangle, \quad \forall\dot{\phi} \in T_\psi\mathcal{H}, \quad \forall\psi \in \mathcal{D}(H).$$

\square

8. On t -dependent Hamiltonian operators

To make a differential geometric approach to t -dependent SEs possible, we present a specific type of t -dependent SEs to be studied. In particular, we are concerned with the ones of the form

$$\frac{\partial \psi}{\partial t} = -iH(t)\psi, \quad \psi \in \mathcal{H}, \quad (8.1)$$

where the t -dependent operator $H(t)$ are determined by the following proposition:

Definition 8.1. A t -dependent Hamiltonian operator is a t -parametrised family of self-adjoint operators $H(t)$ on a Hilbert space \mathcal{H} such that all the $H(t)$ admit a common invariant domain \mathcal{D} of analytic vectors in \mathcal{H} .

Note that we do not impose each $H(t)$ to be self-adjoint on \mathcal{D} . Since \mathcal{D} will be a subset of the domains $D(H(t))$, with $t \in \mathbb{R}$, where each $H(t)$, for a fixed t , is self-adjoint for a certain domain $D(H(t))$, the operators $H(t)$ are only symmetric on \mathcal{D} . Definition 8.1 is satisfied by relevant physical systems. The following example shows one of them.

Example 8.2. The quantum operators

$$H_1 := \hat{p}^2, \quad H_2 := \hat{x}^2, \quad H_3 := (\hat{x}\hat{p} + \hat{p}\hat{x}), \quad H_4 := \hat{p}, \quad H_5 := \hat{x}, \quad H_6 := \text{Id}$$

are self-adjoint operators on $L^2(\mathbb{R})$ with respect to appropriate domains [27, 38, 57]. The operators H_1, \dots, H_6 span a Lie algebra of symmetric operators on a common domain \mathcal{D} of functions of the form $e^{-x^2/2}P(x)$, where $P(x)$ is any polynomial [57]. The space \mathcal{D} admits an algebraic basis $e^{-x^2/2}H_n^p(x)$ with $n = 0, 1, 2, \dots$, where the $H_n^p(x)$ are the so-called *Hermite probabilistic polynomials*. The functions $e^{-x^2/2}H_n^p(x)$ constitute a basis of $L^2(\mathbb{R})$ and, then, \mathcal{D} is dense in $L^2(\mathbb{R})$. The operators H_1, \dots, H_6 are symmetric on \mathcal{D} . Moreover, \mathcal{D} is also invariant relative to the action of H_1, \dots, H_6 . Hence, every t -dependent Hamiltonian operator on $L^2(\mathbb{R})$ of the form

$$H(t) := \sum_{\alpha=1}^6 b_{\alpha}(t)H_{\alpha}, \quad (8.2)$$

where the functions $b_{\alpha}(t)$ are any real t -dependent functions, is a t -dependent Hamiltonian operator with a domain \mathcal{D} .

The t -dependent Hamiltonian (8.2) embraces t -dependent harmonic oscillators and other related systems [17]. Note that

$$V = \langle iH_1, \dots, iH_6 \rangle$$

is a Lie algebra of skew-symmetric operators on \mathcal{D} . Moreover, if one restricts to the functions

$$\psi_m(x) := x^m e^{-x^2/2},$$

where $m = 0, 1, 2, 3, \dots$, one finds after a simple calculation that these elements are analytic for H_4, H_5, H_6 . The constant bound for a vector ψ_m according to the Definition 6.1 of analytic vector could be, for instance

$$C_{\psi_m} = 2^{m+1}m!$$

for all the operators iH_4-iH_6 . Then, \mathcal{D} consists of analytic vectors for iH_4-iH_6 . Moreover, it can be proved that $H_m^p(x)e^{-x^2/2}$ are also analytic for iH_1-iH_3 . Hence, the elements of \mathcal{D} are analytic for iH_1, \dots, iH_6 . Therefore, iH_1, \dots, iH_6 can be integrated giving rise to the so-called *metaplectic representation* of the *metaplectic group*.

It is worth noting that \mathcal{D} is not invariant relative to the operators of the form $\exp(i\lambda H_\alpha)$ for any real λ . For instance,

$$\exp(i\lambda H_4)(x^m e^{-x^2/2}) = (x + \lambda)^m e^{-(x+\lambda)^2/2}$$

and the latter does not belong to \mathcal{D} . Recall also that the operators $\exp(i\lambda H_4)$ exist because the given operators are self-adjoint for an appropriate domain.

