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Research article

The connection between the magical coloring of trees

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Abstract: Let f be a set-ordered edge-magic labeling of a graph G from $V(G)$ and $E(G)$ to $[0, p-1]$ and $[1, p-1]$, respectively; it also satisfies the following conditions: $|f(V(G))| = p$, $\max f(X) < \min f(Y)$, and $f(x) + f(y) + f(xy) = C$ for each edge $xy \in E(G)$. In this paper, we removed the restriction that the labeling of vertices could not be repeated, and presented the concept of magical colorings including edge-magic coloring, edge-difference coloring, felicitous-difference coloring, and graceful-difference coloring. We studied the magical colorings on the tree and proved the existence of four kinds of magical colorings on the tree from a set-ordered edge-magic labeling. Further, we revealed the transformation relationship between these kinds of colorings.

Keywords: set-ordered edge-magic labeling; magical coloring; tree
Mathematics Subject Classification: 05C15

1. Introduction

With the rapid development of quantum computing theory and quantum computer technology, the traditional public key cryptosystem is facing great challenges [1–3]. Because of the difficulty in solving the two NP-complete problems of graph coloring and subgraph isomorphism, topological graphs are integrated into the cryptography field to resist quantum computer attacks [4–7], this means that NP-complete problems can be solved in non-deterministic polynomial time, but no efficient deterministic polynomial time algorithm has been found to solve such problems. Sedláček first defined the magic labeling of a graph [8], and his research stems from magic squares in number theory. Based on Bloom and Golomb results [9], Wallis proposed the concept of edge magic total labeling in his research on communication networks and radar pulse coding for assigned address [10]. Wang et al. [11, 12] designed a topological coding consisting of topological structure and graph colorings. Yao et al. [13] arranged the colors of each edge and two end-vertices of a (p,q)-graph G in topological matrix $T_{code} = (X,E,Y)^T_{3\times q}$, where $X = (f(x_1), f(x_2), \cdots, f(x_q))$, $E = (f(e_1), f(e_2), \cdots, f(e_q))$, $Y = (f(y_1), f(y_2), \cdots, f(y_q))$ are three vectors of real numbers, and matrix T_{code} distributes us at least $(3q)!$ different number-based strings in total based on a coloring f. Therefore, if a topology has many types of colorings and these colorings can be transformed into each other, the key space of topology encryption will be increased.

In order to increase the diversity and space of topological keys, we extend the graph labeling and the graph coloring in the mathematical function to obtain a coloring function with more restrictive conditions, which is used to generate the encryption key and decryption key of the topological model. According to the topological coding diagram, we give some methods of obtaining the topological key by combining the graph base and graph operation. In addition, we generate large-scale graphs on the basis of small-scale graphs for key generation, and study the relationship between graph labeling and graph coloring in these models.

The graphs mentioned here are undirected, simple finite graphs. A graph with p vertices and q edges is called a (p,q)-graph. A symbol $[a,b]$ represents a set $\{x \in Z : a \leq x \leq b\}$ with $a, b \in N$ and $a < b$. The number of elements of a set X is denoted as $|X|$, $N(u)$ is the set of vertices adjacent with a vertex u, and the number $d(u) = |N(u)|$ is called the degree of the vertex u. If $d(u) = 1$ then the vertex u is called a leaf. The notations and terminologies not mentioned here can be found in [14, 15].

Definition 1. *Let G be a (p,q)-graph with a vertex bipartition (X,Y) holding $V(G) = X \cup Y$ and $X \cap Y = \emptyset$. Suppose that G admits a labeling $f : V(G) \cup E(G) \rightarrow [0, M]$, and $f(P) = \{f(w) : w \in P\}$ represents a set composed of the coloring of all elements in set P. These are the following restrictions:*

1) $|f(V(G))| = p$;

2) $f(V(G)) \subseteq [0, p-1], \min f(V(G)) = 0$;

3) $f(E(G)) = \{f(xy) : xy \in E(G)\} = [1, p-1]$;

4) $\max f(X) < \min f(Y)$;

5) $f(V(G)) \bigcup f(E(G)) \subset [0, p+q]$;

6) $f(x) + f(xy) + f(y) = C_1$ *for each edge* $xy \in E(G)$;

7) $|f(x) - f(y)| + f(xy) = C_2$ *for each edge* $xy \in E(G)$;

8) $|f(x) + f(y) - f(xy)| = C_3$ *for each edge* $xy \in E(G)$;

9) $||f(x) - f(y)| - f(xy)| = C_4$ *for each edge* $xy \in E(G)$.

We call f a set-ordered edge-magic labeling if conditions 1), 2), 3), 4), 6) are met. If we allow that there is at least a pair of vertices colored with the same color in the above, it means that $f(u) = f(v)$ for any two vertices u, and $v \in V(G)$ is allowed to be true. We will obtain four types of new colorings:

(1) f is an edge-magic coloring when conditions 5), 6) are met;

(2) f is an edge-difference coloring when conditions 5), 7) are met;

(3) f is a felicitous-difference coloring when conditions 5), 8) are met;

(4) f is a graceful-difference coloring when conditions 5), 9) are met.

The above $C_i (i = 1, 2, 3, 4)$ are constants, called magic constants, and these four colorings are known as magic-type colorings.

