AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(10): 27871–27895.
DOI:10.3934/math.20241353
Received: 24 August 2024
Revised: 13 September 2024
Accepted: 18 September 2024
Published: 26 September 2024

Research article

On coupled non-linear Schrödinger systems with singular source term

Saleh Almuthaybiri and Tarek Saanouni*

Department of Mathematics, College of Science, Qassim University, Saudi Arabia

* **Correspondence:** Email: s.almuthybiri@qu.edu.sa, t.saanouni@qu.edu.sa.

Abstract: This work studies a coupled non-linear Schrödinger system with a singular source term. First, we investigate the question of the local existence of solutions. Second, one proves the existence of global solutions which scatter in some Sobolev spaces. Finally, one establishes the existence of non-global solutions. The main difficulty here is to overcome the regularity problem in the non-linearity. Indeed, because of the singularity of the source term, the classical contraction method in the energy space fails in such a regime. So, this paper is to fill such a gap in the literature. The argument follows ideas in T. Cazenave and I. Naumkin (*Comm. Contemp. Math.*, **19** (2017), 1650038). This consists to remark that the singularity problem is only near the origin. So, one needs to impose that the solution stays away from zero. This is not trivial, since there is no maximum principle for the Schrödinger equation. The existence of global solutions which scatter follows with the pseudo-conformal transformation via the existence of local solutions. Finally, the existence of non-global solutions follows with the classical variance method.

Keywords: Schrödinger system; nonlinear equations; fixed point method; existence; uniqueness; scattering; approximation; blow-up
Mathematics Subject Classification: 35Q55

1. Introduction

This paper considers the Cauchy problem for the coupled non-linear Schrödinger system, shortened to CNLS,

$$\begin{cases} i\partial_t u_j + \Delta u_j = \tau\Big(\sum_{1\le k\le n} a_{jk}|u_k|^{\sigma}\Big)|u_j|^{\sigma-2}u_j; \\ u_{|t=0} = u_0. \end{cases} \tag{CNLS}$$

Here and hereafter, for $j \in [1, n]$, u_j is a complex valued function of the variable $(t, y) \in \mathbb{R}_+ \times \mathbb{R}^N$, and $u := (u_1, \dots, u_n)$. The constant $0 \neq \tau \in \mathbb{C}$ and the coupling parameters satisfy:

$$a_{jk} = a_{kj} \ge 0 \quad \text{and} \quad a_{jj} > 0 \quad \text{for any} \quad j, k \in [1, n].$$

The coupled Schrödinger system (CNLS) models many physical phenomena, such as the propagation in birefringent optical fibers, Kerr-like photorefractive media in optic and Bose-Einstein condensates [1, 4, 7], see also [2, 3, 5, 6]. The passage of a light beam with two components along an optical fiber produces the decomposition of the ray into two. Then, the model of a scalar Schrödinger equation can be improved by a system of coupled Shrödinger equations [8]. As a consequence, new kinds of solutions appear, first observed by Manakov [9] in a Kerr medium.

In mathematical point of view, many authors focused their attention on coupled nonlinear equations of Schrödinger type. So, the list of references is necessarily incomplete. The local well-posedness in the energy space was proved in [10, 11]. The existence of ground states was investigated in [12–14]. The scattering of defocusing global solutions was treated in [15]. The scattering in the repulsive regime with small data was obtained in [16, 17]. See also [18–21] for the inhomogeneous case and [22] for the case of harmonic potential and [23] for the parabolic context.

To the author's knowledge, the above works treat only the case $\sigma \geq 2$ in (CNLS). This is to avoid a singularity of the source term for $\sigma < 2$. The contribution of this paper is to try to fill in this gap in the literature.

The purpose of this paper is to prove the existence of a local solution to the coupled Schrödinger problem (CNLS) in the case $\frac{3}{2} < \sigma < 2$. Then, one establishes the existence of global solutions which scatter in some Sobolev spaces and the existence of non-global solutions with finite variance.

The paper is organized as follows: Section 2 contains the main results and some standard estimates needed in the sequel. Section 3 proves the local existence of solutions to (CNLS). Section 4 establishes the scattering of global solutions and the existence of non-global solutions to (CNLS).

Throughout this paper, we denote the spaces and norms

$$W^{s,p} := W^{s,p}(\mathbb{R}^N), \quad H^s := W^{s,2}, \quad L^r := L^r(\mathbb{R}^N);$$
$$\|\cdot\|_r := \|\cdot\|_{L^r}, \quad \|\cdot\| := \|\cdot\|_2;$$
$$\|(u_1, \dots, u_n)\|_p := \Big(\sum_{1\leq j\leq n} \|u_j\|_p^p\Big)^{\frac{1}{p}}, \quad \|(u_1, \dots, u_n)\| := \|(u_1, \dots, u_n)\|_2.$$

Lastly, $T^* > 0$ denotes the lifespan for an eventual solution to (CNLS).

2. Background and main results

In this section, we give the main results and some standard estimates.

2.1. Preliminary

Solutions of (CNLS) formally satisfy the conservation of the following real quantities, respectively the mass and the energy

$$M(u) := \sum_{1\leq j\leq n} \int_{\mathbb{R}^N} |u_j(t,y)|^2\, dy;$$
$$E(u) := \sum_{1\leq j\leq n} \Big(|u_j(t,y)|^2 + |\nabla u_j(t,y)|^2\, dy + \frac{\tau}{\sigma} \sum_{1\leq k\leq n} a_{jk} \Big|u_j(t,y)u_k(t,y)\Big|^\sigma \Big)\, dy.$$

Let the non-linear terms

$$\mathcal{F} := \mathcal{F}(u) := \Big[\Big(\sum_{1\leq k\leq n} a_{1k}|u_k|^{\sigma}\Big)|u_1|^{\sigma-2}u_1, \dots, \Big(\sum_{1\leq k\leq n} a_{nk}|u_k|^{\sigma}\Big)|u_n|^{\sigma-2}u_n\Big];$$
$$\mathcal{F}_j := \mathcal{F}_j(u) := \Big(\sum_{1\leq k\leq n} a_{jk}|u_k|^{\sigma}\Big)|u_j|^{\sigma-2}u_j;$$
$$\mathcal{F}_{j,k} := \mathcal{F}_{j,k}(u) := |u_k|^{\sigma}|u_j|^{\sigma-2}u_j.$$

Take for $s \in \mathbb{R}$,

$$\langle y\rangle^2 := 1 + |y|^2 \quad \text{and} \quad I_\gamma := (1-\Delta)^{\frac{s}{2}}.$$

Here and hereafter, one defines some real numbers

$$2-\sigma < \alpha < \min\{\sigma - \frac{N}{2m}, p-1\}; \tag{2.1}$$
$$\max\{\frac{N}{4(\sigma-1)}, \frac{N}{2(\sigma-1-\alpha)}, \frac{N}{2}\} < m < \frac{N}{2(2-\sigma)}; \tag{2.2}$$
$$M_0 > N + m; \tag{2.3}$$
$$M > 4E[\frac{N}{2}] + 5 + m, \tag{2.4}$$

where $E[\cdot]$ refers to the integer part. Take the Banach space

$$\mathcal{Y} := \Big\{u \in (H^{-N+M_0+M})^n,$$
$$\|u\|_{\mathcal{Y}} := \sum_{1\leq j\leq n}\Big(\sum_{|\alpha|\leq E[\frac{N}{2}]} \|\langle y\rangle^m \partial^\alpha u_j\|_\infty + \sum_{E[\frac{N}{2}]<|\alpha|\leq M} \|\langle y\rangle^m \partial^\alpha u_j\|$$
$$+ \sum_{M<|\alpha|\leq -N+M_0+M} \|\partial^\alpha u_j\|\Big) < \infty\Big\}.$$

Let the centered ball of radius $R > 0$, denoted by $B_T(R) := \{u \in C_T(\mathcal{Y}),\ \|u\|_{L^\infty_T(\mathcal{Y})} \leq R\}$, and for $\nu > 0$, take

$$B_{\nu,T}(R) := \Big\{u \in B_T(R),\ 2\inf_{t,x} |\langle y\rangle^m u_j(t,y)| \geq \nu, \quad \forall j \in [1,n]\Big\}.$$

Finally, we define the vector Schrödinger group

$$e^{i\cdot\Delta}(u_1,\dots,u_n) := (e^{i\cdot\Delta}u_1,\dots,e^{i\cdot\Delta}u_n).$$

2.2. Main results

In what follows, we list the results proved in this paper. The first purpose of this note is to prove the existence of local solutions to (CNLS).

Theorem 2.1. *Let $N \geq 1$, $\frac{3}{2} < \sigma < 2$, and $u_0 \in \mathcal{Y}$, satisfying*

$$\inf_{x\in\mathbb{R}^N} |\langle y\rangle^m u_0(x)| > 0. \tag{2.5}$$

Then, there are $T > 0$ and a unique solution to (CNLS) denoted by $u \in C_T(\mathcal{Y})$. Moreover, the flow is locally continuous.

Remarks 2.1.

1) The condition $\sigma > \frac{3}{2}$, which seems to be technical, is necessary in order to have (2.1);
2) the continuity of the flow follows with standard arguments;
3) assumption (2.5) avoids in some meaning the singularity of the source term for $\sigma < 2$;
4) the proof follows ideas in [24], where the scalar case is treated;
5) some tools needed in the proof are taken from [25];
6) the proof is based on a Picard fixed point argument.

The second result is about the scattering of global solutions to (CNLS) in some Sobolev spaces.

Theorem 2.2. *Let $N \geq 2$ and $\max\{1+\frac{1}{N}, \frac{3}{2}\} < \sigma < 2$. Take $v_0 \in \mathcal{Y}$ satisfying* (2.5)*, and $u_0 := e^{\frac{i\kappa t|x|^2}{4}} v_0$ for some real number $\kappa \gg 1$. Take $0 \leq s < m - \frac{N}{2}$. Then, there is a unique global solution to* (CNLS) *in $C(\mathbb{R}_+, H^s) \cap \langle y\rangle^{-\frac{N}{2}} L^\infty(\mathbb{R}_+ \times \mathbb{R}^N)$, which scatters in H^s.*

Remarks 2.2.

1) The proof is based on the pseudo-conformal transformation [26] and Theorem 2.1;
2) a similar result was first proved [24] in the scalar case;
3) a part of the proof is omitted because it follows [25];
4) the real number κ depends on $v, \|v_0\|_{\mathcal{Y}}, N, m$.

Finally, one proves the existence of non-global solutions to (CNLS).

Proposition 2.1. *Take the assumptions of Theorem 2.1, $m > 1 + \frac{1}{N}$ and $\tau < 0$. Then, the solution to* (CNLS) *is non-global if one of the following statements holds:*

1) $p \geq 1 + \frac{2}{N}$ and $E(u_0) < 0$;
2) $E(u_0) < d$ and $I(u) < 0$.

Remarks 2.3.

1) The blow-up in the first case follows with classical variance method;
2) the second case follows like [18, Theorem 2.8];
3) d denotes the ground state energy $d := \inf_{0\neq u\in(H^1)^m}\{E(u), \quad I(u) = 0\}$, where $I(u) := BE(u) - (B - 2)\|\nabla u\|^2$ and $B := N(\sigma - 1)$.

Some intermediate results are listed in what follows.

2.3. Tools

In order to investigate the blow-up of solutions, one needs the next variance identity [27].

Proposition 2.2. *Assume that $u \in C_T([H^1]^n)$ is a solution to* (CNLS) *for $\tau = -1$, and satisfies $xu_j(t) \in L^2$, for any $1 \leq j \leq n$. Then,*

$$\frac{1}{8}\Big(\sum_{1\leq j\leq n} \|xu_j\|^2\Big)'' = \sum_{1\leq j\leq n} \|\nabla u_j\|^2 - \frac{B}{2\sigma} \sum_{1\leq j,k\leq n} a_{jk} \int_{\mathbb{R}^N} \Big(|u_k u_j|\Big)^\sigma dy.$$

The following estimate will be useful [25, Lemma 2.9].

Lemma 2.1. *Let $\alpha \geq 0$. Then, for all real number: t, we have*

$$\| \langle y \rangle^{\alpha} e^{it\Delta} u \| \leq c_{\alpha} \langle t \rangle^{\alpha} \Big(\| I_{\alpha} u \| + \| \langle y \rangle^{\alpha} u \| \Big).$$

Recall also [25, Proposition 2.10].

Lemma 2.2. *Let $\alpha \geq 0$, $k, K \in \mathbb{N}$, and $s \geq 2k + K + 3 + E[\frac{N}{2}] + \alpha$. Then, for any $t \in \mathbb{R}$ and any $|\mu| \leq K$,*

$$\| \langle y \rangle^{\alpha} \partial^{\mu} e^{it\Delta} u \|_{\infty} \lesssim \langle t \rangle^{k} \sum_{|\gamma| \leq k} \| \langle y \rangle^{\alpha} \partial^{\gamma} u \|_{\infty} + \langle t \rangle^{1+k+\alpha} \Big(\| I_s u \| + \sum_{k < |\gamma| \leq 2k+K+3+E[\frac{N}{2}]} \| \langle y \rangle^{\alpha} \partial^{\gamma} u \| \Big).$$

One recalls the nonlinear estimates [25, Proposition 3.1].

Lemma 2.3. *Let $\alpha \geq 0$, $u \in \mathcal{B}_{\nu,T}(R)$, and $r \in [1, \infty]$. Then, for all $|\mu| \leq -N + M_0 + M$, we have*

1)

$$\begin{aligned} \| \langle y \rangle^{\alpha} \partial^{\mu} (|u|^{\sigma}) \|_{L^{\infty}((0,T),L^r)} \lesssim R^{\sigma} \| \langle y \rangle^{\alpha - m\sigma} \|_r + \sum_{1 \leq k \leq |\mu|} \nu^{-(2k-\sigma)} R^{2k-1} \Big(R \| \langle y \rangle^{\alpha - m\sigma} \|_r \\ + \sum_{E[\frac{N}{2}] < |\gamma| \leq |\mu|} \| \langle y \rangle^{\alpha + m(1-\sigma)} \partial^{\gamma} u \|_{L^{\infty}((0,T),L^r)} \Big). \end{aligned}$$

2)

$$\begin{aligned} \| \langle y \rangle^{\alpha} \partial^{\mu} (|u|^{\sigma-2} u) \|_{L^{\infty}((0,T),L^r)} \lesssim \sum_{0 \leq k \leq |\mu|} \nu^{-(2k+2-\sigma)} R^{2k} \Big(R \| \langle y \rangle^{\alpha - m(\sigma-1)} \|_r \\ + \sum_{E[\frac{N}{2}] < |\gamma| \leq |\mu|} \| \langle y \rangle^{\alpha + m(2-\sigma)} \partial^{\gamma} u \|_{L^{\infty}((0,T),L^r)} \Big). \end{aligned}$$

Finally, let us give some nonlinear estimates [25, Proposition 3.2].

