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Research article

(θ_i, λ)-constacyclic codes and DNA codes over $\mathbb{Z}_4 + u\mathbb{Z}_4 + u^2\mathbb{Z}_4$

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Abstract: In this paper, three new automorphisms were identified over the ring $\mathbb{Z}_4 + u\mathbb{Z}_4 + u^2\mathbb{Z}_4$ where $u^3 = u^2$. With the help of these automorphisms, the characteristic structures of the generator polynomials for the θ_i-cyclic codes and (θ_i, λ)-constacyclic codes of odd length on this ring were investigated. Also, for all the units over the ring, $\mathbb{Z}_4$-images of θ_i-cyclic and (θ_i, λ)-constacyclic codes were reviewed with the associated codes based on determined transformations. Using these observations, new and optimal codes were obtained and presented in the table. In addition, a new transformation was identified that involved DNA base pairs with the elements of $\mathbb{Z}_4$. Moreover, a unit reverse polynomial was created, and in this way a new generation method has been built to construct reversible DNA codes over this ring. Finally, this article was further enhanced with supporting examples of the DNA as a part of the study.

Keywords: codes over rings; constacyclic codes; skew codes; DNA codes
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1. Introduction

Within the coding theory, linear codes and cyclic codes, which have been studied for years on different rings, have a strong algebraic structure. Therefore, a wide range of methods and approaches have been studied in [1–5]. Constacyclic codes, which are an extension of these important codes, were introduced by Eugene Prange for the first time [6], and recently new $\mathbb{Z}_4$-codes were found by using these code families. Dinh et al. and Gao et al. have worked over the ring $\mathbb{Z}_4 + v\mathbb{Z}_4$ when $v^2 = v$. Dinh et al. [7] has illustrated an original Gray map over this ring and has studied cyclic, constacyclic for the units $1+2v$ and $3+2v$, negacyclic, and the self dual of θ-constacyclic codes. They have described a generator polynomial for cyclic and constacyclic codes of odd length. They have given multiple samples and obtained new $\mathbb{Z}_4$ codes. Gao et al. [8] evaluated the linear codes that were placed on this ring and researched the Euclidean self dual codes. They have drawn attention to Hermitian dual codes and discussed the connection to unimodular complex lattice points. By analyzing the cyclic codes over the ring, they have created generator polynomials. Ultimately, based on the quadratic codes, they have achieved good and new $\mathbb{Z}_4$-linear codes.

In addition to the studies in commutative structures, the studies on noncommutative structures gained a rapid acceleration in a short period of time and took its place in the world of literature. The definition of a special multiplication is the most important feature that separates the noncommutative structure from the commutative structure. This structure called skew has been studied mainly over $\mathbb{F}_q$ [9, 10]. Skew cyclic codes, one of the generalizations of cyclic codes and first introduced by Boucher, have attracted the attention of many researchers as they are more advantageous for finding optimal codes. Then, in addition to skew cyclic codes, other families of codes were also researched by many researchers. The articles [10–14] are some examples of skew articles. Gursoy et al. [10], using the decomposition, researched the structural features of the skew cyclic codes over $\mathbb{F}_q$ where $v^2=v$ and created generator polynomials for these codes. They also mentioned idempotent generators and BCH (Bose-Chaudhuri-Hocquenghem) type bounds. Sharma et al. [14] defined a new automorphism over the ring $\mathbb{Z}_4+u\mathbb{Z}_4$ when $u^2=0$ and investigated the characteristic structure of skew constacyclic codes. They also mentioned double constacyclic codes and found good codes over $\mathbb{Z}_4$.

Adleman, who successfully solved the NP-hard problem (non-deterministic polynomial-time problem. For example, travelling salesman problem.) using DNA molecules, proposed the first computation on the structure of DNA [15]. For many years now, the structure of the DNA cyclic codes has been studied by many researchers and a large number of articles have been written about it. Notable contributions in this field include the works of [16–20]. In addition to these, you can also find several important articles in more detail here: In [21], an analysis of skew-constacyclic codes over the ring ${}^{\mathbb{F}_4[v]}/_{\langle v^2-v\rangle}$ was performed by Bayram et al. They also searched for reversible codes and obtained DNA codes using Griesmer bound. Dinh et al. [22] studied the reversible codes and the reversible-complement codes over the ring $\mathbb{F}_2+u\mathbb{F}_2+v\mathbb{F}_2+uv\mathbb{F}_2+v^2\mathbb{F}_2+uv^2\mathbb{F}_2$, where $u^2=0$ and $v^3=v$, and explored the binary image of the cyclic DNA codes over this ring. In [23], the authors searched for cyclic DNA codes with the help of ${}^{\mathbb{F}_2[u]}/_{\langle u^2-1\rangle}$ and studied the CG-content (The CG-content (or GC-content) of DNA codes refers to the percentage of nucleotides in a DNA molecule that are either cytosine (C) or guanine (G).) of these codes. In [24], Yildiz and Siap investigated the algebraic structure of cyclic DNA codes of odd length. They did so by associating the elements of the ring with the DNA pairs. In [25], the authors studied DNA codes of odd length over the ring $\mathbb{Z}_4+v\mathbb{Z}_4$ with $v^2=v$. They also characterized cyclic codes of odd length and presented a new method of constructing DNA codes. Hence, they found some DNA codes with 256 code words.

Our specific focus in this article is on cyclic, θ_i-cyclic, (θ_i,λ)-constacyclic, and DNA codes over $\mathbb{Z}_4+u\mathbb{Z}_4+u^2\mathbb{Z}_4$ with $u^3=u^2$. Throughout this paper, we will represent the 64-element commutative ring $\mathbb{Z}_4+u\mathbb{Z}_4+u^2\mathbb{Z}_4$ with $u^3=u^2$ via T_3 and also search the structure of T_3 for odd length ϖ. This paper is divided into the following sections: In Section 2, we deal with the basic concepts of the ring T_3. In Section 3, we give the most important descriptions of the skew codes and determine all the automorphisms of T_3. In the following, we define the generator polynomials for skew cyclic codes and skew λ-constacyclic codes over this ring. Finally, we concentrate on the $\mathbb{Z}_4$-images of skew λ-constacyclic codes for each defined automorphisms. For all units over this ring, there are cyclic codes or quasi-cyclic codes of index 2 over $\mathbb{Z}_4$. We present a number of new and optimal codes as a result of this observation and we present them in tables. In Section 4, we relate the components of the T_3 ring to the DNA 2-mers through the new transformation identified, with the ϕ_1 Gray map. Furthermore, we have created a new generation method for generating a reversible code over the ring T_3 by defining a unit reverse polynomial. In addition, we have also provided examples of how to strengthen the operation of this method.

2. Preliminary informations for the ring T_3

T_3 is isomorphic to the quotient ring ${}^{\mathbb{Z}_4[u]}/_{\langle u^3-u^2\rangle}$ and is a non-chain ring. Moreover, this ring is a nonlocal ring because it does not have a single maximal ideal. As well, the set of units of T_3 are $\{1,3,1+2u,3+2u,1+u+u^2,3+u+u^2,1+3u+u^2,3+3u+u^2,1+2u^2,3+2u^2,1+2u+2u^2,3+2u+2u^2,1+u+3u^2,3+u+3u^2,1+3u+3u^2,3+3u+3u^2\}$. The ring is a Frobenius ring because, through the instrument of the Chinese remainder theorem (CRT), it can be stated as a direct sum of the local rings with a single minimal ideal. Nonlocal Frobenius rings are expressed as the direct sum of local rings with the help of the CRT.

Let $\mathfrak{y}$ be any element of T_3 demonstrated as $\mathfrak{y} = a_0+ua_1+u^2a_2$ for $a_0,a_1,a_2 \in \mathbb{Z}_4$. A code of length ϖ over T_3 is a subset of T_3^{ϖ}. $\mathfrak{C}_v$ is a linear if, and only if, $\mathfrak{C}_v$ is a sub-module of T_3. The elements of the linear code are called code words.

Each code word $\mathfrak{y} = (\mathfrak{y}_0,\mathfrak{y}_1,\dots,\mathfrak{y}_{\varpi-1})$ is qualified via its polynomial form $\mathfrak{y}(x) = \mathfrak{y}_0+\mathfrak{y}_1x+\cdots+\mathfrak{y}_{\varpi-1}x^{\varpi-1}$ for each $\mathfrak{y}_i = a_0^i+ua_1^i+u^2a_2^i$ with $i = 0,1,\dots,\varpi-1$.

Using these explanations, we can define the cyclic code and λ-constacyclic code definitions needed in this study as follows:

(i) Let ρ_λ be a λ-constacyclic shift operator. A linear code $\mathfrak{C}_v$ is said to be λ-constacyclic code of length ϖ over T_3 if $\rho_\lambda(\mathfrak{y}_0,\mathfrak{y}_1,\dots,\mathfrak{y}_{\varpi-1}) = (\lambda\mathfrak{y}_{\varpi-1},\mathfrak{y}_0,\mathfrak{y}_1,\dots,\mathfrak{y}_{\varpi-2}) \in \mathfrak{C}_v$ while $(\mathfrak{y}_0,\mathfrak{y}_1,\dots,\mathfrak{y}_{\varpi-1}) \in \mathfrak{C}_v$. In other words, $\mathfrak{C}_v$ is a λ-constacyclic code over T_3 if, and only if, $\mathfrak{C}_v$ is an ideal of ${}^{T_3[x]}/_{\langle x^{\varpi}-\lambda\rangle}$.

(ii) In the above definition, if 1 is written instead of λ, this code is called a cyclic code. In other words, $\sigma(\mathfrak{y}_0,\mathfrak{y}_1,\dots,\mathfrak{y}_{\varpi-1}) = (\mathfrak{y}_{\varpi-1},\mathfrak{y}_0,\mathfrak{y}_1,\dots,\mathfrak{y}_{\varpi-2})$ is an element in $\mathfrak{C}_v$ where $(\mathfrak{y}_0,\mathfrak{y}_1,\dots,\mathfrak{y}_{\varpi-1}) \in \mathfrak{C}_v$ such that σ is a cyclic shift operator.

Nonlocal rings can be represented by local rings, which have an important position in coding theory with the help of CRT. For detailed information, see [26,27]. From this point of view, motivated by our work in [1], we obtain the decomposition of T_3. Recall that the ring $\mathbb{Z}_4+u\mathbb{Z}_4$ works with $u^2 = 0$.

$$T_3 = u^2T_3 \oplus (1+3u^2)T_3 = u^2\mathbb{Z}_4 \oplus (1+3u^2)(\mathbb{Z}_4+u\mathbb{Z}_4).$$

Moreover, the linear code $\mathfrak{R}$ over $\mathbb{Z}_4$ with length ϖ is defined as $\mathfrak{R} = \{z+c+t \in \mathbb{Z}_4^m,\ z+uc+u^2t \in \mathfrak{C}_v\}$ and the linear code $\mathfrak{I}$ over $\mathbb{Z}_4+u\mathbb{Z}_4$ with length ϖ is defined as $\mathfrak{I} = \{z+uc \in (\mathbb{Z}_4+u\mathbb{Z}_4)^{\varpi},\ z+uc+u^2t \in \mathfrak{C}_v$ for some $t \in \mathbb{Z}_4^{\varpi}\}$. Based on this, the linear code $\mathfrak{C}_v$ of odd length ϖ over T_3 can be uniquely shown as $\mathfrak{C}_v = u^2\mathfrak{R} \oplus (1+3u^2)\mathfrak{I}$.