Definition 2. *The graph G obtained by adding a new edge $uv \notin E(G)$ is denoted as $G_A = G + uv$, $u \in V(G), v \notin V(G)$. We call the process of obtaining the graph G_A randomly adding leaf operation, and say that G_A is a leaf-added graph and v is a leaf of G. Let G be a (p,q)-graph, and leaf-added* *graph G_A is the result of adding randomly m leaves to graph G, then G_A is a (p_A, q_A)-graph, where $p_A = p + m$ and $q_A = q + m$.*

2. Connections between the four colorings of trees

Lemma 1. *Each tree admits a set-ordered edge-magic labeling defined in Definition 1.*

Proof. Let the number of vertices and edges of tree T be p and q, respectively. Since the vertex of each tree T can be divided into two parts X and Y, (X, Y) represents the bipartition of vertices of the tree T, then $X = \{x_i | i \in [1, s]\}$ and $Y = \{y_j | j \in [1, t]\}$ holding $s + t = |V(T)| = p$. Clearly, $x_i \in X$ and $y_j \in Y$ for each edge x_iy_j of a tree T. Without loss of generality, we let $f(x_i) = i - 1$ with $i \in [1, s]$, $f(y_j) = p - j$ with $j \in [1, t]$, and $f(x_iy_j) = s + j - i$ for every edge $x_iy_j \in E(T)$, so the labels of each vertex satisfy the following relation,

$$f(x_1) < f(x_2) < \cdots < f(x_s) < f(y_t) < f(y_{t-1}) < \cdots < f(y_1), \tag{2.1}$$

so condition $s - 1 = \max f(X) < \min f(Y) = s$ is true. In addition, we can find $f(x_i) + f(x_iy_j) + f(y_i) = i - 1 + s + j - i + p - j = s + p - 1$ for $x_iy_j \in E(T)$. According to Definition 1, f is a set-ordered edge-magic labeling of tree T. An example for illustrating the proof of Lemma 1 (see Figure 1). □

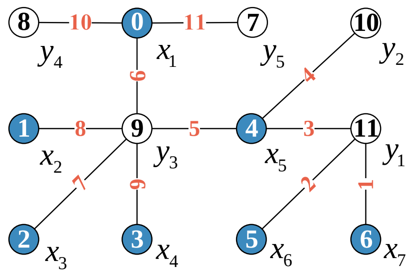

Figure 1. An example for illustrating the proof of Lemma 1.

Theorem 1. *If a tree admits a set-ordered edge-magic labeling, then the tree admits edge-magic coloring, edge-difference coloring, felicitous-difference coloring, and graceful-difference coloring, one of which can be converted from the other magical colorings.*

Proof. Below we show the proof process of the conversion between the edge-magic coloring, edge-difference coloring, felicitous-difference coloring, and graceful-difference coloring. Let the above four types of colorings be f_1, f_2, f_3, and f_4, and the relationship between the four colorings of tree T is given below.

(1) ⇔ (2). Let $f_2(x_i) = f_1(x_{s+1-i})$, $f_2(y_j) = f_1(y_j)$, $f_2(x_iy_j) = f_1(x_iy_j)$. Since f_1 is an edge-magic coloring, f_1 satisfies $f_1(V(t)) \cup f_1(E(T)) \subset [0, p + q]$ and $f_1(x_i) + f_1(x_iy_j) + f_1(y_j) = s + p - 1$. We can deduce that $|f_2(x_i) - f_2(y_j)| + f_2(x_iy_j) = |f_1(x_{s+1-i}) - f_1(y_j)| + f_1(x_iy_j) = f_1(y_j) - s + 1 + f_1(x_i) + f_1(x_iy_j) = p$. It is clear that the coloring of $V(T)$ and $E(T)$ satisfies $f_2(V(T)) \cup f_2(E(T)) \subset [0, p + q]$ when f_1 is transformed to f_2.

(2) ⇔ (3). Let $f_3(x_i) = f_2(x_{s+1-i})$, $f_3(y_j) = f_2(y_j)$, $f_3(x_iy_j) = q + 1 - f_2(x_iy_j)$. Similar to the above proof, we can derive $|f_3(x_i) + f_3(y_j) - f_3(x_iy_j)| = |f_2(x_{s+1-i}) + f_2(y_j) - q - 1 + f_2(x_iy_j)| = p + s - q - 2 = s - 1$ is a constant; also, $f_3(V(t)) \cup f_3(E(T)) \subset [0, p + q]$, so we get to coloring f_3 by coloring f_2.

(3) ⇔ (4). We give the following transformation: $f_4(x_i) = f_3(x_{s+1-i})$, $f_4(y_j) = f_3(y_j)$, $f_4(x_iy_j) = f_3(x_iy_j)$. We can further derive that $||f_4(x_i) - f_4(y_j)| - f_4(x_iy_j)| = ||s - 1 - f_3(x_i) - f_3(y_j)| - f_3(x_iy_j)| = |f_3(x_i) + f_3(y_j) - f_3(x_iy_j)| - s + 1$ is equal to 0, and $f_4(V(t)) \cup f_4(E(T)) \subset [0, p+q]$ is satisfied, which shows that coloring f_3 and coloring f_4 can be transformed into each other.