Lemma 2.4. *Let $\alpha \geq 0$, $u, v \in \mathcal{B}_{\nu,T}(R)$, and $r \in [1, \infty]$. Then, for all $|\mu| \leq -N + M_0 + M$, we have*

1)

$$\begin{aligned} & \| \langle y \rangle^{\alpha} \partial^{\mu} (|u|^{\sigma} - |v|^{\sigma}) \|_{L^{\infty}((0,T),L^r)} \\ \lesssim \;& (R^{\sigma-1} + \nu^{-2(3-\sigma)} R^{5-\sigma}) \| \langle y \rangle^{\alpha - m\sigma} \|_r \| u - v \|_{L^{\infty}_T(\mathcal{Y})} \\ + & \sum_{1 \leq k \leq |\mu|} \nu^{-2(2k-\sigma)} R^{4k-\sigma-2} \Big(R \| \langle y \rangle^{\alpha - m\sigma} \|_r + \sum_{E[\frac{N}{2}] < |\gamma| \leq |\mu|} \| \langle y \rangle^{\alpha + m(1-\sigma)} \partial^{\gamma} (u - v) \|_{L^{\infty}((0,T),L^r)} \Big) \\ + & \sum_{1 \leq k \leq |\mu|} \nu^{-(2k-\sigma)} \Big(R^{2k-1} [\| u - v \|_{L^{\infty}_T(\mathcal{Y})} \| \langle y \rangle^{\alpha - m\sigma} \|_r + \sum_{E[\frac{N}{2}] < |\gamma| \leq |\mu|} \| \langle y \rangle^{\alpha + m(1-\sigma)} \partial^{\gamma} (u - v) \|_{L^{\infty}((0,T),L^r)}] \\ + & R^{2(k-1)} \| u - v \|_{L^{\infty}_T(\mathcal{Y})} [\sum_{E[\frac{N}{2}] < |\gamma| \leq |\mu|} \| \langle y \rangle^{\alpha + m(1-\sigma)} \partial^{\gamma} u \|_{L^{\infty}((0,T),L^r)} + \| \langle y \rangle^{\alpha + m(1-\sigma)} \partial^{\gamma} v \|_{L^{\infty}((0,T),L^r)}] \Big); \end{aligned}$$

2)

$$
\begin{aligned}
& \| \langle y\rangle^{\alpha} \partial^{\mu}(|u|^{\sigma-2}u - |v|^{\sigma-2}v)\|_{L^{\infty}_T(L^2)} \\
\lesssim\ & \nu^{-2(3-\sigma)}R^{4-\sigma}\| \langle y\rangle^{\alpha-m\sigma} \|_r \|u-v\|_{L^{\infty}_T(\mathcal{Y})} + \sum_{1\le k\le|\mu|} \nu^{-2(2k+2-\sigma)}R^{4k-\sigma+1}\Big(R\| \langle y\rangle^{\alpha-m(\sigma-1)} \|_r \\
+\ & \sum_{E[\frac{N}{2}]<|\gamma|\le|\mu|} \| \langle y\rangle^{\alpha+m(2-\sigma)} \partial^{\gamma}(u-v)\|_{L^{\infty}((0,T),L^r)}\Big) \\
+\ & \sum_{1\le k\le|\mu|} \nu^{-(2(1+k)-\sigma)}\Big(R^{2k}[\|u-v\|_{L^{\infty}_T(\mathcal{Y})}\| \langle y\rangle^{\alpha-m(\sigma-1)} \|_r + \sum_{E[\frac{N}{2}]<|\gamma|\le|\mu|} \| \langle y\rangle^{\alpha+m(2-\sigma)} \partial^{\gamma}(u-v)\|_{L^{\infty}((0,T),L^r)}] \\
+\ & R^{2k-1}\|u-v\|_{L^{\infty}_T(\mathcal{Y})}[\sum_{E[\frac{N}{2}]<|\gamma|\le|\mu|} \| \langle y\rangle^{\alpha+m(2-\sigma)} \partial^{\gamma}u\|_{L^{\infty}((0,T),L^r)} + \| \langle y\rangle^{\alpha+m(2-\sigma)} \partial^{\gamma}v\|_{L^{\infty}((0,T),L^r)}]\Big).
\end{aligned}
$$

3. Local well-posedness

This section is devoted to Theorem 2.1, which deals with the existence of a unique solution of (CNLS) in $C_T(\mathcal{Y})$. Take the function

$$f(u) := e^{i\cdot\Delta}u_0 - i\tau\int_0^{\cdot} e^{i(\cdot-s)\Delta}\mathcal{F}\,ds := (f_1(u),\dots,f_m(u)),$$

where $u_0 := (u_{0,1},\dots,u_{0,n})$, and

$$f_j(u) := e^{i\cdot\Delta}u_{0,j} - i\tau\int_0^{\cdot} e^{i(\cdot-s)\Delta}\mathcal{F}_j\,ds.$$

Let us prove that $f(B_{\nu,T}(R)) \subset B_{\nu,T}(R)$. By Lemma 2.2, for the choice

$$k := E[\frac{N}{2}],\quad s = -N + M_0 + M := K + 2k + 3 + E[\frac{N}{2}] + M_0 - N,$$

and for $|\alpha| \le E[\frac{N}{2}]$, we have

$$
\begin{aligned}
& \| \langle y\rangle^{m} \partial^{\alpha} f_j(u(t,y))\|_{L^{\infty}_T(L^{\infty})} \\
\lesssim\ & \langle T\rangle^{E[\frac{N}{2}]+1+m}\Big(\sum_{|\gamma|\le E[\frac{N}{2}]} \| \langle y\rangle^{m} \partial^{\gamma}u_0\|_{\infty} + \|I_{-N+M_0+M}u_0\| + \sum_{E[\frac{N}{2}]<|\gamma|\le M} \| \langle y\rangle^{m} \partial^{\gamma}u_0\|\Big) \\
+\ & T\langle T\rangle^{E[\frac{N}{2}]+1+m}\Big(\sum_{|\gamma|\le E[\frac{N}{2}]} \| \langle y\rangle^{m} \partial^{\gamma}\mathcal{F}\|_{L^{\infty}_T(L^{\infty})} + \|I_{-N+M_0+M}\mathcal{F}\|_{L^{\infty}_T(L^2)} \\
+\ & \sum_{E[\frac{N}{2}]<|\gamma|\le M} \| \langle y\rangle^{m} \partial^{\gamma}\mathcal{F}\|_{L^{\infty}_T(L^2)}\Big).
\end{aligned}
$$

Because $p \ge 1$, by Lemma 2.3, for $|\gamma_1| \le E[\frac{N}{2}]$, we have

$$\| \langle y\rangle^{m(2-\sigma)} \partial^{\gamma_1}(|u_k|^{\sigma})\|_{L^{\infty}_T(L^{\infty})} \lesssim R^{\sigma}\| \langle y\rangle^{-2m(\sigma-1)} \|_{\infty} + \sum_{1\le l\le E[\frac{N}{2}]} \nu^{-(2l-\sigma)}R^{2l-1}\Big(R\| \langle y\rangle^{-2m(\sigma-1)} \|_{\infty}$$

$$+\sum_{E[\frac{N}{2}]<|\gamma|\le|\gamma_1|}\|\langle y\rangle^{m(3-2\sigma)}\,\partial^{\gamma}u_k\|_{L^\infty_T(L^\infty)}\Big)$$
$$\lesssim R^{\sigma}+\sum_{1\le l\le E[\frac{N}{2}]}\nu^{-(2l-\sigma)}R^{2l}.$$

Also, by Lemma 2.3, we get

$$\|\langle y\rangle^{m(\sigma-1)}\,\partial^{\gamma_2}(|u_j|^{\sigma-2}u_j)\|_{L^\infty_T(L^\infty)}\lesssim\sum_{0\le l\le E[\frac{N}{2}]}\nu^{-(2l+2-\sigma)}R^{1+2l}.$$

Thus, with the Leibniz rule, if $|\gamma|\le E[\frac{N}{2}]$, we get

$$\begin{aligned}
&\|\langle y\rangle^{m}\,\partial^{\gamma}\mathcal{F}\|_{L^\infty_T(L^\infty)}\\
\lesssim\;&\sum_{j,k=1}^{n}\sum_{\gamma=\gamma_1+\gamma_2}\|\langle y\rangle^{m(2-\sigma)}\,\partial^{\gamma_1}(|u_k|^{p})\|_{L^\infty_T(L^\infty)}\|\langle y\rangle^{m(\sigma-1)}\,\partial^{\gamma_2}(|u_j|^{\sigma-2}u_j)\|_{L^\infty_T(L^\infty)}\\
\lesssim\;&\Big(R^{\sigma}+\sum_{1\le l\le E[\frac{N}{2}]}\nu^{-(2l-\sigma)}R^{2l}\Big)\Big(\sum_{0\le l\le E[\frac{N}{2}]}\nu^{-(2l+2-\sigma)}R^{1+2l}\Big).
\end{aligned}\tag{3.1}$$

Now, with the Leibniz rule via Lemma 2.3, for $E[\frac{N}{2}]\le|\gamma|\le M$, $\frac{1}{2}=\frac{1}{r_1}+\frac{1}{r_2}$, and $\alpha\in\mathbb{R}$,

$$\begin{aligned}
&\|\langle y\rangle^{m}\,\partial^{\gamma}\mathcal{F}\|_{L^\infty_T(L^2)}\\
\lesssim\;&\sum_{j,k=1}^{n}\sum_{\gamma=\gamma_1+\gamma_2,|\gamma_2|\le N}\|\langle y\rangle^{m\alpha}\,\partial^{\gamma_1}(|u_k|^{p})\|_{L^\infty((0,T),L^{r_1})}\|\langle y\rangle^{m(1-\alpha)}\,\partial^{\gamma_2}(|u_j|^{\sigma-2}u_j)\|_{L^\infty((0,T),L^{r_2})}\\
+\;&\sum_{j,k=1}^{n}\sum_{\gamma=\gamma_1+\gamma_2,|\gamma_2|\ge N}\|\langle y\rangle^{m\alpha}\,\partial^{\gamma_1}(|u_k|^{p})\|_{L^\infty((0,T),L^{r_1})}\|\langle y\rangle^{m(1-\alpha)}\,\partial^{\gamma_2}(|u_j|^{\sigma-2}u_j)\|_{L^\infty((0,T),L^{r_2})}\\
:=\;&(I_{|\gamma_2|\le N})+(I_{|\gamma_2|\ge N}).
\end{aligned}$$

Now, (2.2) enables to take $2-\sigma<\alpha<\sigma-\frac{N}{2m}$, $r_1=2$, and $r_2=\infty$. This implies that $\langle y\rangle^{m(\alpha-\sigma)}\in L^2$. Thus, we have

$$\begin{aligned}
&(I_{|\gamma_2|\le N})\\
\lesssim\;&\sum_{j,k=1}^{n}\sum_{\gamma=\gamma_1+\gamma_2}\Big[R^{\sigma}\|\langle y\rangle^{m(\alpha-\sigma)}\|+\sum_{1\le l\le|\gamma_1|}\nu^{-(2l-\sigma)}R^{2l-1}\Big(R\|\langle y\rangle^{m(\alpha-\sigma)}\|\\
+\;&\sum_{E[\frac{N}{2}]<|\mu|\le|\gamma_1|}\|\langle y\rangle^{m(\alpha+1-\sigma)}\,\partial^{\mu}u_k\|_{L^\infty_T(L^2)}\Big)\Big]\Big[\sum_{0\le l\le|\gamma_2|}\nu^{-(2l+2-\sigma)}R^{2l}\Big(R\|\langle y\rangle^{m(2-\alpha-\sigma)}\|_\infty\\
+\;&\sum_{E[\frac{N}{2}]<|\mu|\le|\gamma_2|}\|\langle y\rangle^{m(3-\alpha-\sigma)}\,\partial^{\mu}u_j\|_{L^\infty_T(L^\infty)}\Big]\\
\lesssim\;&\sum_{j,k=1}^{n}\sum_{\gamma=\gamma_1+\gamma_2}\Big[R^{\sigma}+\sum_{1\le l\le|\gamma_1|}\nu^{-(2l-\sigma)}R^{2l-1}\Big(R+\sum_{E[\frac{N}{2}]<|\mu|\le|\gamma_1|}\|\langle y\rangle^{m}\,\partial^{\mu}u_k\|_{L^\infty_T(L^2)}\Big)\Big]
\end{aligned}$$

$$\Big[\sum_{0\le l\le|\gamma_2|}\nu^{-(2l+2-\sigma)}R^{2l}\Big(R+\sum_{E[\frac{N}{2}]<|\mu|\le|\gamma_2|}\|\langle y\rangle^m\,\partial^\mu u_j\|_{L^\infty_T(L^\infty)}\Big)\Big]$$

$$\begin{aligned}\lesssim\ &\sum_{j,k=1}^{n}\sum_{\gamma=\gamma_1+\gamma_2}\Big[R^\sigma+\sum_{1\le l\le|\gamma_1|}\nu^{-(2l-\sigma)}R^{2l}\Big]\Big[\sum_{0\le l\le|\gamma_2|}\nu^{-(2l+2-\sigma)}R^{2l}\Big(R\\ +\ &\sum_{E[\frac{N}{2}]<|\mu|\le|\gamma_2|+E[\frac{N}{2}]+1}\|\langle y\rangle^m\,\partial^\mu u_j\|_{L^\infty_T(L^2)}\Big)\Big].\end{aligned}$$

Assuming that $|\gamma_2|\le N$, $M\ge N+E[\frac{N}{2}]+1$ gives

$$\begin{aligned}&(I_{|\gamma_2|\le N})\\ \lesssim\ &\sum_{j,k=1}^{n}\sum_{\gamma=\gamma_1+\gamma_2}\Big[R^\sigma+\sum_{1\le l\le|\gamma_1|}\nu^{-(2l-\sigma)}R^{2l}\Big]\Big[\sum_{0\le l\le|\gamma_2|}\nu^{-(2l+2-\sigma)}R^{2l}\Big(R\\ +\ &\sum_{E[\frac{N}{2}]<|\mu|\le|\gamma_2|+E[\frac{N}{2}]+1}\|\langle y\rangle^m\,\partial^\mu u_j\|_{L^\infty_T(L^2)}\Big)\Big]\\ \lesssim\ &\sum_{j,k=1}^{n}\sum_{\gamma=\gamma_1+\gamma_2}\Big[R^\sigma+\sum_{1\le l\le|\gamma_1|}\nu^{-(2l-\sigma)}R^{2l}\Big]\Big[\sum_{1\le l\le|\gamma_2|}\nu^{-(2l+1-\sigma)}R^{2l}\Big(R\\ +\ &\sum_{E[\frac{N}{2}]<|\mu|\le M}\|\langle y\rangle^m\,\partial^\mu u_j\|_{L^\infty_T(L^2)}\Big)\Big]\\ \lesssim\ &\sum_{j,k=1}^{n}\sum_{\gamma=\gamma_1+\gamma_2}\Big[R^\sigma+\sum_{1\le l\le M}\nu^{-(2l-\sigma)}R^{2l}\Big]\Big[\sum_{0\le l\le N}\nu^{-(2l+2-\sigma)}R^{1+2l}\Big].\end{aligned}\tag{3.2}$$