Consider the same three Gray maps in [1] for θ_i-cyclic codes over T_3. Recall these maps, which are linear and preserve the Euclidean, Lee, and Hamming distances from T_3 to $\mathbb{Z}_4^{2\varpi}$:

$$\phi_1 : T_3 \longrightarrow \mathbb{Z}_4^2,$$
$$(a_0 + ua_1 + u^2a_2) \to (a_0 + a_1 + 3a_2,\, 3a_0 + 3a_1 + a_2),$$

$$\phi_2 : T_3 \longrightarrow \mathbb{Z}_4^2,$$
$$(a_0 + ua_1 + u^2a_2) \to (a_0 + a_1 + 3a_2,\, a_0 + 3a_1 + a_2),$$

$$\phi_3 : T_3 \longrightarrow \mathbb{Z}_4^2,$$
$$(a_0 + ua_1 + u^2a_2) \to (a_0 + a_1 + 3a_2,\, 3a_0 + a_1 + 3a_2).$$

Identifying the elements $\phi_i(\mathfrak{y}(x)) = \mathfrak{y} = (\mathfrak{y}_0, \mathfrak{y}_1, \ldots, \mathfrak{y}_{\varpi-1})$ in T_3^{ϖ} with polynomials $\mathfrak{y}(x) = \mathfrak{y}_0 + \mathfrak{y}_1 x + \cdots + \mathfrak{y}_{\varpi-1}x^{\varpi-1}$ for each $\mathfrak{y}_i = a_0^i + ua_1^i + u^2a_2^i$ with $i = 0, 1, \ldots, \varpi - 1$, we get

$$\Phi_i : T_3^{\varpi} \to \mathbb{Z}_4^{2\varpi},$$
$$\Phi_i : (\mathfrak{y}_0, \mathfrak{y}_1, \ldots, \mathfrak{y}_{m-1}) \to (\phi_i(\mathfrak{y}_0), \phi_i(\mathfrak{y}_1), \ldots, \phi_i(\mathfrak{y}_{\varpi-1})).$$

Based on the information presented, we will now examine the skew cyclic, skew constacyclic and DNA codes over T_3. By constructing generator polynomials, our aim is to acquire new and optimal codes via the Gray maps that have been defined earlier. Furthermore, we are attempting a unique perspective by constructing a new polynomial that is exclusive to DNA codes over the ring. This polynomial will be enriched with examples to serve our purpose.

3. θ_i-cyclic and (θ_i, λ)-constacylic codes over T_3

In this section, skew cyclic and skew constacyclic codes over T_3 are analyzed. To begin, all nontrivial automorphisms of T_3 are identified. These maps θ_i on T_3 for $i = 1, 2, 3$ are defined such that

$$\theta_1(a_0 + ua_1 + u^2a_2) = a_0 + (2 + 3u)a_1 + u^2a_2,$$
$$\theta_2(a_0 + ua_1 + u^2a_2) = a_0 + (2u^2 + u + 2)a_1 + u^2a_2 \text{ and}$$
$$\theta_3(a_0 + ua_1 + u^2a_2) = a_0 + (2u^2 + 3u)a_1 + u^2a_2,$$

from T_3 to T_3. This ring $T_3[x, \theta_i] = \{a_o + a_1x + \cdots + a_{\varpi-1}x^{\varpi-1} : a_i \in T_3,\ i = 0, 1, \ldots, \varpi - 1,\ \varpi \in \mathbb{N}\}$ is called a skew polynomial ring. Note that this ring is a noncommutative ring. Herewith the multiplication is described using the precise normal size which is well-known to be $(fx^r)(yx^k) = f\theta_i^r(y)x^{r+k}$ while the addition in this ring is the usual polynomial addition. The order of all defined automorphisms θ_i is 2.

An element $d(x) \in T_3[x, \theta_i]$ is said to be a right divisor of $l(x)$ if there exists $q(x) \in T_3[x, \theta_i]$ such that $l(x) = q(x)d(x)$. Thus, $l(x)$ is called a left multiple of $d(x)$, and a left divisor of $l(x)$ can be defined similarly. In this paper, division stands for right division, and if $l(x) \in T_3[x, \theta_i]$, then we put to use the notation $\langle l(x) \rangle$ for the left ideal generated by $l(x)$.

Throughout this section, the quotient ring ${}^{T_3[x,\theta_i]}/_{\langle x^{\varpi}-1\rangle}$ will be represented by $T_{3,\varpi_{\theta_i}}$ and ${}^{T_3[x,\theta_i]}/_{\langle x^{\varpi}-\lambda\rangle}$ will be represented by $T_{3,\varpi_{\theta_i,\lambda}}$. These quotient rings are left-$T_3[x,\theta_i]$ module with the multiplication identified by $d(x)(l(x)+\langle x^{\varpi}-1\rangle)=d(x)l(x)+\langle x^{\varpi}-1\rangle$ and $d(x)(l(x)+\langle x^{\varpi}-\lambda\rangle)=d(x)l(x)+\langle x^{\varpi}-\lambda\rangle$ for any $d(x),l(x)\in T_3[x,\theta_i]$. We characterize a T_3-module isomorphism from T_3^{ϖ} to $T_{3,\varpi_{\theta_i}}$ such that $(\mathfrak{y}_0,\mathfrak{y}_1,\dots,\mathfrak{y}_{\varpi-1})\to\mathfrak{y}_0+\mathfrak{y}_1x+\cdots+\mathfrak{y}_{\varpi-1}x^{\varpi-1}$.

The fundamental definition and theorems that underlie the structure of the skew codes are outlined below.

Definition 1. *A skew linear code $\mathfrak{C}_v$ of odd length ϖ over the ring T_3 is a left $T_3[x,\theta_i]$-sub-module of the left module ${}^{T_3[x,\theta_i]}/_{\langle l(x)\rangle}$, where $l(x)$ is a polynomial of degree ϖ over $T_3[x,\theta_i]$.*

Theorem 1. *$T_{3,\varpi_{\theta_i}}$ is a $T_3[x,\theta_i]$-left module where multiplication is defined as above.*

Theorem 2. *A code $\mathfrak{C}_v$ of length ϖ in $T_{3,\varpi_{\theta_i}}$ is a θ_i-cyclic code if, and only if, $\mathfrak{C}_v$ is a left $T_3[x,\theta_i]$-sub-module of the left $T_3[x,\theta_i]$-module $T_{3,\varpi_{\theta_i}}$.*

Theorem 3. *A code $\mathfrak{C}_v$ of length ϖ in $T_{3,\varpi_{\theta_i}}$ is a (θ_i,λ)-constacyclic code if, and only if, $\mathfrak{C}_v$ is a left $T_{3,\varpi_{\theta_i,\lambda}}$-sub-module of the left $T_3[x,\theta_i]$-module over $T_{3,\varpi_{\theta_i,\lambda}}$.*

Note that throughout this paper we represent skew cyclic codes by θ_i-cyclic codes and skew λ-constacyclic codes by (θ_i,λ)-constacyclic codes. So, if $T_{3\,\theta_i,\lambda}(\mathfrak{C}_v)=\mathfrak{C}_v$ then a T_3-sub-module of T_3^{ϖ} is a (θ_i,λ)-constacyclic code. In particular, if $\lambda=1$, then $\mathfrak{C}_v$ is said to be a θ_i-cyclic code.

Definition 2. *A subset $\mathfrak{C}_v$ of T_3 is called a (θ_i,λ)-constacyclic code of length ϖ over T_3 if $\mathfrak{C}_v$ is a T_3-sub-module of T_3^{ϖ}, and for any $(\mathfrak{y}_0,\mathfrak{y}_1,\dots,\mathfrak{y}_{\varpi-1})\in\mathfrak{C}_v$, we have $(\lambda\theta_i(\mathfrak{y}_{\varpi-1}),\theta_i(\mathfrak{y}_0),\dots,\theta_i(\mathfrak{y}_{\varpi-2}))\in\mathfrak{C}_v$. It should not be forgotten that if λ is chosen as 1, then the (θ_i,λ)-constacyclic code of length ϖ over T_3 is a θ_i-cyclic code of length ϖ over T_3.*

With the help of all these descriptions, let's construct the generator polynomial for θ_i-cyclic and (θ_i,λ)-constacyclic codes over T_3.

Theorem 4. *Let $\mathfrak{C}_v$ be a linear code over T_3 of length ϖ and $\mathfrak{C}_v=u^2\mathfrak{R}\oplus(1+3u^2)\mathfrak{I}$ be its decomposition, where $\mathfrak{R}$ is a code of length ϖ over $\mathbb{Z}_4$ and $\mathfrak{I}$ is a code of length ϖ over $\mathbb{Z}_4+u\mathbb{Z}_4$ where $u^2=0$. Then, $\mathfrak{C}_v$ is a θ_i-cyclic code as regards to the automorphism θ_i if, and only if, $\mathfrak{R}$ and $\mathfrak{I}$ are both θ_i-cyclic codes over $\mathbb{Z}_4$ and $\mathbb{Z}_4+u\mathbb{Z}_4$, respectively, as regards to the automorphism θ_i.*

Proof. For $i=0,1,\dots,\varpi-1$, let $\mathfrak{y}=(\mathfrak{y}_0,\mathfrak{y}_1,\dots,\mathfrak{y}_{\varpi-1})\in\mathfrak{C}_v$ and $\mathfrak{y}_i=u^2p_i+(1+3u^2)v_i$. Assume that $p=(p_0,\dots,p_{\varpi-1})\in\mathfrak{R}$ and $v=(v_0,\dots,v_{\varpi-1})\in\mathfrak{I}$ such that $v_i=a_i+ub_i$ where $i=0,1,\dots,\varpi-1$. Due to $\mathfrak{C}_v$ being θ_i-cyclic if $(u^2p_0\oplus(1+3u^2)v_0,\dots,u^2p_{\varpi-1}\oplus(1+3u^2)v_{\varpi-1})\in\mathfrak{C}_v$, then $(\theta_i(u^2p_{\varpi-1}\oplus(1+3u^2)v_{\varpi-1}),\theta_i(u^2p_0\oplus(1+3u^2)v_0),\dots,\theta_i(u^2p_{\varpi-2}\oplus(1+3u^2)v_{\varpi-2}))\in\mathfrak{C}_v$. Herefrom, $u^2\sigma_{\theta_i}p\oplus(1+3u^2)\sigma_{\theta_i}v\in\mathfrak{C}_v$. Because of $\sigma_{\theta_i}(u^2p\oplus(1+3u^2)v)=u^2\sigma_{\theta_i}p\oplus(1+3u^2)\sigma_{\theta_i}v$, then $\mathfrak{R}$ and $\mathfrak{I}$ are θ_i-cyclic. Conversely, if $\mathfrak{R}$ and $\mathfrak{I}$ is θ_i-cyclic, $\sigma_{\theta_i}p\in\mathfrak{R}$ while $p\in\mathfrak{R}$ and $\sigma_{\theta_i}v\in\mathfrak{R}$ while $v\in\mathfrak{I}$. So $u^2\sigma_{\theta_i}p\oplus(1+3u^2)\sigma_{\theta_i}v\in\mathfrak{C}_v$. Hence, $\mathfrak{C}_v$ is θ_i-cyclic. □

Let us compose the generator polynomial of the θ_i-cyclic code with the assistance of this theorem.

