



Research article

The signature of a monomial ideal

Jovanny Ibarguen, Daniel S. Moran, Carlos E. Valencia* and Rafael H. Villarreal

Departamento de Matemáticas, Centro de Investigación y de Estudios Avanzados del IPN, Apartado Postal 14–740, 07000, México

* **Correspondence:** Email: cvalencia@math.cinvestav.edu.mx.

Abstract: The irreducible decomposition of a monomial ideal has played an important role in combinatorial commutative algebra, with applications beyond pure mathematics, such as biology. Given a monomial ideal I of a polynomial ring $S = k[\mathbf{x}]$ over a field k and variables $\mathbf{x} = \{x_1, \dots, x_n\}$, its incidence matrix, is the matrix whose rows are indexed by the variables \mathbf{x} and whose columns are indexed by its minimal generators. The main contribution of this paper is the introduction of a novel invariant of a monomial ideal I , termed its signature, which could be thought of as a type of canonical form of its incidence matrix, and the proof that two monomial ideals with the same signature have essentially the same irreducible decomposition.

Keywords: irreducible decomposition; monomial ideal; signature; incidence matrix; regular sequences; minimal free resolution

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1. Introduction

The main contribution of this paper is the introduction of a novel invariant of a monomial ideal, termed its signature, which can be thought of as a type of canonical form of its incidence matrix, and the proof that two monomial ideals with the same signature have essentially the same irreducible decomposition.

In general, monomial ideals are central objects in commutative algebra through which a fruitful bridge has been established between commutative algebra and combinatorics; see, for instance, [1–3] and the references contained there. In particular, the irreducible decomposition of a monomial ideal has played an important role in combinatorial commutative algebra. It has an increasing number of applications, from pure mathematics to biology. Its applications in pure mathematics go from the integer programming gap, the Frobenius problem, secants of monomial ideals, tropical convex hulls and cyclic polytopes, differential powers of monomial ideals, and in biology as the reverse engineering

of biochemical networks; see, for instance, the references contained in [4]. There are also available efficient specialized algorithms to compute the irredundant irreducible decomposition of a monomial ideal; see, for instance, [4, 5].

The irreducible decomposition of an ideal is very close to its primary decomposition, in which primary ideals are replaced by irreducible ideals. One difference with a primary decomposition is that the irreducible decomposition is unique and generally simpler to compute. Moreover, when the ideal is monomial, several aspects of the irreducible decomposition can be simplified, leading us naturally toward a more combinatorial context. The primary decomposition of an ideal, introduced by Lasker and Noether, is one of the seminal ideas of commutative algebra. It was inspired by the fact that any ring element can be factorized as the product of some irreducible factors, such as in the Fundamental Theorem of Arithmetic for the ring of integers at the beginning of the 1900s. A key fact to factorizing an ideal is that it is more convenient to consider the intersection of irreducible ideals instead of its product, as in the factorization of integers.

Let's fix some of the notation that we will use throughout the paper. Let $S = k[\mathbf{x}]$ be a polynomial ring over a field k and variables $\mathbf{x} = \{x_1, \dots, x_n\}$. Here, we will stress the bijection between the variables in S and the elements of $[n] = \{1, \dots, n\}$ via the correspondence between x_i and i . Moreover, this bijection can be extended to a bijection between multisets in $[n]$, monomials in $k[\mathbf{x}]$, and vectors in \mathbb{N}^n . The first one can be obtained with the map that sends the multiset A of $[n]$ to the monomial $\mathbf{x}^A = \prod_{a \in A} x_a$ in S , and the second one can be obtained by the map that sends the monomial $\mathbf{x}^{\mathbf{a}} = \prod_{i=1}^n x_i^{a_i}$ to the vector $\mathbf{a} = (a_1, \dots, a_n)$ in \mathbb{N}^n . When no confusion arises, we will identify monomials in $k[\mathbf{x}]$, vectors in \mathbb{N}^n , and multisets in $[n]$. This provides a bridge between commutative algebra, polyhedral geometry, and combinatorics, enriching the interaction between these areas.

In this sense, the set of monomials $\text{Mon}(S)$ of S corresponds to the elements in \mathbb{N}^n . Moreover, antichains \mathbf{G} in \mathbb{N}^n (that is, a set $\mathbf{G} = \{\mathbf{g}_1, \dots, \mathbf{g}_q\} \subset \mathbb{N}^n$ which are incomparable $\mathbf{g}_i \perp \mathbf{g}_j$ by pairs with respect to the order \leq of \mathbb{N}^n) corresponds to minimal generating sets of monomial ideals in S . Thus, given an antichain \mathbf{G} in \mathbb{N}^n , let $I_{\mathbf{G}} = \langle \mathbf{x}^{\mathbf{g}_1}, \dots, \mathbf{x}^{\mathbf{g}_q} \rangle \subset S$ be its antichain ideal, which is simply another way of saying monomial ideals. We use the term antichain ideal to refer to a monomial ideal since the former emphasizes that the order in \mathbb{N}^n plays a crucial role in studying these ideals.

It is well-known that each monomial ideal I in S is minimally generated by a finite set of generators $\mathbf{x}^{\mathbf{G}} = \{\mathbf{x}^{\mathbf{g}_1}, \dots, \mathbf{x}^{\mathbf{g}_q}\}$. Moreover, $\mathbf{x}^{\mathbf{G}}$ is a minimal generating set of a monomial ideal if and only if \mathbf{G} is an antichain in \mathbb{N}^n . Although the algebraic properties of a monomial ideal do not depend on the order of its generators, throughout the paper, we are always assuming that the elements of \mathbf{G} are ordered. This is similar to the difference between a linear transformation and its matrix representation; unlike the linear transformation, the latter depends on the chosen base.

In Section 2, we recall what the irreducible decomposition of a monomial ideal means. More precisely, we will use the ideals generated by the power of a single variable as basic blocks and the union and intersection as operations between them; we get that any monomial ideal has an irreducible decomposition; see Eq (2.1).

After that, we introduce the novel concept of ∞ -cover of the antichain, which generalizes the concept of minimal vertex cover for graphs.

Definition 2.8. An ∞ -cover of the antichain \mathbf{G} is a vector $\mathbf{c} \in \mathbb{N}_{\infty}^n$, $\mathbb{N}_{\infty} = \mathbb{N} \cup \{\infty\}$, such that for any

$$\mathbf{g} \in \mathbf{G} \text{ there exists } 1 \leq j \leq n \text{ such that } \mathbf{c}_j \leq \mathbf{g}_j.$$

Moreover, it is called maximal whenever it is maximal with respect to the order \leq in \mathbb{N}_∞^n .

This leads to the following combinatorial description of the irredundant irreducible decomposition of an antichain ideal I_G in terms of the set $\mathcal{T}(G)$ of its maximal ∞ -covers.

Theorem 2.12. *If G is an antichain in \mathbb{N}^n , then*

$$\bigcap_{\mathbf{a} \in \mathcal{T}(G)} \mathfrak{m}^{\mathbf{a}}$$

is the irredundant irreducible decomposition of the antichain ideal I_G .

In Section 3, we introduce the central concept of this paper, the signature of a monomial ideal. Roughly speaking, the signature of a monomial ideal is a matrix that encodes the way the minimal generators of the ideal are ordered for each variable separately. More precisely, given a vector $\mathbf{c} \in \mathbb{N}^n$, its content is the vector $\text{cont}(\mathbf{c}) = (m_0, \dots, m_r)$ that contains its different entries in ascending order and its signature is given by

$$\sigma(\mathbf{c})_i = k, \text{ where } c_i = m_k, \text{ for some } k \in \{0, \dots, r\}.$$

Definition 3.5. *The signature of a monomial ideal I_G , denoted by $\sigma(I_G)$, is the matrix obtained from its incidence matrix B_G by replacing its rows with their corresponding signatures.*

The main result of this paper gives us a description of the irredundant irreducible decomposition of any monomial with signature σ in terms of the irredundant irreducible decomposition of the antichain ideal I_σ of σ . Given a vector $\mathbf{c} \in \mathbb{N}^n$, the support of \mathbf{c} is given by $\text{supp}(\mathbf{c}) = \{i \in [n] : c_i \neq 0\}$.

Theorem 3.11. *If σ is a signature, I_G is a monomial ideal with signature σ and $\mathbf{m} = \text{gcd}(\mathbf{g}_i \in G)$, then*

$$\left(\bigcap_{\{i : m_i > 0\}} \langle x_i^{m_i} \rangle \right) \cap \left(\bigcap_{J \in \mathcal{J}_\sigma} \mathfrak{m}^{G_J} \right)$$

is the irredundant irreducible decomposition of I_G for some $\mathcal{J}_\sigma \subseteq \prod_{i=1}^q \text{supp}(\mathbf{g}_i)$ and $\mathfrak{m}^{G_J} = \langle x_{J_1}^{(\mathbf{g}_1)_{J_1}}, \dots, x_{J_q}^{(\mathbf{g}_q)_{J_q}} \rangle$ for $J \in \mathcal{J}_\sigma$.

In Section 4, we explore Alexander duality in the context of the signature. If τ_0 is the matrix that encodes the irredundant irreducible decomposition of the antichain ideal associated with a signature σ and interchanging the zero entries by ∞ , σ^\vee and τ_0^\vee are the matrices obtained from σ and τ_0 by reversing the order on its entries (see Definition 4.3), respectively, then the content of its rows is preserved.

Theorem 4.5. *Let σ be the signature of a monomial ideal without the power of a variable as a generator. If \mathbf{s}_i , \mathbf{t}_i , \mathbf{u}_i and \mathbf{v}_i are the i -rows of σ , τ_0 , σ^\vee , and τ_0^\vee , respectively, then*

$$\text{cont}(\mathbf{s}_i) = \text{cont}(\mathbf{t}_i) = \text{cont}(\mathbf{u}_i) = \text{cont}(\mathbf{v}_i) \text{ for all } 1 \leq i \leq n.$$

This leads to show that Alexander duality behaves well with respect to signature.

Theorem 4.12. *Let σ be a signature with no column with only one nonzero entry and D_G be the decomposition matrix of the antichain ideal of G . Then, $\sigma(D_G) = \tau_0$ if and only if $\sigma(I_G) = \sigma$.*

Therefore, it is natural to extend Alexander duality between monomial ideals with the same signature.

We finish this paper by posing a pair of conjectures that claim that monomial ideals with the same signature have essentially the same minimal free resolution and that their quotient modules share

at least a common regular sequence. These pair of conjectures give us an insight into the possible usefulness of the signature of a monomial ideal.

Conjecture 5.2. *Let I and J be monomial ideals with the same signature. If d is the depth of I , then there exists a sequence h_1, \dots, h_d of polynomials such that it is a regular sequence of both S/I and S/J .*

Before establishing the second conjecture, we must introduce the concept of poset resolution. Given a minimal free resolution $\mathbf{F}_\bullet = \{F_i, d_i\}_{i=0}^p$ of a monomial ideal, its poset resolution is a decorated graded poset \mathbf{P}_\bullet given by a triplet (P, s, m) , where $P = (F, \leq)$ is the graded poset on the generators of each of the free modules F_i , where $u \in F_{i-1}$ is covered by $v \in F_i$ whenever $d_i(v) = \sum_{w \in F_{i-1}} s_w w$ with $0 \neq s_u \in k$ for some $1 \leq i \leq p$. The function s comes from the edges of the Hasse diagram of P to the field k given by the coefficient of the term s_u in

$$d_i(v) = \sum_{w \in F_{i-1}} s_w w$$

and $m : F \rightarrow \text{Mon}(S)$ is the function that takes the monomial with the multidegree of v .