Example 8.3. Consider the space

$$\mathcal{D} = \left\{ P(x, y, z) e^{-x^2/2 - y^2/2 - z^2/2} \in L^2(\mathbb{R}^3) \mid P(x, y, z) \text{ is a polynomial} \right\},$$

and the operators

$$\widehat{L}_x = yp_z - zp_y, \quad \widehat{L}_y = zp_x - xp_z, \quad \widehat{L}_z = xp_y - yp_x.$$

Using an appropriate basis of polynomials in x, y, z , one can prove that \mathcal{D} is a space of analytic vectors for the above operators.

The t -dependent SEs on \mathcal{H} describe the dynamics of quantum systems and the elements of \mathcal{H} represent its quantum states. Under the conditions given in Definition 8.1 on the associated $H(t)$, we can prove that the t -dependent SE can be considered as a Hamiltonian system.

Using the diffeomorphism

$$T\mathcal{H} = \mathcal{H} \oplus \mathcal{H},$$

we can understand the solutions to (8.1) as curves in \mathcal{H} whose tangent vectors at $\psi(t)$ are given by

$$X(t, \psi(t)) = (\psi(t), -iH(t)\psi(t)).$$

Note that this equality makes sense only in the space \mathcal{D} which is included in the space of smooth vectors for all the operators $-iH(t)$ for $t \in \mathbb{R}$.

Finally, Proposition 7.4 ensures that the t -dependent vector field X is, for every $t \in \mathbb{R}$, a Hamiltonian vector field with Hamiltonian function $\frac{1}{2}\langle\psi|H(t)\psi\rangle$. As a consequence, we obtain the following theorem:

Theorem 8.4. *A t -dependent SE related to a t -dependent Hamiltonian $H(t)$ is the differential equation for the integral curves of the t -dependent vector field on a domain associated with the t -dependent Hamiltonian system $(\mathcal{H}, \omega, \frac{1}{2}\langle\psi|H(t)\psi\rangle)$.*

The domain \mathcal{D} of analytic vectors is not invariant relative to the exponential of the operators $H(t)$ for $t \in \mathbb{R}$. This happens, for instance, in Example 8.2. There are other many case where this can be assumed.

Definition 8.5. *A quantum t -dependent Hamiltonian system is a Hamiltonian system of the form $(\mathcal{H}, \omega, \frac{1}{2}\langle\psi|H(t)\psi\rangle)$, where $H(t)$ is a t -dependent Hamiltonian operator on \mathcal{H} .*

Hence, the above states that the t -dependent SEs defined by a t -dependent Hamiltonian are the Hamilton equations of the t -dependent Hamiltonian.

9. Reduction process for t -dependent SEs

In this section, we describe the projection of a t -dependent SE determined by a t -dependent Hamiltonian operator, which satisfies appropriate technical conditions, onto a projective space via a MW reduction.

Recall that the projective space, \mathcal{PH} , of a Hilbert space \mathcal{H} is the space of equivalence classes of proportional non-zero elements of \mathcal{H} . Every equivalence class in \mathcal{PH} is called a *ray* of $\mathcal{H}/\{0\}$. Physically, the projective space \mathcal{PH} is what is really relevant, as different elements of \mathcal{H} belonging to the same ray have the same physical meaning [56].

More particularly, let us apply the MW reduction to a t -dependent SE determined by a t -dependent Hamiltonian operator $H(t)$ on a Hilbert space \mathcal{H} to describe its projection onto the projective space \mathcal{PH} as a t -dependent Hamiltonian system relative to a reduced symplectic form. To accomplish the process for t -dependent SEs related to unbounded operators, we first define what an invariant function is. As in previous sections, to define it, one considers an appropriate restriction of the infinite-dimensional Hilbert space to an appropriate normed domain, as it happens in quantum mechanics. Thus, it is convenient to recall what we mean by a symmetry of a function relative to a Lie group action.