Theorem 5. *Let $\mathfrak{C}_v=u^2\mathfrak{R}\oplus(1+3u^2)\mathfrak{I}$ be a θ_i-cyclic code of length ϖ over T_3. In this case, $\mathfrak{R}$ is a cyclic code over $\mathbb{Z}_4$ and $\mathfrak{I}$ is a cyclic code over $\mathbb{Z}_4+u\mathbb{Z}_4$ such that $\mathfrak{C}_v=(u^2\langle\mathfrak{f}_g(x)(\mathfrak{t}_g(x)+2)\rangle)\oplus((1+3u^2)\langle\mathfrak{f}_y(x)(\mathfrak{t}_y(x)+2)+u\mathfrak{f}_{g,y}(x)(\mathfrak{t}_{g,y}(x)+2),\ u\mathfrak{f}_v(x)(\mathfrak{t}_v(x)+2)\rangle)$ where $x^{\varpi}-1=\mathfrak{f}_{\mathfrak{i}}(x)\mathfrak{t}_{\mathfrak{i}}(x)\mathfrak{d}_{\mathfrak{i}}(x)$ for $i=1,2,3$.*

Proof. The proof can easily be done following the methodology outlined in our previous publication [1]. □

Theorem 6. *Let $\mathfrak{C}_v = u^2\mathfrak{R} \oplus (1+3u^2)\mathfrak{I}$ be a θ_i-cyclic code of length ϖ over T_3. Given that the generator polynomial of $\mathfrak{R}$ is $\langle\tau_1(x)\rangle$ and the generator polynomial of $\mathfrak{I}$ is $\langle\tau_2(x), \tau_3(x)\rangle$, then $\mathfrak{C}_v = \langle u^2\tau_1(x),\ (1+3u^2)\langle\tau_2(x), \tau_3(x)\rangle\rangle$. Editing the generator polynomial of $\mathfrak{C}_v$, we obtain $\mathfrak{C}_v = \langle u^2\tau_1(x),\ (1+3u^2)\tau_2(x),\ (1+3u^2)\tau_3(x)\rangle$.*

Proof. Due to $\mathfrak{R} = \langle\tau_1(x)\rangle$ and $\mathfrak{I} = \langle\tau_2(x), \tau_3(x)\rangle$, we can conclude that $\mathfrak{C}_v = u^2\mathfrak{R} \oplus (1+3u^2)\mathfrak{I}$. From this, we claim that $\mathfrak{C}_v = \{\mathfrak{y}(x) = u^2b_1(x)\tau_1(x) + (1+3u^2)b_2(x)\langle\tau_2(x),\ \tau_3(x)\rangle \textit{ such that } b_1(x), b_2(x) \in T_3[x,\theta_i]\}$. We can further infer that $\mathfrak{C}_v \subseteq \langle u^2\tau_1(x) + (1+3u^2)\langle\tau_2(x), \tau_3(x)\rangle\rangle \subseteq T_{3,\varpi_{\theta_i}}$. Conversely, let us consider $u^2y_1(x)\tau_1(x) + (1+3u^2)y_2(x)\langle\tau_2(x), \tau_3(x)\rangle \in \langle u^2\tau_1(x), (1+3u^2)\langle\tau_2(x), \tau_3(x)\rangle\rangle$ with $y_1(x), y_2(x) \in T_{3,\varpi_{\theta_i}}$. We have $u^2y_1(x) = u^2b_1(x)$ and $(1+3u^2)y_2(x) = (1+3u^2)b_2(x)$ for some $b_1(x), b_2(x) \in T_3[x,\theta_i]$. Hence, it can be deduced that $\langle u^2\tau_1(x), (1+3u^2)\langle\tau_2(x), \tau_3(x)\rangle\rangle \subseteq \mathfrak{C}_v$. Therefore, the proof is $\mathfrak{C}_v = \langle u^2\tau_1(x), (1+3u^2)\langle\tau_2(x), \tau_3(x)\rangle\rangle$. □

Theorem 7. *Let $\mathfrak{R}$ and $\mathfrak{I}$ be θ_i-cyclic codes over $\mathbb{Z}_4$ and $\mathbb{Z}_4 + u\mathbb{Z}_4$, respectively. Assume that $\langle\tau_1(x)\rangle$ and $\langle\tau_2(x), \tau_3(x)\rangle$ are the monic generator polynomials of these codes and also $\mathfrak{C}_v = u^2\mathfrak{R} \oplus (1+3u^2)\mathfrak{I}$. In this case, there is a unique polynomial $\tau(x)$ over $T_3[x,\theta_i]$ such that $\mathfrak{C}_v = \langle\tau(x)\rangle$ and $\tau(x)$ is a right divisor of $x^{\varpi}-1$, where $\tau(x) = u^2\tau_1(x) + (1+3u^2)(\tau_2(x) + \tau_3(x))$.*

Proof. Using the previous theorem, we can express $\mathfrak{C}_v = \langle u^2\tau_1(x), (1+3u^2)\langle\tau_2(x), \tau_3(x)\rangle\rangle$. Assume that $\tau(x) = u^2\tau_1(x) + (1+3u^2)(\tau_2(x) + \tau_3(x))$. Then, it's trivial that $\langle\tau(x)\rangle \subseteq \mathfrak{C}_v$. On the other hand, we have that $u^2\tau_1(x) = u^2\tau(x)$ and $(1+3u^2)(\tau_2(x) + \tau_3(x)) = (1+3u^2)\tau(x)$, which implies that $\mathfrak{C}_v \subseteq \langle\tau(x)\rangle$. Hence, $\mathfrak{C}_v = \langle\tau(x)\rangle$. Because $\tau_1(x)$ and $(\tau_2(x) + \tau_3(x))$ are monic divisors $x^{\varpi}-1$ in $\mathbb{Z}_4[x,\theta_i]$ and $\mathbb{Z}_4 + u\mathbb{Z}_4[x,\theta_i]$, respectively, then there exists $b_1(x), b_2(x) \in T_{3,\varpi_{\theta_i}}$ such that $x^{\varpi}-1 = b_1(x)\tau_1(x) = b_2(x)(\tau_2(x) + \tau_3(x))$. Therefore, $(u^2b_1(x) + (1+3u^2)b_2(x))\tau(x) = (u^2b_1(x) + (1+3u^2)b_2(x))(u^2\tau_1(x) + (1+3u^2)(\tau_2(x) + \tau_3(x)) = u^2(x^{\varpi}-1) + (1+3u^2)(x^{\varpi}-1) = x^{\varpi}-1$. From this point of view, $\tau(x)$ is a right divisor of $x^{\varpi}-1$. □

First of all, we define two sets for units. In this case,
$\mathfrak{B}_y = \{1, 3, 1+2u, 3+2u, 1+2u^2, 3+2u^2, 1+2u+2u^2, 3+2u+2u^2\}$,
$\mathfrak{B}_w = \{1+u+u^2, 1+u+3u^2, 1+3u+3u^2, 3+3u+u^2, 3+u+3u^2, 3+3u+3u^2, 1+3u+u^2, 3+u+u^2\}$.
Now we use these sets to describe a ring homomorphism. Therefore, we can state the following propositions and corollaries, whose proofs are trivial.

Proposition 8. *Let $\nu : T_{3,\varpi_{\theta_i}} \to T_{3,\varpi_{\theta_i,\lambda}}$. In this case,*

(i) For each unit $\lambda \in \mathfrak{B}_y$, this map is defined as $\nu(\mathfrak{y}(x)) = \mathfrak{y}(\lambda x)$. Then, ν is a ring isomorphism for all units with an odd length and all automorphisms over the ring T_3.

(ii) For each unit $\lambda \in \mathfrak{B}_w$, define this map with $\nu(\mathfrak{y}(x)) = \mathfrak{y}(\lambda^2 x)$. Then, ν is a ring isomorphism, with the length ϖ as

$$\begin{cases} \textit{odd}, & \textit{for the automorphism } \theta_3, \\ \textit{4k+1} \quad \textit{for } k \in \mathbb{Z}, & \textit{for the automorphisms } \theta_1 \textit{ and } \theta_2. \end{cases}$$

Proof. The proof can be easily observed through the defined automorphisms θ_i and the skew multiplication. □

Corollary 1. *There is a one-to-one relation between the ideals of $T_{3,\varpi_{\theta_i}}$ and $T_{3,\varpi_{\theta_i,\lambda}}$.*

Proposition 9. *Let $\mathfrak{C}_v$ denote a linear code of length ϖ over T_3 and let*

$$\widehat{v}(\mathfrak{y}_0,\mathfrak{y}_1,\ldots,\mathfrak{y}_{\varpi-1})=(\mathfrak{y}_0,\lambda\mathfrak{y}_1,\lambda^2\mathfrak{y}_2,\ldots,\lambda^{\varpi-1}\mathfrak{y}_{\varpi-1}).$$

Then, $\mathfrak{C}_v$ is a cyclic code if, and only if, $\widehat{v}(\mathfrak{C}_v)$ is a λ-constacyclic code of length ϖ over T_3.

Theorem 10. *Let $\mathfrak{C}_v=u^2\mathfrak{R}\oplus(1+3u^2)\mathfrak{I}$ be a (θ_i,λ)-constacyclic code of length ϖ over T_3. We identify the methods to construct the generator polynomial of (θ_i,λ)-constacyclic codes of length ϖ over T_3 as follows:*

(i) We determine this generator polynomial by using $\mathfrak{C}_v=(u^2\langle\mathfrak{f}_g(x)(\mathfrak{t}_g(x)+2)\rangle)\oplus((1+3u^2)\langle\mathfrak{f}_y(x)(\mathfrak{t}_y(x)+2)+u\mathfrak{f}_{g,y}(x)(\mathfrak{t}_{g,y}(x)+2),\ u\mathfrak{f}_v(x)(\mathfrak{t}_v(x)+2)\rangle)$ where $x^{\varpi}-\lambda=\mathfrak{f}_\mathfrak{i}(x)\mathfrak{d}_\mathfrak{i}\mathfrak{t}_\mathfrak{i}(x)$ for $i=1,2,3$. This is the most classical method.

(ii) In another way, with the help of the Proposition 8 and Corollary 1, we construct this generator polynomial via $\mathfrak{C}_3=(u^2\langle\mathfrak{f}_g(\tilde{x})(\mathfrak{t}_g(\tilde{x})+2)\rangle)\oplus((1+3u^2)\langle\mathfrak{f}_y(\tilde{x})(\mathfrak{t}_y(\tilde{x})+2)+u\mathfrak{f}_{g,y}(\tilde{x})(\mathfrak{t}_{g,y}(\tilde{x})+2),\ u\mathfrak{f}_v(\tilde{x})(\mathfrak{t}_v(\tilde{x})+2)\rangle)$ such that

$$\begin{cases}\tilde{x}=\lambda x, & \text{for } \lambda\in\mathfrak{B}_y\\ \tilde{x}=\lambda^2 x, & \text{for } \lambda\in\mathfrak{B}_w\end{cases}$$

and $x^{\varpi}-1=f_i(x)h_i(x)\mathfrak{s}_i(x)$ for $i=1,2,3$.

3.1. $\mathbb{Z}_4$-images of (θ_i,λ)-constacyclic codes over T_3

In this section, we look for $\mathbb{Z}_4$-images of θ_i-cyclic and (θ_i,λ)-constacyclic codes over T_3.

