AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(10): 27797–27833.
DOI: 10.3934/math. 20241350
Received: 07 August 2024
Revised: 07 September 2024
Accepted: 14 September 2024
Published: 26 September 2024

Research article

Selection of artificial neutral networks based on cubic intuitionistic fuzzy Aczel-Alsina aggregation operators

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Abstract: Artificial neural networks (ANNs) are the collection of computational techniques or models encouraged by the shape and purpose of natural or organic neural networks. Furthermore, a cubic intuitionistic fuzzy (CIF) set is the modified or extended form of a Fuzzy set (FS). Our goal was to address or compute the model of Aczel-Alsina operational laws under the consideration of the CIF set as well as Aczel-Alsina t-norm (AATN) and Aczel-Alsina t-conorm (AATCN), where the model of Algebraic norms and Drastic norms were the special parts of the Aczel-Alsina norms. Further, using the above invented operational laws, we aimed to develop the model of Aczel-Alsina average/geometric aggregation operators, called CIF Aczel-Alsina weighted averaging (CIFAAWA), CIF Aczel-Alsina ordered weighted averaging (CIFAAOWA), CIF Aczel-Alsina hybrid averaging (CIFAAHA), CIF Aczel-Alsina weighted geometric (CIFAAWG), CIF Aczel-Alsina ordered weighted geometric (CIFAAOWG), and CIF Aczel-Alsina hybrid geometric (CIFAAHG) operators with some well-known and desirable properties. Moreover, a procedure decision-making technique was presented for finding the best type of artificial neural networks with the help of multi-attribute decision-making (MADM) problems based on CIF aggregation information. Finally, we determined a numerical example for showing the rationality and advantages of the developed method by comparing their ranking values with the ranking values of many prevailing tools.

Keywords: cubic intuitionistic fuzzy sets; Aczel-Alsina averaging/geometric aggregation operators; artificial neural networks; decision-making problems
Mathematics Subject Classification: 03B52, 03E72, 03E73, 28E10, 94D05

1. Introduction

To find the finest or best optimal form from the collection of finite alternatives, different techniques have been proposed by different scholars, such as MADM techniques [1,2], pattern recognition, artificial neural networks, and artificial intelligence. Various individuals have developed the MADM tools in different fields [3,4] based on classical set theory. However, because of complications and complexity in the case of classical set theory, experts have lost a lot of data during the decision-making process. To solve this problem, the fuzzy set (FS) was developed by Zadeh [5] in 1965 by modifying the function $\ddot{\mu}_{\ddot{\mu}}: ℃ \rightarrow \{0,1\}$ into $\ddot{\mu}_{\ddot{\mu}}: ℃ \rightarrow [0,1]$, called truth grade. Furthermore, the FS has a lot of benefits, but it also has some limitations, such as in the presence of truth grade and falsity grade, FS has failed because it deals only with truth information and not with falsity information. For this, the IFS was proposed by Atanassov [6], which covered the truth and falsity grades, such as $\ddot{\mu}_{\ddot{\mu}}: ℃ \rightarrow [0,1]$ and $\ddot{\eta}_{\ddot{\eta}}: ℃ \rightarrow [0,1]$ with a condition $0 \leq \ddot{\mu}_{\ddot{\mu}}(\alpha) + \ddot{\eta}_{\ddot{\eta}}(\alpha) \leq 1$. The FS is a special part of the IFS if we exclude the falsity of information $\ddot{\eta}_{\ddot{\eta}}: ℃ \rightarrow [0,1]$. Additionally, to increase the ratio of correctness, we have also the best option to take the shape of an interval instead of a real number. For example, during any cricket match between Team A and Team B, we provided our opinion in the shape of the interval, and we decided that Team A would score between 150 to 180 in the T20 match. For such type of problem, the IFS and FS are not good; therefore, the idea of interval-valued IFS (IVIFS) was given by Atanassov [7], with a characteristic $0 \leq \ddot{\mu}^{+}_{\ddot{\mu}}(\alpha) + \ddot{\eta}^{+}_{\ddot{\eta}}(\alpha) \leq 1$, where $\left[\ddot{\mu}^{-}_{\ddot{\mu}}(\alpha), \ddot{\mu}^{+}_{\ddot{\mu}}(\alpha)\right]$ and $\left[\ddot{\eta}^{-}_{\ddot{\eta}}(\alpha), \ddot{\eta}^{+}_{\ddot{\eta}}(\alpha)\right]$ represents the interval-valued truth and interval-valued falsity information. Moreover, Jun et al. [8] developed the cubic set, which is the combination of FS and interval-valued FS (IVFS) [9,10]. Moreover, Kaur and Garg [11] developed the cubic IFS (CIFS), which is the combination of IFS and IVIFS. The geometric representation of the FSs and their extensions are described in Figure 1. The model of cubic intuitionistic fuzzy sets is more extensive compared to other existing models. Further, a detailed review of the above existing models is described in the next sub-sections.