Definition 9.1. Let

$$\Phi : G \times \mathcal{H} \rightarrow \mathcal{H}$$

be a Lie group action and let D be a dense complex subspace of \mathcal{H} . A t -dependent function

$$f : (t, \psi) \in \mathbb{R} \times D \mapsto f_t(\psi) := f(t, \psi) \in \mathbb{R}$$

is said to be *invariant* under Φ (or G -invariant when G is known from context) if

$$f_t \circ \Phi_g = f_t$$

for all $g \in G$ and every $t \in \mathbb{R}$.

Implicitly, Definition 9.1 implies that $\Phi_g(D) \subset D$ for every $g \in G$. Otherwise, if $\Phi_g(\psi) \notin D$ for certain $g \in G$ and $\psi \in D$, it would follow that

$$f(\Phi_g(\psi)) \neq f(\psi)$$

since the left-hand side of the equality would not be defined. Naturally, one says that a t -independent function

$$f : D \rightarrow \mathbb{R}$$

is G -invariant relative to a Lie group action

$$G \times \mathcal{H} \rightarrow \mathcal{H}$$

if j^*f , where

$$j : (t, \psi) \in \mathbb{R} \times D \mapsto \psi \in D$$

is G -invariant in the sense given in Definition 9.1.

Example 9.2. Consider the multiplicative group

$$U(1) = \{e^{i\theta} \mid \theta \in [0, 2\pi)\}$$

and the Lie group action

$$\Phi : (e^{i\theta}, \psi) \in U(1) \times \mathcal{H} \mapsto e^{i\theta}\psi \in \mathcal{H}.$$

Let us prove that if H is a symmetric operator with domain $D(H)$, then its average function f_H is $U(1)$ -invariant. In fact, the domain $D(H)$ is a \mathbb{C} -linear subspace of \mathcal{H} . Moreover,

$$f_H \circ \Phi_{e^{i\theta}}(\psi) = \langle e^{i\theta}\psi \mid H e^{i\theta}\psi \rangle = e^{-i\theta} e^{i\theta} \langle \psi \mid H \psi \rangle = f_H(\psi), \quad \forall \psi \in D(H).$$

Therefore, f_H is invariant relative to $\Phi_{e^{i\theta}}$ for every $e^{i\theta} \in U(1)$. Recall that since H is symmetric, f_H is a real function. Consequently, f_H is a $U(1)$ -invariant function. It is worth noting that, if $\psi \in D(H)$, then $e^{i\theta}\psi \in D(H)$ for every $\theta \in \mathbb{R}$ and $\Phi_g(\psi) \in D(H)$ for every

$$g = e^{i\theta} \in U(1).$$

Hence, $D(H)$ is invariant under Φ .

Let us now reduce a t -dependent SE (8.1) associated with a t -dependent Hamiltonian operator $H(t)$ on a separable Hilbert space \mathcal{H} onto its projective Hilbert space, \mathcal{PH} , via a MW symplectic reduction.

Consider global real-valued coordinates $\{q^j, p_j\}_{j \in \mathbb{N}}$ on \mathcal{H} as mentioned in Section 5. The initial data for the reduction is given by

$$\omega := \sum_{j \in \mathbb{N}} dq^j \wedge dp_j, \quad H(t) := \sum_{\alpha=1}^r b_\alpha(t) H_\alpha, \quad (9.1)$$

where the $b_1(t), \dots, b_r(t)$ are arbitrary t -dependent real-valued functions and H_α , with $\alpha = 1, \dots, r$, are self-adjoint operators such that all elements of $\langle H_1, \dots, H_r \rangle$ share a common domain of analytic vectors $D \subset \mathcal{H}$. We stress that we do not require H_1, \dots, H_r to be self-adjoint on D . Moreover, note that $\{dq^j, dp_j\}_{j \in \mathbb{N}}$ is a basis of $T^*\mathcal{H}$ and every element of each $T_{(q,p)}^*\mathcal{H}$ is the limit of a sum of finite linear combinations of elements of $\{dq^j, dp_j\}_{j \in \mathbb{N}}$.

The t -dependent SE (8.1) induced by $H(t)$ defines a t -dependent Hamiltonian vector field on \mathcal{H} with a domain $D \subset \mathcal{H}$ of the form

$$X(t, \psi) := \sum_{\alpha=1}^r b_\alpha(t) X_\alpha(\psi), \quad \forall \psi \in D, \quad \forall t \in \mathbb{R},$$

such that

$$X_\alpha(\psi) := -iH_\alpha\psi$$

for all $\alpha = 1, \dots, r$ and $\psi \in D$. We will hereafter consider the reduction of ω to the projective space, and then the reduction of the t -dependent SE associated with $H(t)$.