Definition 3. *Let $\mathfrak{y}\in\mathbb{Z}_4^{2\varpi}$ with $\mathfrak{y}=(\mathfrak{y}_0,\mathfrak{y}_1,\ldots,\mathfrak{y}_\delta)$ where $\mathfrak{y}_i\in\mathbb{Z}_4$ for $i=0,1$. Let υ_δ be a map from $\mathbb{Z}_4^{2\varpi}$ to $\mathbb{Z}_4^{2\varpi}$ defined by $\upsilon_\delta(\mathfrak{y})=(\sigma(\mathfrak{y}_0),\sigma(\mathfrak{y}_1),\ldots,\sigma(\mathfrak{y}_\delta))$, where σ is the cyclic shift from $\mathbb{Z}_4^{2\varpi}$ to $\mathbb{Z}_4^{2\varpi}$ provided by $\sigma(\mathfrak{y}_i)=({\mathfrak{y}_i}^{\varpi-1},{\mathfrak{y}_i}^0,\ldots,{\mathfrak{y}_i}^{\varpi-2})$ for each $\mathfrak{y}_i=({\mathfrak{y}_i}^0,\ldots,{\mathfrak{y}_i}^{\varpi-1})$ where ${\mathfrak{y}_i}^j\in\mathbb{Z}_4$ and $j=0,1,\ldots,\varpi-1$. The 2ϖ-length code over $\mathbb{Z}_4$ is called a quasi-cyclic code with an index of δ if $\upsilon_\delta(\mathfrak{C})=\mathfrak{C}$.*

Let Φ_j be defined Gray maps from ${T_3}^{\varpi}$ to ${\mathbb{Z}_4}^{2\varpi}$, σ_{θ_i} be the θ_i-cyclic shift, $\rho_{\theta_i,\lambda}$ be the (θ_i,λ)-constacyclic shift, and υ_2 be the quasi-cyclic shift operator with index 2. Thus, the following proposition and theorem can be stated as the result of crucial observations.

Proposition 11. *(i) We have $\Phi_j\sigma_{\theta_i}(\mathfrak{y})=\upsilon_2\Phi_j(\mathfrak{y})$ for any $\mathfrak{y}\in{T_3}^{\varpi}$ and $i,j=1,2,3$.*

(ii) We have $\Phi_1\rho_{\theta_i,\lambda}(\mathfrak{y})=\sigma\Phi_1(\mathfrak{y})$ for any $\mathfrak{y}\in{T_3}^{\varpi}$ and $i=1,2,3$ where $\lambda=3,1+2u,1+2u^2,3+2u+2u^2$.

(iii) *We have* $\Phi_2\rho_{\theta_i,\lambda}(\mathfrak{y}) = \sigma\Phi_2(\mathfrak{y})$ *for any* $\mathfrak{y} \in T_3{}^{\varpi}$ *and* $i = 1,2,3$ *when* $\lambda = 1+u+u^2, 3+3u+u^2, 3+u+3u^2, 1+3u+3u^2$.

(iv) *We have* $\Phi_3\rho_{\theta_i,\lambda}(\mathfrak{y}) = \sigma\Phi_3(\mathfrak{y})$ *for any* $\mathfrak{y} \in T_3{}^{\varpi}$ *and* $i = 1,2,3$ *where* $\lambda = 3+u+u^2, 1+3u+u^2, 1+u+3u^2, 3+3u+3u^2$.

(v) *We have* $\Phi_j\rho_{\theta_i,\lambda}(\mathfrak{y}) = \upsilon_2\Phi_j(\mathfrak{y})$ *for any* $\mathfrak{y} \in T_3{}^{\varpi}$ *and* $i,j = 1,2,3$ *where* $\lambda = 1+2u+2u^2, 3+2u, 3+2u^2$.

Proof. Determine $\mathfrak{y}$ which consists of $(\mathfrak{y}_0, \mathfrak{y}_1, \ldots, \mathfrak{y}_{\varpi-1})$ in $T_3[x,\theta_i]^{\varpi}$, where $\mathfrak{y}_j$ is calculated by the formula $a_0^i + ua_1^i + u^2a_2^i$ for values of j that are from 0 to $\varpi-1$. We know that the images of $\mathfrak{y}$ under the specified Gray maps are as follows.

$\Phi_1(\mathfrak{y}) = (a_0{}^0 + a_1{}^0 + 3a_2{}^0, \ldots, a_0{}^{\varpi-1} + a_1{}^{\varpi-1} + 3a_2{}^{\varpi-1}, 3a_0{}^0 + 3a_1{}^0 + a_2{}^0, \ldots, 3a_0{}^{\varpi-1} + 3a_1{}^{\varpi-1} + a_2{}^{\varpi-1})$,

$\Phi_2(\mathfrak{y}) = (a_0^0 + a_1^0 + 3a_2^0, \ldots, a_0{}^{\varpi-1} + a_1{}^{\varpi-1} + 3a_2^{\varpi-1}, a_0^0 + 3a_1^0 + a_2^0, \ldots, a_0^{\varpi-1} + 3a_1^{\varpi-1} + a_2{}^{\varpi-1})$,

$\Phi_3(\mathfrak{y}) = (a_0{}^0 + a_1{}^0 + 3a_2{}^0, \ldots, a_0{}^{\varpi-1} + a_1{}^{\varpi-1} + 3a_2{}^{\varpi-1}, 3a_0{}^0 + a_1^0 + 3a_2{}^0, \ldots, 3a_0{}^{\varpi-1} + a_1{}^{\varpi-1} + 3a_2{}^{\varpi-1})$.

To demonstrate that $\Phi_j\sigma_{\theta_i}(\mathfrak{y}) = \upsilon_2\Phi_j(\mathfrak{y})$ for all $\mathfrak{y} \in T_3^{\varpi}$ and $i,j = 1,2,3$, let's obtain $\upsilon_2\Phi_1(\mathfrak{y}), \upsilon_2\Phi_2(\mathfrak{y})$ and $\upsilon_2\Phi_3(\mathfrak{y})$ first. We have

$\upsilon_2\Phi_1(\mathfrak{y}) = (a_0{}^{\varpi-1} + a_1{}^{\varpi-1} + 3a_2{}^{\varpi-1}, a_0{}^0 + a_1{}^0 + 3a_2{}^0, \ldots, a_0{}^{\varpi-2} + a_1{}^{\varpi-2} + 3a_2{}^{\varpi-2}, 3a_0{}^{\varpi-1} + 3a_1{}^{\varpi-1} + a_2{}^{\varpi-1}, 3a_0{}^0 + 3a_1{}^0 + a_2{}^0, \ldots, 3a_0{}^{\varpi-2} + 3a_1{}^{\varpi-2} + a_2{}^{\varpi-2})$,

$\upsilon_2\Phi_2(\mathfrak{y}) = (a_0{}^{\varpi-1} + a_1{}^{\varpi-1} + 3a_2{}^{\varpi-1}, a_0{}^0 + a_1{}^0 + 3a_2{}^0, \ldots, a_0{}^{\varpi-2} + a_1{}^{\varpi-2} + 3a_2{}^{\varpi-2}, a_0{}^{\varpi-1} + 3a_1{}^{\varpi-1} + a_2{}^{\varpi-1}, a_0{}^0 + 3a_1{}^0 + a_2{}^0, \ldots, a_0{}^{\varpi-2} + 3a_1{}^{\varpi-2} + a_2{}^{\varpi-2})$,

$\upsilon_2\Phi_3(\mathfrak{y}) = (a_0^{\varpi-1} + a_1^{\varpi-1} + 3a_2^{\varpi-1}, a_0^0 + a_1^0 + 3a_2^0, \ldots, a_0{}^{\varpi-2} + a_1^{\varpi-2} + 3a_2^{\varpi-2}, 3a_0^{\varpi-1} + a_1{}^{\varpi-1} + 3a_2{}^{\varpi-1}, 3a_0{}^0 + a_1{}^0 + 3a_2{}^0, \ldots, 3a_0{}^{\varpi-2} + a_1{}^{\varpi-2} + 3a_2{}^{\varpi-2})$.

On the other hand, we obtain

$\sigma_{\theta_1}(\mathfrak{y}) = (\theta_1(\mathfrak{y}_{\varpi-1}), \theta_1(\mathfrak{y}_0), \theta_1(\mathfrak{y}_1), \ldots, \theta_1(\mathfrak{y}_{\varpi-2})) = (a_0{}^{\varpi-1} + 2a_1{}^{\varpi-1} + 3ua_1 + u^2a_2{}^{\varpi-1}, a_0{}^0 + ua_1{}^0 + u^2a_2{}^0, \ldots, a_0{}^{\varpi-2} + ua_1{}^{\varpi-2} + u^2a_2{}^{\varpi-2})$,

$\sigma_{\theta_2}(\mathfrak{y}) = (\theta_2(\mathfrak{y}_{\varpi-1}), \theta_2(\mathfrak{y}_0), \theta_2(\mathfrak{y}_1), \ldots, \theta_2(\mathfrak{y}_{\varpi-2})) = (a_0{}^{\varpi-1} + 2a_1 + ua_1{}^{\varpi-1} + u^2(2a_1{}^{\varpi-1} + a_2{}^{\varpi-1}), a_0{}^0 + ua_1{}^0 + u^2a_2{}^0, \ldots, a_0{}^{\varpi-2} + ua_1{}^{\varpi-2} + u^2a_2{}^{\varpi-2})$,

$\sigma_{\theta_3}(\mathfrak{y}) = (\theta_3(\mathfrak{y}_{\varpi-1}), \theta_3(\mathfrak{y}_0), \theta_3(\mathfrak{y}_1), \ldots, \theta_3(\mathfrak{y}_{\varpi-2})) = (a_0{}^{\varpi-1} + 3ua_1 + u^2(2a_1{}^{\varpi-1} + a_2{}^{\varpi-1}), a_0{}^0 + ua_1{}^0 + u^2a_2{}^0, \ldots, a_0{}^{\varpi-2} + ua_1{}^{\varpi-2} + u^2a_2{}^{\varpi-2})$.

The image of them under Φ_1:

$\Phi_1\sigma_{\theta_1}(\mathfrak{y}) = \Phi_1\sigma_{\theta_2}(\mathfrak{y}) = \Phi_1\sigma_{\theta_3}(\mathfrak{y}) = (a_0^{\varpi-1} + a_1^{\varpi-1} + 3a_2^{\varpi-1}, a_0^0 + a_1^0 + 3a_2^0, \ldots, a_0^{\varpi-2} + a_1^{\varpi-2} + 3a_2^{\varpi-2}, 3a_0^{\varpi-1} + 3a_1^{\varpi-1} + a_2^{\varpi-1}, 3a_0^0 + 3a_1^0 + a_2^0, \ldots, 3a_0^{\varpi-2} + 3a_1^{\varpi-2} + a_2^{\varpi-2})$,

The image of them under Φ_2:

$\Phi_2\sigma_{\theta_1}(\mathfrak{y}) = \Phi_2\sigma_{\theta_2}(\mathfrak{y}) = \Phi_2\sigma_{\theta_3}(\mathfrak{y}) = (a_0{}^{\varpi-1} + a_1{}^{\varpi-1} + 3a_2{}^{\varpi-1}, a_0{}^0 + a_1{}^0 + 3a_2{}^0, \ldots, a_0{}^{\varpi-2} + a_1{}^{\varpi-2} + 3a_2{}^{\varpi-2}, a_0{}^{\varpi-1} + 3a_1{}^{\varpi-1} + a_2{}^{\varpi-1}, a_0{}^0 + 3a_1{}^0 + a_2{}^0, \ldots, a_0{}^{\varpi-2} + 3a_1{}^{\varpi-2} + a_2{}^{\varpi-2})$.

The image of them under Φ_3:

$\Phi_3\sigma_{\theta_1}(z\mathfrak{y}) = \Phi_3\sigma_{\theta_2}(\mathfrak{y}) = \Phi_3\sigma_{\theta_3}(\mathfrak{y}) = (a_0^{\varpi-1} + a_1^{\varpi-1} + 3a_2^{\varpi-1}, a_0^0 + a_1^0 + 3a_2^0, \ldots, a_0^{\varpi-2} + a_1^{\varpi-2} + 3a_2^{\varpi-2}, 3a_0^{\varpi-1} + a_1^{\varpi-1} + 3a_2^{\varpi-1}, 3a_0^0 + a_1^0 + 3a_2^0, \ldots, 3a_0^{\varpi-2} + a_1^{\varpi-2} + 3a_2^{\varpi-2})$.