Conjecture 5.3. *If I and J are monomial ideals with the same signature, then there exists a poset resolution $\mathbf{P}_\bullet = (P, s, m)$ of I such that $\mathbf{P}'_\bullet = (P, s, m')$ is a poset resolution of J for some m' .*

2. Irredundant irreducible decomposition of a monomial ideal

We begin by recalling what the irreducible decomposition of a monomial ideal means. The reader can consult, for instance, Section 3 of [6] for a very accessible exposition of the subject.

Although many of the ideas included in this section are found (explicitly or implicitly) in the literature, the way they are presented here differs from how they are commonly presented. More importantly, it provides a point of view that will be useful for our purposes, and we believe it contains some new ideas. For instance, one of the novel elements introduced here is the maximal ∞ -cover of an antichain, which generalizes the concept of the minimal vertex cover of a graph.

A monomial ideal has at least two minimal ways to be represented. The most common form of these, which gives rise to its name, is as the ideal generated by a set of monomials. A second way is the so-called irreducible decomposition, which expresses it as the intersection of ideals generated by the powers of some subsets of variables, called irreducible.

In what follows, we will show that all the monomials in a monomial ideal I can be obtained from ideals generated by the power of a single variable as basic blocks and the union and intersection as operations between them. Given an ideal I in the polynomial ring S , let $\text{Mon}(I)$ denote its set of monomials. We recall that any monomial ideal is determined by its set of monomials. That is, if I and J are monomial ideals of S , then $I = J$ if and only if $\text{Mon}(I) = \text{Mon}(J)$.

The irreducible decomposition of a monomial ideal can begin with the following very simple observation: the ideal generated by the monomial $\mathbf{x}^{\mathbf{g}}$ is equal to

$$\langle \mathbf{x}^{\mathbf{g}} \rangle = \bigcap_{j \in \text{supp}(\mathbf{g})} \langle x_j^{\mathbf{g}^j} \rangle,$$

where $\text{supp}(\mathbf{g}) = \{i \in [n] : \mathbf{g}_i \neq 0\}$ is the support of \mathbf{g} . Moreover, since the set of monomials of the

antichain ideal I_G is equal to $\text{Mon}(I_G) = \text{Mon}(\langle \mathbf{x}^{\mathbf{g}_1}, \dots, \mathbf{x}^{\mathbf{g}_q} \rangle) = \bigcup_{k=1}^q \text{Mon}(\langle \mathbf{x}^{\mathbf{g}_k} \rangle)$, then

$$\text{Mon}(I_G) = \bigcup_{k=1}^q \bigcap_{j \in \text{supp}(\mathbf{g}_k)} \text{Mon}(\langle x_j^{(\mathbf{g}_k)_j} \rangle).$$

Distributing the union over the intersection, we get that

$$\text{Mon}(I_G) = \bigcap_{J \in \prod_{k=1}^q \text{supp}(\mathbf{g}_k)} \bigcup_{k=1}^q \text{Mon}(\langle x_{J_k}^{(\mathbf{g}_k)_{J_k}} \rangle) = \bigcap_{J \in \prod_{k=1}^q \text{supp}(\mathbf{g}_k)} \text{Mon}(\langle \mathbf{x}_{J_1}^{(\mathbf{g}_1)_{J_1}}, \dots, \mathbf{x}_{J_q}^{(\mathbf{g}_q)_{J_q}} \rangle)$$

and therefore

$$I_G = \bigcap_{J \in \prod_{k=1}^q \text{supp}(\mathbf{g}_k)} \langle x_{J_1}^{(\mathbf{g}_1)_{J_1}}, \dots, x_{J_q}^{(\mathbf{g}_q)_{J_q}} \rangle. \quad (2.1)$$

We remark that each J is a vector in $[n]^q$. Thus, using this very simple observation, we have proved that any monomial ideal has at least one irreducible decomposition. However, the decomposition given in (2.1) has several redundancies. The simplest of which is that the factors

$$\langle x_{j_1}^{(\mathbf{g}_1)_{j_1}}, \dots, x_{j_q}^{(\mathbf{g}_q)_{j_q}} \rangle$$

can have powers on the same variable. This can be easily solved by recalling that $\langle x^a, x^b \rangle = \langle x^{\min(a,b)} \rangle$. More precisely:

Definition 2.1. Given $J \in \prod_{i=1}^q \text{supp}(\mathbf{g}_i) \subseteq [n]^q$, let $\mathbf{G}_J \in \mathbb{N}^n$ be such that

$$\mathfrak{m}^{\mathbf{G}_J} = \langle x_{J_1}^{(\mathbf{g}_1)_{J_1}}, \dots, x_{J_q}^{(\mathbf{g}_q)_{J_q}} \rangle,$$

where $\mathfrak{m} = \langle x_1, \dots, x_n \rangle$ and $\mathfrak{m}^{\mathbf{a}} = \langle \{x_i^{\mathbf{a}_i} : \mathbf{a}_i > 0\} \rangle$ for any $\mathbf{a} \in \mathbb{N}^n$.

To simplify the notation, let $\Pi_G = \prod_{i=1}^q \text{supp}(\mathbf{g}_i)$. Thus, by Definition 2.1, establish a mapping from Π_G to \mathbb{N}^n that sends J to \mathbf{G}_J .

Remark 2.2. In the definition of $\mathfrak{m}^{\mathbf{a}}$ there are only considered positive powers of the variables.

Thus, the decomposition (2.1) can be simplified as

$$I_G = \bigcap_{J \in \Pi_G} \mathfrak{m}^{\mathbf{G}_J}. \quad (2.2)$$

A second type of redundancy that appears in this decomposition is that, in general, many of the factors in (2.2) are not necessary; that is, we can omit them from the intersection preserving I_G . This leads us to introduce the concept of irredundancy of an irreducible decomposition.

Definition 2.3. An irreducible decomposition with factors $\mathcal{Q} = \{Q_1, \dots, Q_m\}$ of an ideal I is called irredundant whenever

$$I \neq \bigcap_{k \in [m] \setminus j} Q_k \text{ for all } 1 \leq j \leq m,$$

and redundant otherwise.

Although irreducible decompositions of a monomial ideal are not necessarily unique, it is not difficult to check that an irredundant irreducible decomposition must be unique.

The irreducible decomposition given in (2.2) has $|\Pi_G| = \prod_{i=1}^q |\text{supp}(\mathbf{g}_i)|$ factors, which can be huge. Even so, there exist cases where it is actually irredundant. For instance, let $I_{qK_2} = \langle x_1x_2, x_3x_4, \dots, x_{2q-1}x_{2q} \rangle$ be the edge ideal of the disjoint union of q edges. It can be checked that $|\Pi_{qK_2}| = \prod_{i=1}^q |\text{supp}(\mathbf{g}_i)| = 2^q$ and its irredundant irreducible decomposition is equal to

$$\bigcap_{J \in [2]^q} \langle \{x_{2(i-1)+J_i} : 1 \leq i \leq q\} \rangle,$$

where $E(qK_2) = \{v_1v_2, v_3v_4, \dots, v_{2q-1}v_{2q}\}$.

In order to get an irredundant irreducible decomposition from (2.2), we need to omit the factors that are not minimal with respect to the inclusion of sets. In the literature, several strategies to achieve this can be found. In our case, we choose to use the partially ordered extended natural numbers \mathbb{N}_∞^n .

2.1. The partially ordered extended natural numbers $\overline{\mathbb{N}}^n$.

Let $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ be the extended natural numbers and \leq be the extension on it of the usual order in \mathbb{N} with $a \leq \infty$ for any $a \in \mathbb{N}$. The reader can think of $\overline{\mathbb{N}}$ as a type of compactification of \mathbb{N} . In a similar way, let $\overline{\mathbb{N}}^n$ be the set of vectors of size n with entries in $\overline{\mathbb{N}}$, and \leq be the partial order on $\overline{\mathbb{N}}^n$ induced by \leq componentwise. Additionally, let $\mathbb{N}_\infty = \overline{\mathbb{N}} \setminus 0$.

Consider $(\overline{\mathbb{N}}^n, \leq)$ instead of only (\mathbb{N}^n, \leq) , at least in our case, to do this has some advantages. For instance, it is not difficult to check that for \mathbf{a} and \mathbf{b} in \mathbb{N}^n (see, for instance, [7, Proposition 1.4] for a very similar result),

$$\mathfrak{m}^{\mathbf{a}} \subseteq \mathfrak{m}^{\mathbf{b}} \text{ if and only if } \mathbf{a}_\infty \geq \mathbf{b}_\infty,$$

where $\mathbf{c}_\infty \in \overline{\mathbb{N}}^n$ is obtained from $\mathbf{c} \in \mathbb{N}^n$ by exchanging its zero entries by ∞ . In a similar way, in this context it is very simple to describe the complement of $\text{Mon}(\mathfrak{m}^{\mathbf{a}})$ in $\text{Mon}(S)$ as

$$\text{Mon}(S) \setminus \text{Mon}(\mathfrak{m}^{\mathbf{a}}) = \{\mathbf{x}^{\mathbf{b}} : \mathbf{b} + \mathbf{1} \leq \mathbf{a}_\infty\}.$$

Remark 2.4. In the same way that it is natural to consider that $x_i^0 = 1$, it is also natural to consider that $\langle x_i^\infty \rangle = 0$ because $0 \subsetneq \cdots \subsetneq \langle x_i^n \rangle \subsetneq \cdots \subsetneq \langle x_i^2 \rangle \subsetneq \langle x_i \rangle$.

Since $\mathbf{x}^{\mathbf{a}} = 0$ for any $\mathbf{a} \in \mathbb{N}_\infty^n$ with some entry equal to ∞ and $\mathfrak{m}^{\mathbf{a}} = S$ for any $\mathbf{a} \in \mathbb{N}^n$ with some entry equal to 0, then from here on when we write $\mathbf{x}^{\mathbf{a}}$ we are assuming that $\mathbf{a} \in \mathbb{N}^n$ and when we write $\mathfrak{m}^{\mathbf{a}}$ we are assuming that $\mathbf{a} \in \mathbb{N}_\infty^n$. This convention is also adopted in [4], which helps distinguish between irreducible ideals and ideals generated by a monomial. This is a glimpse of a duality between 0 and ∞ into the two ways of expressing a monomial ideal, as generated by monomials and as the intersection of irreducible ideals.

Remark 2.5. If $\mathbf{a} \in \mathbb{N}_\infty^n$, then $\mathfrak{m}^{\mathbf{a}} = \langle x_1^{\mathbf{a}_1}, \dots, x_n^{\mathbf{a}_n} \rangle = \langle \{x_i^{\mathbf{a}_i} : \mathbf{a}_i < \infty\} \rangle \neq S$, which is equivalent to the previous definition of an irreducible ideal given in 2.1 for \mathbf{a}_0 , where \mathbf{a}_0 is obtained from \mathbf{a} by exchanging its ∞ entries by zero.