(4) ⇔ (1). By the following transformation: $f_1(x_i) = f_4(x_{s+1-i})$, $f_1(y_j) = f_4(y_j)$, $f_1(x_iy_j) = q + 1 - f_4(x_iy_j)$, we can get $f_1(x_i) + f_1(x_iy_j) + f_1(y_j) = s + 1 - f_4(x_i) + q + 1 - f_4(x_iy_j) + f_4(y_j) = s + q - f_4(y_j) - f_4(x_i) - f_4(x_iy_j) = s + q$. Also, we get $f_1(V(t)) \cup f_1(E(T)) \subset [0, p+q]$, then f_1 can be further obtained by coloring f_4.

Therefore, Theorem 1 is proved. An example for illustrating the proof of Theorem 1 (see Figure 2). □

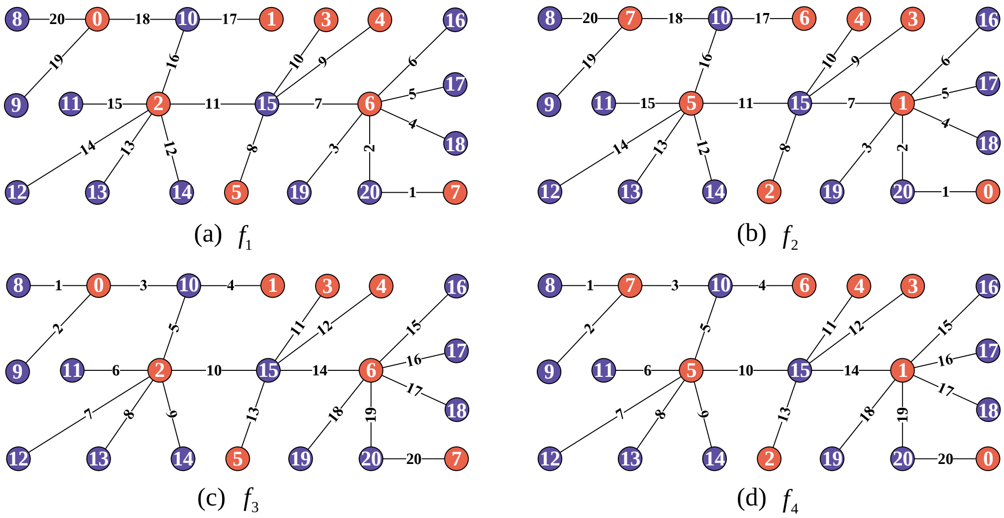

Figure 2. An example for illustrating the proof of Theorem 1.

Theorem 2. *Let a tree T admit a set-ordered edge-magic labeling and (X, Y) represents the bipartition of vertices of T, then $|X| = s$, $|Y| = t$, and its leaf-added graph T_A is obtained by adding K leaves to tree T. Let $|V(T_A)| = p'$, $|E(T_A)| = q'$, then T_A admits an edge-magic coloring and its magic constant is $C_A = s + q' + K$.*

Proof. Lemma 1 proves that every tree T admits a set-ordered edge-magic labeling f, then we prove the existence of magical colorings of the leaf-added graph T_A based on the labeling f of T. Since T has a vertex set $V(T) = X \cup Y$ with $X \cap Y = \emptyset$, where $X = \{x_1, x_2, \cdots, x_s\}$ and $Y = \{y_1, y_2, \cdots, y_t\}$ with $s + t = p = |V(T)|$, according to the definition of a set-ordered edge-magic labeling, we get the set-ordered restriction

$$0 = f(x_1) < f(x_2) < \cdots < f(x_s) < f(y_t) < f(y_{t-1}) < \cdots < f(y_1) = p - 1. \tag{2.2}$$

Let $C = s + q$, then the sum of the labels of each edge $x_iy_j \in E(T)$ and its two end vertices $x_i, y_j \in V(T)$ satisfies

$$f(x_i) + f(x_iy_j) + f(y_j) = s + q = C > 0, \tag{2.3}$$

as well as $f(E(G)) = \{f(x_iy_j) | x_iy_j \in E(T)\} = [1, p - 1]$.

Next, we consider the topology of leaf-added graph T_A of T. Adding randomly m_i new leaves $a_{i,k}$ to each vertex $x_i \in X \subset V(T)$ by joining $a_{i,k}$ with x_i together by new edges $x_i a_{i,k}$ for $k \in [1, m_i]$ and $i \in [1, s]$, the set of new leaves $a_{i,k}$ is denoted by the symbol $L(x_i) = \{a_{i,k} | k \in [1, m_i], i \in [1, s]\}$. Meanwhile, adding randomly n_j new leaves $b_{j,r} \in L(y_j) = \{b_{j,r} | r \in [1, n_j], j \in [1, t]\}$ to each vertex $y_j \in Y \subset V(T)$ by joining $b_{j,r}$ with y_j together by new edges $y_j b_{j,r}$, when $m_i = 0$ or $n_j = 0$ exist, it means that no new leaves are added to a vertex x_i or y_j. Let $M = \sum_{c=1}^{s} m_c$ and $N = \sum_{c=1}^{t} n_c$. Obviously, we have $K = M + N$, so the number of vertices and edges of T_A is $p' = p + M + N$ and $q' = q + M + N$, respectively.