Therefore, we have $\Phi_j\sigma_{\theta_i}(\mathfrak{y}) = \upsilon_2\Phi_j(\mathfrak{y})$ for $i,j = 1,2,3$.

The proof of others can be achieved using the same methodology. □

Theorem 12. *(i) Let $\mathfrak{C}_v$ denote a θ_i-cyclic code of length ϖ over T_3, where i and j range from* 1 *to* 3. *In this regard, the Gray image of a θ_i-cyclic code over T_3 with a length ϖ is equal to a QC code of index* 2 *over $\mathbb{Z}_4$ with a length 2ϖ.*

(ii) Let $\mathfrak{C}_v$ denote (θ_i,λ)-constacyclic codes of length m over T_3 for $\lambda = 3, 1+2u, 1+2u^2, 3+2u+2u^2$, where i takes on the values 1, 2, *and* 3. *By taking the $\mathbb{Z}_4$-images of $\Phi_1(\mathfrak{C}_v)$, the cyclic codes over $\mathbb{Z}_4$ are obtained.*

(iii) Let $\mathfrak{C}_v$ denote (θ_i,λ)-constacyclic codes of length ϖ over T_3 for $\lambda = 1+u+u^2, 1+3u+3u^2, 3+u+3u^2, 3+3u+u^2$, and $i = 1,2,3$. So their Gray images $\Phi_2(\mathfrak{C}_v)$ are observed as cyclic codes over $\mathbb{Z}_4$.

(iv) If $\mathfrak{C}_v$ is a (θ_i,λ)-constacyclic code of length ϖ over T_3 for $\lambda = 1+u+3u^2, 1+3u+u^2, 3+u+u^2$, and $3+3u+3u^2$, then their Gray images $\Phi_3(\mathfrak{C}_v)$ are cyclic codes over $\mathbb{Z}_4$.

(v) Let $\mathfrak{C}_v$ denote (θ_i,λ)-constacyclic codes of length ϖ over T_3 for $i, j = 1,2,3$ and $\lambda = 3+2u, 3+2u^2, 1+2u+2u^2$. So their Gray images $\Phi_j(\mathfrak{C}_v)$ are QC codes of index 2 *with a length of 2ϖ over $\mathbb{Z}_4$.*

Proof. Assume that $\mathfrak{C}_v$ is a θ_i-cyclic code of length ϖ over T_3 for $i, j = 1,2,3$. This is $\sigma_{\theta_i}(\mathfrak{C}_v) = \mathfrak{C}_v$. According to the previous proposition, $\Phi_j\sigma_{\theta_i}(\mathfrak{C}_v) = \upsilon_2\Phi_j(\mathfrak{C}_v) = \Phi_j(\mathfrak{C}_v)$. Therefore, $\Phi_j(\mathfrak{C}_v)$ is equivalent to a QC code of index 2 over $\mathbb{Z}_4$ with a length of 2ϖ.

The evidence from others is similarly conducted. □

3.2. *Computational results for θ_i-cyclic codes*

In this section, we search for θ_i-cyclic and (θ_i,λ)-constacyclic codes over T_3 and their $\mathbb{Z}_4$-images. In particular, the automorphism θ_1 and the Gray image ϕ_1 for length 7 is studied. Based on Theorem 6, Theorem 10, and using MAGMA software [28], we present the results of a computational study on θ_1-cyclic codes over T_3. Here, we express each term of the generator polynomial given in this theorem with τ_i, where $i = 1,2,3$. In the present case, $\langle \tau_1, \tau_2, \tau_3 \rangle$ will be the representative of the generator polynomial of T_3. Thus, we have many θ_1-cyclic codes over T_3, whose $\mathbb{Z}_4$-images are new, optimal, and the best-known linear codes. We present all of these codes in the tables that follow. Note that the representation of the elements of T_3 is the same as in Table 1 in [1]. In these tables, the Lee, Euclidean, and Hamming weights are determined for each generator polynomial. While giving information about new and optimal parameters in $\mathbb{Z}_4$, the online database [29] has been checked. In addition, the "*" sign is used to indicate new parameters and the "**" sign is used to indicate optimal parameters. To express the spelling more clearly, the polynomial's coefficients will be listed in decreasing order starting with the highest order x. It is important to note that this length increases to 14 in according to the defined Gray maps since 7 lengths of a code are scanned. For example, the polynomial $(3u^2+3u)x^7 + (2u+1)x^6 + ux^2 + 3$ will be written as $9'E0^3403$. It should be noted that the 0^3 notation indicates that the 0 expression is repeated 3 times. The $\mathbb{Z}_4$ images of some θ_1-cyclic codes over T_3 are given in Tables 1–3, and the $\mathbb{Z}_4$ images of some $(\theta_1, 3+3u+u^2)$-constacyclic codes over T_3 are given in Tables 4 and 5.

Table 1. Some θ_1-cyclic codes over T_3 with $\mathbb{Z}_4$-images.

$\tau_1(\mathbf{x})$	$\tau_2(\mathbf{x})$	$\tau_3(\mathbf{x})$	**Type**	$\mathbf{W_L}$	$\mathbf{W_E}$	$\mathbf{W_H}$
808^3	$v'5'v'b$	$(7')^6$	4^02^7	4^{**}	8	2^{**}
78979	$b^3v'b(v')^2$	$7'3'5'7'$	4^32^1	8^{**}	8	6^{**}
7987	$3'rDv'H$	$(7')^33'7'(3')^2$	4^42^0	6^*	6^*	6^{**}
7^3979^2	$R^3v'000$	$7'7'$	4^12^6	4^{**}	8	2^{**}
7^2989	$7'Dv'NH$	$(5')^7$	4^32^1	8^{**}	8	6^{**}
77	$(5')^3F5'F^2$	$(5')^305'$	4^62^0	4^*	4^*	4^{**}
70779	$U^3e'000$	$(7')^3(3')^27'3'$	4^32^1	8^{**}	8	6^{**}
9897	U^2rUr^2e'	$7'5'(3')^3$	4^42^3	4^{**}	6^{**}	2^{**}
77909	$s'v'de's'$	$(7')^63'$	4^32^4	4^{**}	8	2^{**}
79909	$N^3(e')^4$	$3'5'3'7'$	4^32^1	8^{**}	8	6^{**}
77909	$s'v'de's'$	$(7')^63'$	4^32^4	4^{**}	8	2^{**}
79909	$b(v')^20v'$	$7'5'(3')^3$	4^32^3	8^{**}	8	4^{**}
79909	U^2drHds'	$(7')^23'5'3'$	4^32^2	4^{**}	8^{**}	2^{**}
8808	$dUD7's'$	$(7')^63'$	4^02^7	4^{**}	8	2^{**}
8088	$H^3(s')^2Hs'$	$7'0(7')^23'$	4^02^7	4^{**}	8	2^{**}
77989	$(3')^2Hv'(s')^3$	$3'5'7'(3')^2$	4^32^4	4^{**}	8	2^{**}
78979	$(5')^2e'Fr(e')^2$	$7'5'(7')^2$	4^32^4	4	8	2
9787	U^3DrD^2	$7'3'5'7'$	4^42^1	4^*	6^*	2^*
8088	$dbs'e's'$	$(7')^33'7'(3')^2$	4^02^7	4^{**}	8	2^{**}
8808	d^3H^2ds'	$(5')^32$	4^02^4	12	24^{**}	6^{**}
99789	$3's'be'H$	$(5')^7$	4^32^1	8^{**}	8	6^{**}
99789	$3's'be'H$	$(5')^7$	4^32^1	8^{**}	8	6^{**}
8808	$(7')^2dHs'7's'$	$7'5'(3')^3$	4^02^{5*}	4^*	8^*	2^*
70779	$(7')^3(3')^2DA$	$5'0(5')^3$	4^32^2	4^{**}	8^{**}	2^{**}
8^7	$7'v'3'Rv'$	$(7')^3(3')^27'3'$	4^02^4	12	24^{**}	6^{**}
79909	$HrDv's'$	$(5')^305'$	4^32^3	8^{**}	8	4^{**}
8808	$U3'rHD$	$(7')^33'7'(3')^2$	4^02^7	4^{**}	8	2^{**}
7987	$(7')^2Ds'dv's'$	$(7')^23'5'3'$	4^42^3	4^{**}	6^{**}	2^{**}
9787	$(3')^27'rH$	$(7')^3(3')^27'3'$	4^42^3	4^{**}	6^{**}	2^{**}

Table 2. Some θ_1-cyclic codes over T_3 with $\mathbb{Z}_4$-images.

$\tau_1(\mathbf{x})$	$\tau_2(\mathbf{x})$	$\tau_3(\mathbf{x})$	**Type**	$\mathbf{W_L}$	$\mathbf{W_E}$	$\mathbf{W_H}$
77989	$(3')^2Hv'(s')^3$	$(5')^7$	4^32^2	4^{**}	8^{**}	2^{**}
70779	$(7')^3(3')^2DH$	$7'3'5'7'$	4^32^2	4^{**}	8^{**}	2^{**}
70779	$(7')^2D^2dbs'$	$(7')^2(3')^27'3'$	4^32^2	4^{**}	8^{**}	2^{**}
9787	$(5')^3Ur$	$(7')^63'$	4^42^3	4^{**}	6^{**}	2^{**}
88	$(7')^2b3'v'bv'$	$(3')^27'5'3'$	4^02^7	4^{**}	8	2^{**}
7^3979^2	$v'UFr$	$(3')^7$	4^12^3	12^{**}	14^{**}	6^{**}
8088	$H^3(s')^2Hs'$	$(5')^305'$	4^02^4	12	24^{**}	6^{**}
79909	U^2drHds'	$(7')^23'5'3'$	4^32^2	4^{**}	8^{**}	2^{**}
77989	$(7')^23'Fv'$	$(7')^23'03'$	4^32^0	8	8^{**}	8^{**}
99789	$(e')^7$	$(7')^3(3')^27'3'$	4^32^1	8^{**}	8	6^{**}
77989	$UHe'v'D$	$7'5'(3')^3$	4^32^4	4^{**}	8	2^{**}
7^39^279	$Ube'D^2$	$(7')^23'03'$	4^12^6	4^{**}	8	2^{**}
7877	$(7')^2dv's'ds'$	$(5')^205'$	4^42^1	4^*	6^*	2^*
7^39^279	$De'Hv's'$	$(5')^305'$	4^12^6	4^{**}	8	2^{**}
79909	$Ude'7'D$	$(7')^63'$	4^32^1	8^{**}	8	6^{**}
99789	$(e')^7$	$7'(3')^203'$	4^32^1	8^{**}	8	6^{**}
8^308	$5'rFUr$	$3'5'7'(3')^2$	4^02^7	4^{**}	8	2^{**}
789^3	$U^3Ds'rD$	$(7')^23'5'3'$	4^32^4	4^{**}	8	2^{**}
70779	$(7')^2D^2dbs'$	$(5')^20(5')^2$	4^32^2	4^{**}	8^{**}	2^{**}
79909	U^2drHds'	$7'5'7'7'$	4^32^2	4^{**}	8^{**}	2^{**}
99789	$3's'be'H$	$(3')^7$	4^32^1	8^{**}	8	6^{**}
7^63	$7'DHFH$	$7'5'3'7'3'$	4^12^6	4^{**}	8	2^{**}
78979	$3'5'7'H^2$	$7'5'3'7'3'$	4^32^3	8^{**}	8	4^{**}
9897	$7'F3'bv'$	$(7')^33'7'(3')^2$	4^42^3	4^{**}	6	2^{**}
77909	$R^27'b$	$(7')^33'7'(3')^2$	4^32^4	4^{**}	8	2^{**}
78999	$R^3v'Fv'b$	$3'7'5'7'$	4^32^2	4^{**}	8^{**}	2^{**}
78999	U^2dH^2Us'	$3'5'7'(3')^2$	4^32^2	4^{**}	8^{**}	2^{**}
77989	$UrNFe'$	$(5')^7$	4^32^0	8	8^{**}	8^{**}
8808	d^3H^2ds'	$7'5'(7')^2$	4^02^7	4^{**}	8	2^{**}

Table 3. Some θ_1-cyclic codes over T_3 with $\mathbb{Z}_4$-images.