Besides, in this new context, the containment relation between two irreducible ideals is expressed simply as

$$\mathfrak{m}^{\mathbf{a}} \subseteq \mathfrak{m}^{\mathbf{b}} \text{ if and only if } \mathbf{a} \geq \mathbf{b}$$

and $\text{Mon}(\mathfrak{m}^{\mathbf{a}})^c = \{\mathbf{x}^{\mathbf{b}} : \mathbf{b} + \mathbf{1} \leq \mathbf{a}\}$. In what follows, we characterize when an intersection of irreducible ideals is irredundant.

Lemma 2.6. *If $\mathcal{T} \subset \mathbb{N}_{\infty}^n$, then \mathcal{T} is an antichain in $(\mathbb{N}_{\infty}^n, \leq)$ if and only if*

$$I = \bigcap_{\mathbf{a} \in \mathcal{T}} \mathfrak{m}^{\mathbf{a}} \quad (2.3)$$

is irredundant.

Proof. We will proceed by contradiction, that is, we will assume that $\bigcap_{\mathbf{a} \in \mathcal{T}} \mathfrak{m}^{\mathbf{a}}$ is redundant. The decomposition $\bigcap_{\mathbf{a} \in \mathcal{T}} \mathfrak{m}^{\mathbf{a}}$ is redundant if and only if there exists $\mathbf{b} \in \mathcal{T}$ such that $\bigcap_{\mathbf{a} \in \mathcal{T}} \mathfrak{m}^{\mathbf{a}} = \bigcap_{\mathbf{a} \in \mathcal{T} \setminus \mathbf{b}} \mathfrak{m}^{\mathbf{a}}$ if and only if $\bigcap_{\mathbf{a} \in \mathcal{T}} \text{Mon}(\mathfrak{m}^{\mathbf{a}}) = \bigcap_{\mathbf{a} \in \mathcal{T} \setminus \mathbf{b}} \text{Mon}(\mathfrak{m}^{\mathbf{a}})$ if and only if

$$\left[\bigcap_{\mathbf{a} \in \mathcal{T}} \text{Mon}(\mathfrak{m}^{\mathbf{a}}) \right]^c = \bigcup_{\mathbf{a} \in \mathcal{T}} \text{Mon}(\mathfrak{m}^{\mathbf{a}})^c = \bigcup_{\mathbf{a} \in \mathcal{T} \setminus \mathbf{b}} \text{Mon}(\mathfrak{m}^{\mathbf{a}})^c = \left[\bigcap_{\mathbf{a} \in \mathcal{T} \setminus \mathbf{b}} \text{Mon}(\mathfrak{m}^{\mathbf{a}}) \right]^c,$$

if and only if

$$\text{Mon}(\mathfrak{m}^{\mathbf{b}})^c = \{\mathbf{x}^{\mathbf{d}} : \mathbf{d} + \mathbf{1} \leq \mathbf{b}\} \subseteq \bigcup_{\mathbf{a} \in \mathcal{T} \setminus \mathbf{b}} \{\mathbf{x}^{\mathbf{d}} : \mathbf{d} + \mathbf{1} \leq \mathbf{a}\} = \bigcup_{\mathbf{a} \in \mathcal{T} \setminus \mathbf{b}} \text{Mon}(\mathfrak{m}^{\mathbf{a}})^c,$$

if and only if there exists $\mathbf{a} \in \mathcal{T} \setminus \mathbf{b}$ such that $\mathbf{b} \leq \mathbf{a}$ if and only if \mathcal{T} is not an antichain of $(\mathbb{N}_{\infty}^n, \leq)$, which is a contradiction. \square

Remark 2.7. *If A is a subset of \mathbb{N}_{∞}^n whose elements are maximal with respect to the order in \mathbb{N}_{∞}^n , then A is an antichain in $(\mathbb{N}_{\infty}^n, \leq)$. Thus, taking only maximal elements of \mathcal{T} from the decomposition (2.2), we can get an irredundant irreducible decomposition of an antichain ideal.*

2.2. maximal ∞ -covers

In this subsection, we will express the irredundant irreducible decomposition of a monomial ideal in combinatorial terms. More precisely, we generalize the concept of a minimal vertex cover given in graph theory, which we call a maximal ∞ -cover, to the context of monomial ideals.

Definition 2.8. *An ∞ -cover of the antichain \mathbf{G} is a vector $\mathbf{c} \in \mathbb{N}_{\infty}^n$ such that for any*

$$\mathbf{g} \in \mathbf{G} \text{ there exists } 1 \leq j \leq n \text{ such that } \mathbf{c}_j \leq \mathbf{g}_j.$$

Moreover, it is called maximal whenever it is maximal with respect to the order \leq in \mathbb{N}_{∞}^n .

It is well-known that the concept of minimal vertex cover has played a significant role in graph theory and combinatorial commutative algebra; see, for instance, [3] and [8], respectively. To make reading easier, we recall that a minimal vertex cover of a graph G is a set of vertices C such that it is incident (or covers) to all the edges of G , which is also minimal with respect to the inclusion of sets.

Since the edge ideal of a graph is the antichain ideal of the characteristic vectors of its edges and $\mathbf{a} \leq \mathbf{b}$ for $\mathbf{a}, \mathbf{b} \in \{0, 1\}^n$ if and only if $\mathbf{b}_{\infty} \leq \mathbf{a}_{\infty}$. Then, it is not difficult to check that $\mathbf{a} \in \mathbb{N}^n$ is the characteristic vector of a minimal vertex cover of the graph G if and only if $\mathbf{a}_{\infty} \in \mathbb{N}_{\infty}^n$ is a maximal ∞ -cover of \mathbf{G} .

In what follows, we characterize the maximal ∞ -cover of a given antichain. Recalling Definition 2.1, for any $J \in \Pi_{\mathbf{G}}$, let $\mathbf{G}_J \in \mathbb{N}^n$ be such that

$$\mathbf{m}^{\mathbf{G}_J} = \langle x_{J_1}^{(\mathbf{g}_1)_{J_1}}, \dots, x_{J_q}^{(\mathbf{g}_q)_{J_q}} \rangle.$$

In other words,

$$0 \neq (\mathbf{G}_J)_{J_k} = \min\{(\mathbf{g}_i)_{J_i} : J_i = J_k\} \leq (\mathbf{g}_k)_{J_k} \text{ for all } 1 \leq k \leq q,$$

and therefore we have that \mathbf{G}_J is always an ∞ -cover of \mathbf{G} . We recall that when $i \neq J_k$ for all $1 \leq k \leq q$, then $(\mathbf{G}_J)_i = \infty$. Moreover, as shown in the following lemma, every maximal ∞ -cover of an antichain \mathbf{G} is of this form.

Lemma 2.9. *Let \mathbf{G} be an antichain in \mathbb{N}^n . If $\mathbf{c} \in \mathbb{N}_{\infty}^n$ is a maximal ∞ -cover of \mathbf{G} , then there exists $J \in \Pi_{\mathbf{G}}$ such that*

$$\mathbf{c} = \mathbf{G}_J.$$

Proof. To start, we will prove that for any $1 \leq i \leq n$ such that $\mathbf{c}_i \neq \infty$, there exists $1 \leq k_i \leq q$ such that $\mathbf{c}_i = (\mathbf{g}_{k_i})_i$. Otherwise, assume that $\mathbf{c}_i \neq (\mathbf{g}_k)_i$ for some $1 \leq i \leq n$ and for any $1 \leq k \leq q$. Thus, let $1 \leq j \leq q$ such that $\mathbf{c}_i < (\mathbf{g}_j)_i$ and $(\mathbf{g}_k)_i \notin [\mathbf{c}_i, (\mathbf{g}_j)_i - 1]$ for any $1 \leq k \leq q$.

Now, let \mathbf{e}_i be the vector of size n whose entries are zero, except its i -th entry, which equals one. It is not difficult to check that if \mathbf{c} is an ∞ -cover of \mathbf{G} , then also $\mathbf{c} + \mathbf{e}_i$ is an ∞ -cover of \mathbf{G} , which is a contradiction to the maximality of \mathbf{c} .

Let J be a vector in $[n]^q$ such that $J_{k_i} = i$ and its other entries are defined using any of the values i already added to J . Moreover, for all $i \in [n]$, we have $0 \neq \mathbf{c}_i \leq (\mathbf{g}_k)_i$. Since \mathbf{c} is an ∞ -cover, then J is well defined and $\mathbf{c} = \mathbf{G}_J$. We remark that J is not necessarily unique. \square

Remark 2.10. *In the proof of [7, Proposition 1.2], one can find similar arguments to those used in the proof of Lemma 2.9.*

Note that the converse of Lemma 2.9 is not necessarily true; that is, although always \mathbf{G}_J is an ∞ -cover, it is not necessarily maximal. For instance, if

$$I_{\mathbf{G}} = \langle xy^3, x^2y^2 \rangle \subseteq k[x, y],$$

then $\mathbf{G}_{(1,2)} = (1, 2)$ is not maximal because $\mathbf{G}_{(2,1)} = (2, 3) \geq (1, 2) = \mathbf{G}_{(1,2)}$.

Remark 2.11. *Moreover, if $J \neq K \in \Pi_{\mathbf{G}}$, then not necessarily $\mathbf{G}_J \neq \mathbf{G}_K$. That is, different J 's can define the same ∞ -cover of \mathbf{G} . For instance, if $I_{\mathbf{G}}$ is the edge ideal of the complete graph with three vertices K_3 with $\mathbf{g}_1 = (1, 1, 0)$, $\mathbf{g}_2 = (0, 1, 1)$, and $\mathbf{g}_3 = (1, 0, 1)$, then $\mathbf{G}_{(1,2,1)} = \mathbf{G}_{(2,2,1)} = (1, 1, \infty)$, which correspond to the minimal vertex cover $\{v_1, v_2\}$ of K_3 .*

Lemma 2.9 gives an insight into how maximal ∞ -covers of \mathbf{G} are related to the irreducible factors of the irreducible decomposition of $I_{\mathbf{G}}$ given in (2.2).

We are now ready to generalize the classical result, which gives an irredundant irreducible decomposition of the edge ideal of a graph in terms of its minimal vertex covers [9, p. 279]. Let $\mathcal{T}(\mathbf{G})$ be the set of maximal ∞ -covers of an antichain \mathbf{G} .

Theorem 2.12. *If \mathbf{G} is an antichain in \mathbb{N}^n , then*

$$\bigcap_{\mathbf{a} \in \mathcal{T}(\mathbf{G})} \mathfrak{m}^{\mathbf{a}} \quad (2.4)$$

is the irredundant irreducible decomposition of the antichain ideal $I_{\mathbf{G}}$.

Proof. Since $\mathfrak{m}^{\mathbf{a}} \subseteq \mathfrak{m}^{\mathbf{b}}$ if and only if $\mathbf{a} \geq \mathbf{b}$, we get the result by applying Lemmas 2.6 and 2.9 to the irreducible decomposition given in (2.2). \square

This result is similar to [5, Proposition 3], which establishes a bijection between the maximal standard monomial of S/I and the irredundant irreducible components of I . However, Theorem 2.12 is more direct and less mysterious in the way that it works. Even more important, it generalizes the result that describes the irredundant irreducible decomposition of an edge ideal I_G in terms of the minimal vertex covers of G .

The next example illustrates before constructions.