We define a coloring f_1 of the leaf-added graph T_A in the following steps:

Step 1. Color edges $y_j b_{j,r}$ for leaves $b_{j,r} \in L(y_j)$ with $r \in [1, n_j]$ and $j \in [1, t]$ as follows:

$f_1(y_1 b_{1,r}) = r$ for $r \in [1, n_1]$, then the maximum colors of the newly added edge connected to y_1 is $f_1(y_1 b_{1,n_1}) = n_1$;

$f_1(y_2 b_{2,r}) = n_1 + r$ for $r \in [1, n_2]$, so the largest number of colors in the leaves of vertex y_2 is equal to $f_1(y_2 b_{2,n_2}) = n_1 + n_2$;

For $j \in [3, t-1]$, $f_1(y_j b_{j,r}) = \sum_{c=1}^{j-1} n_c + r$ for $r \in [1, n_j]$, we have $f_1(y_j b_{j,n_j}) = \sum_{c=1}^{j-1} n_c + n_j = \sum_{c=1}^{j} n_c$;

$f_1(y_t b_{t,r}) = \sum_{c=1}^{t-1} n_c + r$ for $r \in [1, n_t]$, so the edge $y_t b_{t,n_t}$ is colored by

$$f_1(y_t b_{t,n_t}) = \sum_{c=1}^{t-1} n_c + n_t = \sum_{c=1}^{t} n_c = N. \tag{2.4}$$

The above indicates that all new edges associated with $y_j (1 \leq j \leq t)$ are given the corresponding color.

Step 2. Next, we focus on the coloring of the newly added edge connected to $x_i \in X$. Color edges $x_i a_{i,k}$ for leaves $a_{i,k} \in L(x_i)$ with $k \in [1, m_i]$ and $i \in [1, s]$ as follows:

$f_1(x_s a_{s,k}) = N + k$ for $k \in [1, m_s]$, therefore, the coloring of edge $x_s a_{s,m_s}$ is $f_1(x_s a_{s,m_s}) = N + m_s$;

$f_1(x_{s-1} a_{s-1,k}) = N + m_s + k$ for $k \in [1, m_{s-1}]$; according to coloring rule, we get the largest color $f_1(x_{s-1} a_{s-1,m_{s-1}}) = N + m_s + m_{s-1}$ for the newly added adjacent edge of vertex x_{s-1};

For $i \in [2, s-2]$, $f_1(x_{s-i} a_{s-i,k}) = N + \sum_{c=s-i+1}^{s} m_c + k$ for $k \in [1, m_{s-i}]$, we get $f_1(x_{s-i} a_{s-i,m_{s-i}}) = N + \sum_{c=s-i}^{s} m_c$;

$f_1(x_1 a_{1,k}) = N + \sum_{c=2}^{s} m_c + k$ for $k \in [1, m_1]$ and the last edge $x_1 a_{1,m_1}$ is colored with

$$f_1(x_1 a_{1,m_1}) = N + \sum_{c=2}^{s} m_c + m_1 = M + N = K. \tag{2.5}$$

Based on the above two steps, all the new leaves have gained their colors.

Step 3. In this step, we recolor the edges and vertices that already exist in T as the following way: $f_1(x_i y_j) = f(x_i y_j) + 2K$ for $x_i y_j \in E(T)$, and $f_1(x_i) = f(x_i)$ for $x_i \in V(T)$, $f_1(y_j) = f(y_j)$ for $y_j \in V(T)$. Therefore, each edge $x_i y_j \in E(T)$ holds

$$f_1(x_i) + f_1(x_i y_j) + f_1(y_j) = f(x_i) + f(x_i y_j) + 2K + f(y_j) = C + 2K. \tag{2.6}$$

We have the edge color set $f_1(E(T_A))$ of the leaf-added graph T_A as follows:

$$f_1(E(T_A)) = [1, K] \cup [2K + 1, 2K + q] \subset [0, p' + q']. \tag{2.7}$$

Step 4. Finally, we color the newly added vertices including the added leaves $b_{j,r} \in L(y_j)$ and $a_{i,k} \in L(x_i)$ with $i \in [1, s]$ and $j \in [1, t]$. Let $C_A = C + 2K = s + q' + K$, and each leaf $b_{j,r} \in L(y_j)$ is colored by

$$f_1(b_{j,r}) = C_A - f_1(y_j) - f_1(y_j b_{j,r}), r \in [1, n_j], j \in [1, t]. \tag{2.8}$$

Obviously, the equation $f_1(y_j) + f_1(y_j b_{j,r}) + f_1(b_{j,r}) = C_A$ is satisfied on every leaf connected to y_j, and C_A is a constant. In the same way, each leaf $a_{i,k} \in L(x_i)$ is colored by

$$f_1(a_{i,k}) = C_A - f_1(x_i) - f_1(x_i a_{i,k}). \tag{2.9}$$

Immediately, $f_1(x_i) + f_1(x_i a_{i,k}) + f_1(a_{i,k}) = C_A$ holds for $k \in [1, m_i]$ and $i \in [1, s]$. Also, we can get $f_1(V(T_A)) \cup f_1(E(T_A)) \subset [0, p' + q']$, then f_1 is an edge-magic coloring of leaf-added graph T_A. □

Theorem 3. *Let a tree T admits a set-ordered edge-magic labeling, and its leaf-added graph T_A is a (p', q')-graph, then T_A admits an edge-difference coloring and its magic constant is $C_B = p'$.*

Proof. Let (X, Y) represents the bipartition of vertices of T and $|X| = s$, $|Y| = t$, and its leaf-added graph T_A is obtained by adding K leaves to tree T. We still define a coloring f_2 for T_A and prove that f_2 is an edge-difference coloring.