$\tau_1(\mathbf{x})$	$\tau_2(\mathbf{x})$	$\tau_3(\mathbf{x})$	**Type**	$\mathbf{W_L}$	$\mathbf{W_E}$	$\mathbf{W_H}$
7^39^279	D^2HRs'	$3'5'3'7'$	4^12^3	12^{**}	14^{**}	6^{**}
8808	R^5F^2	$(3')^27'5'3'$	4^02^7	4^{**}	8	2^{**}
77	d^3D^2ds'	$7'(3')^203'$	4^62^1	4^*	4^*	2^*
7^3979^2	$(3')^5H^2$	$(7')^23'03'$	4^12^4	4^*	8^*	2^*
7987	$(7')^2Ds'dv's'$	$(7')^3(3')^27'3'$	4^42^1	4^*	6^*	2^*

Table 4. Some $(\theta_1, 3+3u+u^2)$-constacyclic codes over T_3 with $\mathbb{Z}_4$-images.

$\tau_1(\mathbf{x})$	$\tau_2(\mathbf{x})$	$\tau_3(\mathbf{x})$	**Type**	$\mathbf{W_L}$	$\mathbf{W_E}$	$\mathbf{W_H}$
8^7	$FR7'7'7'Rb$	$7'7'23'7'3'7'7'$	$4^0 2^{2^*}$	12^*	24^*	6^*
98779	$FR3'b3'5'v'$	$7'03'(7')^2$	$4^3 2^1$	8^{**}	8^{**}	$4*$
79	$FR0^3R^2$	$7'5'7'3'7'$	$4^6 2^0$	4^{**}	4^{**}	4^{**}
70799	$3'7'7'Hbs'U$	$5'5'05'005'$	$4^3 62^2$	4^{**}	8^{**}	2^{**}
98779	$7'3'3'7'7'v'v'$	$7'03'3'7'$	$4^3 2^1$	8^{**}	8^{**}	6^{**}
7979799	$7'DUFe'$	$7'5'3'7'$	$4^1 2^6$	4^{**}	8	2^{**}
9797979	$UDs'rdDd$	$7'03'3'7'$	$4^1 2^4$	4^{**}	8^{**}	2^{**}
97789	$rde'de'Hr$	$5'$	$4^3 2^2$	4^{**}	8^{**}	2^{**}
9899	$7'7'HUdr'D$	$3'7'$	$4^4 2^3$	4^{**}	6^{**}	2^{**}
9989	$FR5'b7'0v'$	$7'5'7'3'7'$	$4^4 2^0$	8	8	6^{**}
78999	$FR3'3'b5'b$	$7'03'7'7'$	$4^3 2^1$	8^{**}	8^{**}	4^*
9899	$rs'e'HUDU$	$7'3'5'3'$	$4^4 2^3$	4^{**}	6^{**}	2^{**}
79989	$3'7'Fb$	$3'7'7'3'3'7'7'$	$4^3 2^4$	4^{**}	8	2^{**}
98779	$5'5'UdU5'N$	$5'5'0^3 5'5'$	$4^3 2^2$	4^{**}	8^{**}	2^{**}
7789	$3'Ue'v'e'$	$7'7'3'7'3'7'7'$	$4^4 2^3$	4^{**}	6^{**}	2^{**}
8088	$HUs'rDUd$	$7'5'7'3'7'$	$4^0 2^7$	4^{**}	8	2^{**}
77909	$FR0^3bv'$	$7'3'5'3'$	$4^3 2^4$	4^{**}	8	2^{**}
78979	$d7's'Dd$	$5'5'005'05'$	$4^3 2^3$	4^*	8	2^*

Table 5. Some $(\theta_1, 3+3u+u^2)$-constacyclic codes over T_3 with $\mathbb{Z}_4$-images.

$\tau_1(\mathbf{x})$	$\tau_2(\mathbf{x})$	$\tau_3(\mathbf{x})$	**Type**	$\mathbf{W_L}$	$\mathbf{W_E}$	$\mathbf{W_H}$
79779^3	$7'de'FN$	$7'3'3'7'7'3'7'$	$4^1 2^6$	4^{**}	8	2^{**}
7879	$7'3'HrH7's'$	$3'7'7'3'3'7'7'$	$4^4 2^1$	4^{**}	6^{**}	2^{**}
98779	$7'3'3'7'7'v'v'$	$5'5'5'05'$	$4^3 2^1$	8^{**}	8^{**}	6^{**}
$797^4 9$	$FR(7')^3Rb$	$3'7'5'3'$	$4^1 2^4$	4^{**}	8^{**}	2^{**}
88808	$7'03'bv'$	$7'03'3'7'$	$4^0 2^6$	8^{**}	16^{**}	4^{**}

4. DNA codes over T_3

DNA forms the genes that carry the code for biological processes in living organisms. The information needed to make the substances that cells need is stored in DNA. The double helix structure that forms the physical shape of the DNA structure consists of bases. There are 4 fundamental bases in living genetics. These are Adenine (A), Guanine (G), Cytosine (C), and Thymine (T). These bases are arranged on the double helix of DNA by a normal size. This is called the Watson-Crick complement (WCC). In relation to this normality, A and T, G and C are connected. There are also two hydrogen bonds between the bases A and T, and three hydrogen bonds between the bases G and C.

Now, we will first talk about some notations and give some basic definitions. Then we will explain the DNA reversibility problem, define a unit reverse polynomial, and relate the elements of the ring T_3 to the DNA codons.

Definition 4. *Let D be a code of arbitrary length ϖ over a finite set Λ.*

(i) If for all $\mathfrak{y}^R = (\mathfrak{y}_{\varpi-1}, \mathfrak{y}_{\varpi-2}, \dots, \mathfrak{y}_0) \in D$ for $\mathfrak{y} = (\mathfrak{y}_0, \dots, \mathfrak{y}_{\varpi-1}) \in D$, then D is called reversible code.

(ii) If for all $\mathfrak{y}^{RC} = (\mathfrak{y}_{\varpi-1}, \mathfrak{y}_{\varpi-2}, \dots, \mathfrak{y}_0)^C \in D$ for $z = (\mathfrak{y}_0, \dots, \mathfrak{y}_{\varpi-1}) \in D$, then D is called reversible complement code.

Definition 5. *Let $r(x) \in T_3$ be a polynomial of degree b and let $r(x)$ be expressed as $r_0 + r_1x + \cdots + r_{b-1}x^{b-1}$. For the coefficients of the polynomial $r(x)$ where $j = 0, 1, \dots, b-1$, if $r_j = r_{b-j}$, then the polynomial $r(x)$ is referred to as a palindromic polynomial.*

The DNA code of length ϖ is described as a set of code words $(f_0, f_1, \dots, f_{\varpi-1})$ such that $f_i \in \{A, G, C, T\}$. The following restrictions for these code words exist in DNA.

Let D be a DNA code word and d be a positive integer,

(i) Hamming Distance Constraint: $\forall \mathfrak{y}, \mathfrak{g} \in D$ and $\mathfrak{y} \neq \mathfrak{g} : d_H(\mathfrak{y}, \mathfrak{g}) \geq d$,

(ii) Reverse Constraint: $\forall \mathfrak{y}, \mathfrak{g} \in D$ and $\mathfrak{y} \neq \mathfrak{g} : d_H(\mathfrak{y}^R, \mathfrak{g}) \geq d$,

(iii) Reverse Complement Constraint: $\forall \mathfrak{y}, \mathfrak{g} \in D$ and $\mathfrak{y} \neq \mathfrak{g} : d_H(\mathfrak{y}^{RC}, \mathfrak{g}) \geq d$ such that $\mathfrak{y}^{RC}$ is the WC-complement of $\mathfrak{y}^R$,

(iv) GC-content Constraint: $\forall \mathfrak{y} \in D :$ The total number of G and C bases contained in each $\mathfrak{y}$ code word is equal.

We will use Hamming distance, reverse, and reverse complement constraints here. The GC-content constraint will be left as an open problem.

To explain the reversibility problem; let $(z_1, z_2, z_3, z_4, z_5)$ be a code word corresponding to $GGTCCTGGAA$ as a DNA strain where $z_1 = 3u + u^2, z_2 = u^2 + 3u + 3, z_3 = 3u^2 + 2, z_4 = 2u^2 + u + 3, z_5 = 2u^2 + u + 1 \in T_3$. The reverse of $(z_1, z_2, z_3, z_4, z_5)$ is $(z_5, z_4, z_3, z_2, z_1)$, and this DNA strain corresponds to $AAGGCTTCGG$. However, the reverse of the DNA strain $GGTCCTGGAA$ is $AAGGTCCTGG$. It is trivial that the DNA strain of the reverse of $(z_1, z_2, z_3, z_4, z_5)$ is not equal to the DNA strain $AAGGCTTCGG$. We have a reversibility problem when we convert the element of a ring to binary or more DNA via the Gray map. Although there are several methods to solve this problem, we have identified unit reverse polynomials and a suitable new generation method for these polynomials to solve the DNA reversibility problem.

First of all, let's define the sets as follows.

$U_A = \{1, 3+2u, 1+u+u^2, 3+3u+u^2, 3+2u^2, 1+2u+2u^2, 3+u+3u^2, 1+3u+3u^2\}$,

$U_B = \{3, 1+2u, 3+u+u^2, 1+3u+u^2, 1+2u^2, 3+2u+2u^2, 1+u+3u^2, 3+3u+3u^2\}$,

$\kappa = \{0, 2\}$.

Now we define the unit reverse polynomial with the help of these sets. This polynomial will help us to find a reversible DNA code.

Definition 6. *[Unit reverse polynomial]*

Let $g(x)$ be a polynomial of degree t over T_3 and $\mathfrak{y}$ be an element of T_3. In this case, *(i) If the degree of the polynomial $g(x)$ is even, then the unit reverse polynomial is*

$$U_R(x) = \mathfrak{y}_S + \mathfrak{y}_Y x^t + (\sum_{i=1}^{(t/2)-1} \beta_S x^i + \beta_Y x^{t-i}) + \kappa_{0,2} x^{t/2}.$$

(ii) If the degree of the polynomial $g(x)$ is odd, then the unit reverse polynomial is

$$U_R(x) = \sum_{i=0}^{(t-1)/2} \beta_S x^i + \beta_Y x^{t-i},$$

where $\mathfrak{y}_S \in U_A$, $\mathfrak{y}_Y \in U_B$, $\beta_S \in U_A$, $\beta_Y \in U_B$. Here, if $\mathfrak{y}_S = U_B$, then $\mathfrak{y}_Y = U_A$, and if $\beta_S = U_B$, then $\beta_Y = U_A$.

Example 1. $1+(1+u+u^2)x+(2u+3)x^2+2x^3+(3u^2+3u+3)x^4+(u^2+3u+1)x^5+(2u+1)x^6$ *is a unit reverse polynomial with even degree in $T_3[x]$.* $(3+2u^2)+(2u+3)x+(2u+1)x^2+(3u^2+u+1)x^3$ *is a unit reverse polynomial with odd degree in $T_3[x]$.*

Now, first of all, we define a $\mathfrak{T}-$module code with the help of the paper by Oztas et.al. that motivates us. We also remaind that $\mathfrak{T}-$module code is called an x-module code if $\mathfrak{T}$ is generated by an $x \in \mathbb{R}$.