Now, we assume that $|\gamma_2|\ge N$, thus $|\gamma_1|\le M-N\le M-E[\frac{N}{2}]-1$. Since by (2.2) we have $m>\frac{N}{2(\sigma-1-\alpha)}$ and $\|\langle y\rangle^{m(1+\alpha-\sigma)}\|<\infty$. So, with Lemma 2.3, we get

$$\begin{aligned}&(I_{|\gamma_2|\ge N})\\ \lesssim\ &\sum_{j,k=1}^{n}\sum_{\gamma=\gamma_1+\gamma_2}\Big[R^\sigma\|\langle y\rangle^{m(\alpha-\sigma)}\|_\infty+\sum_{1\le l\le|\gamma_1|}\nu^{-(2l-\sigma)}R^{2l-1}\Big(R\|\langle y\rangle^{m(\alpha-\sigma)}\|_\infty\\ +\ &\sum_{E[\frac{N}{2}]<|\mu|\le|\gamma_1|}\|\langle y\rangle^{m(\alpha+1-\sigma)}\,\partial^\mu u_k\|_{L^\infty_T(L^\infty)}\Big)\Big]\Big[\sum_{0\le l\le|\gamma_2|}\nu^{-(2l+2-\sigma)}R^{2l}\Big(R\|\langle y\rangle^{m(1+\alpha-\sigma)}\|\\ +\ &\sum_{E[\frac{N}{2}]<|\mu|\le|\gamma_2|}\|\langle y\rangle^{m(3-\alpha-\sigma)}\,\partial^\mu u_j\|_{L^\infty_T(L^2)}\Big)\Big]\\ \lesssim\ &\sum_{j,k=1}^{n}\sum_{\gamma=\gamma_1+\gamma_2}\Big[R^\sigma+\sum_{1\le l\le|\gamma_1|}\nu^{-(2l-\sigma)}R^{2l-1}\Big(R+\sum_{E[\frac{N}{2}]<|\mu|\le|\gamma_1|}\|\langle y\rangle^{m(\alpha+1-\sigma)}\,\partial^\mu u_k\|_{L^\infty_T(L^\infty)}\Big)\Big]\\ &\Big[\sum_{0\le l\le|\gamma_2|}\nu^{-(2l+2-\sigma)}R^{2l}\Big(R+\sum_{E[\frac{N}{2}]<|\mu|\le|\gamma_2|}\|\langle y\rangle^{m(3-\alpha-\sigma)}\,\partial^\mu u_j\|_{L^\infty_T(L^2)}\Big)\Big].\end{aligned}$$

So,

$$(I_{|\gamma_2|\ge N})$$

$$
\begin{aligned}
&\lesssim \sum_{j,k=1}^{n}\sum_{\gamma=\gamma_1+\gamma_2}\Big[R^{\sigma}+\sum_{1\le l\le|\gamma_1|}\nu^{-(2l-\sigma)}R^{2l}\Big(R+\sum_{1+2E[\frac{N}{2}]<|\mu|\le M}\|\partial^{\mu}u_k\|_{L^{\infty}_T(L^2)}\Big)\Big]\\
&\quad\Big[\sum_{0\le l\le|\gamma_2|}\nu^{-(2l+2-\sigma)}R^{1+2l}\Big]\\
&\lesssim \sum_{j,k=1}^{n}\sum_{\gamma=\gamma_1+\gamma_2}\Big[R^{\sigma}+\sum_{1\le l\le M}\nu^{-(2l-\sigma)}R^{1+2l}\Big]\Big[\sum_{0\le l\le M}\nu^{-(2l+2-\sigma)}R^{1+2l}\Big].
\end{aligned}
$$

It follows that

$$
\|\langle y\rangle^{m}\,\partial^{\gamma}\mathcal{F}\|_{L^{\infty}_T(L^2)}\lesssim\sum_{j,k=1}^{n}\sum_{\gamma=\gamma_1+\gamma_2}\Big[R^{\sigma}+\sum_{1\le l\le M}\nu^{-(2l-\sigma)}R^{1+2l}\Big]\Big[\sum_{0\le l\le M}\nu^{-(2l+2-\sigma)}R^{1+2l}\Big]. \tag{3.3}
$$

We break down the next term as follows:

$$
\begin{aligned}
&\|\partial^{\gamma}\mathcal{F}\|_{L^{\infty}_T(L^2)}\\
\lesssim\;& \sum_{j,k=1}^{n}\sum_{\gamma=\gamma_1+\gamma_2,|\gamma_1|<N}\|\langle y\rangle^{m(2-\sigma)}\,\partial^{\gamma_1}(|u_k|^{p})\|_{L^{\infty}_T(L^{\infty})}\|\langle y\rangle^{-m(2-\sigma)}\,\partial^{\gamma_2}(|u_j|^{\sigma-2}u_j)\|_{L^{\infty}_T(L^2)}\\
+\;& \sum_{j,k=1}^{n}\sum_{\gamma=\gamma_1+\gamma_2,N\le|\gamma_1|\le M}\|\langle y\rangle^{m(2-\sigma)}\,\partial^{\gamma_1}(|u_k|^{p})\|_{L^{\infty}_T(L^2)}\|\langle y\rangle^{-m(2-\sigma)}\,\partial^{\gamma_2}(|u_j|^{\sigma-2}u_j)\|_{L^{\infty}_T(L^{\infty})}\\
+\;& \sum_{j,k=1}^{n}\sum_{\gamma=\gamma_1+\gamma_2,M<|\gamma_1|\le-N+M_0+M}\|\partial^{\gamma_1}(|u_k|^{p})\|_{L^{\infty}_T(L^2)}\|\partial^{\gamma_2}(|u_j|^{\sigma-2}u_j)\|_{L^{\infty}_T(L^{\infty})}\\
:=\;& (A_{|\gamma_1|<N})+(A_{N\le|\gamma_1|\le M})+(A_{M<|\gamma_1|\le-N+M_0+M}).
\end{aligned} \tag{3.4}
$$

Now, with the Leibniz rule and Lemma 2.3, for $|\gamma|\le -N+M_0+M$, we have

$$
\begin{aligned}
&(A_{|\gamma_1|<N})\\
\lesssim\;& \sum_{j,k=1}^{n}\sum_{\gamma=\gamma_1+\gamma_2}\|\langle y\rangle^{m(2-\sigma)}\,\partial^{\gamma_1}(|u_k|^{p})\|_{L^{\infty}_T(L^{\infty})}\|\langle y\rangle^{-m(2-\sigma)}\,\partial^{\gamma_2}(|u_j|^{\sigma-2}u_j)\|_{L^{\infty}_T(L^2)}\\
\lesssim\;& \sum_{j,k=1}^{n}\sum_{\gamma=\gamma_1+\gamma_2}\Big[R^{\sigma}\|\langle y\rangle^{2m(1-\sigma)}\|_{\infty}+\sum_{1\le l\le|\gamma_1|}\nu^{-(2l-\sigma)}R^{2l-1}\Big(R\|\langle y\rangle^{2m(1-\sigma)}\|_{\infty}\\
+\;& \sum_{E[\frac{N}{2}]<|\mu|\le|\gamma_1|}\|\langle y\rangle^{m(3-2\sigma)}\,\partial^{\mu}u_k\|_{L^{\infty}_T(L^{\infty})}\Big)\Big]\Big[\sum_{0\le l\le|\gamma_2|}\nu^{-(2l+2-\sigma)}R^{2l}\Big(R\|\langle y\rangle^{-m}\|\\
+\;& \sum_{E[\frac{N}{2}]<|\mu|\le|\gamma_2|}\|\partial^{\mu}u_j\|_{L^{\infty}_T(L^2)}\Big].
\end{aligned} \tag{3.5}
$$

(2.2) gives $\langle y\rangle^{-m}\in L^2$, and

$$(A_{|\gamma_1|<N})$$

$$\lesssim \sum_{j,k=1}^{n}\sum_{\gamma=\gamma_1+\gamma_2}\Big[R^{\sigma}+\sum_{1\le l\le|\gamma_1|}\nu^{-(2l-\sigma)}R^{2l-1}\Big(R+\sum_{E[\frac{N}{2}]<|\mu|\le|\gamma_1|}\|\langle y\rangle^{m(3-2\sigma)}\partial^{\mu}u_k\|_{L^\infty_T(L^\infty)}\Big)\Big]$$
$$\Big[\sum_{0\le l\le|\gamma_2|}\nu^{-(2l+2-\sigma)}R^{2l}\Big(R+\sum_{E[\frac{N}{2}]<|\mu|\le M}\|\langle y\rangle^{m}\partial^{\mu}u_j\|_{L^\infty_T(L^2)}+\sum_{M<|\mu|\le -N+M_0+M}\|\partial^{\mu}u_j\|_{L^\infty_T(L^2)}\Big)\Big]$$
$$\lesssim \sum_{j,k=1}^{n}\sum_{\gamma=\gamma_1+\gamma_2}\Big[R^{\sigma}+\sum_{1\le l\le|\gamma_1|}\nu^{-(2l-\sigma)}R^{2l-1}\Big(R+\sum_{E[\frac{N}{2}]<|\mu|\le|\gamma_1|}\|\langle y\rangle^{m(3-2\sigma)}\partial^{\mu}u_k\|_{L^\infty_T(L^\infty)}\Big)\Big]$$
$$\Big[\sum_{0\le l\le|\gamma_2|}\nu^{-(2l+2-\sigma)}R^{1+2l}\Big]. \tag{3.6}$$

By Sobolev injection, we have

$$\|\langle y\rangle^{m(3-2\sigma)}\partial^{\mu}u_k\|_{L^\infty_T(L^\infty)}\lesssim\|\partial^{\mu}u_k\|_{L^\infty_T(L^\infty)}\lesssim\|\partial^{\mu}u_k\|_{L^\infty((0,T),\dot H^{1+E[\frac{N}{2}]})}.$$

Thus, by (2.4), we get

$$(A_{|\gamma_1|<N})$$
$$\lesssim \sum_{j,k=1}^{n}\sum_{\gamma=\gamma_1+\gamma_2}\Big[R^{\sigma}+\sum_{1\le l\le|\gamma_1|}\nu^{-(2l-\sigma)}R^{2l-1}\Big(R+\sum_{1+2E[\frac{N}{2}]<|\mu|\le 1+N+E[\frac{N}{2}]}\|\partial^{\mu}u_k\|_{L^\infty_T(L^2)}\Big)\Big]$$
$$\Big[\sum_{0\le l\le|\gamma_2|}\nu^{-(2l+2-\sigma)}R^{1+2l}\Big]$$
$$\lesssim \sum_{j,k=1}^{n}\sum_{\gamma=\gamma_1+\gamma_2}\Big[R^{\sigma}+\sum_{1\le l\le N}\nu^{-(2l-\sigma)}R^{2l}\Big]\Big[\sum_{0\le l\le -N+M_0+M}\nu^{-(2l+2-\sigma)}R^{1+2l}\Big]. \tag{3.7}$$

Now, with Lemma 2.3, we have

$$(A_{N\le|\gamma_1|\le M})$$
$$\lesssim \sum_{j,k=1}^{n}\sum_{\gamma=\gamma_1+\gamma_2}\|\langle y\rangle^{m(2-\sigma)}\partial^{\gamma_1}(|u_k|^p)\|_{L^\infty_T(L^2)}\|\langle y\rangle^{-m(2-\sigma)}\partial^{\gamma_2}(|u_j|^{\sigma-2}u_j)\|_{L^\infty_T(L^\infty)}$$
$$\lesssim \sum_{j,k=1}^{n}\sum_{\gamma=\gamma_1+\gamma_2}\Big[R^{\sigma}\|\langle y\rangle^{2m(1-\sigma)}\|+\sum_{1\le l\le|\gamma_1|}\nu^{-(2l-\sigma)}R^{2l-1}\Big(R\|\langle y\rangle^{2m(1-\sigma)}\|$$
$$+\sum_{E[\frac{N}{2}]<|\mu|\le|\gamma_1|}\|\langle y\rangle^{m(3-2\sigma)}\partial^{\mu}u_k\|_{L^\infty_T(L^2)}\Big)\Big]\Big[\sum_{0\le l\le|\gamma_2|}\nu^{-(2l+2-\sigma)}R^{2l}\Big(R\|\langle y\rangle^{-m}\|_{\infty}$$
$$+\sum_{E[\frac{N}{2}]<|\mu|\le|\gamma_2|}\|\partial^{\mu}u_j\|_{L^\infty_T(L^\infty)}\Big].$$

Moreover, (2.2) gives $\langle y\rangle^{2m(1-\sigma)}\in L^2$, and so

$$(A_{N\le|\gamma_1|\le M})$$

$$\lesssim \sum_{j,k=1}^{n}\sum_{\gamma=\gamma_1+\gamma_2}\Big[R^{\sigma}+\sum_{1\leq l\leq|\gamma_1|}\nu^{-(2l-\sigma)}R^{2l-1}\Big(R+\sum_{E[\frac{N}{2}]<|\mu|\leq M}\|\langle y\rangle^{m(3-2\sigma)}\partial^{\mu}u_k\|_{L^{\infty}_T(L^2)}\Big)\Big]$$
$$\Big[\sum_{0\leq l\leq|\gamma_2|}\nu^{-(2l+2-\sigma)}R^{2l}\Big(R+\sum_{E[\frac{N}{2}]<|\mu|\leq|\gamma_2|}\|\partial^{\mu}u_j\|_{L^{\infty}_T(L^{\infty})}\Big)\Big]$$
$$\lesssim \sum_{j,k=1}^{n}\sum_{\gamma=\gamma_1+\gamma_2}\Big[R^{\sigma}+\sum_{1\leq l\leq|\gamma_1|}\nu^{-(2l-\sigma)}R^{2l}\Big]\Big[\sum_{0\leq l\leq|\gamma_2|}\nu^{-(2l+2-\sigma)}R^{2l}\Big(R$$
$$+\sum_{E[\frac{N}{2}]<|\mu|\leq 1+E[\frac{N}{2}]+|\gamma_2|}\|\partial^{\mu}u_j\|_{L^{\infty}_T(L^2)}\Big)\Big]. \tag{3.8}$$