To start, color edges $y_j b_{j,r}$ for leaves $b_{j,r} \in L(y_j)$ with $r \in [1, n_j]$ and $j \in [1, t]$ as follows: $f_2(y_1 b_{1,r}) = K + 1 - r$ for $r \in [1, n_1]$, and the color of the new leaf connected to vertex y_1 decreases in turn until $f_2(y_1 b_{1,n_1}) = K + 1 - n_1$ is obtained; $f_2(y_2 b_{2,r}) = K + 1 - n_1 - r$ for $r \in [1, n_2]$, so we get $f_2(y_2 b_{2,n_2}) = K + 1 - (n_1 + n_2)$. For $j \in [3, t - 1]$, $f_2(y_j b_{j,r}) = K + 1 - \sum_{c=1}^{j-1} n_c - r$ for $r \in [1, n_j]$, then we have $f_2(y_j b_{j,n_j}) = K + 1 - \sum_{c=1}^{j} n_c$; and $f_2(y_t b_{t,r}) = K + 1 - \sum_{c=1}^{t-1} n_c - r$ for $r \in [1, n_t]$. The last edge $y_t b_{t,n_t}$ is colored by

$$f_2(y_t b_{t,n_t}) = K + 1 - \sum_{c=1}^{t} n_c = K + 1 - N = M + 1. \tag{2.10}$$

Next, color edges $x_i a_{i,k}$ for leaves $a_{i,k} \in L(x_i)$ with $k \in [1, m_i]$ and $i \in [1, s]$ as follows: $f_2(x_s a_{s,k}) = k$ for $k \in [1, m_s]$, so we have $f_2(x_s a_{s,m_s}) = m_s$; $f_2(x_{s-1} a_{s-1,k}) = m_s + k$ for $k \in [1, m_{s-1}]$. We follow this coloring rule until we get the last coloring $f_2(x_{s-1} a_{s-1,m_{s-1}}) = m_s + m_{s-1}$. For $i \in [3, s-1]$, $f_2(x_{s-i} a_{s-i,k}) = \sum_{c=s-i+2}^{s} m_c + k$ for $k \in [1, m_{s-i}]$, then we get the coloring of edge $x_{s-i} a_{s-i,m_{s-i}}$ is $f_2(x_{s-i} a_{s-i,m_{s-i}}) = \sum_{c=s-i+1}^{s} m_c + k$, $f_2(x_1 a_{1,k}) = \sum_{c=2}^{s} m_c + k$ for $k \in [1, m_1]$, and the last edge $x_1 a_{1,m_1}$ is colored with

$$f_2(x_1 a_{1,m_1}) = \sum_{c=2}^{s} m_c + m_1 = M. \tag{2.11}$$

The first two steps have given the colors of all the newly added edges.

Then, we recolor the vertices and edges that already exist in tree T as the following way: $f_2(x_i y_j) = f(x_i y_j) + K$ for $x_i y_j \in E(T)$, and $f_2(x_i) = f(x_{s+1-i}) = s - 1 - f(x_i)$ for $x_i \in V(T)$, $f_2(y_j) = f(y_j)$ for $y_j \in V(T)$, so, the $x_i y_j \in E(T_A)$ holds

$$\begin{aligned} |f_2(x_i) - f_2(y_j)| + f_2(x_i y_j) &= |s - 1 - f(x_i) - f(y_j)| + f(x_i y_j) + K \\ &= f(y_j) - s + 1 + f(x_i) + f(x_i y_j) + K \\ &= C - s + 1 + K \\ &= q' + 1 = p'. \end{aligned} \tag{2.12}$$

We have the edge color set $f_2(E(T_A))$ of the leaf-added graph T_A as follows:

$$f_2(E(T_A)) = [1, K] \cup [K + 1, q'] = [1, q']. \tag{2.13}$$

Finally, we color the added leaves of $b_{j,r} \in L(y_j)$ and $a_{i,k} \in L(x_i)$ with $i \in [1, s]$ and $j \in [1, t]$. For the sake of simplicity, let $C_B = p'$. Each leaf $b_{j,r} \in L(y_j)$ with $r \in [1, n_j]$ and $j \in [1, t]$ is colored by

$$f_2(b_{j,r}) = C_B - f_2(y_j b_{j,r}) + f_2(y_j), \tag{2.14}$$

so we have $|f_2(y_j) - f_2(b_{j,r})| + f_2(y_j b_{j,r}) = C_B$ for $r \in [1, n_j]$ and $j \in [1, t]$.