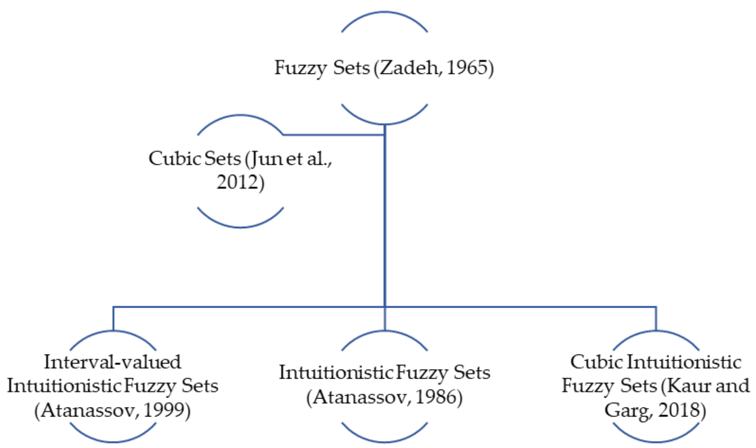

Figure 1. Geometrical representation of the fuzzy sets and their extensions.

1.1. Literature review

FS and its extensions have many applications in different fields, and because of their valuable and dominant structure, FS is better than the classical set, and IFS is more beneficial than FS, but the CIFS is more advanced and reliable than FS because it is a combination of two different structures, such as IFS and IVIFS. Some valuable applications are given; for instance, Mardani et al. [12] explored the aggregation operators for FS. Moreover, Merigo and Casanovas [13] developed the generalized hybrid aggregation operators for FS and their applications. Additionally, Xu [14] derived the simple aggregation operators for IFSs, whereas the prioritized aggregation operators for IFS were used by Yu and Xu [15]. Moreover, Xu and Yager [16] examined the geometric aggregation operators for IFS and their application in decision-making problems. Garg et al. [17] explored the Schweizer-Sklar prioritized aggregation operators for IFSs. Wang et al. [18] developed the aggregation operators for IVIFSs and their applications. Senapati et al. [19] proposed the Aczel-Alsina aggregation operators for IVIFSs. Shi et al. [20] evaluated the power aggregation operators based on Aczel-Alsina operational laws for IVIFSs. Wei and Wang [21] studied the geometric aggregation operators for IVIFSs and their application in decision-making problems. Xu and Chen [22] proposed the geometric aggregation operators for IVIFSs. Fahmi et al. [23] developed the Einstein aggregation operators for cubic fuzzy sets. Khan et al. [24] studied the cubical fuzzy aggregation operators and their application in decision-making problems. Kaur and Garg [25] examined the simple aggregation operators based on cubic IFSs and their applications. Additionally, Kaur and Garg [26] derived the generalized aggregation operators (AOs) for cubic IFS and their application in decision-making problems.