Since $N+|\gamma_2|\leq|\gamma_1|+|\gamma_2|\leq -N+M_0+M$, we have $|\gamma_2|\leq M+M_0-2N$, and

$$(A_{N\leq|\gamma_1|\leq M})$$
$$\lesssim \sum_{j,k=1}^{n}\sum_{\gamma=\gamma_1+\gamma_2}\Big[R^{\sigma}+\sum_{1\leq l\leq|\gamma_1|}\nu^{-(2l-\sigma)}R^{2l}\Big]\Big[\sum_{0\leq l\leq|\gamma_2|}\nu^{-(2l+2-\sigma)}R^{2l}\Big(R+\sum_{E[\frac{N}{2}]<|\mu|\leq 1+E[\frac{N}{2}]+|\gamma_2|}\|\partial^{\mu}u_j\|_{L^{\infty}_T(L^2)}\Big)\Big]$$
$$\lesssim \sum_{j,k=1}^{n}\sum_{\gamma=\gamma_1+\gamma_2}\Big[R^{\sigma}+\sum_{1\leq l\leq|\gamma_1|}\nu^{-(2l-\sigma)}R^{2l}\Big]\Big[\sum_{0\leq l\leq|\gamma_2|}\nu^{-(2l+2-\sigma)}R^{2l}\Big(R+\sum_{E[\frac{N}{2}]<|\mu|\leq -N+M_0+M}\|\partial^{\mu}u_j\|_{L^{\infty}_T(L^2)}\Big)\Big]$$
$$\lesssim \sum_{j,k=1}^{n}\sum_{\gamma=\gamma_1+\gamma_2}\Big[R^{\sigma}+\sum_{1\leq l\leq M}\nu^{-(2l-\sigma)}R^{2l}\Big]\Big[\sum_{0\leq l\leq -N+M_0+M}\nu^{-(2l+2-\sigma)}R^{1+2l}\Big]. \tag{3.9}$$

Finally, assume that $M<|\gamma_1|\leq -N+M_0+M$. Thus, $|\gamma_2|\leq M_0-N\leq M-E[\frac{N}{2}]-1$. So, with Sobolev injections via Lemma 2.3, we have

$$(A_{M<|\gamma_1|\leq -N+M_0+M})$$
$$\lesssim \sum_{j,k=1}^{n}\sum_{\gamma=\gamma_1+\gamma_2}\|\partial^{\gamma_1}(|u_k|^p)\|_{L^{\infty}_T(L^2)}\|\partial^{\gamma_2}(|u_j|^{\sigma-2}u_j)\|_{L^{\infty}_T(L^{\infty})}$$
$$\lesssim \Big[R^{\sigma}\|\langle y\rangle^{-m\sigma}\|+\sum_{1\leq k\leq|\gamma_1|}\nu^{-(2k-\sigma)}R^{2k-1}\Big(R\|\langle y\rangle^{-m\sigma}\|+\sum_{E[\frac{N}{2}]<|\gamma|\leq|\gamma_1|}\|\langle y\rangle^{m(1-\sigma)}\partial^{\gamma}u\|_{L^{\infty}_T(L^2)}\Big)\Big]$$
$$\Big[\sum_{0\leq k\leq|\gamma_2|}\nu^{-(2k+2-\sigma)}R^{2k}\Big(R\|\langle y\rangle^{-m(\sigma-1)}\|_{\infty}+\sum_{E[\frac{N}{2}]<|\gamma|\leq|\gamma_2|}\|\langle y\rangle^{m(2-\sigma)}\partial^{\gamma}u\|_{L^{\infty}_T(L^{\infty})}\Big)\Big]$$
$$\lesssim \Big[R^{\sigma}+\sum_{1\leq k\leq|\gamma_1|}\nu^{-(2k-\sigma)}R^{2k-1}\Big(R+\sum_{E[\frac{N}{2}]<|\gamma|\leq|\gamma_1|}\|\langle y\rangle^{m(1-\sigma)}\partial^{\gamma}u\|_{L^{\infty}_T(L^2)}\Big)\Big]$$
$$\Big[\sum_{0\leq k\leq|\gamma_2|}\nu^{-(2k+2-\sigma)}R^{2k}\Big(R+\sum_{E[\frac{N}{2}]<|\gamma|\leq|\gamma_2|+1+E[\frac{N}{2}]}\|\langle y\rangle^{m}\partial^{\gamma}u\|_{L^{\infty}_T(L^2)}\Big)\Big].$$

Thus,

$$(A_{M<|\gamma_1|\leq -N+M_0+M})$$
$$\lesssim \Big[R^{\sigma}+\sum_{1\leq k\leq|\gamma_1|}\nu^{-(2k-\sigma)}R^{2k-1}\Big(R+\sum_{E[\frac{N}{2}]<|\gamma|\leq M}\|\langle y\rangle^{m}\partial^{\gamma}u\|_{L^{\infty}_T(L^2)}$$

$$
\begin{aligned}
&+ \sum_{M<|\gamma|\le -N+M_0+M} \partial^\gamma u\|_{L^\infty_T(L^2)}\Big)\Big]\Big[\sum_{0\le k\le M-E[\frac{N}{2}]-1} \nu^{-(2k+2-\sigma)}R^{1+2k}\Big]\\
&\lesssim \Big[R^\sigma + \sum_{1\le k\le -N+M_0+M} \nu^{-(2k-\sigma)}R^{2k}\Big]\Big[\sum_{0\le k\le M-E[\frac{N}{2}]-1} \nu^{-(2k+2-\sigma)}R^{1+2k}\Big]. \qquad (3.10)
\end{aligned}
$$

Collecting (3.7), (3.9), and (3.10), we get

$$
\|I_{M+M_0-N}\mathcal{F}\|_{L^\infty_T(L^2)} \lesssim \Big[R^\sigma + \sum_{1\le k\le -N+M_0+M} \nu^{-(2k-\sigma)}R^{2k}\Big]\Big[\sum_{0\le k\le -N+M_0+M} \nu^{-(2k+2-\sigma)}R^{1+2k}\Big]. \qquad (3.11)
$$

Thus, with (3.1)–(3.3) and (3.11), we get

$$
\begin{aligned}
&\| \langle y\rangle^m \partial^\alpha f_j(u)\|_{L^\infty_T(L^\infty)}\\
\lesssim\ & \langle T\rangle^{E[\frac{N}{2}]+1+m}\Big(\sum_{|\gamma|\le E[\frac{N}{2}]} \| \langle y\rangle^m \partial^\gamma u_0\|_\infty + \|I_{-N+M_0+M}u_0\| + \sum_{E[\frac{N}{2}]<|\gamma|\le M} \| \langle y\rangle^m \partial^\gamma u_0\|\Big)\\
+\ & T\langle T\rangle^{E[\frac{N}{2}]+1+m}\Big[R^\sigma + \sum_{1\le l\le -N+M_0+M} \nu^{-(2l-\sigma)}R^{2l}\Big]\Big[\sum_{0\le l\le -N+M_0+M} \nu^{-(2l+2-\sigma)}R^{1+2l}\Big].
\end{aligned}
$$

Now, by Lemma 2.1, (3.3), and (3.11), we write

$$
\begin{aligned}
&\| \langle y\rangle^m \partial^\alpha f_j(u)\|_{L^\infty_T(L^2)}\\
\lesssim\ & \langle T\rangle^m \Big(\|I_{-N+M_0+M}u_0\| + \| \langle y\rangle^m \partial^\alpha u_0\|\Big)\\
+\ & T\langle T\rangle^m \Big(\|I_{-N+M_0+M}\mathcal{F}\|_{L^\infty_T(L^2)} + \| \langle y\rangle^m \partial^\alpha \mathcal{F}\|_{L^\infty_T(L^2)}\Big)\\
\lesssim\ & \langle T\rangle^m \Big(\|I_{-N+M_0+M}u_0\| + \| \langle y\rangle^m \partial^\alpha u_0\|\Big)\\
+\ & T\langle T\rangle^m \Big[R^\sigma + \sum_{1\le l\le -N+M_0+M} \nu^{-(2l-\sigma)}R^{2l}\Big]\Big[\sum_{0\le l\le -N+M_0+M} \nu^{-(2l+2-\sigma)}R^{1+2l}\Big]. \qquad (3.12)
\end{aligned}
$$

Moreover, with Lemma 2.1 and (3.11), for $|\alpha| \le -N + M_0 + M$, yields

$$
\begin{aligned}
&\|\partial^\alpha f_j(u)\|_{L^\infty_T(L^2)}\\
\lesssim\ & \|I_{-N+M_0+M}u_0\| + T\|\partial^\alpha \mathcal{F}\|_{L^\infty_T(L^2)}\\
\lesssim\ & \|u_0\|_{H^{-N+M_0+M}} + T\Big[R^\sigma + \sum_{1\le l\le -N+M_0+M} \nu^{-(2l-\sigma)}R^{2l}\Big]\Big[\sum_{0\le l\le -N+M_0+M} \nu^{-(2l+2-\sigma)}R^{1+2l}\Big].
\end{aligned}
$$

So, taking $R := 2c\|u_0\|_{\mathcal{Y}}$, we get

$$
\begin{aligned}
&\|f(u)\|_{L^\infty_T(\mathcal{Y})}\\
\le\ & c\langle T\rangle^{E[\frac{N}{2}]+1+m}\|u_0\|_{\mathcal{Y}} + cT\langle T\rangle^{E[\frac{N}{2}]+1+m}\Big[R^\sigma + \sum_{1\le l\le -N+M_0+M} \nu^{-(2l-\sigma)}R^{2l}\Big]\Big[\sum_{0\le l\le -N+M_0+M} \nu^{-(2l+2-\sigma)}R^{1+2l}\Big]\\
\le\ & \langle T\rangle^{E[\frac{N}{2}]+1+m}\frac{R}{2} + cT\langle T\rangle^{E[\frac{N}{2}]+1+m}\Big[R^\sigma + \sum_{1\le l\le -N+M_0+M} \nu^{-(2l-\sigma)}R^{2l}\Big]\Big[\sum_{0\le l\le -N+M_0+M} \nu^{-(2l+2-\sigma)}R^{1+2l}\Big]\\
:=\ & \langle T\rangle^{E[\frac{N}{2}]+1+m}\frac{R}{2} + cT\langle T\rangle^{E[\frac{N}{2}]+1+m}F_1(\nu,R)F_2(\nu,R). \qquad (3.13)
\end{aligned}
$$

Then, choosing $0 < T << 1$, it follows that $f(B_T(R)) \subset B_T(R)$. Now, we prove that

$$2 \inf_{(t,y)\in[0,T]\times\mathbb{R}^N} |\langle y\rangle^m f(u(t,y))| \geq \nu.$$

Using the time derivative identity $(e^{it\Delta})^{(k)} = (i\Delta)^k e^{it\Delta}$, we get

$$e^{it\Delta} = \sum_{0\leq k\leq E[\frac{N}{2}]} \frac{(it)^k}{k!}\Delta^k + \frac{i^{1+E[\frac{N}{2}]}}{(E[\frac{N}{2}])!}\int_0^t (t-s)^{E[\frac{N}{2}]}\Delta^{1+E[\frac{N}{2}]}(e^{is\Delta})ds.$$

Thus, with Lemma 2.1 and Sobolev injections via (2.4), we write

$$\begin{aligned}
&\|\langle y\rangle^m (e^{it\boldsymbol{\Delta}}u_0 - u_0)\|_\infty \\
\lesssim\ & \sum_{1\leq j\leq n}\sum_{1\leq k\leq E[\frac{N}{2}]} t^k\|\langle y\rangle^m \Delta^k u_{0,j}\|_\infty + \int_0^t (t-s)^{E[\frac{N}{2}]}\|\langle y\rangle^m \Delta^{1+E[\frac{N}{2}]}(e^{is\Delta}u_{0,j})\|_\infty ds \\
\lesssim\ & t\langle t\rangle^{E[\frac{N}{2}]}\Big(\sum_{1\leq|\alpha|\leq E[\frac{N}{2}]}\|\langle y\rangle^m \partial^\alpha u_0\|_\infty + \sum_{E[\frac{N}{2}]<|\alpha|\leq N}\|\langle y\rangle^m \partial^\alpha u_0\|_\infty\Big) \\
+\ & \int_0^t (t-s)^{E[\frac{N}{2}]}\|\langle y\rangle^m \Delta^{1+E[\frac{N}{2}]}(e^{is\boldsymbol{\Delta}}u_0)\|_\infty ds \\
\lesssim\ & t\langle t\rangle^{E[\frac{N}{2}]}\Big(\sum_{1\leq|\alpha|\leq E[\frac{N}{2}]}\|\langle y\rangle^m \partial^\alpha u_0\|_\infty + \sum_{E[\frac{N}{2}]<|\alpha|\leq M}\|\langle y\rangle^m \partial^\alpha u_0\|\Big) \\
+\ & \int_0^t (t-s)^{E[\frac{N}{2}]}\|\langle y\rangle^m \Delta^{1+E[\frac{N}{2}]}(e^{is\boldsymbol{\Delta}}u_0)\|_\infty ds.
\end{aligned}$$

Now, by Lemma 2.2, for $s = 5 + m + 5E[\frac{N}{2}]$ and $K = 2(1 + E[\frac{N}{2}])$,

$$\begin{aligned}
&\|\langle y\rangle^m \Delta^{1+E[\frac{N}{2}]}(e^{is\boldsymbol{\Delta}}u_0)\|_\infty \\
\lesssim\ & \langle t\rangle^{E[\frac{N}{2}]}\sum_{|\gamma|\leq E[\frac{N}{2}]}\|\langle y\rangle^m \partial^\gamma u\|_\infty + \langle t\rangle^{1+E[\frac{N}{2}]+m}\Big(\|I_s u\| + \sum_{E[\frac{N}{2}]<|\gamma|\leq 2(1+E[\frac{N}{2}])+3+3E[\frac{N}{2}]}\|\langle y\rangle^m \partial^\gamma u\|\Big).
\end{aligned}$$

Now, since $M \geq 4E[\frac{N}{2}] + 5 + m$ and $M_0 - N > m > \frac{N}{2}$, we get

$$\begin{aligned}
&\|\langle y\rangle^m \Delta^{1+E[\frac{N}{2}]}(e^{is\boldsymbol{\Delta}}u_0)\|_\infty \\
\lesssim\ & \langle t\rangle^{E[\frac{N}{2}]}\sum_{|\gamma|\leq E[\frac{N}{2}]}\|\langle y\rangle^m \partial^\gamma u\|_\infty + \langle t\rangle^{1+E[\frac{N}{2}]+m}\Big(\|I_{-N+M_0+M}u\| + \sum_{E[\frac{N}{2}]<|\gamma|\leq 5+5E[\frac{N}{2}]}\|\langle y\rangle^m \partial^\gamma u\|\Big) \\
\lesssim\ & \langle t\rangle^{E[\frac{N}{2}]}\sum_{|\gamma|\leq E[\frac{N}{2}]}\|\langle y\rangle^m \partial^\gamma u\|_\infty + \langle t\rangle^{1+E[\frac{N}{2}]+m}\Big(\|I_{-N+M_0+M}u\| + \sum_{E[\frac{N}{2}]<|\gamma|\leq M}\|\langle y\rangle^m \partial^\gamma u\|\Big).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\|\langle y\rangle^m (e^{it\boldsymbol{\Delta}}u_0 - u_0)\|_\infty \\
\lesssim\ & t\langle t\rangle^{E[\frac{N}{2}]}\Big(\sum_{1\leq|\alpha|\leq E[\frac{N}{2}]}\|\langle y\rangle^m \partial^\alpha u_0\|_\infty + \sum_{E[\frac{N}{2}]<|\alpha|\leq M}\|\langle y\rangle^m \partial^\alpha u_0\|\Big)
\end{aligned}$$

$$+ \quad t\langle t\rangle^{E[\frac{N}{2}]}\Big[\langle t\rangle^{E[\frac{N}{2}]}\sum_{|\gamma|\le E[\frac{N}{2}]}\|\langle y\rangle^m\partial^\gamma u\|_\infty + \langle t\rangle^{1+E[\frac{N}{2}]+m}\Big(\|I_{-N+M_0+M}u\| + \sum_{E[\frac{N}{2}]<|\gamma|\le M}\|\langle y\rangle^m\partial^\gamma u\|\Big)\Big].$$