Example 2.13. *Let $\mathbf{G} = \{(2, 0, 2), (0, 2, 0), (1, 1, 3)\}$ be an antichain in \mathbb{N}^3 and*

$$I_{\mathbf{G}} = \langle x_1^2 x_3^2, x_2^2, x_1 x_2 x_3^3 \rangle$$

be its antichain ideal. Thus, the decomposition (2.1) has the next six factors

$$\langle x_1^2, x_2^2, x_1 \rangle, \langle x_1^2, x_2^2, x_2 \rangle, \langle x_1^2, x_2^2, x_3^3 \rangle, \langle x_3^2, x_2^2, x_1 \rangle, \langle x_3^2, x_2^2, x_2 \rangle, \langle x_3^2, x_2^2, x_3^3 \rangle,$$

which can be simplified to

$$\langle x_1, x_2^2 \rangle, \langle x_1^2, x_2 \rangle, \langle x_1^2, x_2^2, x_3^3 \rangle, \langle x_1, x_2^2, x_3^3 \rangle, \langle x_2, x_3^2 \rangle, \langle x_2^2, x_3^2 \rangle.$$

On the other hand, since $(2, 2, 3) \geq (1, 2, 2)$ and $(0, 2, 2) \geq (0, 1, 2)$, then $\langle x_1^2, x_2^2, x_3^3 \rangle \subseteq \langle x_1, x_2^2, x_3^2 \rangle$ and $\langle x_2^2, x_3^2 \rangle \subseteq \langle x_2, x_3^2 \rangle$. That is, $\langle x_1, x_2^2, x_3^2 \rangle$ and $\langle x_2, x_3^2 \rangle$ are redundant and, therefore,

$$\langle x_1, x_2^2 \rangle \cap \langle x_1^2, x_2 \rangle \cap \langle x_2^2, x_3^2 \rangle \cap \langle x_1^2, x_2^2, x_3^3 \rangle$$

is the irredundant irreducible decomposition of $I_{\mathbf{G}}$ because $(1, 2, \infty)$, $(2, 1, \infty)$, $(\infty, 2, 2)$, and $(2, 2, 3)$ are an antichain in \mathbb{N}_{∞}^3 . Moreover, it is not difficult to check that these last vectors are the maximal ∞ -covers of \mathbf{G} .

Since the concept of primary decomposition has played a significant role in commutative algebra, we finish this section by recalling how to recover the primary decomposition of a monomial ideal from its irreducible decomposition. We recall that a monomial ideal is primary if and only if it is generated by the powers and some monomials in a set of variables.

2.3. The primary decomposition of a monomial ideal

It is not difficult to recover the primary decomposition of a monomial ideal from its irreducible decomposition. To get the primary factors, we only need to intersect the irreducible factors with the same radical, which is very simple to check. The next example illustrates this procedure.

Example 2.14. Let I_G be the monomial ideal given in Example 2.13. Thus,

$$\langle x_1, x_2^2 \rangle \cap \langle x_1^2, x_2 \rangle \cap \langle x_2^2, x_3^2 \rangle \cap \langle x_1^2, x_2^2, x_3^3 \rangle$$

is its irredundant irreducible decomposition. Since $\langle x_1, x_2^2 \rangle$ and $\langle x_1^2, x_2 \rangle$ have the same radical ideal $\langle x_1, x_2 \rangle$ and $\langle x_1, x_2^2 \rangle \cap \langle x_1^2, x_2 \rangle = \langle x_1^2, x_1 x_2, x_2^2 \rangle$,

$$I_G = \langle x_1^2, x_1 x_2, x_2^2 \rangle \cap \langle x_2^2, x_3^2 \rangle \cap \langle x_1^2, x_2^2, x_3^3 \rangle$$

is its primary decomposition.

3. The signature of a monomial ideal

In this section, we will introduce the central concept of this paper, the signature of a monomial ideal. Roughly speaking, the signature of a monomial ideal is a matrix that encodes the way the minimal generators of the ideal are ordered for each variable separately. Although the signature is a simple concept, many of the algebraic and combinatorial properties of the monomial ideals with the same signature are surprisingly preserved.

After introducing the signature formally, we will show that given a signature σ , there exists a polyhedral cone P_σ such that its integral points correspond to the monomial ideals with signature σ . Even more important, we will give a description of the irredundant irreducible decomposition of any monomial with signature σ in terms of the irredundant irreducible decomposition of the antichain ideal I_σ of σ .

Before defining the signature, we need to introduce the incidence matrix of a monomial ideal, which is another way to represent a monomial ideal as a matrix. Here, we will use the matrix representations of a monomial ideal and its irredundant irreducible decomposition instead of its usual representations because it is more convenient to work with.

Definition 3.1. The incidence matrix of an antichain \mathbf{G} (or of the antichain ideal I_G) is the matrix whose columns are its vectors.

That is, the columns of B_G are indexed by the minimal generators of I_G , and their rows are indexed by the variables of the polynomial ring S .

Remark 3.2. A matrix is the incidence matrix of a monomial ideal whenever its entries are nonnegative integers and its columns form an antichain. Moreover, it is not difficult to check that there exists a bijection between the set of monomial ideals in S and the set of nonnegative integer matrices whose columns form an antichain.

Now, the content of a nonnegative integer vector $\mathbf{c} \in \mathbb{N}^n$, denoted by $\text{cont}(\mathbf{c})$, is the vector that contains its different entries in ascending order. Note that a vector and its content do not necessarily have the same dimension. When the content of a vector is equal to $(0, 1, 2, \dots, r)$ for some $r \in \mathbb{N}$, we say that it is tight. That is, a tight vector can be thought of as a vector without gaps.

Definition 3.3. A nonnegative integer matrix B is row-tight or simply tight whenever its rows are tight.

The following example illustrates previous concepts.

Example 3.4. Let $\mathbf{G} = \{(0, 5, 9), (2, 2, 1), (1, 2, 9)\}$ be an antichain in \mathbb{N}^3 . Then the incidence matrix of \mathbf{G} is given by

$$B_{\mathbf{G}} = \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{ccc} \mathbf{g}_1 & \mathbf{g}_2 & \mathbf{g}_3 \\ \left[\begin{array}{ccc} 0 & 2 & 1 \\ 5 & 2 & 2 \\ 9 & 1 & 9 \end{array} \right] \end{array}.$$

Besides, it is not difficult to check that $\text{cont}(\mathbf{r}_2) = (2, 5)$ and, therefore, $B_{\mathbf{G}}$ is not row-tight.

We are now almost ready to define the signature of a matrix. However, given a vector \mathbf{c} in \mathbb{N}^n with $\text{cont}(\mathbf{c}) = (m_0, \dots, m_r)$ its signature is the vector in \mathbb{N}^n given by

$$\sigma(\mathbf{c})_i = k, \text{ where } c_i = m_k, \text{ for some } k \in \{0, \dots, r\}.$$

Thus, by definition, the signature of a vector is a tight vector.

Definition 3.5. The signature of a monomial ideal $I_{\mathbf{G}}$, denoted by $\sigma(I_{\mathbf{G}})$, is the matrix obtained from its incidence matrix $B_{\mathbf{G}}$ by replacing its rows with their corresponding signatures.

Similarly, it can be defined the signatures of any nonnegative integer matrix. For instance, if B is the matrix given in Example 3.4, then its signature is equal to

$$\sigma(B) = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

As we mentioned before, the signature is a very simple concept; even calculating it is straightforward. Actually, the computational complexity of calculating it is linear. However, as we will show in the following, many of the algebraic and combinatorial properties of the monomial ideals with the same signature are surprisingly preserved.

Remark 3.6. Getting a monomial ideal with the same signature as a given one is very simple. For instance, it is not difficult to check that the ideal obtained from $I_{\mathbf{G}}$ by replacing each variable with a power of it has the same signature as $I_{\mathbf{G}}$.

Again, in general, a matrix is not necessarily tight, but its signature always is. Clearly, if B is tight, then $\sigma(B) = B$. The idea behind the signature is that looking at each variable of the minimal generators only matters by the relative position of its powers.

The next result shows that the signature of a monomial ideal is well-defined. That is, it preserves the fact that the columns of a matrix form an antichain.

Lemma 3.7. The columns of a nonnegative integer matrix B form an antichain if and only if the columns of its signature σ are also an antichain.

Proof. It follows from the fact that $\mathbf{a} \leq \mathbf{b}$ if and only if $\mathbf{a}_i \leq \mathbf{b}_i$ for all $1 \leq i \leq n$ and $B_{i,j} \leq B_{i,k}$ if and only if $\sigma_{i,j} \leq \sigma_{i,k}$ for all $1 \leq i \leq n$ and $1 \leq j \neq k \leq q$. \square

From here on, when we say matrix, we mean a nonnegative integer matrix whose columns form an antichain unless otherwise stated. In a similar way, when we say a signature, we mean a tight matrix. Thus, Lemma 3.7 implies that if σ is a signature, then the columns of any matrix with signature σ form an antichain.

In what follows, we will define a polyhedral cone P_σ whose integral points correspond to the monomial ideals with signature σ .

3.1. The polyhedral cone of a signature

Monomial ideals with the same signature can easily be characterized as those whose incidence matrix are the integral points of a polyhedral cone. More precisely, given a signature σ , let

$$P_\sigma = \{B \in \text{Mat}_{n,q}(\mathbb{R}_{\geq 0}) : B_{i,j} + \sigma_{i,j} \leq B_{i,k} + \sigma_{i,k} \text{ for all } 1 \leq i \leq n \text{ and } 1 \leq j, k \leq q \text{ such that } \sigma_{i,j} \leq \sigma_{i,k}\}.$$

Many of the inequalities used here are clearly redundant. The polyhedral cone P_σ can be defined alternatively as (without redundant inequalities): $B_{i,j} \geq 0$ for all $1 \leq i \leq n$ and $1 \leq j \leq q$, $B_{i,j} = B_{i,k}$ if and only if $\sigma_{i,j} = \sigma_{i,k}$ and $B_{i,j} \leq B_{i,k} + 1$ for all $1 \leq i \leq n$ and $1 \leq j \neq k \leq q$ such that $\sigma_{i,k} = \sigma_{i,j} + 1$.

The next result establishes that there is a bijection between integral points of the polyhedral cone P_σ and monomial ideals with signature σ .

Proposition 3.8. *If σ is a signature and B is a nonnegative integral matrix, then $\sigma(B) = \sigma$ if and only if B is an integral point of the polyhedral cone P_σ .*

Proof. It is followed directly from the definition of the signature and Lemma 3.7. \square

Directly for the definition of P_σ , it is not difficult to get a formula for its dimension in terms of the content of the rows of σ .

Proposition 3.9. *If σ is a signature and $(\mathbf{r}_1, \dots, \mathbf{r}_q)$ are its rows, then*

$$\dim(P_\sigma) = \sum_{i=1}^n |\text{cont}(\mathbf{r}_i)|.$$

The next example illustrates previous concepts.

Example 3.10. *If $\sigma = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & 2 \\ 2 & 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 4 & 3 \end{bmatrix}$ is a signature, then*

$$P_\sigma = \{[b_{i,j}] \in \text{Mat}_{n,q}(\mathbb{R}_{\geq 0}) : \begin{aligned} 0 \leq b_{1,4} = b_{1,5} \leq b_{1,1} + 1 = b_{1,3} + 1 \leq b_{1,2} + 2, \\ 0 \leq b_{2,2} = b_{2,3} \leq b_{2,4} + 1 \leq b_{2,1} + 2 = b_{2,5} + 2, \\ 0 \leq b_{3,2} = b_{3,3} = b_{3,4} \leq b_{3,5} + 1 \leq b_{3,1} + 2, \\ 0 \leq b_{4,1} \leq b_{4,2} + 1 \leq b_{4,3} + 2 \leq b_{4,5} + 3 \leq b_{4,4} + 4 \end{aligned}\}$$

is the polyhedral cone associated to σ and its dimension is equal to $14 = 3 + 3 + 3 + 5$.