On the other hand, for each leaf $a_{i,k} \in L(x_i)$, $k \in [1, m_i]$, and $i \in [1, s]$, its coloring is

$$f_2(a_{i,k}) = C_B - f_2(x_i a_{i,k}) + f_2(x_i). \tag{2.15}$$

Immediately, $|f_2(x_i) - f_2(a_{i,k})| + f_2(x_i a_{i,k}) = C_B$ is true for $k \in [1, m_i]$ and $i \in [1, s]$. In summary, we know $f_2(V(T_A)) \cup f_2(E(T_A)) \subset [0, p' + q']$, so leaf-added graph T_A has an edge-difference coloring f_2, and its magic constant is p'. □

Theorem 4. *Let a tree T admits a set-ordered edge-magic labeling and (X, Y) represents the bipartition of vertices of T, then $|X| = s$, $|Y| = t$. Its leaf-added graph T_A is obtained by adding K leaves to tree T, then T_A admits a felicitous-difference coloring and its magic constant is $C_C = s - 1 + 2K$.*

Proof. Like the previous proof, we start by coloring the edges of the newly added leaves $b_{j,r}$ and $a_{i,k}$, where $r \in [1, n_j]$, $j \in [1, t]$, $k \in [1, m_i]$, and $i \in [1, s]$.

First, color edges $y_j b_{j,r}$ for leaves $b_{j,r} \in L(y_j)$ as follows: $f_3(y_1 b_{1,r}) = K + 1 - r$ for $r \in [1, n_1]$, then the minimum color of the newly added edge associated with y_1 is $f_3(y_1 b_{1,n_1}) = K + 1 - n_1$; $f_3(y_2 b_{2,r}) = K + 1 - n_1 - r$ for $r \in [1, n_2]$, so we have $f_3(y_2 b_{2,n_2}) = K + 1 - (n_1 + n_2)$. For $j \in [3, t - 1]$, $f_3(y_j b_{j,r}) = K + 1 - \sum_{c=1}^{j-1} n_c - r$ for $r \in [1, n_j]$, therefore, the minimum color of each adjacent edge of y_j is equal to $f_3(y_j b_{j,n_j}) = K + 1 - \sum_{c=1}^{j} n_c$; $f_3(y_t b_{t,r}) = K + 1 - \sum_{c=1}^{t-1} n_c - r$ for $r \in [1, n_t]$, and the last edge $y_t b_{t,n_t}$ is colored by

$$f_3(y_t b_{t,n_t}) = K + 1 - \sum_{c=1}^{t} n_c = M + 1. \tag{2.16}$$

Second, color edges $x_i a_{i,k}$ for leaves $a_{i,k} \in L(x_i)$ with $k \in [1, m_i]$ and $i \in [1, s]$ as follows: $f_3(x_1 a_{1,k}) = k$ for $k \in [1, m_1]$, which means that $f_3(x_1 a_{1,m_1}) = m_1$. What we need to note is that if no new leaves are added to vertex x_1, then $m_1 = 0$; $f_3(x_2 a_{2,k}) = m_1 + k$ for $k \in [1, m_2]$. According to this rule, we get the maximum color of the added edge connected to x_2, $f_3(x_2 a_{2,m_2}) = m_1 + m_2$. When $i \in [3, s - 1]$, $f_3(x_i a_{i,k}) = \sum_{c=1}^{i-1} m_c + k$ for $k \in [1, m_i]$, so we have $f_3(x_i a_{i,m_i}) = \sum_{c=1}^{i} m_c$. Further, $f_3(x_s a_{s,k}) = \sum_{c=1}^{s-1} m_c + k$ for $k \in [1, m_s]$ and the last edge $x_s a_{s,m_s}$ is colored with

$$f_3(x_s a_{s,m_s}) = m_s + \sum_{c=1}^{s-1} m_c = M. \tag{2.17}$$

Third, consider the color of the edges and vertices that already exist in T, which corresponds to the coloring $f_3(x_i y_j)$, $f_3(x_i)$, $f_3(y_j)$ in T_A. We recolor each element of $V(T) \cup E(T)$ in the following way:

$f_3(x_iy_j) = p - f(x_iy_j) + 2K$ for $x_iy_j \in E(T)$, $f_3(x_i) = f(x_i) + 2K$ for $x_i \in V(T)$, and $f_3(y_j) = f(y_j) + 2K$ for $y_j \in V(T)$. The colors of these edges $x_iy_j \in E(T)$ satisfy

$$\begin{aligned} |f_3(x_i) + f_3(y_j) - f_3(x_iy_j)| &= f(x_i) + f(y_j) + 4K - p + f(x_iy_j) - 2K \\ &= C + 2K - p \\ &= s - 1 + 2K. \end{aligned} \tag{2.18}$$

We have the edge color set $f_3(E(T_A))$ of the leaf-added graph T_A as follows:

$$f_3(E(T_A)) = [1, K] \cup [2K + 1, 2K + p - 1] \subset [0, p' + q']. \tag{2.19}$$

In the last step, we color the added leaves of $b_{j,r} \in L(y_j)$ and $a_{i,k} \in L(x_i)$ with $i \in [1, s]$ and $j \in [1, t]$. Let $C_C = s - 1 + 2K$, then each leaf $b_{j,r} \in L(y_j)$ with $r \in [1, n_j]$ and $j \in [1, t]$ is colored by

$$f_3(b_{j,r}) = C_C + f_3(y_jb_{j,r}) - f_3(y_j), \tag{2.20}$$

so $|f_3(y_j) + f_3(b_{j,r}) - f_3(y_jb_{j,r})| = C_C$ for $r \in [1, n_j]$, $j \in [1, t]$, where C_C is a constant. Each leaf $a_{i,k} \in L(x_i)$ with $k \in [1, m_i]$ and $i \in [1, s]$ is colored by

$$f_3(a_{i,k}) = C_C + f_3(x_ia_{i,k}) - f_3(x_i). \tag{2.21}$$

Obviously, $|f_3(x_i) + f_3(a_{i,k}) - f_3(x_ia_{i,k})| = C_C$ for $k \in [1, m_i]$ and $i \in [1, s]$, and we have $f_3(V(T_A)) \cup f_3(E(T_A)) \subset [0, p' + q']$. The above shows that f_3 is a felicitous-difference coloring. Thus, the Theorem 4 is proved. □