9.1. The MW reduction of the symplectic form on \mathcal{H}

Consider the Lie group action

$$\Phi : U(1) \times \mathcal{H} \ni (e^{i\theta \text{Id}_{\mathcal{H}}}, \psi) \mapsto e^{i\theta} \psi \in \mathcal{H}.$$

This Lie group action is symplectic relative to the symplectic form (5.2), as proved in Example 4.14.

To obtain the momentum map associated with Φ , we must relate every fundamental vector field of Φ to a Hamiltonian function. The Lie algebra $\mathfrak{u}(1)$ of $U(1)$ consists of pure imaginary numbers, that is

$$\mathfrak{u}(1) = \{ai \mid a \in \mathbb{R}\}.$$

The fundamental vector field of Φ generated by $i \in \mathfrak{u}(1)$ reads

$$X_i(\psi) = \left. \frac{d}{dt} \right|_{t=0} \Phi(e^{it}, \psi) = i\psi, \quad \forall \psi \in \mathcal{H}.$$

This is the vector field associated with the skew-self-adjoint and bounded operator

$$A = i\text{Id}.$$

In view of these results and Proposition 7.4, an appropriate momentum map J corresponding to the action Φ is given by

$$J : \mathcal{H} \ni \psi \mapsto -\frac{1}{2} \langle \psi | \psi \rangle \in \mathfrak{u}^*(1) \simeq \mathbb{R}^*,$$

so that the values of J at every $\psi \in \mathcal{H}$ are proportional to the square of the norm of ψ . Let us prove that J is smooth. The directional derivative of J in the direction ψ at ψ_0 , namely $\dot{\psi}_{\psi_0} J$, becomes

$$\dot{\psi}_{\psi_0} J = \lim_{t \rightarrow 0} \frac{J(\psi_0 + t\psi) - J(\psi_0)}{t} = -\Re \langle \psi_0 | \psi \rangle, \quad \forall \psi_0 \in \mathcal{H}, \quad \forall \dot{\psi}_{\psi_0} \in T_{\psi_0} \mathcal{H}.$$

This gives rise to a linear and bounded Gateaux differential of J on \mathcal{H} for each $\psi_0 \in \mathcal{H}$. Hence, J is differentiable (see [19, 25, 50] for details). Let us write $\nabla_{\dot{\psi}} J$ for the partial derivative of J in the direction $\dot{\psi}$. The second-order directional derivative of J along $\dot{\phi}$ and $\dot{\psi}$ is constant:

$$\begin{aligned} [\nabla_{\dot{\phi}} \nabla_{\dot{\psi}} J]_{\psi_0} &= \lim_{t \rightarrow 0} \frac{1}{t} ((\nabla_{\dot{\psi}} J)(\psi_0 + t\dot{\phi}) - (\nabla_{\dot{\psi}} J)(\psi_0)) \\ &= -\lim_{t \rightarrow 0} \frac{1}{t} (\Re \langle \psi_0 + t\dot{\phi} | \psi \rangle - \Re \langle \psi_0 | \psi \rangle) = -\Re \langle \dot{\phi} | \psi \rangle, \quad \forall \psi_0 \in \mathcal{H}, \quad \forall \dot{\psi}, \dot{\phi} \in T_{\psi_0} \mathcal{H}. \end{aligned}$$

Hence, all second-order directional derivatives are constant and continuous relative to $\dot{\psi}, \dot{\phi}$. Higher-order derivatives are zero. Thus, J is smooth.

Let us show that Φ is Ad^* -equivariant. Since $U(1)$ is an Abelian Lie group, its adjoint action Ad is trivial, that is

$$\text{Ad}_g = \text{Id}_{\mathfrak{u}(1)}$$

for every $g \in U(1)$. Then, the coadjoint action is also trivial. Moreover, J is invariant with respect to the action of $U(1)$ on \mathcal{H} , since unitary transformations do not change the norm of elements of the Hilbert space \mathcal{H} . Therefore,

$$J(\psi) = J \circ \Phi_g(\psi) = \text{Id}_{\mathfrak{u}^*(1)} \circ J(\psi) = (\text{Ad}^*)_g \circ J(\psi), \quad \forall \psi \in \mathcal{H} \quad \forall g \in U(1).$$

Thus, the momentum map J is Ad^* -equivariant.