1.2. Aczel-Alsina t-norm and t-conorm

Aggregating the collection of finite information is a very challenging task for scholars. The triangular norms were proposed by Klement and Mesiar [27], which are very valuable and dominant for evaluating any kind of aggregation operator. Furthermore, Aczel and Alsina [28] developed the Aczel-Alsina t-norm and t-conorm, which are the modified versions of the algebraic norms. Many scholars developed different types of aggregation operators based on Aczel-Alsina operational laws, for instance, Senapati et al. [29] presented the Aczel-Alsina AOs for IFSs. Further, Senapati et al. [30] developed the Aczel-Alsina AOs for IVIFSs. Moreover, Aczel-Alsina AOs based on hesitant FS were given by Senapati et al. [31]. Mahmood et al. [32] presented the Aczel-Alsina AOs for complex IFSs and their application. Senapati et al. [33] examined the geometric AOs for IFSs and their applications. Ahmad et al. [34] derived the Aczel-Alsina AOs for the intuitionistic fuzzy rough set. Sarfraz et al. [35] proposed the prioritized Aczel-Alsina AOs for IFSs. Mahmood et al. [36] explored the Aczel-Alsina power AOs for complex IFSs. Recently, Hussain et al. [37] introduced the intuitionistic fuzzy rough Aczel-Alsina AOs and their application in decision-making problems. Further, many types of operators were constructed by well-known scholars, for instance, the model of Dombi operators [38], Archimedean operators [39], and Frank operators [40]. Moreover, some scholars have modified the model of IFSs and invented the model of quasirung orthopair fuzzy sets [41], the model of (3, 4)-quasirung orthopair fuzzy sets [42], q-rung orthopair fuzzy prioritized operators [43], linear Diophantine fuzzy sets [44], cubic picture fuzzy topology [45], and picture fuzzy soft-max Einstein operators [46].

1.3. Main problems/research gaps/motivations of the proposed techniques

The model for FSs theory and their modifications are very flexible because of their features, where these techniques are very reliable; however, due to ambiguity and problems, experts have lost a lot of information during the decision-making procedure. During decision-making assessments, all decision-makers have faced the following dilemmas, such as

1) How we define a new aggregation operator.
2) How we aggregate the collection of information into a singleton set.
3) How we rank all alternatives to select the best one.

For handling such kinds of problems, the Aczel-Alsina operators based on the CIF set are very beneficial and consistent for assessing uncooperative and vague information in real-life problems. The model of Aczel-Alsina norms is described below, such as

$$\overline{\overline{\pi_{tn}}}^{\hbar\hbar}(\widetilde{\alpha_1},\widetilde{\alpha_2}) = \begin{cases} \overline{\overline{\pi_{tn}}}(\widetilde{\alpha_1},\widetilde{\alpha_2}) & when\ \hbar\hbar = 0, \\ min(\widetilde{\alpha_1},\widetilde{\alpha_2}) & when\ \hbar\hbar = \infty, \\ \mathfrak{E}^{-\left((-\mathbb{I}\mathrm{n}(\widetilde{\alpha_1}))^{\hbar\hbar}+(-\mathbb{I}\mathrm{n}(\widetilde{\alpha_2}))^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}} & otherwise. \end{cases}$$

$$\overline{\overline{\pi_{tcn}}}^{\hbar\hbar}(\widetilde{\alpha_1},\widetilde{\alpha_2}) = \begin{cases} \overline{\overline{\pi_{tcn}}}(\widetilde{\alpha_1},\widetilde{\alpha_2}) & when\ \hbar\hbar = 0, \\ max(\widetilde{\alpha_1},\widetilde{\alpha_2}) & when\ \hbar\hbar = \infty, \\ 1-\mathfrak{E}^{-\left((-\mathbb{I}\mathrm{n}(1-\widetilde{\alpha_1}))^{\hbar\hbar}+(-\mathbb{I}\mathrm{n}(1-\widetilde{\alpha_2}))^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}} & otherwise. \end{cases}$$