In what follows, we will show that all the monomials ideal with the same signatures have essentially the same irredundant irreducible decomposition.

3.2. The irreducible decomposition of a monomial ideal with a given signature

In this subsection, we give a description of the irredundant irreducible decomposition of the monomial ideals with signature σ in terms of the irredundant irreducible decomposition of the antichain ideal I_σ of σ .

We need some definitions before establishing our main result. By Lemma 2.9 we have that any maximal ∞ -cover of I_σ is equal to σ_J for some $J \in \Pi_{\mathbf{G}_\sigma}$, where $\mathbf{G}_\sigma = (\mathbf{g}_1, \dots, \mathbf{g}_q)$ are the columns of σ . Thus, there exists $\mathcal{J}_\sigma \subseteq \Pi_{\mathbf{G}_\sigma}$ such that

$$\bigcap_{J \in \mathcal{J}_\sigma} \mathfrak{m}^{\sigma_J}$$

is the irredundant irreducible decomposition of I_σ . In other words, $\mathcal{T}(\mathbf{G}_\sigma) = \{(\mathbf{G}_\sigma)_J : J \in \mathcal{J}_\sigma\}$. We recall again that \mathcal{J}_σ is not necessarily unique. Finally, given an integer nonnegative matrix B , let $\gcd(B) \in \mathbb{N}^n$ given by

$$\mathbf{x}^{\gcd(B)} = \gcd(\{\mathbf{x}^{\mathbf{b}_i} : 1 \leq i \leq q\}),$$

where $\mathbf{b}_1, \dots, \mathbf{b}_q$ are the columns of B . That is, $\gcd(B)_i = \min\{B_{i,j} : 1 \leq j \leq q\}$.

Theorem 3.11. *If σ is a signature, $I_{\mathbf{G}}$ is a monomial ideal with signature σ and $\mathbf{m} = \gcd(B_{\mathbf{G}})$, then*

$$\left(\bigcap_{\{i: \mathbf{m}_i > 0\}} \langle x_i^{\mathbf{m}_i} \rangle \right) \cap \left(\bigcap_{J \in \mathcal{J}_\sigma} \mathfrak{m}^{\mathbf{G}_J} \right)$$

is the irredundant irreducible decomposition of $I_{\mathbf{G}}$.

Proof. By Theorem 2.12, we only need to prove that

$$\mathcal{T}(\mathbf{G}) = \{\mathbf{m}_i \mathbf{e}_i : \mathbf{m}_i > 0\} \cup \{\mathbf{G}_J : J \in \mathcal{J}_\sigma\}.$$

To begin, it is clear that $\mathbf{m}_i \mathbf{e}_i$ is a maximal ∞ -cover of \mathbf{G} if and only if $\mathbf{m}_i > 0$. On the other hand, \mathbf{G}_J is always an ∞ -cover for any $J \in \Pi_{\mathbf{G}}$.

The rest follows by using similar arguments to those used in the proof of Lemma 3.7. \square

Now, it is not difficult to check that

$$\mathfrak{m}^{\mathbf{G}_J} = \langle x_{J_i}^{(B_{\mathbf{G}})_{J_i, i}} : 1 \leq i \leq q \rangle = \langle x_k^{(B_{\mathbf{G}})_{k, i_k}} : i \in \text{cont}(J) \rangle,$$

where i_k is such that $(B_{\mathbf{G}})_{k, i_k} = \min\{(B_{\mathbf{G}})_{J_i, i} : J_i = k\}$. Thus, $\mathfrak{m}^{\mathbf{G}_J}$ can be expressed as the irreducible ideal whose exponent on each variable is some entry of the matrix $B_{\mathbf{G}}$.

The next example illustrates the previous process for obtaining the irredundant irreducible decomposition of a monomial ideal with a given signature.

Example 3.12. Let $\sigma = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & 2 \\ 2 & 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 4 & 3 \end{bmatrix}$ be the signature given in Example 3.10 and

$$I_\sigma = \langle x_1 x_2^2 x_3^2, x_1^2 x_4, x_1 x_4^2, x_2 x_4^4, x_2^2 x_3 x_4^3 \rangle$$

be its antichain ideal. Using Macaulay2 [10], we obtain that

$$\langle x_1, x_2 \rangle \cap \langle x_1, x_4^3 \rangle \cap \langle x_2^2, x_4 \rangle \cap \langle x_3^2, x_4 \rangle \cap \langle x_1, x_2^2, x_4^4 \rangle \cap \langle x_1, x_3, x_4^4 \rangle \cap \langle x_1^2, x_2^2, x_4^2 \rangle \cap \langle x_1^2, x_3^2, x_4^2 \rangle$$

is its irredundant irreducible decomposition. Moreover, it is not difficult to check that

$$\begin{aligned} m^{\sigma(1,1,1,2,2)} &= \langle x_1, x_2 \rangle, & m^{\sigma(1,1,1,4,4)} &= \langle x_1, x_4^3 \rangle, & m^{\sigma(1,1,1,4,2)} &= \langle x_1, x_2^2, x_4^4 \rangle, \\ m^{\sigma(2,4,4,4,4)} &= \langle x_2^2, x_4 \rangle, & m^{\sigma(3,4,4,4,4)} &= \langle x_3^2, x_4 \rangle, & m^{\sigma(1,1,1,4,3)} &= \langle x_1, x_3, x_4^4 \rangle, \\ m^{\sigma(2,1,4,4,4)} &= \langle x_1^2, x_2^2, x_4^2 \rangle, & m^{\sigma(3,1,4,4,4)} &= \langle x_1^2, x_3^2, x_4^2 \rangle. \end{aligned}$$

That is, $\mathcal{J}_\sigma = \{(1, 1, 1, 2, 2), (1, 1, 1, 4, 4), (1, 1, 1, 4, 2), (1, 1, 1, 4, 3), (2, 4, 4, 4, 4), (3, 4, 4, 4, 4), (2, 1, 4, 4, 4), (3, 1, 4, 4, 4)\}$ is a subset of $\Pi_{\mathbf{G}_\sigma}$ such that $\bigcap_{J \in \mathcal{J}_\sigma} m^{\sigma_J}$ is the irredundant irreducible decomposition of I_σ .

Now, if $B = \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} & 0 & 0 \\ b_{2,1} & 0 & 0 & b_{2,4} & b_{2,5} \\ b_{3,1} & 0 & 0 & 0 & b_{3,5} \\ 0 & b_{4,2} & b_{4,3} & b_{4,4} & b_{4,5} \end{bmatrix} \in P_\sigma$ and \mathbf{B} is the set of columns of B , then

$$\begin{aligned} m^{\mathbf{B}(1,1,1,2,2)} &= \langle x_1^{b_{1,1}}, x_2^{b_{2,4}} \rangle, & m^{\mathbf{B}(1,1,1,4,4)} &= \langle x_1^{b_{1,1}}, x_4^{b_{4,5}} \rangle, & m^{\mathbf{B}(1,1,1,4,2)} &= \langle x_1^{b_{1,1}}, x_2^{b_{2,5}}, x_4^{b_{4,4}} \rangle, \\ m^{\mathbf{B}(3,4,4,4,4)} &= \langle x_3^{b_{3,1}}, x_4^{b_{4,2}} \rangle, & m^{\mathbf{B}(2,4,4,4,4)} &= \langle x_2^{b_{2,1}}, x_4^{b_{4,2}} \rangle, & m^{\mathbf{B}(1,1,1,4,3)} &= \langle x_1^{b_{1,1}}, x_3^{b_{3,5}}, x_4^{b_{4,4}} \rangle, \\ m^{\mathbf{B}(2,1,4,4,4)} &= \langle x_1^{b_{1,2}}, x_2^{b_{2,1}}, x_4^{b_{4,3}} \rangle, & m^{\mathbf{B}(3,1,4,4,4)} &= \langle x_1^{b_{1,2}}, x_3^{b_{3,1}}, x_4^{b_{4,3}} \rangle. \end{aligned}$$

Thus, by Theorem 3.11,

$$\begin{aligned} &\langle x_1^{b_{1,1}}, x_2^{b_{2,4}} \rangle \cap \langle x_1^{b_{1,1}}, x_4^{b_{4,5}} \rangle \cap \langle x_1^{b_{1,1}}, x_2^{b_{2,5}}, x_4^{b_{4,4}} \rangle \cap \langle x_1^{b_{1,1}}, x_3^{b_{3,5}}, x_4^{b_{4,4}} \rangle \cap \\ &\langle x_2^{b_{2,1}}, x_4^{b_{4,2}} \rangle \cap \langle x_3^{b_{3,1}}, x_4^{b_{4,2}} \rangle \cap \langle x_1^{b_{1,2}}, x_2^{b_{2,1}}, x_4^{b_{4,3}} \rangle \cap \langle x_1^{b_{1,2}}, x_3^{b_{3,1}}, x_4^{b_{4,3}} \rangle \end{aligned}$$

is the irredundant irreducible decomposition of the antichain ideal

$$I_B = \langle x_1^{b_{1,1}} x_2^{b_{2,1}} x_3^{b_{3,1}}, x_1^{b_{1,2}} x_4^{b_{4,2}}, x_1^{b_{1,3}} x_4^{b_{4,3}}, x_2^{b_{2,4}} x_4^{b_{4,4}}, x_2^{b_{2,5}} x_3^{b_{3,5}} x_4^{b_{4,5}} \rangle,$$

where $b_{1,1} = b_{1,3}$ and $b_{2,1} = b_{2,5}$, $b_{1,1}, b_{2,1}, b_{3,1}, b_{1,2}, b_{4,2}, b_{4,3}, b_{2,4}, b_{4,4}, b_{3,5}, b_{4,5} \in \mathbb{N}_+$.

In the most general case, if $B = \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} & b_{1,5} \\ b_{2,1} & b_{2,2} & b_{2,3} & b_{2,4} & b_{2,5} \\ b_{3,1} & b_{3,2} & b_{3,3} & b_{3,4} & b_{3,5} \\ b_{4,1} & b_{4,2} & b_{4,3} & b_{4,4} & b_{4,5} \end{bmatrix} \in P_\sigma$, we need to add the factors $\langle x_i^{b_{i,j_i}} \rangle$

for which $b_{i,j_i} > 0$ to get the irredundant irreducible decomposition of the antichain ideal

$$\begin{aligned} I_B = \langle &mx_1^{b_{1,1}-b_{1,4}} x_2^{b_{2,1}-b_{2,2}} x_3^{b_{3,1}-b_{3,2}}, mx_1^{b_{1,2}-b_{1,4}} x_4^{b_{4,2}-b_{4,1}}, mx_1^{b_{1,3}-b_{1,4}} x_4^{b_{4,3}-b_{4,1}}, \\ &mx_2^{b_{2,4}-b_{2,2}} x_4^{b_{4,4}-b_{4,1}}, mx_2^{b_{2,5}-b_{2,2}} x_3^{b_{3,5}-b_{3,2}} x_4^{b_{4,5}-b_{4,1}} \rangle, \end{aligned}$$

where $m = \gcd(\mathbf{G}) = x_1^{b_{1,4}} x_2^{b_{2,2}} x_3^{b_{3,2}} x_4^{b_{4,1}}$. Let us recall that $b_{1,4} = b_{1,5}$, $b_{2,2} = b_{2,3}$ and $b_{3,2} = b_{3,3} = b_{3,4}$.