Let us prove that every $\mu < 0$ is a regular value of J . Since J is smooth, it is enough to prove that

$$T_\psi J : T_\psi \mathcal{H} \rightarrow T_{J(\psi)} \mathbb{R}^* \simeq \mathbb{R}^*$$

is surjective with a split kernel for all non-zero elements $\psi \in \mathcal{H}$ (see [16]). The tangent map of J at $\psi \in \mathcal{H}$ is

$$T_\psi J(\dot{\phi}) = \dot{\phi}_\psi J = -\Re e \langle \psi | \dot{\phi} \rangle.$$

Note that $T_\psi J \neq 0$ if and only if $\psi \neq 0$. Therefore,

$$\text{Im } T_\psi J = T_{J(\psi)} \mathbb{R}^*$$

for $\psi \neq 0$ and thus $T_\psi J$ is surjective if and only if $\psi \in \mathcal{H} \setminus \{0\}$. Since J is smooth, $T_\psi J$ is a continuous map, $\ker T_\psi J$ is a closed subspace of $T_\psi \mathcal{H}$, which becomes a Hilbert space relative to $\Re e \langle \cdot | \cdot \rangle$. Moreover, $T_\psi J$ has a split kernel, since

$$\ker T_\psi J \oplus (\ker T_\psi J)^\perp = T_\psi \mathcal{H},$$

where $(\ker T_\psi J)^\perp$ is a closed subspace. Hence, all $\mu < 0$ are regular values of J , and, for these values, we have the submanifolds

$$J^{-1}(\mu) = \left\{ \psi \in \mathcal{H} \mid -\frac{1}{2} \langle \psi | \psi \rangle = -\frac{1}{2} \sum_{j \in \mathbb{N}} [q_j(\psi)^2 + p^j(\psi)^2] = \mu \right\}, \quad \forall \mu < 0.$$

As mentioned before,

$$(\text{Ad}^*)_g = \text{Id}_{\mathfrak{u}^*(1)}$$

for every $g \in U(1)$. Therefore, the isotropy group at μ is

$$U(1)_\mu := \{g \in U(1) \mid (\text{Ad}^*)_g(\mu) = \mu\} = U(1).$$

Let us prove that $U(1)$ acts freely and properly on $J^{-1}(\mu)$. The condition

$$\Phi_{e^{i\theta}}(\psi) = \psi,$$

for some $\theta \in [0, 2\pi)$, implies that

$$\theta = 0,$$

so that $e^{i\theta} = 1$ is the identity element of $U(1)$ and, as a result, Φ is a free action. Moreover, $U(1)$ is homeomorphic to the group

$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\} \subset \mathbb{C} \simeq \mathbb{R}^2.$$

Therefore, $U(1)$ is a compact Lie group. In view of Proposition 3.9, the Lie group action Φ is also proper. Consequently, Theorem 3.8 ensures that

$$P_\mu := J^{-1}(\mu)/U(1)$$

is a smooth manifold for all $\mu < 0$. Note that if

$$\pi_\mu : J^{-1}(\mu) \rightarrow P_\mu$$

is such that the images of two elements of $J^{-1}(\mu)$ under π_μ are the same, both elements are the same up to a proportional complex constant with module one. This amounts to saying that the images of two elements of $J^{-1}(\mu)$ are the same if their images under

$$\pi : \mathcal{H}_0 = \mathcal{H} \setminus \{0\} \rightarrow \mathcal{PH}$$

are the same. As

$$\pi_\mu(J^{-1}(\mu)) = \pi(\mathcal{H}_0),$$

one can indeed identify

$$P_\mu = \mathcal{PH}$$

and see that

$$\pi_\mu = \pi|_{J^{-1}(\mu)}.$$

Since the differentiable structure on P_μ is one making

$$\pi_\mu : J^{-1}(\mu) \rightarrow P_\mu$$

into a differentiable submersion and

$$\pi_\mu = \pi|_{J^{-1}(\mu)},$$

the smooth structure on P_μ is the one of

$$\pi(J^{-1}(\mu)) = \mathcal{PH}.$$

Hence,

$$P_\mu = \mathcal{PH}$$

is a smooth manifold for all $\mu < 0$.