Note that $\overline{\overline{\pi_{tn}}}(\widetilde{\alpha_1},\widetilde{\alpha_2}) = \widetilde{\alpha_1}.\widetilde{\alpha_2}$ and $\overline{\overline{\pi_{tcn}}}(\widetilde{\alpha_1},\widetilde{\alpha_2}) = \widetilde{\alpha_1}+\widetilde{\alpha_2}-\widetilde{\alpha_1}.\widetilde{\alpha_2}$ describe the algebraic norms with drastic norms $min(\widetilde{\alpha_1},\widetilde{\alpha_2})$ and $max(\widetilde{\alpha_1},\widetilde{\alpha_2})$, which are the special cases of the Aczel-Alsina norms. After a long assessment, we noticed that the technique of Aczel-Alsina operational laws is based on CIF sets. Further, the technique of averaging and geometric operators based on Aczel-Alsina norms for CIF values are also very reliable but have not been invented yet. These techniques are very capable and strong due to their characteristics and have not been proposed by anyone. The main motivation of the proposed work is that no one can propose it, and the Aczel-Alsina aggregation operators for the CIF set and the Aczel-Alsina AOs were proposed for FSs, IFSs, IVIFSs, but not for cubic IFSs. To propose these operators, many operators are the only parts of the proposed operators because they are the modified version of the existing operators. The limitations of the existing techniques are briefly evaluated and discussed in Table 1.

Table 1. Theoretical comparison between proposed and existing models.

Authors	Methods	Membership function	Non-membership function	Interval-valued function	Cubic information	Aczel-Alsina norms	Algebraic norms	Drastic norms
Zadeh [5]	Fuzzy sets	√	×	×	×	×	√	√
Atanassov [6]	Intuitionistic fuzzy sets	√	√	×	×	×	√	√
Atanassov [7]	Interval-valued Intuitionistic fuzzy sets	√	√	√	×	×	√	√
Jun et al. [8]	Cubic sets	√	×	√	√	×	√	√
Zadeh [9]	Interval-valued fuzzy sets	√	×	√	×	×	√	√
Turksen [10]	Interval-valued fuzzy sets	√	×	√	×	×	√	√
Kaur and Garg [11]	Cubic intuitionistic fuzzy sets	√	√	√	√	×	√	√
Proposed	Aczel-Alsina operators for CIF values	√	√	√	√	√	√	√

Information in Table 1 briefly describes that the proposed models are very effective because of their features, where the symbol "×" represents the "no" and the term "yes" is denoted by "√". Therefore, according to theoretical assessments, we observed the model of Aczel-Alsina operators for CIF values is very reliable and dominant compared to others.

1.4. Advantages and major contributions of the proposed techniques

To compile the solution to the above queries, we aim to simplify the model of Aczel-Alsina operational laws for evaluating the models of average/geometric operators based on it for CIF values. The model of Aczel-Alsina aggregation operators based on CIF values is the modified version of the existing technique of FSs and their related extensions. Some advantages of the proposed operators are as follows: The model of Aczel-Alsina, Algebraic, and Drastic aggregation operators, which are the special cases of the proposed theory.

The above information contains the special cases of the invented theory. The proposed model is superior and more dominant because of the parameters that are involved in the structure of the proposed theory. Inspired by the above observation, we decided to determine the following major contributions of the proposed manuscript, such as

1) To obtain the Aczel-Alsina operational laws based on the CIF set.
2) To develop the CIFAAWA, CIFAAOWA, CIFAAHA, CIFAAWG, CIFAAOWG, and CIFAAHG operators with some well-known and desirable properties.
3) A procedure of decision-making technique is presented for finding the best type of artificial neural networks with the help of MADM problems based on CIF information.
4) A numerical example is provided to show the rationality and advantages of the developed method by comparison with many prevailing tools. The geometrical interpretation of the proposed models is briefly evaluated in Figure 2.