Now, with Lemma 2.2 with (3.13),

$$\|\langle y\rangle^m(f(u)-e^{it\Delta}u_0)\|_{L^\infty_T(L^\infty)} \le cT\langle T\rangle^{E[\frac{N}{2}]+1+m}F_1(\nu,R)F_2(\nu,R).$$

Thus, there is $C(T)\to 0$ as $T\to 0$ such that

$$\begin{aligned} |\langle y\rangle^m f(u)| &\ge |\langle y\rangle^m e^{it\Delta}u_0| - |\langle y\rangle^m(e^{it\Delta}u_0-u_0)| - |\langle y\rangle^m(f(u)-e^{it\Delta}u_0)| \\ &\ge \nu - C(T). \end{aligned} \tag{3.14}$$

So, taking $0<T<<1$, we get

$$\inf_{\{(t,y)\in[0,T]\times\mathbb{R}^N\}}|\langle y\rangle^m f(u)| \ge \frac{\nu}{2}.$$

Thus, $f(B_{\nu,T}(R))\subset B_{\nu,T}(R)$. Now, we prove that f is a contraction. For $u,v\in C_T(\mathcal{Y})$ and $w:=u-v$, we have

$$\begin{aligned} &\|f(u)-f(v)\|_{L^\infty_T(\mathcal{Y})} \\ =\ & \|\int_0^\cdot e^{i(\cdot-s)\Delta}[\mathcal{F}(u)-\mathcal{F}(v)]\,ds\|_{L^\infty_T(\mathcal{Y})} \\ \lesssim\ & \|\int_0^\cdot e^{i(\cdot-s)\Delta}\Big[\Big(|u_k|^\sigma|u_1|^{\sigma-2}u_1-|v_k|^\sigma|v_1|^{\sigma-2}v_1,\dots,|u_k|^\sigma|u_n|^{\sigma-2}u_n-|v_k|^\sigma|v_n|^{\sigma-2}v_n\Big)\Big]\,ds\|_{L^\infty_T(\mathcal{Y})} \\ \lesssim\ & \sum_{1\le j\le n}\Big(\sum_{|\alpha|\le E[\frac{N}{2}]}\|\int_0^\cdot\langle y\rangle^m\partial^\alpha e^{i(\cdot-s)\Delta}\Big[\Big(|u_k|^\sigma|u_j|^{\sigma-2}u_j-|v_k|^\sigma|v_j|^{\sigma-2}v_j\Big)\Big]\|_{L^\infty_T(L^\infty)} \\ +\ & \sum_{E[\frac{N}{2}]<|\alpha|\le M}\|\int_0^\cdot\langle y\rangle^m\partial^\alpha e^{i(\cdot-s)\Delta}\Big[\Big(|u_k|^\sigma|u_j|^{\sigma-2}u_j-|v_k|^\sigma|v_j|^{\sigma-2}v_j\Big)\Big]\|_{L^\infty_T(L^2)} \\ +\ & \sum_{M<|\alpha|\le -N+M_0+M}\|\int_0^\cdot\partial^\alpha e^{i(\cdot-s)\Delta}\Big[\Big(|u_k|^\sigma|u_j|^{\sigma-2}u_j-|v_k|^\sigma|v_j|^{\sigma-2}v_j\Big)\Big]\|_{L^\infty_T(L^2)}\Big). \end{aligned}$$

Let us control the three above terms. By Lemma 2.2 via (2.4), we have

$$\begin{aligned} (I) &:= \sum_{|\alpha|\le E[\frac{N}{2}]}\|\int_0^\cdot\langle y\rangle^m\partial^\alpha e^{i(\cdot-s)\Delta}\Big[\Big(|u_k|^\sigma|u_j|^{\sigma-2}u_j-|v_k|^\sigma|v_j|^{\sigma-2}v_j\Big)\Big]\|_{L^\infty_T(L^\infty)} \\ &\lesssim T\langle T\rangle^{E[\frac{N}{2}]+1+m}\Big(\sum_{|\gamma|\le E[\frac{N}{2}]}\|\langle y\rangle^m\partial^\gamma(\mathcal{F}_j(u)-\mathcal{F}_j(v))\|_{L^\infty_T(L^\infty)} \\ &+ \|I_{-N+M_0+M}(\mathcal{F}_j(u)-\mathcal{F}_j(v))\|_{L^\infty_T(L^2)} + \sum_{E[\frac{N}{2}]<|\gamma|\le M}\|\langle y\rangle^m\partial^\gamma(\mathcal{F}_j(u)-\mathcal{F}_j(v))\|_{L^\infty_T(L^2)}\Big). \end{aligned}$$

Let $|\gamma|\le E[\frac{N}{2}]$, and write

$$\begin{aligned} (I_1) &:= \|\langle y\rangle^m\partial^\gamma(\mathcal{F}_j(u)-\mathcal{F}_j(v))\|_{L^\infty_T(L^\infty)} \\ &= \|\langle y\rangle^m\partial^\gamma\Big((|u_k|^\sigma-|v_k|^\sigma)|u_j|^{\sigma-2}u_j+|v_k|^\sigma(|u_j|^{\sigma-2}u_j-|v_j|^{\sigma-2}v_j)\Big)\|_{L^\infty_T(L^\infty)} \end{aligned}$$

$$
\begin{aligned}
&\lesssim \sum_{\gamma=\gamma_1+\gamma_2} \Big(\| \langle y\rangle^m \, \partial^{\gamma_1}(|u_k|^\sigma - |v_k|^\sigma)\partial^{\gamma_2}(|u_j|^{\sigma-2}u_j)\|_{L^\infty_T(L^\infty)} \\
&+ \| \langle y\rangle^m \, \partial^{\gamma_1}|v_k|^\sigma \partial^{\gamma_2}(|u_j|^{\sigma-2}u_j - |v_j|^{\sigma-2}v_j)\|_{L^\infty_T(L^\infty)} \Big) \\
&\lesssim \sum_{\gamma=\gamma_1+\gamma_2} \Big(\| \langle y\rangle^{m(2-\sigma)} \, \partial^{\gamma_1}(|u_k|^\sigma - |v_k|^\sigma)\|_{L^\infty_T(L^\infty)} \| \langle y\rangle^{m(\sigma-1)} \, \partial^{\gamma_2}(|u_j|^{\sigma-2}u_j)\|_{L^\infty_T(L^\infty)} \\
&+ \| \langle y\rangle^{m(2-\sigma)} \, \partial^{\gamma_1}|v_k|^\sigma\|_{L^\infty_T(L^\infty)} \| \langle y\rangle^{m(\sigma-1)} \, \partial^{\gamma_2}(|u_j|^{\sigma-2}u_j - |v_j|^{\sigma-2}v_j)\|_{L^\infty_T(L^\infty)} \Big).
\end{aligned}
$$

Taking account of Lemma 2.4, we have

$$
\begin{aligned}
& \| \langle y\rangle^{m(2-\sigma)} \, \partial^{\gamma_1}(|u_k|^\sigma - |v_k|^\sigma)\|_{L^\infty_T(L^\infty)} \\
\lesssim\ & (R^{\sigma-1} + \nu^{-2(3-\sigma)}R^{5-\sigma}) \| \langle y\rangle^{2m(1-\sigma)} \, \|_\infty \|w\|_{L^\infty_T(\mathcal{Y})} \\
+ & \sum_{1\le k\le E[\frac{N}{2}]} \nu^{-2(2k-\sigma)}R^{4k-\sigma-1} \| \langle y\rangle^{2m(1-\sigma)} \, \|_\infty \|w\|_{L^\infty_T(\mathcal{Y})} \\
+ & \sum_{1\le k\le E[\frac{N}{2}]} \nu^{-(2k-\sigma)}R^{2k-1} \| \langle y\rangle^{2m(1-\sigma)} \, \|_\infty \|w\|_{L^\infty_T(\mathcal{Y})} \\
+ & \sum_{1\le k\le E[\frac{N}{2}]} \nu^{-(2k-\sigma)}R^{2(k-1)} \|w\|_{L^\infty_T(\mathcal{Y})} \\
\lesssim\ & \Big[R^{\sigma-1} + \nu^{-2(3-\sigma)}R^{5-\sigma} + \sum_{1\le k\le E[\frac{N}{2}]} \Big(\nu^{-2(2k-\sigma)}R^{4k-\sigma-1} + \nu^{-(2k-\sigma)}R^{2k-1} \Big) \Big] \|w\|_{L^\infty_T(\mathcal{Y})}.
\end{aligned}
$$

Moreover, also with Lemma 2.4, we have

$$
\begin{aligned}
& \| \langle y\rangle^{m(\sigma-1)} \, \partial^{\gamma_2}(|u_k|^{\sigma-2}u_k - |v_k|^{\sigma-2}v_k)\|_{L^\infty_T(L^\infty)} \\
\lesssim\ & \Big[\nu^{-2(3-\sigma)}R^{4-\sigma} + \sum_{1\le k\le E[\frac{N}{2}]} \Big(\nu^{-2(2(1+k)-\sigma)}R^{4k-\sigma+2} + \nu^{-(2(1+k)-\sigma)}R^{2k} \Big) \Big] \|w\|_{L^\infty_T(\mathcal{Y})}.
\end{aligned}
$$

Now, by (3.1), it follows that

$$
\begin{aligned}
& (I_1) \\
\lesssim\ & \Big(\Big[R^{\sigma-1} + \nu^{-2(3-\sigma)}R^{5-\sigma} + \sum_{1\le k\le E[\frac{N}{2}]} \Big(\nu^{-2(2k-\sigma)}R^{4k-\sigma-1} + \nu^{-(2k-\sigma)}R^{2k-1} \Big) \Big] F_2(\nu, R) \\
+ & \Big[\nu^{-2(3-\sigma)}R^{4-\sigma} + \sum_{1\le k\le E[\frac{N}{2}]} \Big(\nu^{-2(2(1+k)-\sigma)}R^{4k-\sigma+2} + \nu^{-(2(1+k)-\sigma)}R^{2k} \Big) \Big] F_1(\nu, R) \Big) \\
& \|w\|_{L^\infty_T(\mathcal{Y})}.
\end{aligned} \tag{3.15}
$$

Let $E[\frac{N}{2}] < \gamma \le M$. Then,

$$
\begin{aligned}
(I_2) \ &:= \ \| \langle y\rangle^m \, \partial^\gamma(\mathcal{F}_j(u) - \mathcal{F}_j(v))\|_{L^\infty_T(L^2)} \\
&= \ \| \langle y\rangle^m \, \partial^\gamma\Big((|u_k|^\sigma - |v_k|^\sigma)|u_j|^{\sigma-2}u_j + |v_k|^\sigma(|u_j|^{\sigma-2}u_j - |v_j|^{\sigma-2}v_j) \Big)\|_{L^\infty_T(L^2)} \\
&\lesssim \sum_{\gamma=\gamma_1+\gamma_2} \Big(\| \langle y\rangle^m \, \partial^{\gamma_1}(|u_k|^\sigma - |v_k|^\sigma)\partial^{\gamma_2}(|u_j|^{\sigma-2}u_j)\|_{L^\infty_T(L^2)}
\end{aligned}
$$

$$
\begin{aligned}
&+\quad \| \langle y\rangle^{m}\, \partial^{\gamma_1}|v_k|^{\sigma}\partial^{\gamma_2}(|u_j|^{\sigma-2}u_j-|v_j|^{\sigma-2}v_j)\|_{L^\infty_T(L^2)}\Big)\\
&\lesssim \sum_{\gamma=\gamma_1+\gamma_2}\Big(\| \langle y\rangle^{m\alpha}\, \partial^{\gamma_1}(|u_k|^{\sigma}-|v_k|^{\sigma})\|_{L^\infty_T(L^2)}\| \langle y\rangle^{m(1-\alpha)}\, \partial^{\gamma_2}(|u_j|^{\sigma-2}u_j)\|_{L^\infty_T(L^\infty)}\\
&+\quad \| \langle y\rangle^{m\alpha}\, \partial^{\gamma_1}|v_k|^{\sigma}\|_{L^\infty_T(L^2)}\| \langle y\rangle^{m(1-\alpha)}\, \partial^{\gamma_2}(|u_j|^{\sigma-2}u_j-|v_j|^{\sigma-2}v_j)\|_{L^\infty_T(L^\infty)}\Big).
\end{aligned}
$$

Now, by Lemma 2.4 and using (2.1), we get

$$
\begin{aligned}
&\| \langle y\rangle^{m\alpha}\, \partial^{\gamma_1}(|u_k|^{\sigma}-|v_k|^{\sigma})\|_{L^\infty_T(L^2)}\\
\lesssim\;& (R^{\sigma-1}+\nu^{-2(3-\sigma)}R^{5-\sigma})\| \langle y\rangle^{m(\alpha-\sigma)}\, \|\|w\|_{L^\infty_T(\mathcal{Y})}\\
+\;& \sum_{1\le k\le M}\nu^{-2(2k-\sigma)}R^{4k-\sigma-2}\Big(R\| \langle y\rangle^{m(\alpha-\sigma)}\, \|+\sum_{E[\frac{N}{2}]<|\mu|\le|\gamma_1|}\| \langle y\rangle^{m(1+\alpha-\sigma)}\, \partial^{\mu}u_k\|_{L^\infty_T(L^2)}\Big)\|w\|_{L^\infty_T(\mathcal{Y})}\\
+\;& \sum_{1\le k\le M}\nu^{-(2k-\sigma)}\Big(R^{2k-1}(\| \langle y\rangle^{m(\alpha-\sigma)}\, \|\|w\|_{L^\infty_T(\mathcal{Y})}+\sum_{E[\frac{N}{2}]<|\mu|\le|\gamma_1|}\| \langle y\rangle^{m(1+\alpha-\sigma)}\, \partial^{\mu}(u_k-v_k)\|_{L^\infty_T(L^2)})\\
+\;& R^{2(k-1)}(\sum_{E[\frac{N}{2}]<|\mu|\le|\gamma_1|}\| \langle y\rangle^{m(1+\alpha-\sigma)}\, \partial^{\mu}u_k\|_{L^\infty_T(L^2)}\\
+\;& \sum_{E[\frac{N}{2}]<|\mu|\le|\gamma_1|}\| \langle y\rangle^{m(1+\alpha-\sigma)}\, \partial^{\mu}v_k\|_{L^\infty_T(L^2)})\|w\|_{L^\infty_T(\mathcal{Y})}\Big)\\
\lesssim\;& \Big(R^{\sigma-1}+\nu^{-2(3-\sigma)}R^{5-\sigma}+\sum_{1\le k\le M}\nu^{-2(2k-\sigma)}R^{4k-\sigma-1}+\sum_{1\le k\le M}\nu^{-(2k-\sigma)}R^{2k-1}\Big)\|w\|_{L^\infty_T(\mathcal{Y})}. \qquad (3.16)
\end{aligned}
$$