Note that all assumptions of the MW reduction theorem are satisfied for $\mu < 0$. We may now implement the reduction procedure for the symplectic form. Let us rewrite π_μ as

$$\pi_\mu : J^{-1}(\mu) \ni \psi \mapsto [\psi] \in \mathcal{PH},$$

and consider the natural embedding

$$J_\mu : J^{-1}(\mu) \hookrightarrow \mathcal{H}.$$

The symplectic form on the projective space \mathcal{PH} is the only symplectic form on this latter manifold, let us say ω_μ , such that

$$\pi_\mu^* \omega_\mu = J_\mu^* \omega.$$

In other words,

$$\omega_{\mu|\psi_0}([\dot{\psi}], [\dot{\phi}]) = \Im \langle \psi | \dot{\psi} \dot{\phi} \rangle, \quad \forall \psi_0 \in J^{-1}(\mu), \quad \forall \dot{\psi}, \dot{\phi} \in T_{\psi_0}(J^{-1}(\mu)).$$

9.2. MW reduction of SEs

Consider the t -dependent Schrödinger equation associated with (9.1). The vector fields

$$X_\alpha = X_{-iH_\alpha},$$

with $\alpha = 1, \dots, r$, are tangent to $J^{-1}(\mu)$ at the points where they are defined. In fact, the self-adjointness of H_α for each $\alpha = 1, \dots, r$, implies that, on points of D , we have

$$X_\alpha J(\psi) = -\frac{1}{2} X_\alpha \langle \psi | \psi \rangle = -\frac{d}{dt} \Big|_{t=0} \langle \exp(-itH_\alpha) \psi | \exp(-itH_\alpha) \psi \rangle = 0.$$

Then, the restriction of X_α to $J^{-1}(\mu)$ is a well-defined vector field: it is not only tangent where defined, but it is defined on a subset that is dense in $J^{-1}(\mu)$. Let us prove this.

Since D is dense in \mathcal{H} , for every $\psi \in J^{-1}(\mu)$ with $\mu \neq 0$, there exists a sequence $(\psi_n^D)_{n \in \mathbb{N}}$ of non-zero elements of $\mathcal{H} \setminus \{0\}$ that belong to D , such that $\psi_n^D \rightarrow \psi$. Note that

$$\|\psi\|^2 = \langle \psi | \psi \rangle = -2\mu =: k > 0.$$

If $\psi_n^D \rightarrow \psi$, then $(\sqrt{k}\psi_n^D / \|\psi_n^D\|)_{n \in \mathbb{N}}$ is a sequence in $J^{-1}(\mu) \cap D$, such that

$$\sqrt{k}\psi_n^D / \|\psi_n^D\| \rightarrow \sqrt{k}\psi / \|\psi\| = \psi.$$

Consequently,

$$D_\mu := J^{-1}(\mu) \cap D$$

is dense in $J^{-1}(\mu)$ and

$$\overline{D_\mu} = J^{-1}(\mu).$$

The flow of X_α , namely

$$F : (t, \psi) \in \mathbb{R} \times \mathcal{H} \mapsto F_t \psi := \exp(-itH_\alpha) \psi \in \mathcal{H},$$

leaves invariant $J^{-1}(\mu)$. Additionally, it commutes with every Φ_g , since each Φ_g is a multiplication by a phase and the $\exp(-itH_\alpha)$ are linear mappings for every $t \in \mathbb{R}$. This gives rise to an induced flow on P_μ of the form

$$K : (t, [\psi]) \in \mathbb{R} \times P_\mu \mapsto [F_t \psi] \in P_\mu.$$

Let us write X_α^μ for the restriction of X_α to $J^{-1}(\mu)$, which is tangent to this submanifold. Since the projection π_μ amounts to the restriction of π to $J^{-1}(\mu)$, and X_α^μ is the restriction of X_α to $J^{-1}(\mu)$, one obtains that the projection of X_α^μ onto $J^{-1}(\mu)/U(1)_\mu$ is the projection of X_α on \mathcal{H}_0 onto \mathcal{PH} .