Thus, it is very simple to calculate the irredundant irreducible decomposition of an antichain ideal from the irredundant irreducible decomposition of its signature.

To simplify the notation, we will encode the irredundant irreducible decomposition on its incidence matrix.

Definition 3.13. Given an antichain \mathbf{G} , its irredundant irreducible decomposition matrix (or simply decomposition matrix) is the matrix whose columns are the exponents of the irreducible factors of the irredundant irreducible decomposition $\bigcap_{i=1}^p \mathfrak{m}^{\mathbf{d}_i}$ of $I_{\mathbf{G}}$.

We recall that the entries of $D_{\mathbf{G}}$ are in \mathbb{N}_{∞}^n . In a similar way to the definition of the signature of a monomial ideal, the signature of a matrix D with entries in \mathbb{N}_{∞}^n is defined as $\sigma(D) = \sigma(D_0)_{\infty}$.

Again, similar to the Lemma 3.7, we get the following result:

Lemma 3.14. If D is a matrix with entries in \mathbb{N}_{∞}^n , then its columns are an antichain in \mathbb{N}_{∞}^n if and only if the columns of its signature $\sigma(D)$ are also an antichain in \mathbb{N}_{∞}^n .

Proof. It follows by using similar arguments to those used in the proof of Lemma 2.6. \square

We end this section with an example that illustrates previous concepts.

Example 3.15. Following with Example 3.12, we have that

$$D_{\sigma} = \begin{array}{c} \mathbf{d}_1 \quad \mathbf{d}_2 \quad \mathbf{d}_3 \quad \mathbf{d}_4 \quad \mathbf{d}_5 \quad \mathbf{d}_6 \quad \mathbf{d}_7 \quad \mathbf{d}_8 \\ \begin{array}{l} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \left[\begin{array}{cccccccc} 1 & 1 & 1 & 1 & 2 & 2 & \infty & \infty \\ 1 & 2 & \infty & \infty & 2 & \infty & 2 & \infty \\ \infty & \infty & 1 & \infty & \infty & 2 & \infty & 2 \\ \infty & 4 & 4 & 3 & 2 & 2 & 1 & 1 \end{array} \right]. \end{array}$$

Moreover, if $B = [b_{i,j}] \in P_{\sigma}$, then

$$D_B = \begin{array}{c} \mathbf{d}_1 \quad \mathbf{d}_2 \quad \mathbf{d}_3 \quad \mathbf{d}_4 \quad \mathbf{d}_5 \quad \mathbf{d}_6 \quad \mathbf{d}_7 \quad \mathbf{d}_8 \\ \begin{array}{l} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \left[\begin{array}{cccccccc} b_{1,1} & b_{1,1} & b_{1,1} & b_{1,1} & b_{1,2} & b_{1,2} & \infty & \infty \\ b_{2,4} & b_{2,5} & \infty & \infty & b_{2,1} & \infty & b_{2,1} & \infty \\ \infty & \infty & b_{3,5} & \infty & \infty & b_{3,1} & \infty & b_{3,1} \\ \infty & b_{4,4} & b_{4,4} & b_{4,5} & b_{4,3} & b_{4,3} & b_{4,2} & b_{4,2} \end{array} \right] \end{array}$$

whenever $\gcd(B) = \mathbf{0}$, and it is not difficult to check that $\sigma(D_B) = D_{\sigma}$. Note that the columns of D_{σ} are antichains in \mathbb{N}_{∞}^n ; however, the columns of $(D_{\sigma})_0$ are not necessarily antichains in \mathbb{N}^n . For instance, $(\mathbf{d}_1)_0 = (1, 1, 0, 0) \leq (2, 2, 0, 2) = (\mathbf{d}_5)_0$ and $\mathbf{d}_1 = (1, 1, \infty, \infty) \not\leq (2, 2, \infty, 2) = \mathbf{d}_5$.

On the other hand, if $\gcd(B) \geq \mathbf{1}$, then $\gcd(B) = (b_{1,4}, b_{2,2}, b_{3,2}, b_{4,1})$ and

$$D_B = \left[\begin{array}{cccccccccccc} b_{1,4} & \infty & \infty & \infty & b_{1,1} & b_{1,1} & b_{1,1} & b_{1,1} & b_{1,2} & b_{1,2} & \infty & \infty \\ \infty & b_{2,2} & \infty & \infty & b_{2,4} & b_{2,5} & \infty & \infty & b_{2,1} & \infty & b_{2,1} & \infty \\ \infty & \infty & b_{3,2} & \infty & \infty & \infty & b_{3,5} & \infty & \infty & b_{3,1} & \infty & b_{3,1} \\ \infty & \infty & \infty & b_{4,1} & \infty & b_{4,4} & b_{4,4} & b_{4,5} & b_{4,3} & b_{4,3} & b_{4,2} & b_{4,2} \end{array} \right].$$

4. Alexander duality

Alexander duality has become an important tool in the study of monomial ideals. In this section, after recalling Alexander duality in monomial ideals as in [7], we extend it to the space of polyhedral

cones associated with signatures and use it to prove some properties of signatures. The way it is presented here is slightly different from how it is presented in [7], more consistent with our purposes. Roughly speaking, from our point of view, Alexander dual of the antichain I_G is the monomial ideal whose incidence matrix is obtained from D_G by reversing the order of the entries in each of its rows.

We will try to convince the reader that an essential part of Alexander duality relies on the order in \mathbb{N}^n as well as the reversing and the minimal-maximal duality. There exist several ways to reverse the order of the entries in each row of a matrix; one of the most simple is the one used in [7], which subtracts all nonzero entries from a given constant. In what follows, we will present a slightly more general adaptation of this construction, where additionally we interchange the zero entries with infinity. In [7], a big enough number is used instead of using infinity as an entry in the factors of the irredundant irreducible decomposition.

Given a matrix D of size $n \times p$ with entries in \mathbb{N}_∞ and $\mathbf{a} \in \mathbb{N}^n$, its \mathbf{a} -reversing matrix is given by

$$\text{rev}_{\mathbf{a}}(D)_{i,j} = \begin{cases} \infty & \text{if } D_{i,j} = 0, \\ 0 & \text{if } D_{i,j} = \infty, \\ \mathbf{a}_i - D_{i,j} + 1 & \text{otherwise.} \end{cases}$$

In [7], the \mathbf{a} -reversing operation is used as a column operation on D_G ; however, in our context, it is more clear to see as an operation over its rows. It is not difficult to check that the entries of $\text{rev}_{\mathbf{a}}(D)$ are nonnegative if and only if $\mathbf{a} \geq \text{lcm}(D)$, where $\text{lcm}(D)_i = \max\{D_{i,j} : \text{for all } 1 \leq j \leq q \text{ such that } D_{i,j} \neq \infty\}$.

Definition 4.1. *The Alexander dual $I_G^{\vee \mathbf{a}}$ with respect to $\mathbf{a} \geq \text{lcm}(B_G)$ of the antichain ideal I_G is the monomial ideal whose incidence matrix is equal to the \mathbf{a} -reversing, denoted by $B_G^{\vee \mathbf{a}}$, of the irredundant irreducible decomposition matrix D_G of I_G .*

By Theorem 3.11, it is not difficult to check that $\text{lcm}(B_G) \geq \text{lcm}(D_G)$ and, therefore, the Alexander dual $I_G^{\vee \mathbf{a}}$ is indeed a monomial ideal. Each monomial ideal has many of Alexander duals. Indeed, for any $\mathbf{a} \in \mathbb{N}^n$ such that $\mathbf{a} \geq \text{lcm}(B_G)$ we have one of them. On the other hand, it is not difficult to check that all of Alexander duals of a given monomial ideal have the same signature. Moreover, all the Alexander duals of the Alexander duals $(I_G^{\vee \mathbf{a}})^{\vee \mathbf{b}}$ has the same signature as I_G . That is, Alexander duality sent integral points in P_σ to integral points in $P_{\sigma^{\vee \mathbf{a}}}$ for $\mathbf{a} = \text{lcm}(B_G)$. It is for this reason that it is natural to consider Alexander duality on the polyhedral cones P_σ instead of a single monomial ideal.

The \mathbf{a} -reversing operation preserves the fact that the columns of the matrix are an antichain in \mathbb{N}_∞^n . Thus, since the columns of D_G are an antichain in \mathbb{N}_∞^n , then it is not difficult to check that the columns of $B_G^{\vee \mathbf{a}}$ also are an antichain in \mathbb{N}^n . That is, the columns of $B_G^{\vee \mathbf{a}}$ correspond to the minimal generators of $I_G^{\vee \mathbf{a}}$. For instance, following Example 3.15, we get that

$$D_\sigma = \begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 2 & \infty & \infty \\ 1 & 2 & \infty & \infty & 2 & \infty & 2 & \infty \\ \infty & \infty & 1 & \infty & \infty & 2 & \infty & 2 \\ \infty & 4 & 4 & 3 & 2 & 2 & 1 & 1 \end{bmatrix} \implies B_\sigma^{\vee \mathbf{a}} = \begin{bmatrix} 2 & 2 & 2 & 2 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 & 3 & 3 & 4 & 4 \end{bmatrix}$$

for $\mathbf{a} = \text{lcm}(B_G) = (2, 2, 2, 4)$. To make the notation simpler, when there is no confusion, we write B_G^\vee instead $B_G^{\vee \mathbf{a}}$ for $\mathbf{a} = \text{lcm}(B_G)$.

Moreover, the \mathbf{a} -reversing operation interchanges the fact that the columns of $D_{\mathbf{G}}$ are maximal to the fact that the columns of $B_{\mathbf{G}}^{\vee_{\mathbf{a}}}$ are minimal with respect to the order \leq in \mathbb{N}^n . In a similar way, it also interchanges the fact that the columns of $B_{\mathbf{G}}$ are minimal to the fact that the columns of $\text{rev}_{\mathbf{a}}(B_{\mathbf{G}})$ are maximal with respect to the order \leq in \mathbb{N}_{∞}^n .

Thus, it is very simple to get that the functor defined by \mathbf{a} -reversing operation is contravariant with respect to taking the irredundant irreducible decomposition.

Theorem 4.2 (Corollary 2.14 [7]). *Let \mathbf{G} be an antichain in \mathbb{N}^n and $\mathbf{a} \geq \text{lcm}(B_{\mathbf{G}})$, then $\text{rev}_{\mathbf{a}}(B_{\mathbf{G}})$ is the irredundant irreducible decomposition matrix of $B_{\mathbf{G}}^{\vee_{\mathbf{a}}}$ (see the commutative diagram given in Figure 1).*

$$\begin{array}{ccc}
 B_{\mathbf{G}} & \xrightarrow{\infty\text{-cover}} & D_{\mathbf{G}} \\
 \text{rev}_{\mathbf{a}} \uparrow & \searrow \vee_{\mathbf{a}} & \uparrow \text{rev}_{\mathbf{a}} \\
 \text{rev}_{\mathbf{a}}(B_{\mathbf{G}}) & \xleftarrow{\infty\text{-cover}} & B_{\mathbf{G}}^{\vee_{\mathbf{a}}}
 \end{array}$$

Figure 1. The \mathbf{a} -reversing $\text{rev}_{\mathbf{a}}(B_{\mathbf{G}})$ of $B_{\mathbf{G}}$ is the irredundant irreducible decomposition matrix of its Alexander dual $B_{\mathbf{G}}^{\vee_{\mathbf{a}}}$.