Assume that $|\gamma_2|\le N$, thus, $M\ge N+E[\frac{N}{2}]+1$ gives by Sobolev embedding and Lemma 2.3 via (2.1), we get

$$
\begin{aligned}
&\| \langle y\rangle^{m(1-\alpha)}\, \partial^{\gamma_2}(|u_j|^{\sigma-2}u_j)\|_{L^\infty_T(L^\infty)}\\
\lesssim\;& \sum_{0\le l\le|\gamma_2|}\nu^{-(2l+2-\sigma)}R^{2l}\Big(R\| \langle y\rangle^{m(2-\alpha-\sigma)}\, \|_\infty+\sum_{E[\frac{N}{2}]<|\mu|\le|\gamma_2|}\| \langle y\rangle^{m(3-\alpha-\sigma)}\, \partial^{\mu}u_j\|_{L^\infty_T(L^\infty)}\Big)\\
\lesssim\;& \sum_{0\le l\le|\gamma_2|}\nu^{-(2l+2-\sigma)}R^{2l}\Big(R+\sum_{E[\frac{N}{2}]<|\mu|\le|\gamma_2|+1+E[\frac{N}{2}]}\| \langle y\rangle^{m}\, \partial^{\mu}u_j\|_{L^\infty_T(L^2)}\Big)\\
\lesssim\;& \sum_{0\le l\le M}\nu^{-(2l+2-\sigma)}R^{1+2l}\\
\lesssim\;& F_2(\nu,R).
\end{aligned}
$$

Moreover, with Lemma 2.4, we write

$$
\begin{aligned}
&\| \langle y\rangle^{m(1-\alpha)}\, \partial^{\gamma_2}(|u_j|^{\sigma-2}u_j-|v_j|^{\sigma-2}v_j)\|_{L^\infty_T(L^\infty)}\\
\lesssim\;& \nu^{-3(2-\sigma)}R^{4-\sigma}\| \langle y\rangle^{m(2-\alpha-\sigma)}\, \|_\infty\|w\|_{L^\infty_T(\mathcal{Y})}\\
+\;& \sum_{1\le k\le M}\nu^{-2(2(1+k)-\sigma)}R^{4k-\sigma+1}\Big(R\| \langle y\rangle^{m(2-\alpha-\sigma)}\, \|_\infty+\sum_{E[\frac{N}{2}]<|\mu|\le|\gamma_2|}\| \langle y\rangle^{m(3-\alpha-\sigma)}\, \partial^{\mu}u_k\|_{L^\infty_T(L^\infty)}\Big)\|w\|_{L^\infty_T(\mathcal{Y})}
\end{aligned}
$$

$$+\sum_{1\le k\le M}\nu^{-(2(1+k)-\sigma)}\Big(R^{2k}(\|\langle y\rangle^{m(2-\alpha-\sigma)}\|_\infty\|w\|_{L^\infty_T(\mathcal{Y})}+\sum_{E[\frac{N}{2}]<|\mu|\le|\gamma_2|}\|\langle y\rangle^{m(3-\alpha-\sigma)}\partial^\mu(u_k-v_k)\|_{L^\infty_T(L^\infty)}$$

$$+R^{2k-1}(\sum_{E[\frac{N}{2}]<|\mu|\le|\gamma_2|}\|\langle y\rangle^{m(3-\alpha-\sigma)}\partial^\mu u_k\|_{L^\infty_T(L^\infty)}+\sum_{E[\frac{N}{2}]<|\mu|\le|\gamma_2|}\|\langle y\rangle^{m(3-\alpha-\sigma)}\partial^\mu v_k\|_{L^\infty_T(L^\infty)})\|w\|_{L^\infty_T(\mathcal{Y})}\Big).$$

So, with Sobolev embeddings,

$$\begin{aligned}
&\|\langle y\rangle^{m(1-\alpha)}\partial^{\gamma_2}(|u_j|^{\sigma-2}u_j-|v_j|^{\sigma-2}v_j)\|_{L^\infty_T(L^\infty)}\\
\lesssim\;& \nu^{-3(2-\sigma)}R^{4-\sigma}\|w\|_{L^\infty_T(\mathcal{Y})}\\
+\;&\sum_{1\le k\le M}\nu^{-2(2(1+k)-\sigma)}R^{4k-\sigma+1}\Big(R+\sum_{E[\frac{N}{2}]<|\mu|\le|\gamma_2|+1+E[\frac{N}{2}]}\|\langle y\rangle^m\partial^\mu u_k\|_{L^\infty_T(L^2)}\Big)\|w\|_{L^\infty_T(\mathcal{Y})}\\
+\;&\sum_{1\le k\le M}\nu^{-(2(1+k)-\sigma)}\Big(R^{2k}(\|w\|_{L^\infty_T(\mathcal{Y})}+\sum_{E[\frac{N}{2}]<|\mu|\le|\gamma_2|+1+E[\frac{N}{2}]}\|\langle y\rangle^m\partial^\mu(u_k-v_k)\|_{L^\infty_T(L^2)}\\
+\;&R^{2k-1}(\sum_{E[\frac{N}{2}]<|\mu|\le|\gamma_2|+1+E[\frac{N}{2}]}\|\langle y\rangle^m\partial^\mu u_k\|_{L^\infty_T(L^2)}+\sum_{E[\frac{N}{2}]<|\mu|\le|\gamma_2|+1+E[\frac{N}{2}]}\|\langle y\rangle^m\partial^\mu v_k\|_{L^\infty_T(L^2)})\|w\|_{L^\infty_T(\mathcal{Y})}\Big)\\
\lesssim\;&\Big(\nu^{-3(2-\sigma)}R^{4-\sigma}+\sum_{1\le k\le M}\nu^{-2(2(1+k)-\sigma)}R^{4k-\sigma+2}+\sum_{1\le k\le M}\nu^{-(2(1+k)-\sigma)}R^{2k}\Big)\|w\|_{L^\infty_T(\mathcal{Y})}.
\end{aligned}\tag{3.17}$$

Finally, with Lemma 2.3, we have

$$\begin{aligned}
&\|\langle y\rangle^{m\alpha}\partial^{\gamma_1}|v_k|^\sigma\|_{L^\infty_T(L^2)}\\
\lesssim\;& R^\sigma\|\langle y\rangle^{m(\alpha-\sigma)}\|+\sum_{1\le k\le|\gamma_1|}\nu^{-(2k-\sigma)}R^{2k-1}\Big(R\|\langle y\rangle^{m(\alpha-\sigma)}\|+\sum_{E[\frac{N}{2}]<|\gamma|\le|\gamma_1|}\|\langle y\rangle^{m(1+\alpha-\sigma)}\partial^\gamma u\|_{L^\infty_T(L^2)}\Big)\\
\lesssim\;& R^\sigma+\sum_{1\le k\le|\gamma_1|}\nu^{-(2k-\sigma)}R^{2k-1}\Big(R+\sum_{E[\frac{N}{2}]<|\gamma|\le|\gamma_1|}\|\langle y\rangle^{m(1+\alpha-\sigma)}\partial^\gamma u\|_{L^\infty_T(L^2)}\Big)\\
\lesssim\;& R^\sigma+\sum_{1\le k\le|\gamma_1|}\nu^{-(2k-\sigma)}R^{2k}\\
\lesssim\;& F_1(\nu,R).
\end{aligned}$$

Collecting the above estimates, we get

$$\begin{aligned}
&\sum_{\gamma=\gamma_1+\gamma_2,|\gamma_2|\le N}\Big(\|\langle y\rangle^{m\alpha}\partial^{\gamma_1}(|u_k|^\sigma-|v_k|^\sigma)\|_{L^\infty_T(L^2)}\|\langle y\rangle^{m(1-\alpha)}\partial^{\gamma_2}(|u_j|^{\sigma-2}u_j)\|_{L^\infty_T(L^\infty)}\\
+\;&\|\langle y\rangle^{m\alpha}\partial^{\gamma_1}|v_k|^\sigma\|_{L^\infty_T(L^2)}\|\langle y\rangle^{m(1-\alpha)}\partial^{\gamma_2}(|u_j|^{\sigma-2}u_j-|v_j|^{\sigma-2}v_j)\|_{L^\infty_T(L^\infty)}\Big)\\
\lesssim\;&\Big[\Big(R^{\sigma-1}+\nu^{-2(3-\sigma)}R^{5-\sigma}+\sum_{1\le k\le M}\nu^{-2(2k-\sigma)}R^{4k-\sigma-1}+\sum_{1\le k\le M}\nu^{-(2k-\sigma)}R^{2k-1}\Big)F_2(\nu,R)\\
+\;&F_1(\nu,R)\Big(\nu^{-3(2-\sigma)}R^{4-\sigma}+\sum_{1\le k\le M}\nu^{-2(2(1+k)-\sigma)}R^{4k-\sigma+2}+\sum_{1\le k\le M}\nu^{-(2(1+k)-\sigma)}R^{2k}\Big)\Big]\\
&\|w\|_{L^\infty_T(\mathcal{Y})}.
\end{aligned}\tag{3.18}$$

Now, assume that $M\ge|\gamma_2|\ge N$. Thus, arguing as in (3.3), we have

$$\sum_{\gamma=\gamma_1+\gamma_2,M\ge|\gamma_2|\ge N}\Big(\|\langle y\rangle^{m\alpha}\partial^{\gamma_2}(|u_k|^\sigma-|v_k|^\sigma)\|_{L^\infty_T(L^2)}\|\langle y\rangle^{m(1-\alpha)}\partial^{\gamma_1}(|u_j|^{\sigma-2}u_j)\|_{L^\infty_T(L^\infty)}$$

$$+ \quad \| \langle y\rangle^{m\alpha}\, \partial^{\gamma_2}|v_k|^{\sigma}\|_{L^\infty_T(L^2)}\| \langle y\rangle^{m(1-\alpha)}\, \partial^{\gamma_1}(|u_j|^{\sigma-2}u_j - |v_j|^{\sigma-2}v_j)\|_{L^\infty_T(L^\infty)}\Big)$$
$$\lesssim \sum_{\gamma=\gamma_1+\gamma_2, M\geq|\gamma_2|\geq N} \Big(\| \langle y\rangle^{m\alpha}\, \partial^{\gamma_2}(|u_k|^{\sigma} - |v_k|^{\sigma})\|_{L^\infty_T(L^2)} F_2(\nu, R)$$
$$+ \quad \| \langle y\rangle^{m(1-\alpha)}\, \partial^{\gamma_1}(|u_j|^{\sigma-2}u_j - |v_j|^{\sigma-2}v_j)\|_{L^\infty_T(L^\infty)} F_1(\nu, R)\Big).$$

Moreover, since $|\gamma_1| \leq M - N \leq M - E[\frac{N}{2}] - 1$, by (3.16) and (3.17), we write

$$\begin{aligned}
&\sum_{\gamma=\gamma_1+\gamma_2, M\geq|\gamma_2|\geq N} \Big(\| \langle y\rangle^{m\alpha}\, \partial^{\gamma_2}(|u_k|^{\sigma} - |v_k|^{\sigma})\|_{L^\infty_T(L^2)}\| \langle y\rangle^{m(1-\alpha)}\, \partial^{\gamma_1}(|u_j|^{\sigma-2}u_j)\|_{L^\infty_T(L^\infty)} \\
+ \quad & \| \langle y\rangle^{m\alpha}\, \partial^{\gamma_2}|v_k|^{\sigma}\|_{L^\infty_T(L^2)}\| \langle y\rangle^{m(1-\alpha)}\, \partial^{\gamma_1}(|u_j|^{\sigma-2}u_j - |v_j|^{\sigma-2}v_j)\|_{L^\infty_T(L^\infty)}\Big) \\
\lesssim \quad & \sum_{\gamma=\gamma_1+\gamma_2} \Big(\Big(R^{\sigma-1} + \nu^{-2(3-\sigma)}R^{5-\sigma} + \sum_{1\leq k\leq M} \nu^{-2(2k-\sigma)}R^{4k-\sigma-1} + \sum_{1\leq k\leq M} \nu^{-(2k-\sigma)}R^{2k-1}\Big) \\
& F_2(\nu, R)\|w\|_{L^\infty_T(\mathcal{Y})} \\
+ \quad & \Big(\nu^{-3(2-\sigma)}R^{4-\sigma} + \sum_{1\leq k\leq M} \nu^{-2(2(1+k)-\sigma)}R^{4k-\sigma+2} + \sum_{1\leq k\leq M} \nu^{-(2(1+k)-\sigma)}R^{2k}\Big) \\
& F_1(\nu, R)\Big)\|w\|_{L^\infty_T(\mathcal{Y})}.
\end{aligned} \tag{3.19}$$

Now, with the Leibniz rule via Lemma 2.3, for $|\gamma| \leq -N + M_0 + M$, we have

$$\begin{aligned}
& (I_3) := \|\partial^{\gamma}(\mathcal{F}_j(u) - \mathcal{F}_j(v))\|_{L^\infty_T(L^2)} \\
\lesssim \quad & \sum_{\gamma=\gamma_1+\gamma_2} \Big(\| \langle y\rangle^{m(2-\sigma)}\, \partial^{\gamma_1}(|u_k|^{\sigma} - |v_k|^{\sigma})\|_{L^\infty_T(L^\infty)}\| \langle y\rangle^{-m(2-\sigma)}\, \partial^{\gamma_2}(|u_j|^{\sigma-2}u_j)\|_{L^\infty_T(L^2)} \\
+ \quad & \| \langle y\rangle^{m(2-\sigma)}\, \partial^{\gamma_1}|v_k|^{\sigma}\|_{L^\infty_T(L^2)}\| \langle y\rangle^{-m(2-\sigma)}\, \partial^{\gamma_2}(|u_j|^{\sigma-2}u_j - |v_j|^{\sigma-2}v_j)\|_{L^\infty_T(L^\infty)}\Big).
\end{aligned}$$

Let us start with estimating the first term. By (3.6),

$$\begin{aligned}
(A_1) \quad := \quad & \| \langle y\rangle^{m(2-\sigma)}\, \partial^{\gamma_1}(|u_k|^{\sigma} - |v_k|^{\sigma})\|_{L^\infty_T(L^\infty)}\| \langle y\rangle^{-m(2-\sigma)}\, \partial^{\gamma_2}(|u_j|^{\sigma-2}u_j)\|_{L^\infty_T(L^2)} \\
\lesssim \quad & \| \langle y\rangle^{m(2-\sigma)}\, \partial^{\gamma_1}(|u_k|^{\sigma} - |v_k|^{\sigma})\|_{L^\infty_T(L^\infty)} F_2(\nu, R).
\end{aligned}$$