To prove that the projection of X_α^μ on \mathcal{H}_0 onto P_μ is a vector field K_α^μ , one has to prove that such a projection is well defined on a dense subset of P_μ . In fact, Theorem 3.8 shows that

$$\pi_\mu : J^{-1}(\mu) \rightarrow J^{-1}(\mu)/U(1)_\mu \simeq \mathcal{PH}$$

is a surjective submersion. Therefore,

$$\overline{\pi_\mu(D_\mu)} = \pi_\mu(\overline{D_\mu}) = \pi_\mu(J^{-1}(\mu)) = J^{-1}(\mu)/U(1) \simeq \mathcal{PH},$$

and $\pi_\mu(D_\mu)$ is dense in \mathcal{PH} . We conclude that K_α^μ is well-defined with a domain being a dense subset of \mathcal{PH} .

Let us prove that K_α^μ is a Hamiltonian vector field on P_μ with a Hamiltonian function given by

$$f_\alpha^\mu([\psi]) = \frac{1}{2} \langle \psi | H_\alpha \psi \rangle, \quad \forall \psi \in D(H_\alpha) \setminus \{0\}, \quad J(\psi) = \mu,$$

for $\alpha = 1, \dots, r$. In fact,

$$\begin{aligned} \omega_\mu(K_\alpha^\mu, K_\beta^\mu)([\psi]) &= (\omega_\mu)(\pi_{\mu*} X_\alpha^\mu, \pi_{\mu*} X_\beta^\mu)([\psi]) = \pi_\mu^* \omega_\mu(X_\alpha^\mu, X_\beta^\mu)(\psi) \\ &= J_\mu^* \omega(X_\alpha^\mu, X_\beta^\mu)(\psi) = (dJ_\mu^* f_\alpha)(X_\beta^\mu)(\psi) \\ &= (d\pi_\mu^* f_\alpha^\mu)(X_\beta^\mu)(\psi) = (df_\alpha^\mu)(K_\beta^\mu)([\psi]). \end{aligned}$$

Then, f_α^μ is the Hamiltonian function of K_α^μ .

The above shows how the t -dependent vector field X on the domain D gives rise to the t -dependent vector field

$$X^\mu = \sum_{\alpha=1}^r b_\alpha(t) X_\alpha^\mu$$

on $J^{-1}(\mu) \cap D$ projecting onto a t -dependent vector field

$$K^\mu = \sum_{\alpha=1}^r b_\alpha(t) K_\alpha^\mu$$

on

$$\pi_\mu(J^{-1}(\mu) \cap D) \subset P^\mu,$$

whose integral curves are determined by the t -dependent projective SE.

Since K^μ is defined on a manifold D_μ modeled on a normed space, we cannot ensure straightforwardly the existence of its integral curves. Nevertheless, their existence is inferred from the existence of the integral curves of the t -dependent vector field X .

10. Conclusions and outlook

We present a careful symplectic geometric approach to the SEs on separable Hilbert spaces determined through t -dependent unbounded self-adjoint Hamiltonian operators satisfying a quite general condition. The projection of such equations onto the projective space has been described as a MW reduction. We describe carefully the necessary technical geometric and functional analysis details to accomplish the description of the results given. This has been accomplished in a mainly differential geometric way and by avoiding complicated functional analysis techniques.

In particular, the most relevant novelty of our paper is to propose the theory of analytic vectors to deal with the problem of non-smoothness of geometric structures in the infinite-dimensional setting, which is intrinsic to t -dependent Schrödinger equations. We have also shown that analytic vectors can be used in a practical manner to study rigorously particular relevant physical examples. Moreover, our work provides a brief introduction and state-of-the-art to the study of symplectic manifolds modeled on infinite-dimensional topological spaces of different types and their application to quantum mechanics.

There is much room for additional developments. In the future, we aim to study how to extend to the infinite-dimensional context using several geometric techniques to deal with SEs (see [4, 60]). Moreover, other types of MW reductions will be contemplated. We expect to analyze the Hamilton-Jacobi theory in infinite-dimensional symplectic manifolds.

Author contributions

Lucas suggested most initial ideas, techniques, and examples for the work. He also collaborated in writing the work. Rivas contributed to improving all the ideas, techniques, and examples provided by Lucas. He also collaborated in writing the work. Lange contributed to improving all the ideas, techniques, and examples provided by Lucas. She also carefully verified the calculations. Additionally, she collaborated in writing all parts of the work. All authors have read and agreed to the published version of the manuscript.

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Conflict of interest

The authors have no competing interests relevant to the content of this article.

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