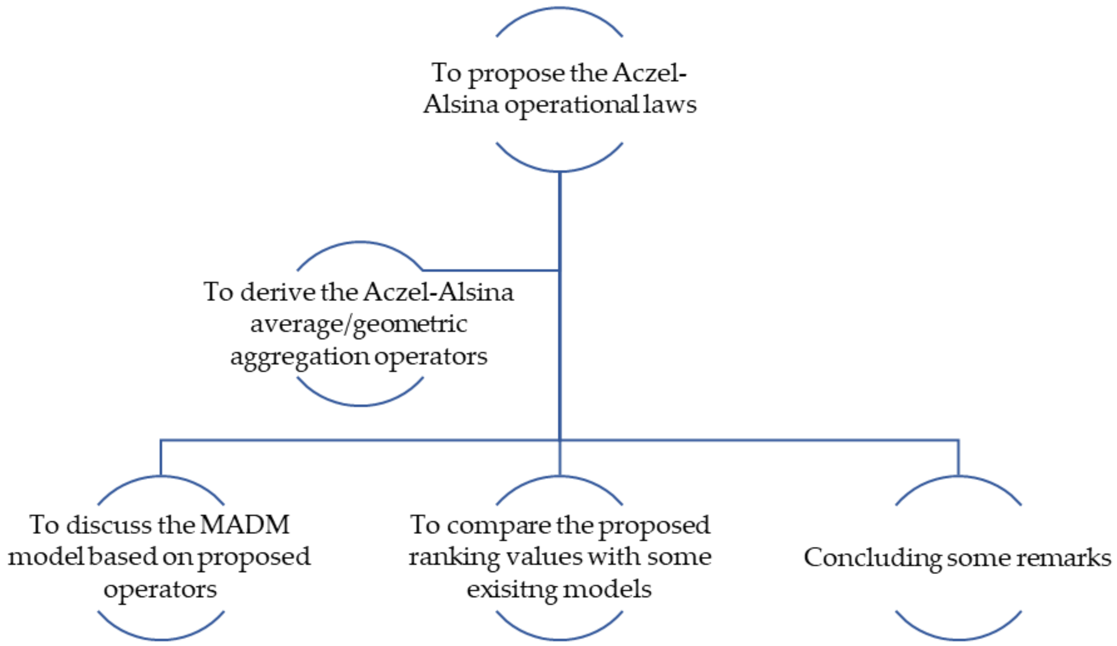

Figure 2. Geometrical interpretation of the proposed models.

1.5. The Summary of the proposed theory

This manuscript is arranged as:

In Section 2, we introduce the valuable IFSs, IVIFSs, CIFSs, and their operational laws.

In Section 3, we develop the Aczel-Alsina operational laws and their related results.

In Section 4, we propose the CIFAAWA, CIFAAOWA, CIFAAHA, CIFAAWG, CIFAAOWG, and CIFAAHG operators. Moreover, some well-known and desirable properties and special cases of them are discussed.

In Section 5, a procedure of decision-making technique is presented for finding the best type of artificial neural networks with the help of MADM problems based on CIF information, and a numerical or practical example is provided to show the rationality and advantages of the developed method by comparison with many prevailing tools.

In Section 6, we conclude final remarks about the proposed theory.

2. Preliminaries

In this section, we introduce the valuable IFSs, IVIFSs, CIFSs, and their operational laws. The main goal is to obtain the Aczel-Alsina operational laws and Aczel-Alsina operators based on CIFSs. For this, we used a universal set °C to state the existing ideas.

Definition 1: [6] Consider a fixed set °C, then the IFS $°\mathrm{F}^{/}_{IF}$ is given below:

$$°\mathrm{F}^{/}_{IF} = \left\{\left(\ddot{\mu}_{\ddot{\mu}}(\alpha), \ddot{\eta}_{\ddot{\eta}}(\alpha)\right) : \alpha \in °\mathrm{C}\right\}. \tag{1}$$

With a characteristic $0 \leq \ddot{\mu}_{\ddot{\mu}}(\alpha) + \ddot{\eta}_{\ddot{\eta}}(\alpha) \leq 1$, where $\ddot{\mu}_{\ddot{\mu}}(\alpha)$ and $\ddot{\eta}_{\ddot{\eta}}(\alpha)$ represents the truth and falsity degrees with a neutral grade $\ddot{\vartheta}_{\ddot{\vartheta}}(\alpha) = 1 - \left(\ddot{\mu}_{\ddot{\mu}}(\alpha) + \ddot{\eta}_{\ddot{\eta}}(\alpha)\right)$. Moreover, the simple form of the IF number (IFN) is shown by: $°\mathrm{F}^{/}_{IF_{\omega}} = \left(\ddot{\mu}_{\ddot{\mu}_{\omega}}, \ddot{\eta}_{\ddot{\eta}_{\omega}}\right), \omega = 1,2,\dots,z$.