Proof. Let \mathbf{c} be a column of $D_{\mathbf{G}}$, that is, \mathbf{c} is a maximal ∞ -cover of \mathbf{G} . A vector \mathbf{c} is ∞ -cover of \mathbf{G} whenever for any $\mathbf{g} \in \mathbf{G}$ there exists $1 \leq j \leq n$ such that $\mathbf{c}_j \leq \mathbf{g}_j$. Moreover, the fact that \mathbf{c} being maximal implies that for any $1 \leq j \leq n$, such that $\mathbf{c}_j \neq \infty$ there exists $\mathbf{g} \in \mathbf{G}$ such that $\mathbf{c}_j \leq \mathbf{g}_j$ (moreover, it can be stated that the \mathbf{g} satisfies that $\mathbf{c}_i \not\leq \mathbf{g}_i$ for any $i \neq j$).

Since $\text{rev}_{\mathbf{a}}(\mathbf{g})_j \leq \text{rev}_{\mathbf{a}}(\mathbf{c})_j$, it follows that $\text{rev}_{\mathbf{a}}(\mathbf{g})$ is an ∞ -cover of the columns of $B_{\mathbf{G}}^{\vee_{\mathbf{a}}} = \text{rev}_{\mathbf{a}}(D_{\mathbf{G}})$. Additionally, it is maximal because $\text{rev}_{\mathbf{a}}$ interchanges minimality and maximality (see Figure 2(a)). Finally, the result follows by Theorem 2.12. \square

It is not difficult to check that Theorem 4.2 can be easily generalized for any reversing function.

Definition 4.3. A function $f : A \rightarrow \overline{\mathbb{N}}$ for some $A \subseteq \overline{\mathbb{N}}$ with the zero and infinity, is called reversing whenever $\text{rev}(0) = \infty$, $\text{rev}(\infty) = 0$, and

$$\text{rev}(a) \leq \text{rev}(b) \text{ for all } a, b \in A \text{ such that } a \geq b.$$

Given a matrix D with entries in \mathbb{N} and rows \mathbf{r}_i for $1 \leq i \leq n$, let $\text{cont}(D) = \bigcup_{i=1}^n \text{cont}(\mathbf{r}_i)$.

Corollary 4.4. *Let \mathbf{G} be an antichain in \mathbb{N}^n and rev be a reversing function for $\text{cont}(B_{\mathbf{G}}) \cup \text{cont}(D_{\mathbf{G}})$, then $\text{rev}(B_{\mathbf{G}})$ is the irredundant irreducible decomposition matrix of $\text{rev}(D_{\mathbf{G}})$.*

In order to simplify the notation, let

$$\tau := D_{\sigma}, \sigma^{\vee} := \text{rev}_{\text{lcm}(\sigma)}(\tau) \text{ and } \tau^{\vee} := (\sigma^{\vee})^* := \text{rev}_{\text{lcm}(\sigma)}(\sigma),$$

see Figure 2(b).

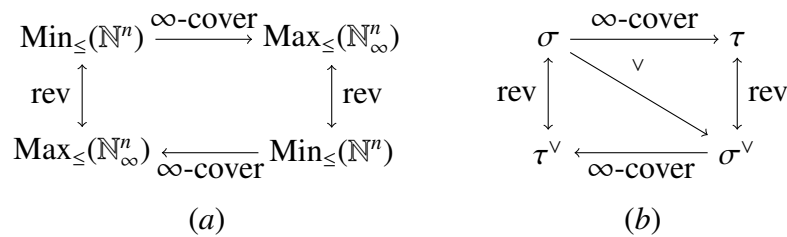


Figure 2. (a) Taking maximal ∞ -covers (irreducible irredundant decomposition) and reversing the order in $\overline{\mathbb{N}}$ interchanges minimality and maximality. Here, rev is any reversing function. (b) A signature σ , its decomposition matrix, its Alexander dual, and the decomposition matrix of the last one. Here, the reversing function is given by $\text{rev}_{\text{lcm}(\sigma)}$.

In what follows, we will show that the content of the rows of the matrices σ and τ and their duals σ^{\vee} and τ^{\vee} are equal when σ is the signature of a monomial ideal without the power of a variable as a generator.

Theorem 4.5. *Let σ be the signature of a monomial ideal without the power of a variable as a generator. If \mathbf{s}_i , \mathbf{t}_i , \mathbf{u}_i , and \mathbf{v}_i are the i -rows of σ , τ_0 , σ^{\vee} , and τ_0^{\vee} , respectively, then*

$$\text{cont}(\mathbf{s}_i) = \text{cont}(\mathbf{t}_i) = \text{cont}(\mathbf{u}_i) = \text{cont}(\mathbf{v}_i) \text{ for all } 1 \leq i \leq n.$$

Proof. For any $\mathbf{r} \in \overline{\mathbb{N}}$, let $\text{cont}_+(\mathbf{r}) = \text{cont}(\mathbf{r}) \cap \mathbb{N}_+$ be its positive content and $\mathbf{a} = \text{lcm}(\sigma)$. To start, by Theorem 4.2 we have that τ^{\vee} is the decomposition matrix of σ^{\vee} ; see Figure 2(b). Thus, by Theorem 3.11 we have that $\text{cont}_+(\mathbf{t}_i) \subseteq \text{cont}_+(\mathbf{s}_i)$ and $\text{cont}_+(\mathbf{v}_i) \subseteq \text{cont}_+(\mathbf{u}_i)$.

Since $\text{cont}(\mathbf{s}_i) = \{0, 1, \dots, \mathbf{a}_i\}$ and $\mathbf{v}_i = \text{rev}_{\mathbf{a}_i}(\mathbf{s}_i)_0$, then $\text{cont}(\mathbf{v}_i) = \text{cont}(\mathbf{s}_i) = \{0, 1, \dots, \mathbf{a}_i\}$. Moreover, since $\mathbf{u}_i = \text{rev}_{\mathbf{a}_i}(\mathbf{t}_i)_0$, then

$$\text{cont}(\mathbf{t}_i)_+ \subseteq \text{cont}(\mathbf{s}_i)_+ = \text{cont}_+(\mathbf{v}_i) \subseteq \text{cont}_+(\mathbf{u}_i) = \text{cont}_+(\text{rev}_{\mathbf{a}_i}(\mathbf{t}_i)_0) \subseteq \text{cont}(\mathbf{s}_i)_+$$

and, therefore, $\text{cont}_+(\mathbf{s}_i) = \text{cont}_+(\mathbf{t}_i) = \text{cont}_+(\mathbf{u}_i) = \text{cont}_+(\mathbf{v}_i) = \{1, \dots, \mathbf{a}_i\}$ for all $1 \leq i \leq n$. Lastly, if $0 \notin \text{cont}(\mathbf{t}_i)$, then $0 \notin \text{cont}(\mathbf{u}_i)$. Applying Theorem 3.11 to $I_{\sigma^{\vee}}$, we get that τ_0^{\vee} has a column with only one nonzero entry, and therefore I_{σ} has a generator which is the power of a variable; a contradiction. \square

Remark 4.6. *Using similar arguments to those used in the proof of Theorem 4.5, it can be proved it is true whenever one of the matrices σ , τ_0 , σ^{\vee} , and τ_0^{\vee} is a signature.*

Example 4.7. *Let σ be a nonnegative matrix, τ be its decomposition matrix, σ^{\vee} the Alexander dual of σ and τ^{\vee} be its decomposition matrix as shown in Figure 3. Besides, let \mathbf{s}_i , \mathbf{t}_i , \mathbf{u}_i , and \mathbf{v}_i be the i -rows of σ , τ_0 , σ^{\vee} , and τ_0^{\vee} , respectively.*

$$\begin{array}{cccc}
\begin{array}{c} \sigma \\ \parallel \\ \begin{bmatrix} 3 & 0 & 2 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} \end{array} & \xrightarrow{\infty\text{-cover}} & \begin{array}{c} \tau \\ \parallel \\ \begin{bmatrix} 1 & 2 & 3 & 3 \\ 2 & 2 & 1 & 2 \\ \infty & 2 & 2 & 1 \end{bmatrix} \end{array} & & \begin{array}{c} \tau' \\ \parallel \\ \begin{bmatrix} 1 & 2 & \infty & \infty \\ \infty & \infty & 1 & \infty \\ \infty & 2 & 2 & 1 \end{bmatrix} \end{array} & \xleftarrow{\infty\text{-cover}} & \begin{array}{c} \sigma' \\ \parallel \\ \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} \end{array} \\
\text{rev}_a \uparrow & & \text{rev}_a \downarrow & & \text{rev}_a \downarrow & & \text{rev}_a \uparrow \\
\begin{array}{c} \begin{bmatrix} 1 & \infty & 2 & 3 \\ \infty & 1 & 2 & \infty \\ \infty & \infty & 2 & 1 \end{bmatrix} & \xleftarrow{\infty\text{-cover}} & \begin{array}{c} \begin{bmatrix} 3 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix} & \xrightarrow{\text{signature}} & \begin{array}{c} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \end{bmatrix} & \xrightarrow{\infty\text{-cover}} & \begin{array}{c} \begin{bmatrix} 1 & 2 \\ 1 & \infty \\ 2 & 1 \end{bmatrix} \end{array} \\
\parallel & & \parallel & & \parallel & & \parallel \\
\tau^\vee & & \sigma^\vee & & (\sigma^\vee)' & & (\tau^\vee)'
\end{array}
\end{array}$$

Figure 3. A signature σ , its decomposition matrix τ , the Alexander dual σ^\vee of σ , and the decomposition matrix τ^\vee of τ .

Since σ is a signature and σ^\vee is not a signature, then it cannot be removed the hypothesis that the monomial ideal does not have minimal generators that are the power of a variable in the Theorem 4.5. Moreover, $0 \in \text{cont}(\mathbf{v}_1) \not\subseteq \text{cont}(\mathbf{u}_1)$, proving that not necessarily the content of a row of a nonnegative integer matrix is contained in the corresponding row of its decomposition matrix.

Since x_i^a is a minimal generator of an antichain ideal I_G if and only if $a = \max(\mathbf{G})_i$, then x_i divides the generators of its Alexander dual I_G^\vee but not x_i^2 and, therefore, the positive content of its i -th row has no gaps. The generators in a monomial ideal that are the power of a variable play a dual role to the factors which are the power of a variable in its irreducible decomposition. For instance, if $(\sigma^\vee)'$ is the signature of σ^\vee , then its Alexander dual σ' corresponds to the original monomial ideal without the generators that are the power of a variable, as shown in Figure 3.

By the symmetry between σ and σ^\vee and between τ_0 and τ_0^\vee , it is not difficult to prove that any one of them are signatures without a column with only one nonzero entry, then the others are also a signature.