Definition 7. *[19] Let $\mathfrak{C}$ be a code generated by $p(x)$ in $R[x]/\langle x^n - 1\rangle$ where $\mathfrak{T}$ is a subring of R and E is a generator set for $\mathfrak{T}$. Here $\mathfrak{C} = \{(y_0 + y_1 x + \cdots + y_{n-1} x^{n-1}) p(x) \mid y_i \in \mathfrak{T}\}$ or $\mathfrak{C} = \{(y_0 c_1 + y_1 c_2 + \ldots, y_a c_a) p(x) \mid y_i \in \mathfrak{T}\}$ is a subset of R^n.*

By finding a ring in which any $k-$base of DNA lives, they identified the $k-$base of the DNA strain with an element of the ring they were studying. It was observed that the problem of reversibility arises with the definition of $k-$bases. To solve this reversibility problem, they presented new notations and new definitions, as mentioned above. Using the $\mathfrak{T}-$module code, they give some notations for n-tuples of DNA k-bases. These notations help to find the reverse of the DNA k-bases, which is provided in the ring structure.

Although the T_3 ring we are working with has 64 elements, it cannot be decomposed into three separate parts. Therefore, the ring elements cannot correspond to DNA 3-mers. Hence, due to the defined Gray structure, the ring elements correspond to the DNA 2-mers, and this happens with restricted elements.

Consider in this strategy we define the function ζ to describe the components of T_3 and 2-mers. To create the map ζ, we match the elements of $\mathbb{Z}_4$ and DNA bases according to the following methodology.

$$\zeta : \mathbb{Z}_4 \longrightarrow \{A, G, T, C\}.$$

Here, we define $\zeta(0) = A$, $\zeta(1) = T$, $\zeta(2) = G$, $\zeta(3) = C$. The ζ map can be mapped in 24 different ways between $\mathbb{Z}_4$ and DNA sequences. For example, $\zeta(0) = C, \zeta(1) = A, \zeta(2) = T, \zeta(3) = G$ or $\zeta(0) = A, \zeta(1) = G, \zeta(2) = C, \zeta(3) = T$, etc. The use of this type of multi-map also provides a variety of examples obtained with the Theorem 13. Using the Gray map ϕ_1 and the transformation ζ, which pairs DNA bases with elements of $\mathbb{Z}_4$, this paper presents a description of the map $\vartheta = \zeta o \phi_1$ to match the elements of T_3 with DNA.

$$\vartheta : T_3 \longrightarrow \{A, G, T, C\}^2,$$

$$a_0 + ua_1 + u^2a_2 \longrightarrow (\zeta(a_0 + a_1 + 3a_2),\ \zeta(3a_0 + 3a_1 + a_2)).$$

This map extended component-wise to

$$\vartheta : T_3^{\varpi} \mapsto \{A, G, T, C\}^{2\varpi},$$

$$(\mathfrak{y}_0, \mathfrak{y}_1, \ldots, \mathfrak{y}_{\varpi-1}) \mapsto (\zeta(\phi_1(\mathfrak{y}_0)), \zeta(\phi_1(\mathfrak{y}_1)), \ldots, \zeta(\phi_1(\mathfrak{y}_{\varpi-1})),$$

where $\mathfrak{y}_i = a_0^i + ua_1^i + u^2a_2^i$ for $i = 0, \ldots, \varpi - 1$.

Now, we create a generation method for the unit reverse polynomial to construct reversible codes over T_3.

Definition 8. *[Generation of $\mathfrak{H}_w$-Module with Unit Reverse Polynomial] $\mathfrak{H}_w(U_R(x))$ and ${\mathfrak{H}_w}^{+1}(U_R(x))$ are generator matrices defined by $U_R(x)$ over T_3 for codes of length ϖ.*

$$\mathfrak{H}_w(U_R(x)) = \begin{bmatrix} U_R(x) \\ xU_R(x) \\ \vdots \\ \\ x^{\varpi-t-1}U_R(x) \end{bmatrix},$$

and

$${\mathfrak{H}_w}^{+1}(U_R(x)) = \begin{bmatrix} U_R(x) \\ xU_R(x) \\ \vdots \\ \\ x^{\varpi-t-1}U_R(x) \\ \mathfrak{p}_3(x) \end{bmatrix},$$

such that $\mathfrak{K} = \{\mathfrak{b}, \mathfrak{b}'\}$ and the polynomial

$$\mathfrak{p}_v(x) = \begin{cases} \sum_{i=0}^{(\varpi-2)/2} \mathfrak{b}x^i + \mathfrak{b}'x^{\varpi-i-1}, & \textit{if} \quad \varpi \quad \textit{is even}, \\ \\ \sum_{i=0}^{(\varpi-1)/2} \mathfrak{b}x^i + \mathfrak{b}'x^{\varpi-i-1} + \mathfrak{a}x^{(\varpi-1)/2} \quad \textit{where} \quad \mathfrak{a} \in \kappa, & \textit{if} \quad \varpi \quad \textit{is odd}, \end{cases}$$

where $\mathfrak{K} = \{1, 3\}$.

Let us consider the polynomial $U_R(x) = \mathfrak{s} + \mathfrak{s}_g x + \cdots + \mathfrak{s}_t x^t$ such that $\mathfrak{s}_i \in T_3$. In this case, the generator matrix $\mathfrak{H}_w(U_R(x))$ of the polynomial $U_R(x)$ is

$$\begin{bmatrix} \mathfrak{s} & \mathfrak{s}_g & \mathfrak{s}_y & \ldots & \mathfrak{s}_t & 0 & 0 & \ldots & 0 \\ 0 & \mathfrak{s} & \mathfrak{s}_g & \mathfrak{s}_y & \ldots & \mathfrak{s}_t & 0 & \ldots & 0 \\ \vdots & & & \ddots & & & \vdots & & \\ 0 & & \ldots & & 0 & \mathfrak{s} & \mathfrak{s}_g & \ldots & \mathfrak{s}_t \end{bmatrix},$$

and the generator matrix ${\mathfrak{H}_w}^{+1}(U_R(x))$ of the polynomial $U_R(x)$ is

$$\begin{bmatrix} \mathfrak{s} & \mathfrak{s}_g & \mathfrak{s}_y & \dots & \mathfrak{s}_{\mathfrak{t}} & 0 & 0 & \dots & 0 \\ 0 & \mathfrak{s} & \mathfrak{s}_g & \mathfrak{s}_y & \dots & \mathfrak{s}_{\mathfrak{t}} & 0 & \dots & 0 \\ \vdots & & & \ddots & & & & \vdots & \\ 0 & & \dots & & 0 & \mathfrak{s} & \mathfrak{s}_g & \dots & \mathfrak{s}_{\mathfrak{t}} \\ \mathfrak{p}_3(x) & \dots & & \mathfrak{p}_3(x) & \dots & \mathfrak{p}_3(x) & \dots & & \mathfrak{p}_3(x) \end{bmatrix}.$$

Theorem 13. *If the code $\mathfrak{C}_v$ (or ${\mathfrak{C}_v}^{+1}$) is generated by a $U_R(x)$ with the generator matrix $\mathfrak{H}_w(U_R(x))$ (or ${\mathfrak{H}_w}^{+1}(U_R(x))$, $\phi_1(\mathfrak{C}_v)$ (or $\phi_1({\mathfrak{C}_v}^{+1})$) and is a reversible $\mathbb{Z}_4-$code, then $\vartheta(\mathfrak{C}_v)$ and $\vartheta({\mathfrak{C}_v}^{+1})$ are reversible DNA codes.*

Proof. For the polynomial $U_R(x) = \mathfrak{s}x + \mathfrak{s}_g x^2 + \cdots + \mathfrak{s}_{\mathfrak{t}} x^t$, as you can see from the definition, let $x^a U_R(x)$ be any row of the generator matrix $\mathfrak{H}_w(U_R(x))$ where $a \in \{0, \dots, \varpi - t - 1\}$. We get

$$x^a U_R(x) = \mathfrak{s}x^a + \mathfrak{s}_g x^{a+1} + \cdots + \mathfrak{s}_{\mathfrak{t}} x^{a+t},$$

and

$$x^{\varpi-t-a} U_R(x) = \mathfrak{s}x^{\varpi-t-a} + \mathfrak{s}_g x^{\varpi-t-a} + \cdots + \mathfrak{s}_{\mathfrak{t}} x^{\varpi-a}.$$

If these polynomials are multiplied by any scalar $\mathfrak{q} \in \mathbb{Z}_4^*$, we obtain

$$\mathfrak{q}x^a U_R(x) = \mathfrak{q}\mathfrak{s}x^a + \mathfrak{q}\mathfrak{s}_g x^{a+1} + \cdots + \mathfrak{q}\mathfrak{s}_{\mathfrak{t}} x^{a+t},$$

and

$$\mathfrak{q}x^{\varpi-t-a} U_R(x) = \mathfrak{q}\mathfrak{s}x^{\varpi-t-a} + \mathfrak{q}\mathfrak{s}_g x^{\varpi-t-a} + \cdots + \mathfrak{q}\mathfrak{s}_{\mathfrak{t}} x^{\varpi-a}.$$

In this case, we attain

$$\phi_1(\mathfrak{q}x^a U_R(x))^R = \phi_1(\mathfrak{q}x^{\varpi-t-1-a} U_R(x)),$$

due to the choice of the $\mathfrak{s}_{\mathfrak{i}}$'s. Since $\vartheta = \zeta o \phi_1$ and $\mathbb{Z}_4-$reverse is found, DNA reverses can also be found as desired. Therefore,

$$\vartheta(\mathfrak{q}x^a U_R(x))^R = \vartheta(\mathfrak{q}x^{\varpi-t-a} U_R(x))$$

equality is obtained.

□

For the complement of the DNA code, we can say the following:

DNA bases are normalized to correspond to elements of $\mathbb{Z}_4$. For example, if we choose 1 and 3, they are complements of each other. Also, 0 and 2 are complements of each other. Based on this information, if we add a row that has all 2 components for the generator matrix $\mathfrak{H}_w(U_R(x))$, then we can obtain a reversible and complement DNA code using the defined DNA correspondence normalized.