Theorem 5. *Let a tree T admits a set-ordered edge-magic labeling, and its leaf-added graph T_A is obtained by adding K leaves to tree T, then T_A admits a graceful-difference coloring and its magic constant is $C_D = 2K$.*

Proof. We define a graceful-difference coloring f_4 of the leaf-added graph T_A.

We color edges $y_jb_{j,r}$ for leaves $b_{j,r} \in L(y_j)$ with $r \in [1, n_j]$ and $j \in [1, t]$ as follows: $f_4(y_1b_{1,r}) = r$ for $r \in [1, n_1]$, then we get $f_4(y_1b_{1,n_1}) = n_1$; $f_4(y_2b_{2,r}) = n_1 + r$ for $r \in [1, n_2]$, $f_4(y_2b_{2,n_2}) = n_1 + n_2$; When $j \in [3, t - 1]$, $f_4(y_jb_{j,r}) = \sum_{c=1}^{j-1} n_c + r$ for $r \in [1, n_j]$, $f_4(y_jb_{j,n_j}) = \sum_{c=1}^{j} n_c$. So, we have $f_4(y_tb_{t,r}) = \sum_{c=1}^{t-1} n_c + r$ for $r \in [1, n_t]$, and the last edge y_tb_{t,n_t} is colored by

$$f_4(y_tb_{t,n_t}) = n_t + \sum_{c=1}^{t-1} n_c = N. \tag{2.22}$$

Color edges $x_ia_{i,k}$ for leaves $a_{i,k} \in L(x_i)$ with $k \in [1, m_i]$ and $i \in [1, s]$ as follows: $f_4(x_sa_{s,k}) = N + k$ for $k \in [1, m_s]$, $f_4(x_sa_{s,m_s}) = N + m_s$, and $f_4(x_{s-1}a_{s-1,k}) = N + m_s + k$ for $k \in [1, m_{s-1}]$, $f_4(x_{s-1}a_{s-1,m_{s-1}}) = N + m_s + m_{s-1}$. For $i \in [2, s - 2]$, $f_4(x_{s-i}a_{s-i,k}) = N + \sum_{c=s-i+1}^{s} m_c + k$ for $k \in [1, m_i]$, $f_4(x_{s-i}a_{s-i,m_{s-i}}) = N + \sum_{c=s-i}^{s} m_c$; Then, $f_4(x_1a_{1,k}) = N + k + \sum_{c=2}^{s} m_c$ for $k \in [1, m_1]$, and the last edge x_1a_{1,m_1} is colored with

$$f_4(x_1a_{1,m_1}) = m_1 + \sum_{c=2}^{s} m_c + N = K. \tag{2.23}$$

Recolor each element of $V(T) \cup E(T)$ in the following way: $f_4(x_i) = f(x_{s+1-i}) = s - 1 - f(x_i)$ for $x_i \in X$, and $f_4(y_j) = f(y_j)$ for $y \in Y$, and these edges $x_iy_j \in E(T_A)$ hold $f_4(x_iy_j) = p - f(x_iy_j) + 2K$, then we have

$$\begin{aligned} ||f_4(x_i) - f_4(y_j)| - f_4(x_iy_j)| &= f(y_j) - s + 1 + f(x_i) - p + f(x_iy_j) + 2K \\ &= C - s + 1 - p + 2K \\ &= 2K. \end{aligned} \tag{2.24}$$

We have the edge color set $f_4(E(T_A))$ of tree T_A as follows:

$$f_4(E(T_A)) = [1, K] \cup [2K + 1, 2K + p - 1] \subset [0, p' + q']. \tag{2.25}$$

For last step, color the added leaves of $L(y_j)$ and $L(x_i)$ with $i \in [1, s]$ and $j \in [1, t]$. Let $C_D = 2K$, and each leaf $b_{j,r} \in L(y_j)$ with $r \in [1, n_j]$ and $j \in [1, t]$ is colored by

$$f_4(b_{j,r}) = C_D + f_4(y_jb_{j,r}) + f_4(y_j), \tag{2.26}$$

so we get $||f_4(b_{j,r}) - f_4(y_j)| - f_4(y_jb_{j,r})| = C_D$ for $y_jb_{j,r} \in E(T_A)$, where $r \in [1, n_j]$, $j \in [1, t]$.

On the other hand, each leaf $a_{i,k} \in L(x_i)$ with $k \in [1, m_i]$ and $i \in [1, s]$ is colored by

$$f_4(a_{i,k}) = C_D + f_4(x_ia_{i,k}) + f_4(x_i). \tag{2.27}$$

Immediately, $||f_4(a_{i,k}) - f_4(x_i)| - f_4(x_ia_{i,k})| = C_D$ for $x_ia_{i,k} \in E(T_A)$, $k \in [1, m_i]$, and $i \in [1, s]$. In addition, we can get $f_4(V(T_A)) \cup f_4(E(T_A)) \subset [0, p' + q']$, so f_4 is a graceful-difference coloring of leaf-added graph T_A, and the magic constant is $2K$. □

Figure 3 shows an example for illustrating the proof of Theorems 2–5.