With Lemma 2.4, we get

$$\begin{aligned}
& \| \langle y\rangle^{m(2-\sigma)}\, \partial^{\gamma_1}(|u_k|^{\sigma} - |v_k|^{\sigma})\|_{L^\infty_T(L^\infty)} \\
\lesssim \quad & (R^{\sigma-1} + \nu^{-2(3-\sigma)}R^{5-\sigma})\| \langle y\rangle^{2m(1-\sigma)}\, \|_\infty \|w\|_{L^\infty_T(\mathcal{Y})} \\
+ \quad & \sum_{1\leq k\leq M} \nu^{-2(2k-\sigma)}R^{4k-\sigma-2}\Big(R\| \langle y\rangle^{2m(1-\sigma)}\, \|_\infty + \sum_{E[\frac{N}{2}]<|\mu|\leq|\gamma_1|} \| \langle y\rangle^{m(3-2\sigma)}\, \partial^{\mu}u_k\|_{L^\infty_T(L^\infty)}\Big)\|w\|_{L^\infty_T(\mathcal{Y})} \\
+ \quad & \sum_{1\leq k\leq M} \nu^{-(2k-\sigma)}\Big(R^{2k-1}(\| \langle y\rangle^{2m(1-\sigma)}\, \|_\infty \|w\|_{L^\infty_T(\mathcal{Y})} + \sum_{E[\frac{N}{2}]<|\mu|\leq|\gamma_1|} \| \langle y\rangle^{m(3-2\sigma)}\, \partial^{\mu}(u_k - v_k)\|_{L^\infty_T(L^\infty)} \\
+ \quad & R^{2(k-1)}(\sum_{E[\frac{N}{2}]<|\mu|\leq|\gamma_1|} \| \langle y\rangle^{m(3-2\sigma)}\, \partial^{\mu}u_k\|_{L^\infty_T(L^\infty)} + \sum_{E[\frac{N}{2}]<|\mu|\leq|\gamma_1|} \| \langle y\rangle^{m(3-2\sigma)}\, \partial^{\mu}v_k\|_{L^\infty_T(L^\infty)})\|w\|_{L^\infty_T(\mathcal{Y})}\Big).
\end{aligned}$$

Since $\sigma > \frac{3}{2}$, we get

$$\| \langle y\rangle^{m(2-\sigma)}\, \partial^{\gamma_1}(|u_k|^{\sigma} - |v_k|^{\sigma})\|_{L^\infty_T(L^\infty)}$$

$$\begin{aligned}
&\lesssim (R^{\sigma-1}+\nu^{-2(3-\sigma)}R^{5-\sigma})\|w\|_{L_T^\infty(\mathcal{Y})}\\
&+ \sum_{1\le k\le M}\nu^{-2(2k-\sigma)}R^{4k-\sigma-2}\Big(R+\sum_{E[\frac{N}{2}]<|\mu|\le|\gamma_1|}\|\langle y\rangle^{m(3-2\sigma)}\,\partial^\mu u_k\|_{L_T^\infty(L^\infty)}\Big)\|w\|_{L_T^\infty(\mathcal{Y})}\\
&+ \sum_{1\le k\le M}\nu^{-(2k-\sigma)}\Big(R^{2k-1}(\|w\|_{L_T^\infty(\mathcal{Y})}+\sum_{E[\frac{N}{2}]<|\mu|\le|\gamma_1|}\|\langle y\rangle^{m(3-2\sigma)}\,\partial^\mu(u_k-v_k)\|_{L_T^\infty(L^\infty)}\\
&+ R^{2(k-1)}(\sum_{E[\frac{N}{2}]<|\mu|\le|\gamma_1|}\|\langle y\rangle^{m(3-2\sigma)}\,\partial^\mu u_k\|_{L_T^\infty(L^\infty)}+\sum_{E[\frac{N}{2}]<|\mu|\le|\gamma_1|}\|\langle y\rangle^{m(3-2\sigma)}\,\partial^\mu v_k\|_{L_T^\infty(L^\infty)})\|w\|_{L_T^\infty(\mathcal{Y})}\Big).
\end{aligned}$$

If $|\gamma_1|<N$, we write by Sobolev injections,

$$\begin{aligned}
\sum_{E[\frac{N}{2}]<|\mu|\le|\gamma_1|}\|\langle y\rangle^{m(3-2\sigma)}\,\partial^\mu(u_k-v_k)\|_{L_T^\infty(L^\infty)} &\lesssim \sum_{E[\frac{N}{2}]<|\mu|\le N+1+E[\frac{N}{2}]}\|\partial^\mu(u_k-v_k)\|_{L_T^\infty(L^2)}\\
&\lesssim \sum_{E[\frac{N}{2}]<|\mu|\le -N+M_0+M}\|\langle y\rangle^{m}\,\partial^\mu(u_k-v_k)\|_{L_T^\infty(L^2)}\\
&\lesssim \|w\|_{L_T^\infty(\mathcal{Y})}. \qquad (3.20)
\end{aligned}$$

Thus,

$$\begin{aligned}
(A_1) \;\lesssim\; &\Big(R^{\sigma-1}+\nu^{-2(3-\sigma)}R^{5-\sigma}+\sum_{1\le k\le M}\nu^{-2(2k-\sigma)}R^{4k-\sigma-1}+\sum_{1\le k\le M}\nu^{-(2k-\sigma)}R^{2k-1}\Big)\\
&F_2(\nu,R)\|u-v\|_{L_T^\infty(\mathcal{Y})}. \qquad (3.21)
\end{aligned}$$

If $|\gamma_1|\ge N$, it is sufficient to estimate the term

$$\begin{aligned}
(A_1') &:= \|\langle y\rangle^{m(2-\sigma)}\,\partial^{\gamma_1}(|u_k|^\sigma-|v_k|^\sigma)\|_{L_T^\infty(L^2)}\|\langle y\rangle^{-m(2-\sigma)}\,\partial^{\gamma_2}(|u_j|^{\sigma-2}u_j)\|_{L_T^\infty(L^\infty)}\\
&\lesssim \|\langle y\rangle^{m(2-\sigma)}\,\partial^{\gamma_1}(|u_k|^\sigma-|v_k|^\sigma)\|_{L_T^\infty(L^2)}F_2(\nu,R),
\end{aligned}$$

where we used (3.9). Moreover, Lemma 2.4 via (2.2) gives

$$\begin{aligned}
&\|\langle y\rangle^{m(2-\sigma)}\,\partial^{\gamma_1}(|u_k|^\sigma-|v_k|^\sigma)\|_{L_T^\infty(L^2)}\\
&\lesssim (R^{\sigma-1}+\nu^{-2(3-\sigma)}R^{5-\sigma})\|u-v\|_{L_T^\infty(\mathcal{Y})}\\
&+ \sum_{1\le k\le M}\nu^{-2(2k-\sigma)}R^{4k-\sigma-2}\Big(R+\sum_{E[\frac{N}{2}]<|\mu|\le|\gamma_1|}\|\langle y\rangle^{m(3-2\sigma)}\,\partial^\mu u_k\|_{L_T^\infty(L^2)}\Big)\|u-v\|_{L_T^\infty(\mathcal{Y})}\\
&+ \sum_{1\le k\le M}\nu^{-(2k-\sigma)}\Big(R^{2k-1}(\|u-v\|_{L_T^\infty(\mathcal{Y})}+\sum_{E[\frac{N}{2}]<|\mu|\le|\gamma_1|}\|\langle y\rangle^{m(3-2\sigma)}\,\partial^\mu(u_k-v_k)\|_{L_T^\infty(L^2)})\\
&+ R^{2(k-1)}(\sum_{E[\frac{N}{2}]<|\mu|\le|\gamma_1|}\|\langle y\rangle^{m(3-2\sigma)}\,\partial^\mu u_k\|_{L_T^\infty(L^2)}+\sum_{E[\frac{N}{2}]<|\mu|\le|\gamma_1|}\|\langle y\rangle^{m(3-2\sigma)}\,\partial^\mu v_k\|_{L_T^\infty(L^2)})\|u-v\|_{L_T^\infty(\mathcal{Y})}\Big).
\end{aligned}$$

Using the fact that

$$\begin{aligned}
\sum_{E[\frac{N}{2}]<|\mu|\le|\gamma_1|}\|\langle y\rangle^{m(3-2\sigma)}\,\partial^\mu u_k\|_{L_T^\infty(L^2)} &\lesssim \sum_{E[\frac{N}{2}]<|\mu|\le M}\|\langle y\rangle^{m}\,\partial^\mu u_k\|_{L_T^\infty(L^2)}\\
&+ \sum_{M<|\mu|\le -N+M_0+M}\|\partial^\mu u_k\|_{L_T^\infty(L^2)},
\end{aligned}$$

we have

$$
\begin{aligned}
&\| \langle y\rangle^{m(2-\sigma)} \partial^{\gamma_1}(|u_k|^{\sigma}-|v_k|^{\sigma})\|_{L_T^\infty(L^2)}\\
\lesssim\ & \Big(R^{\sigma-1}+\nu^{-2(3-\sigma)}R^{5-\sigma}+\sum_{1\le k\le M}\nu^{-2(2k-\sigma)}R^{4k-\sigma-1}+\sum_{1\le k\le M}\nu^{-(2k-\sigma)}R^{2k-1}\Big)\|u-v\|_{L_T^\infty(\mathcal{Y})}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
(A_1') \ \lesssim\ & \Big(R^{\sigma-1}+\nu^{-2(3-\sigma)}R^{5-\sigma}+\sum_{1\le k\le M}\nu^{-2(2k-\sigma)}R^{4k-\sigma-1}+\sum_{1\le k\le M}\nu^{-(2k-\sigma)}R^{2k-1}\Big)\\
& F_2(\nu,R)\|u-v\|_{L_T^\infty(\mathcal{Y})}.
\end{aligned}
\tag{3.22}
$$

By (3.9), let us estimate the term

$$
\begin{aligned}
(A_2) \ :=\ & \| \langle y\rangle^{m(2-\sigma)} \partial^{\gamma_1}|v_k|^{\sigma}\|_{L_T^\infty(L^2)} \| \langle y\rangle^{-m(2-\sigma)} \partial^{\gamma_2}(|u_j|^{\sigma-2}u_j-|v_j|^{\sigma-2}v_j)\|_{L_T^\infty(L^\infty)}\\
\lesssim\ & F_1(\nu,R)\| \langle y\rangle^{-m(2-\sigma)} \partial^{\gamma_2}(|u_j|^{\sigma-2}u_j-|v_j|^{\sigma-2}v_j)\|_{L_T^\infty(L^\infty)}.
\end{aligned}
$$

If $|\gamma_2|\le N$, by a similar reasoning to (3.17), (for $\alpha=3-\sigma$), we have

$$
\begin{aligned}
&\| \langle y\rangle^{m(1-\alpha)} \partial^{\gamma_2}(|u_j|^{\sigma-2}u_j-|v_j|^{\sigma-2}v_j)\|_{L_T^\infty(L^\infty)}\\
\lesssim\ & \Big(\nu^{-3(2-\sigma)}R^{4-\sigma}+\sum_{1\le k\le M}\nu^{-2(2(1+k)-\sigma)}R^{4k-\sigma+2}+\sum_{1\le k\le M}\nu^{-(2(1+k)-\sigma)}R^{2k}\Big)\|u-v\|_{L_T^\infty(\mathcal{Y})}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
(A_2) \ \lesssim\ & \Big(\nu^{-3(2-\sigma)}R^{4-\sigma}+\sum_{1\le k\le M}\nu^{-2(2(1+k)-\sigma)}R^{4k-\sigma+2}+\sum_{1\le k\le M}\nu^{-(2(1+k)-\sigma)}R^{2k}\Big)\\
& \Big(R^{\sigma}+\sum_{1\le l\le|\gamma_1|}\nu^{-(2l-\sigma)}R^{2l}\Big)\|u-v\|_{L_T^\infty(\mathcal{Y})}.
\end{aligned}
$$

If $|\gamma_2|\ge N$, it is sufficient to estimate the term

$$
(A_2') \ :=\ \| \langle y\rangle^{m(2-\sigma)} \partial^{\gamma_1}|v_k|^{\sigma}\|_{L_T^\infty(L^\infty)} \| \langle y\rangle^{-m(2-\sigma)} \partial^{\gamma_2}(|u_j|^{\sigma-2}u_j-|v_j|^{\sigma-2}v_j)\|_{L_T^\infty(L^2)}.
\tag{3.23}
$$

Moreover, with Lemma 2.4 via (2.2), we write

$$
\begin{aligned}
&\| \langle y\rangle^{-m(\sigma-2)} \partial^{\gamma_2}(|u_j|^{\sigma-2}u_j-|v_j|^{\sigma-2}v_j)\|_{L_T^\infty(L^2)}\\
\lesssim\ & \nu^{-3(2-\sigma)}R^{4-\sigma}\|u-v\|_{L_T^\infty(\mathcal{Y})}\\
+\ & \sum_{1\le k\le M}\nu^{-2(2(1+k)-\sigma)}R^{4k-\sigma+1}\Big(R+\sum_{E[\frac{N}{2}]<|\mu|\le|\gamma_2|}\| \langle y\rangle^{m} \partial^{\mu}u_k\|_{L_T^\infty(L^2)}\Big)\|u-v\|_{L_T^\infty(\mathcal{Y})}\\
+\ & \sum_{1\le k\le M}\nu^{-(2(1+k)-\sigma)}\Big(R^{2k}(\|u-v\|_{L_T^\infty(\mathcal{Y})}+\sum_{E[\frac{N}{2}]<|\mu|\le|\gamma_2|}\| \langle y\rangle^{m} \partial^{\mu}(u_k-v_k)\|_{L_T^\infty(L^2)}\\
+\ & R^{2k-1}(\sum_{E[\frac{N}{2}]<|\mu|\le|\gamma_2|}\| \langle y\rangle^{m} \partial^{\mu}u_k\|_{L_T^\infty(L^2)}+\sum_{E[\frac{N}{2}]<|\mu|\le|\gamma_2|}\| \langle y\rangle^{m} \partial^{\mu}v_k\|_{L_T^\infty(L^2)})\|u-v\|_{L_T^\infty(\mathcal{Y})}\Big)\\
\lesssim\ & \Big(\nu^{-3(2-\sigma)}R^{4-\sigma}+\sum_{1\le k\le M}\nu^{-2(2(1+k)-\sigma)}R^{4k-\sigma+2}+\sum_{1\le k\le M}\nu^{-(2(1+k)-\sigma)}R^{2k}\Big)\|u-v\|_{L_T^\infty(\mathcal{Y})}.
\end{aligned}
$$

Since $N + |\gamma_1| \leq |\gamma_1| + |\gamma_2| \leq -N + M_0 + M$, we have $|\gamma_1| \leq M + M_0 - 2N$. So, arguing as in (3.6), we get

$$\begin{aligned} \| \langle y\rangle^{m(2-\sigma)} \partial^{\gamma_1}(|v_k|^p)\|_{L^\infty_T(L^\infty)} &\lesssim R^\sigma + \sum_{1\leq l\leq|\gamma_1|} \nu^{-(2l-\sigma)} R^{2l-1}\Big(R + \sum_{E[\frac{N}{2}]<|\mu|\leq|\gamma_1|} \| \langle y\rangle^{m(3-2\sigma)} \partial^\mu v_k\|_{L^\infty_T(L^\infty)}\Big) \\ &\lesssim F_1(\nu, R). \end{aligned}$$