Corollary 4.8. *If any of the σ , τ_0 , σ^\vee , and τ_0^\vee are a signature with no column with only one nonzero entry, then the others are also a signature.*

In consequence, if any of the σ , τ_0 , σ^\vee , and τ_0^\vee is a signature, then

$$\text{lcm}(\sigma) = \text{lcm}(\tau_0) = \text{lcm}(\sigma^\vee) = \text{lcm}(\tau_0^\vee).$$

If the matrices B_G and D_G are considered as vectors in \mathbb{R}^{nq} and \mathbb{R}^{np} , then Theorem 3.11 can be interpreted as that there exists a linear transformation $T : \mathbb{R}^{nq} \rightarrow \mathbb{R}^{np}$ such that $T(B_G) = D_G$. Moreover, the entries of the matrix representation of the linear transformation T are either zero or one.

Corollary 4.9. *Let G be an antichain with no generator equal to the power of a variable, and B_G and D_G be its incidence and decomposition matrices, respectively. If \mathbf{r}_i and \mathbf{s}_i are the rows of B_G and $(D_G)_0$, respectively, then*

$$\text{cont}(\mathbf{r}_i) = \text{cont}(\mathbf{s}_i) \text{ for all } 1 \leq i \leq n.$$

Proof. Let σ be the signature of I_G and $\tau = \sigma((D_G)_0)$. It is not difficult to check that there exists $f : \text{cont}(\sigma) \rightarrow \text{cont}(B_G)$ such that $f(\sigma_{i,j}) = (B_G)_{i,j}$. Considering the matrices B_G and D_G as vectors in

\mathbb{R}^{nq} and \mathbb{R}^{np} , by Theorem 3.11, there exists a linear transformation $T : \mathbb{R}^{nq} \rightarrow \mathbb{R}^{np}$ such that $T(B_G) = D_G$ and whose matrix representation has only zero and one entries. Thus, $f(\tau_{i,j}) = (D_G)_{i,j}$, as shown Figure 4.

$$\begin{array}{ccc} \sigma & \xrightarrow{T} & \tau \\ f \downarrow & & \downarrow f \\ B_G & \xrightarrow{T} & D_G \end{array}$$

Figure 4. The commutative diagram between σ , τ , B_G , and D_G .

Let \mathbf{r}'_i and \mathbf{s}'_i be the rows of σ and τ , respectively. By Theorem 4.5, $\text{cont}(\mathbf{r}'_i) = \text{cont}(\mathbf{s}'_i)$ for all $1 \leq i \leq n$, and, therefore,

$$\text{cont}(\mathbf{r}_i) = f(\text{cont}(\mathbf{r}'_i)) = f(\text{cont}(\mathbf{s}'_i)) = \text{cont}(\mathbf{s}_i)$$

for all $1 \leq i \leq n$. □

Since, as we have seen in Subsection 2.3, the primary decomposition of a monomial ideal can be obtained by intersecting the factors of the irreducible decomposition with the same radical, then it is not complicated to check that a monomial ideal with the same signature has essentially the same primary decomposition.

Remark 4.10. *It is important to note that throughout the article, a fixed order is used over the variables and generators of the monomial ideal.*

On the other hand, given a signature σ , let

$$P_\sigma^0 = \{B \in P_\sigma : \text{gcd}(B) = \mathbf{0}\} = \{B \in P_\sigma : B_{i,j} = 0 \text{ for all } i, j \text{ such that } \sigma_{i,j} = 0\}$$

be a face of the polyhedral cone P_σ whose integral points correspond to the monomial ideals with signature σ with irredundant irreducible decomposition that does not contain a factor of the form $\langle x_i^{\text{gcd}(B)_i} \rangle$. As the next result shows, it is not difficult to calculate the dimension of P_σ^0 in terms of the dimension of P_σ .

Proposition 4.11. *If σ is a signature, then $\dim(P_\sigma^0) = \dim(P_\sigma) - n$.*

Proof. By the definition of P_σ^0 we have that $\dim(P_\sigma^0) = \sum_{i=1}^n (|\text{cont}(\mathbf{r}_i)| - 1) = \dim(P_\sigma) - n$, where $(\mathbf{r}_1, \dots, \mathbf{r}_q)$ are the rows of σ . □

In a similar way, let

$$P_\tau = \{B \in \text{Mat}_{n,p}(\mathbb{R}_{\geq 0} \cup \infty) : B_{i,j} + \tau_{i,j} \leq B_{i,k} + \tau_{i,k} \text{ for all } 1 \leq i \leq n \text{ and } 1 \leq j, k \leq q \text{ such that } \tau_{i,j} \leq \tau_{i,k}\}.$$

In other words, $P_\tau \cap \mathbb{Z}^{np} = \{D \in \text{Mat}_{n,p}(\mathbb{N}_\infty \setminus 0) : \sigma(D) = \tau\}$. It is not difficult to check that P_τ can be identified with the polyhedral cone $P_{\tau_0} = (P_\tau)_0$.

As the following result proves that there exists a depth relation between P_σ^0 and P_τ .

Theorem 4.12. *Let σ be a signature with no column with only one nonzero entry and D_G be the decomposition matrix of the antichain ideal of \mathbf{G} . Then $\sigma(D_G) = \tau_0$ if and only if $\sigma(I_G) = \sigma$.*

Proof. It follows directly from Corollary 4.9. \square

In other words, there exists a bijection between P_σ^0 and P_τ . Thus, we get that P_σ^0 and $(P_\tau)_0$ have the same dimension.

Corollary 4.13. *If σ is a signature, then $\dim(P_\sigma^0) = \dim(P_{\tau_0})$.*

Thus, it is natural to consider that any monomial ideal whose incidence matrix is in P_σ^0 is an Alexander dual of a monomial ideal whose incidence matrix is in P_{τ_0} .

5. Future work

We finish by posing a pair of conjectures, stating that monomial ideals with the same signature have essentially the same minimal free resolution and that their quotient modules share at least a common regular sequence. These conjectures put into perspective the depth of the concept of the signature of a monomial ideal.

5.1. Regular sequences

In what follows, we recall what a regular sequence of an S -module means. Given a module M over a ring S , we say that nonzero element r of M is regular whenever $rm \neq 0_M$ for all $m \in M \setminus 0_M$. In other words, r is regular whenever it is not a zero divisor of M . Moreover, a sequence h_1, \dots, h_d of polynomials in S is called regular in M whenever

$$h_i \text{ is regular in } M/\langle h_1, \dots, h_{i-1} \rangle M \text{ for all } 1 \leq i \leq d$$

and $\langle h_1, \dots, h_d \rangle M \neq M$. The depth of M , denoted by $\text{depth}(M)$, is the maximal length of its regular sequences.

We begin by raising the weakest version of our conjectures, which claims that the depth of the quotient module of a monomial ideal is preserved under the signature.

Conjecture 5.1. *If I and J are monomial ideals with the same signature, then*

$$\text{depth}(S/I) = \text{depth}(S/J).$$

A stronger version claims that the S -modules S/I and S/J have a common regular sequence.

Conjecture 5.2. *Let I and J be monomial ideals with the same signature. If d is the depth of I , then there exists a sequence h_1, \dots, h_d of polynomials such that it is a regular sequence of both S/I and S/J .*

A stronger version of the previous conjecture asserts that this common regular sequence can be chosen with linear forms.

5.2. Minimal free resolutions

Roughly speaking, a graded free resolution of a finitely generated graded S -module M is a sequence $\mathbf{F}_\bullet = \{F_i, d_i\}_{i=0}^p$ of graded free modules F_i and homogeneous homomorphism $d_i : F_{i+1} \rightarrow F_i$ such that

$$0 \leftarrow M \xleftarrow{\pi=d_0} F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} \dots \xleftarrow{d_p} F_p \leftarrow 0$$

is an exact complex, that is, $\text{Im}(d_i) = \text{Ker}(d_{i-1})$ for all $1 \leq i \leq p$, $\text{Im}(\pi) = M$, and $\text{Ker}(d_p) = 0$. Besides, it is called minimal whenever

$$\text{Im}(d_i) \subseteq \mathfrak{m} := \langle x_1, \dots, x_n \rangle \text{ for all } 0 \leq i \leq p;$$

see for instance [11, Section I.7]. When $M = S/I$ for some ideal I , $F_0 = S$, and we say that \mathbf{F}_\bullet is a minimal free resolution of the ideal I . Although there are many ways to grade an S -module, the most common are with the standard grading and the standard multigrading. To simplify the notation, from here on, when we say multigraded minimal free resolution, we mean that it is graded with the standard multigrading and when we say graded minimal free resolution, we mean that it is graded with the standard grading.

Alternative ways exist to represent a minimal free resolution. For instance, in [12], the minimal resolution of a monomial ideal is encoded as the sequence $\mathbf{M}_\bullet = \{M_i\}_{i=1}^p$ of the matrix representation of the differentials d_i . Thus, it is clear that the entries of these matrices are terms in S and M_1 is a matrix with only one row whose entries are the minimal generators of the ideal. To establish what it means that two monomial ideals have essentially the same minimal free resolution, we will introduce a new way of representing it, which we will call its poset resolution.

The poset resolution of a multigraded minimal free resolution $\mathbf{F}_\bullet = \{F_i, d_i\}_{i=0}^p$ of a monomial ideal is what we call decorated graded poset \mathbf{P}_\bullet . More precisely, a decorated poset means a triplet (P, s, m) , where $P = (F, \leq)$ is the graded poset on the generators of each of the free modules F_i , where $u \in F_{i-1}$ is covered by $v \in F_i$ whenever $d_i(v) = \sum_{w \in F_{i-1}} s_w w$ with $0 \neq s_u \in k$ for some $1 \leq i \leq p$. The second element of the triplet \mathbf{P}_\bullet is the function from the edges of the Hasse diagram H of P to the field k given by the coefficient of the term s_u in $d_i(v) = \sum_{w \in F_{i-1}} s_w w$. Finally,

$$m : F \rightarrow \text{Mon}(S),$$

is the function that takes the monomial with the multidegree of v . The concept of poset resolution was developed during the Ph.D. studies of Javier A. Moreno and David C. Molano; see in [13, 14].

Conjecture 5.3. *If I and J are monomial ideals with the same signature, then there exists a poset resolution $\mathbf{P}_\bullet = (P, s, m)$ of I such that $\mathbf{P}'_\bullet = (P, s, m')$ is a poset resolution of J for some m' .*

According to the previous conjectures, the signature is an invariant fine enough to preserve some of the most important algebraic properties of a monomial ideal.

6. Conclusions

The signature of a monomial ideal is introduced. It is proved that two monomial ideals with the same signature have essentially the same irreducible decomposition. Moreover, using Alexander duality, it is proved that the irredundant irreducible decomposition matrix of a monomial ideal whose incidence matrix is tight is also tight. This leads to enlarging the monomial ideals, which are considered Alexander Duals of a given monomial ideal. Additionally, it is conjectured that monomial ideals with the same signature have essentially the same minimal free resolution and that their quotient modules share at least a common regular sequence.

Author contributions

J.I, D.S.M., C.E.V. and R.H.V. contributed equally to this work regarding conceptualization, methodology, formal analysis, investigation, software, writing—original draft preparation, writing—review and editing. All authors have read and agreed to the published version of the manuscript.

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Conflict of interest

The authors declare no conflicts of interest.

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