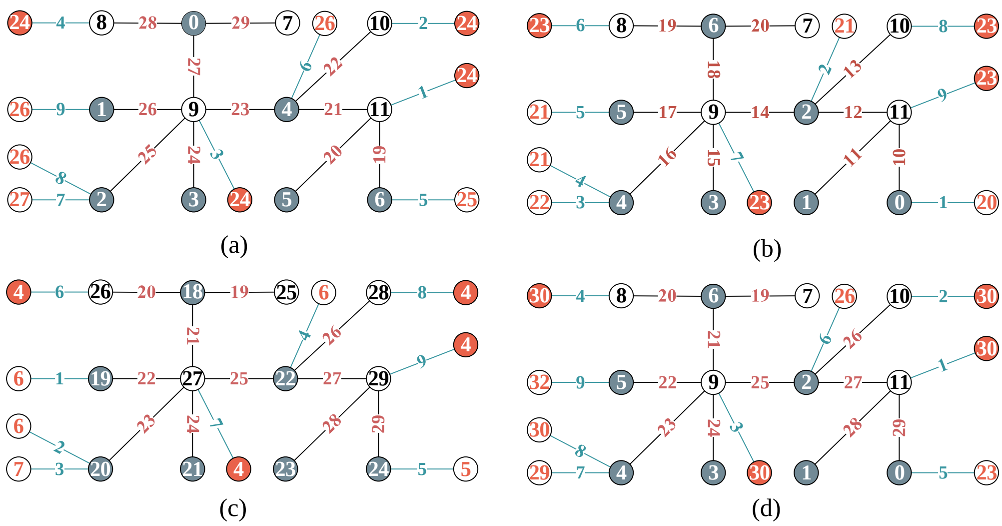

Figure 3. (a) $f_1(x_i) + f_1(x_iy_j) + f_1(y_j) = 36$; (b) $|f_2(x_i) - f_2(y_j)| + f_2(x_iy_j) = 21$; (c) $|f_3(x_i) + f_3(y_j) - f_3(x_iy_j)| = 24$; (d) $||f_4(x_i) - f_4(y_j)| - f_4(x_iy_j)| = 18$.

The size and strength of the topological key are determined by the following aspects: The length of bytes of the topological key; the dimension of the mathematical constraint of the topological key;

the topology structure of the used topological graph must meet the strong constraint and randomness; the base of graph space for constructing topological key is at least 2^{200}; the number of vertices of graph is not less than 50; and so on. Theorem 2 theoretically extends the diversity of colorings on a graph. Theorems 3–6 construct a larger graph by adding leaf operation, enriching the topology of the graph, increasing the length of the topological key, and increasing the difficulty of deciphering the key. Deciphering a string $S = c_1c_2 \cdots c_n$ produced by our algorithm will do the following: (i) Finding out the coding graph G of p vertices and q edges; as known, the numbers of graphs of 23 vertices and 24 vertices are as follows: $N_{23} \sim 2^{179}$, $N_{24} \sim 2^{197}$. (ii) Finding a particular coloring f of the coding graph G. (iii) Finding $S = c_1c_2 \cdots c_n$ from $(p \times p)!$ number strings (for topological signature), or from $(3q)!$ number strings (for encrypting files). However, we can find there is no polynomial algorithm to construct the coding graph G as p and q are quite large; also, there is no polynomial algorithm to distinguish isomorphism between coding graphs, since it has been proven the subgraph isomorphism is NP-complete. It is known that there are thousands of colorings in which there are hundreds of conjectures and open problems; there is no polynomial algorithm to find a particular coloring f for the coding graph G.

3. Conclusions

In this paper, inspired by the edge-magic labeling, we propose the concepts of edge-magic coloring, edge-difference coloring, felicitous-difference coloring, and graceful-difference coloring. As a corollary, we prove the transformation relationship $f_1 \sim f_2 \sim f_3 \sim f_4$ of four types of coloring on the structure of trees. Based on the operation of adding leaves, we obtain the existence of four types of magic coloring for constructing larger trees from smaller trees. It is worth noting that the security of the topological key proposed in this paper is based on the difficulty of the two NP-complete problems of subgraph isomorphism and graph coloring. The challenge it faces is that the security of the topological key will be threatened when the two problems of subgraph isomorphism and graph coloring can be effectively solved within the scope of current computer capability. In fact, we also find that the conclusions of this paper are valid for bipartite graphs, where the vertices of bipartite graphs can also be divided into two disjoint vertices, and other magical colorings can be identified by their set-ordered edge-magic labelings. However, whether the magical colorings of general graphs exist and find their corresponding rules is the direction of our future research. In addition, the use of magic coloring to generate topological keys can make the conversion method between keys simple and easy to implement and generate a wide variety of topological keys, which provides a theoretical guarantee for the diversity of topological keys and the expansion of key space.

Author contributions

J. Su and B. Yao: Conceptualization; J. Su: Writing–original draft; J. Su and Q. Y. Zhang: Writing–review & editing. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

The authors declare that they have no conflict of interest.

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