Then,

$$\begin{aligned} (A_2') \lesssim & \Big(\nu^{-3(2-\sigma)} R^{4-\sigma} + \sum_{1\leq k\leq M} \nu^{-2(2(1+k)-\sigma)} R^{4k-\sigma+2} + \sum_{1\leq k\leq M} \nu^{-(2(1+k)-\sigma)} R^{2k}\Big) \\ & F_1(\nu, R)\|u - v\|_{L^\infty_T(\mathcal{Y})}. \end{aligned} \tag{3.24}$$

Taking (3.15), (3.18), (3.19), (3.21), (3.22), (3.23), and (3.24), we get

$$(I_1) + (I_2) + (I_3) \lesssim F(\nu, R)\|u - v\|_{L^\infty_T(\mathcal{Y})}. \tag{3.25}$$

Now, taking Lemma 2.1, (3.2), (3.3), and (3.11) via the estimates of (I), we write

$$\begin{aligned} & \| \langle y\rangle^m \partial^\alpha(f_j(u) - f_j(v))\|_{L^\infty_T(L^2)} \\ \lesssim\ & T \langle T\rangle^m \Big(\|I_{-N+M_0+M}(\mathcal{F}_j(u) - \mathcal{F}_j(v))\|_{L^\infty_T(L^2)} + \| \langle y\rangle^m \partial^\alpha(\mathcal{F}_j(u) - \mathcal{F}_j(v))\|_{L^\infty_T(L^2)}\Big) \\ \lesssim\ & T \langle T\rangle^m F(\nu, R)\|u - v\|_{L^\infty_T(\mathcal{Y})}. \end{aligned}$$

Moreover, taking Lemma 2.1, and the estimate of (I_3), for $|\alpha| \leq -N + M_0 + M$, we get

$$\begin{aligned} \|\partial^\alpha(f_j(u) - f_j(v))\|_{L^\infty_T(L^2)} &\lesssim T\|\partial^\alpha(\mathcal{F}_j(u) - \mathcal{F}_j(v))\|_{L^\infty_T(L^2)} \\ &\lesssim TF(\nu, R)\|u - v\|_{L^\infty_T(\mathcal{Y})}. \end{aligned}$$

Finally, f is a contraction of $B_T(R)$ for small $T > 0$, and the result follows with a classical Picard argument.

4. Global and non-global solutions

In this section, we prove Theorem 2.2 and Proposition 2.1.

4.1. Global solutions and scattering

Let us start with the next auxiliary result.

Proposition 4.1. *Let* $\max\{1 + \frac{1}{N}, \frac{3}{2}\} < \sigma \leq 2$, $\kappa \gg 1$*, and* $v_0 \in \mathcal{Y}$ *such that* $\inf_{\mathbb{R}^N} |v_0(x)| \geq \nu$*. Then, there is a unique* $v \in C([0, \frac{1}{\kappa}], \mathcal{Y})$ *solution to*

$$iv_j + \Delta v_j = \tau(1 - \kappa t)^{-(2-N(\sigma-1))}\Big(\sum_{1\leq k\leq n} a_{jk}|v_k|^\sigma\Big)|v_j|^{\sigma-2} v_j, \quad \forall j \in [1, n]. \tag{4.1}$$

Proof of Proposition 4.1: For simplicity and without loss of generality, let us fix $\tau = 1$. One applies a Picard fixed point argument. Let the function

$$g(v) := e^{i\cdot\Delta}v_0 + i\int_0^{\cdot}(1-\kappa s)^{-(2-N(\sigma-1))}e^{i(\cdot-s)\Delta}\mathcal{F}\,ds := (g_1(v),\dots,g_m(v)),$$

on the space $B_{\nu,\frac{1}{\kappa}}(R)$. Taking (3.1), (3.3), and (3.11), via $\sigma > 1+\frac{1}{N}$, we write

$$\|g(v)\|_{L^\infty_T(\mathcal{Y})} \lesssim \left\langle\kappa^{-1}\right\rangle^{E[\frac{N}{2}]+1+m}\|v_0\|_{\mathcal{Y}} + \kappa^{-1}\left\langle\kappa^{-1}\right\rangle^{E[\frac{N}{2}]+1+m}F_1(\nu,R)F_2(\nu,R).$$

Moreover, arguing as in (3.14), we write

$$|\langle y\rangle^m g(v)| \geq \nu - c\kappa^{-1}\left\langle\kappa^{-1}\right\rangle^{E[\frac{N}{2}]+1+m}\|v_0\|_{\mathcal{Y}} - c\kappa^{-1}\left\langle\kappa^{-1}\right\rangle^{E[\frac{N}{2}]+1+m}F_1(\nu,R)F_2(\nu,R).$$

Moreover, arguing as in (3.25), and the last lines of the previous section, we get

$$\|g(u)-g(v)\|_{L^\infty_T(\mathcal{Y})} \leq c\kappa^{-1}\left\langle\kappa^{-1}\right\rangle^{E[\frac{N}{2}]+1+m}F(\nu,R)\|u-v\|_{L^\infty_T(\mathcal{Y})}.$$

This implies that, for large $\kappa \gg 1$, an application of the Picard Theorem finishes the proof. □

Take the pseudo conformal transformation [26],

$$v_j(t,y) := (1-\kappa t)^{-\frac{N}{2}}e^{\frac{i\kappa|x|^2}{4(1-\kappa t)}}u_j\left(\frac{t}{1-\kappa t},\frac{x}{1-\kappa t}\right), \quad \forall j\in[1,n].$$

Here, $0 < T < \frac{1}{\kappa}$, and v, given by Proposition 4.1, is a solution to (4.1). Thus, u resolves (CNLS). Now, let define $u_j^+ := e^{\frac{i\kappa|x|^2}{4}-\frac{i}{\kappa}\Delta}v_j(\frac{1}{\kappa},\cdot)$. So, following [25, Section 4], we have $u_j \in C(\mathbb{R}_+,H^s)\cap L^\infty(\mathbb{R}_+\times\mathbb{R}^N,\langle y\rangle^{\frac{N}{2}}\,dy\,dt)$, and

$$\lim_{t\to+\infty}\|e^{-it\Delta}u_j(t)-u_j^+\|_{H^s} = 0.$$

4.2. Non-global solutions

Here, one proves Proposition 2.1. Let $u \in C_T(\mathcal{Y})$ be a solution to (CNLS). Thus, $m > 1+\frac{N}{2}$ implies that $\langle y\rangle^{1-m}\in L^2$, and

$$\|\langle y\rangle u\| \leq \|\langle y\rangle^m u\|_\infty\|\langle y\rangle^{1-m}\| \lesssim \|\langle y\rangle^m u\|_\infty.$$

This implies that the solution has a finite variance $\int_{\mathbb{R}^N}|u(t,y)|^2|x|^2\,dy$. Let us check that the energy is well-defined. Indeed,

$$\begin{aligned}\|\bar{u}_j\mathcal{F}_{j,k}\|_1 &= \||u_k|^\sigma|u_j|^p\|_1\\ &= \|\langle y\rangle^{-2m\sigma}\langle y\rangle^{2m\sigma}|u_k|^\sigma|u_j|^p\|_1\\ &\leq \|\langle y\rangle^{-m\sigma}\|^2\|\langle y\rangle^m u\|_\infty^{2\sigma}\\ &\lesssim \|u\|^{2\sigma}_{L^\infty_T(\mathcal{Y})}.\end{aligned}$$

Thus, one can apply [18, Theorem 2.8].

5. Conclusions

This work examines the singular coupled non-linear Schrödinger system (CNLS) with three main objectives. First, it investigates the local existence of solutions. Second, it establishes the existence of global solutions that scatter in certain Sobolev spaces. Finally, it demonstrates the existence of non-global solutions. The primary difficulty arises from the condition $\sigma < 2$, which introduces a singularity in the term $|u_j|^{\sigma-2}$ near zero. This singularity renders the classical contraction method in the energy space ineffective. This paper aims to address this gap in the literature by leveraging ideas from [24]. The approach highlights that the singularity issue is localized near zero, requiring the solution to avoid this region, see assumption (2.5). This is challenging because the Schrödinger equation lacks a maximum principle. The global solutions that scatter are obtained through a pseudo-conformal transformation based on the local solutions. Lastly, the existence of non-global solutions is demonstrated using the classical variance method.

Author contributions

Saleh Almuthybri and Tarek Saanouni: study conception and design, data collection, analysis and interpretation of results, and manuscript preparation. All authors have read and approved the final version of the manuscript for publication.

Acknowledgments

The researchers would like to thank the Deanship of Graduate Studies and Scientific Research at Qassim University for financial support (QU-APC-2024-9/1).

Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest. No data-sets were generated or analyzed during the current study.

References

1. N. Akhmediev, A. Ankiewicz, Partially coherent solitons on a finite background, *Phys. Rev. Lett.*, **82** (1999), 2661. https://doi.org/10.1103/PhysRevLett.82.2661
2. H. A. Alkhidhr, Closed-form solutions to the perturbed NLSE with Kerr law nonlinearity in optical fibers, *Results Phys.*, **22** (2021), 103875. https://doi.org/10.1016/j.rinp.2021.103875
3. A. Hasegawa, Y. Kodama, *Solitons in optical communications*, Oxford: Oxford University Press, 1995. https://doi.org/10.1093/oso/9780198565079.001.0001
4. A. Hasegawa, F. Tappert, Transmission of stationary non-linear optical pulses in dispersive dielectric fibers II, *Appl. Phys. Lett.*, **23** (1973), 142–144. https://doi.org/10.1063/1.1654836
5. S. Shen, Z. J. Yang, Z. G. Pang, Y. R. Ge, The complex-valued astigmatic cosine-Gaussian soliton solution of the nonlocal nonlinear Schrödinger equation and its transmission characteristics, *Appl. Math. Lett.*, **125** (2022), 107755. https://doi.org/10.1016/j.aml.2021.107755

6. S. Shen, Z. Yang, X. Li, S. Zhang, Periodic propagation of complex-valued hyperbolic-cosine-Gaussian solitons and breathers with complicated light field structure in strongly nonlocal nonlinear media, *Commun. Nonlinear Sci.*, **103** (2021), 106005. https://doi.org/10.1016/j.cnsns.2021.106005
7. V. E. Zakharov, Stability of periodic waves of finite amplitude on the surface of a deep fluid, *J. Appl. Mech. Tech. Phys.*, **9** (1968), 190–194. https://doi.org/10.1007/BF00913182
8. C. Weilnau, M. Ahles, J. Petter, D. Träger, J. Schröder, C. Denz, Spatial optical (2+1)-dimensional scalar and vector-solitons in saturable nonlinear media, *Ann. Phys.*, **514** (2002), 573–629. https://doi.org/10.1002/andp.20025140802
9. S. V. Manakov, On the theory of two-dimensional stationary self-focusing of electromagnetic waves, *Sov. Phys.-JETP*, **38** (1974), 248–253.
10. N. V. Nguyen, R. S. Tian, B. Deconinck, N. Sheils, Global existence for a coupled system of Schrödinger equations with power-type non-linearities, *J. Math. Phys.*, **54** (2013), 011503. https://doi.org/10.1063/1.4774149
11. T. Saanouni, A note on coupled focusing non-linear Schrödinger equations, *Appl. Anal.*, **95** (2016), 2063–2080. https://doi.org/10.1080/00036811.2015.1086757
12. A. Ambrosetti, E. Colorado, Bound and ground states of coupled non-linear Schrödinger equations, *CR Math.*, **342** (2006), 453–458. https://doi.org/10.1016/j.crma.2006.01.024
13. A. Ambrosetti, E. Colorado, Standing waves of some coupled non-linear Schrödinger equations, *J. Lond. Math. Soc.*, **75** (2007), 67–82. https://doi.org/10.1112/jlms/jdl020
14. T.-C. Lin, J. C. Wei, Ground state of N coupled non-linear Schrödinger equations in $\mathbb{R}^n$, $n \leq 3$, *Commun. Math. Phys.*, **255** (2005), 629–653. https://doi.org/10.1007/s00220-005-1313-x
15. B. Cassano, M. Tarulli, H^1-Scattering for systems of N-defocusing weakly coupled NLS equations in low space dimensions, *J. Math. Anal. Appl.*, **430** (2015), 528–548. https://doi.org/10.1016/j.jmaa.2015.05.008
16. T. Saanouni, Scattering threshold for a coupled focusing nonlinear Schrödinger system, *Appl. Anal.*, **101** (2022), 2418–2445. https://doi.org/10.1080/00036811.2020.1808201
17. T. Saanouni, Scattering threshold for the focusing coupled Schrödinger system revisited, *Nonlinear Differ. Equ. Appl.*, **28** (2021), 44. https://doi.org/10.1007/s00030-021-00706-7
18. R. Ghanmi, H. Hezzi, T. Saanouni, A note on inhomogeneous coupled Schrödinger equations, *Ann. Henri Poincaré*, **21** (2020), 2775–2814. https://doi.org/10.1007/s00023-020-00942-0
19. R. Ghanmi, T. Saanouni, Energy-critical scattering for focusing inhomogeneous coupled Schrödinger systems, *Math. Method. Appl. Sci.*, **47** (2024), 9109–9136. https://doi.org/10.1002/mma.10062
20. T. Saanouni, R. Ghanmi, Inhomogeneous coupled non-linear Schrödinger systems, *J. Math. Phys.*, **62** (2021), 101508. https://doi.org/10.1063/5.0047433
21. T. Saanouni, R. Ghanmi, Coupled inhomogeneous nonlinear Schrödinger system with potential in three space diemensions, *Math. Method. Appl. Sci.*, **47** (2024), 5392–5413. https://doi.org/10.1002/mma.9869
22. H. Hezzi, M. M. Nour, T. Saanouni, Coupled non-linear Schrödinger equations with harmonic potential, *Arab. J. Math.*, **7** (2018), 195–218. https://doi.org/10.1007/s40065-017-0192-2

23. Z. Jiao, I. Jadlovská, T. X. Li, Global existence in a fully parabolic attraction-repulsion chemotaxis system with singular sensitivities and proliferation, *J. Differ. Equations*, **411** (2024), 227–267. https://doi.org/10.1016/j.jde.2024.07.005

24. T. Cazenave, I. Naumkin, Local existence, global existence, and scattering for the nonlinear Schrödinger equation, *Commun. Contemp. Math.*, **19** (2017), 1650038. https://doi.org/10.1142/S0219199716500383

25. A. K. Arora, O. Riaño, S. Roudenko, Well-posedness in weighted spaces for the generalized Hartree equation with $p < 2$, *Commun. Contemp. Math.*, **24** (2022), 2150074. https://doi.org/10.1142/S0219199721500747

26. T. Cazenave, *Semilinear Schrödinger equations*, Providence: American Mathematical Society, 2003.

27. R. T. Glassey, On the blowing-up of solutions to the Cauchy problem for the non-linear Schrödinger equation, *J. Math. Phys.*, **18** (1977), 1794–1797. https://doi.org/10.1063/1.523491