Example 2. *Let $U_R(x)=(2u+3)+(u^2+u+1)x+(3u^2+u+3)x^2+2x^3+(2u^2+1)x^4+(3u^2+u+1)x^5+(2u^2+2u+3)x^6$ be a polynomial over T_3 with length 8. Then, the generator matrix $\mathfrak{H}_w(U_R(x))$ of the polynomial $U_R(x)$ is*

$$\begin{bmatrix} 2u+3 & u^2+u+1 & 3u^2+u+3 & 2 & 2u^2+1 & 3u^2+u+1 & 2u^2+2u+3 & 0 \\ 0 & 2u+3 & u^2+u+1 & 3u^2+u+3 & 2 & 2u^2+1 & 3u^2+u+1 & 2u^2+2u+3 \end{bmatrix},$$

and the generator matrix ${\mathfrak{H}_w}^{+1}(U_R(x))$ of the polynomial $U_R(x)$ is

$$\begin{bmatrix} 2u+3 & u^2+u+1 & 3u^2+u+3 & 2 & 2u^2+1 & 3u^2+u+1 & 2u^2+2u+3 & 0 \\ 0 & 2u+3 & u^2+u+1 & 3u^2+u+3 & 2 & 2u^2+1 & 3u^2+u+1 & 2u^2+2u+3 \\ 1 & 1 & 1 & 1 & 3 & 3 & 3 & 3 \end{bmatrix}.$$

Therefore, we get

$$\phi_1(\mathfrak{H}_w(U_R(x)))=\begin{bmatrix} 1 & 3 & 1 & 3 & 1 & 3 & 2 & 2 & 3 & 1 & 3 & 1 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 3 & 1 & 3 & 2 & 2 & 3 & 1 & 3 & 1 & 3 & 1 \end{bmatrix},$$

and

$$\phi_1({\mathfrak{H}_w}^{+1}(U_R(x)))=\begin{bmatrix} 1 & 3 & 1 & 3 & 1 & 3 & 2 & 2 & 3 & 1 & 3 & 1 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 3 & 1 & 3 & 2 & 2 & 3 & 1 & 3 & 1 & 3 & 1 \\ 1 & 3 & 1 & 3 & 1 & 3 & 1 & 3 & 3 & 1 & 3 & 1 & 3 & 1 & 3 & 1 \end{bmatrix}.$$

When the rows in the $\mathbb{Z}_4-$images of $\mathfrak{H}_w(U_R(x))$ and ${\mathfrak{H}_w}^{+1}(U_R(x))$ generator matrices are multiplied by $\mathfrak{q}\in\mathbb{Z}_4^*$, the first row and the second rows are reverses of each other. For example, when $\phi_1(\mathfrak{H}_w(U_R(x)))$ is multiplied by 3, we obtain

$$\begin{bmatrix} 3 & 1 & 3 & 1 & 3 & 1 & 2 & 2 & 1 & 3 & 1 & 3 & 1 & 3 & 0 & 0 \\ 0 & 0 & 3 & 1 & 3 & 1 & 3 & 1 & 2 & 2 & 1 & 3 & 1 & 3 & 1 & 3 \end{bmatrix}.$$

From this, it is clear that the first row and the second row are reversed of each other.

When the first row and the second row in the $\phi_1(\mathfrak{H}_w(U_R(x)))$ matrix are added together, the resulting sequence

$$\begin{bmatrix} 1 & 3 & 2 & 2 & 2 & 2 & 3 & 1 & 1 & 3 & 2 & 2 & 2 & 2 & 3 & 1 \end{bmatrix}$$

is palindromic, so the reverse is equal to itself.

Adding 2 rows to the generator matrix $\mathfrak{H}_w(U_R(x))$ gives the following matrix.

$$\begin{bmatrix} 2u+3 & u^2+u+1 & 3u^2+u+3 & 2 & 2u^2+1 & 3u^2+u+1 & 2u^2+2u+3 & 0 \\ 0 & 2u+3 & u^2+u+1 & 3u^2+u+3 & 2 & 2u^2+1 & 3u^2+u+1 & 2u^2+2u+3 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{bmatrix}.$$

The $\mathbb{Z}_4-$image of this matrix is

$$\begin{bmatrix} 1 & 3 & 1 & 3 & 1 & 3 & 2 & 2 & 3 & 1 & 3 & 1 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 3 & 1 & 3 & 2 & 2 & 3 & 1 & 3 & 1 & 3 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{bmatrix}.$$

In the $\mathbb{Z}_4-$image of this matrix, the sequence

$$\begin{bmatrix} 3 & 1 & 3 & 1 & 3 & 1 & 0 & 0 & 1 & 3 & 1 & 3 & 1 & 3 & 2 & 2 \end{bmatrix}$$

obtained when the first and third rows are added together is the complement of the first row. When the second and third rows are added together in this matrix, the resulting sequence

$$\begin{bmatrix} 2 & 2 & 3 & 1 & 3 & 1 & 3 & 1 & 0 & 0 & 1 & 3 & 1 & 3 & 1 & 3 \end{bmatrix}$$

is the complement of the second row.

In the $\phi_1(\mathfrak{H}_w{}^{+1}(U_R(x)))$ matrix, the

$$\begin{bmatrix} 2 & 2 & 2 & 2 & 2 & 2 & 3 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 1 \end{bmatrix}$$

sequence obtained when the first row and the third row are added together and the

$$\begin{bmatrix} 1 & 3 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 3 & 2 & 2 & 2 & 2 & 2 & 2 \end{bmatrix}$$

sequence obtained when the second row and the third row are summed are the reverse of each other. When it is multiplied by any $\mathfrak{q} \in \mathbb{Z}_4^*$ and the same combined operations are performed, it is seen that the rows are reversed to each other.

Example 3. *Let $U_R(x) = (2u^2+1) + 3x + (2u+1)x^2 + (u^2+3u+3)x^3 + (2u^2+3)x^4 + (2u+3)x^5$ be a polynomial over T_3 with length 9. Then, the generator matrix $\mathfrak{H}_w(U_R(x))$ of the polynomial $U_R(x)$ is*

$$\begin{bmatrix} 2u^2+1 & 3 & 2u+1 & u^2+3u+3 & 2u^2+3 & 2u+3 & 0 & 0 & 0 \\ 0 & 2u^2+1 & 3 & 2u+1 & u^2+3u+3 & 2u^2+3 & 2u+3 & 0 & 0 \\ 0 & 0 & 2u^2+1 & 3 & 2u+1 & u^2+3u+3 & 2u^2+3 & 2u+3 & 0 \\ 0 & 0 & 0 & 2u^2+1 & 3 & 2u+1 & u^2+3u+3 & 2u^2+3 & 2u+3 \end{bmatrix},$$

and the generator matrix $\mathfrak{H}_w{}^{+1}(U_R(x))$ of the polynomial $U_R(x)$ is

$$\begin{bmatrix} 2u^2+1 & 3 & 2u+1 & u^2+3u+3 & 2u^2+3 & 2u+3 & 0 & 0 & 0 \\ 0 & 2u^2+1 & 3 & 2u+1 & u^2+3u+3 & 2u^2+3 & 2u+3 & 0 & 0 \\ 0 & 0 & 2u^2+1 & 3 & 2u+1 & u^2+3u+3 & 2u^2+3 & 2u+3 & 0 \\ 0 & 0 & 0 & 2u^2+1 & 3 & 2u+1 & u^2+3u+3 & 2u^2+3 & 2u+3 \\ 3 & 3 & 3 & 3 & 2 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Therefore, we get

$$\phi_1(\mathfrak{H}_w(U_R(x))) = \begin{bmatrix} 3&1&3&1&3&1&1&3&1&3&1&3&0&0&0&0&0&0 \\ 0&0&3&1&3&1&3&1&1&3&1&3&1&3&0&0&0&0 \\ 0&0&0&0&3&1&3&1&3&1&1&3&1&3&1&3&0&0 \\ 0&0&0&0&0&0&3&1&3&1&3&1&1&3&1&3&1&3 \end{bmatrix},$$

and

$$\phi_1({\mathfrak{H}_w}^{+1}(U_R(x))) = \begin{bmatrix} 3&1&3&1&3&1&1&3&1&3&1&3&0&0&0&0&0&0 \\ 0&0&3&1&3&1&3&1&1&3&1&3&1&3&0&0&0&0 \\ 0&0&0&0&3&1&3&1&3&1&1&3&1&3&1&3&0&0 \\ 0&0&0&0&0&0&3&1&3&1&3&1&1&3&1&3&1&3 \\ 3&1&3&1&3&1&3&1&2&2&1&3&1&3&1&3&1&3 \end{bmatrix}.$$

When the rows in the $\mathbb{Z}_4-$images of $\mathfrak{H}_w(U_R(x))$ and ${\mathfrak{H}_w}^{+1}(U_R(x))$ generator matrices are multiplied by $\mathfrak{q} \in \mathbb{Z}_4^*$, the first row and the fourth row, and the second row and the third row are reverses of each other. For instance, when $\phi_1(\mathfrak{H}_w(U_R(x)))$ is multiplied by 3, we obtain

$$\begin{bmatrix} 1&3&1&3&1&3&3&1&3&1&3&1&0&0&0&0&0&0 \\ 0&0&1&3&1&3&1&3&3&1&3&1&3&1&0&0&0&0 \\ 0&0&0&0&1&3&1&3&1&3&3&1&3&1&3&1&0&0 \\ 0&0&0&0&0&0&1&3&1&3&1&3&3&1&3&1&3&1 \end{bmatrix}.$$

From this, it is clear that the first row and the fourth row, and the second row and third row are reverses of each other.

If the first row and the fourth row in the matrix $\phi_1(\mathfrak{H}_w(U_R(x)))$ are added together, the resulting sequence is

$$\begin{bmatrix} 3&1&3&1&3&1&2&2&0&0&0&0&1&3&1&3&1&3 \end{bmatrix},$$

and when the second row and the third rows in the $\phi_1(\mathfrak{H}_w(U_R(x)))$ matrix are added together, the resulting sequence is

$$\begin{bmatrix} 0&0&3&1&2&2&2&2&0&0&2&2&2&2&1&3&0&0 \end{bmatrix}$$

palindromic, so their reverses are equal to themselves.

In the $\phi_1({\mathfrak{H}_w}^{+1}(U_R(x)))$ matrix, the

$$\begin{bmatrix} 2&2&2&2&2&2&0&0&3&1&2&2&1&3&1&3&1&3 \end{bmatrix}$$

sequence obtained when the first row and the fifth row are added together and the

$$\begin{bmatrix} 3 & 1 & 3 & 1 & 3 & 1 & 2 & 2 & 1 & 3 & 0 & 0 & 2 & 2 & 2 & 2 & 2 & 2 \end{bmatrix}$$

sequence obtained by summing the fourth row and the fifth rows is the reverse of each other. At the same time, the

$$\begin{bmatrix} 3 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 3 \end{bmatrix}$$

sequence is obtained when the second row and the fifth row are added together and the

$$\begin{bmatrix} 3 & 1 & 3 & 1 & 2 & 2 & 2 & 2 & 1 & 3 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 3 \end{bmatrix}$$

sequence is obtained when the third row and the fifth row are summed and are the reverse of each other. When it is multiplied by any $q \in \mathbb{Z}_4^*$ *and the same combinations are performed, it is seen that the rows are reversed to each other.*

5. Conclusions

First, the basic notations are given by reference to the ring structure. Then, all nonobvious automorphisms over T_3 are identified and included in the basic definition and theorems related to the skew polynomial ring. The algebraic structure of the θ_i-cyclic codes of odd length was analyzed using the decomposition method, and the generator polynomial is determined. In addition, an isomorphism between θ_i-cyclic codes and (θ_i, λ)-constacyclic codes is established to obtain the generator polynomial of the (θ_i, λ)-constacyclic codes. Using this isomorphism, the generator polynomial of θ_i-cyclic codes was obtained. Under the described automorphism θ_1, for each unit over the ring T_3, $\mathbb{Z}_4$-images of the (θ_i, λ)-constacyclic codes have been analyzed and significant results have been obtained. Using MAGMA, new and optimal codes have been found and presented in tables. In addition, some basic definitions and theorems about the DNA codes have been included. Through the ϕ_1 Gray map, a relationship between the elements of T_3 and the DNA 2-mers has been established. By defining a unit reverse polynomial, a new generation method has been built. To enhance comprehensibility, supporting examples are provided.

Author contributions

Fatma Zehra UZEKMEK: Conceptualization, formal analysis, investigation, methodology, project administration, validation, visualization, writing-original draft, writing-review-editing; Elif Segah ÖZTAS: Conceptualization, methodology, project administration, software, supervision, validation, visualization, writing-original draft, writing-review-editing; Mehmet ÖZEN: Conceptualization, methodology, project administration, supervision, validation. All authors have read and agreed to the published version of the manuscript.

Conflict of interest

All authors declare no conflicts of interest in this paper.

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