Definition 2: [7] Consider a fixed set °C, then the IVIFS $°\mathrm{F}^{/}_{IVIF}$ is given below:

$$°\mathrm{F}^{/}_{IVIF} = \left\{\left(\left[\ddot{\mu}^{-}_{\ddot{\mu}}(\alpha), \ddot{\mu}^{+}_{\ddot{\mu}}(\alpha)\right], \left[\ddot{\eta}^{-}_{\ddot{\eta}}(\alpha), \ddot{\eta}^{+}_{\ddot{\eta}}(\alpha)\right]\right) : \alpha \in °\mathrm{C}\right\}. \tag{2}$$

With a characteristic $0 \leq \ddot{\mu}^{+}_{\ddot{\mu}}(\alpha) + \ddot{\eta}^{+}_{\ddot{\eta}}(\alpha) \leq 1$, where $\left[\ddot{\mu}^{-}_{\ddot{\mu}}(\alpha), \ddot{\mu}^{+}_{\ddot{\mu}}(\alpha)\right]$ and $\left[\ddot{\eta}^{-}_{\ddot{\eta}}(\alpha), \ddot{\eta}^{+}_{\ddot{\eta}}(\alpha)\right]$ represents the interval-valued truth and interval-valued falsity degrees with a neutral grade $\ddot{\vartheta}_{\ddot{\vartheta}}(\alpha) = \left[\ddot{\vartheta}^{-}_{\ddot{\vartheta}}(\alpha), \ddot{\vartheta}^{+}_{\ddot{\vartheta}}(\alpha)\right] = \left[1 - \ddot{\mu}^{+}_{\ddot{\mu}}(\alpha) + \ddot{\eta}^{+}_{\ddot{\eta}}(\alpha), 1 - \ddot{\mu}^{-}_{\ddot{\mu}}(\alpha) + \ddot{\eta}^{-}_{\ddot{\eta}}(\alpha)\right]$. Moreover, the simple form of the IVIF number (IVIFN) is shown by: $°\mathrm{F}^{/}_{IVIF_{\omega}} = \left(\left[\ddot{\mu}^{-}_{\ddot{\mu}_{\omega}}, \ddot{\mu}^{+}_{\ddot{\mu}_{\omega}}\right], \left[\ddot{\eta}^{-}_{\ddot{\eta}_{\omega}}, \ddot{\eta}^{+}_{\ddot{\eta}_{\omega}}\right]\right), \omega = 1,2,\dots,z$.

Definition 3: [11] Consider a fixed set °C, then the CIFS $°\mathrm{F}^{/}_{CuIF}$ is given below:

$$°F'_{CuIF} = \left\{\left(\left(\ddot{\mu}_{\ddot{\mu}}(\alpha), \ddot{\eta}_{\ddot{\eta}}(\alpha)\right), \left([\ddot{\mu}^{-}_{\ddot{\mu}}(\alpha), \ddot{\mu}^{+}_{\ddot{\mu}}(\alpha)], [\ddot{\eta}^{-}_{\ddot{\eta}}(\alpha), \ddot{\eta}^{+}_{\ddot{\eta}}(\alpha)]\right)\right) : \alpha \in °C\right\}. \tag{3}$$

With a characteristic $0 \leq \ddot{\mu}_{\ddot{\mu}}(\alpha) + \ddot{\eta}_{\ddot{\eta}}(\alpha) \leq 1$ and $0 \leq \ddot{\mu}^{+}_{\ddot{\mu}}(\alpha) + \ddot{\eta}^{+}_{\ddot{\eta}}(\alpha) \leq 1$, where $[\ddot{\mu}^{-}_{\ddot{\mu}}(\alpha), \ddot{\mu}^{+}_{\ddot{\mu}}(\alpha)]$ and $[\ddot{\eta}^{-}_{\ddot{\eta}}(\alpha), \ddot{\eta}^{+}_{\ddot{\eta}}(\alpha)]$ represents the interval-valued truth and interval-valued falsity degrees with a neutral grade $\ddot{\vartheta}_{\ddot{\vartheta}}(\alpha) = [\ddot{\vartheta}^{-}_{\ddot{\vartheta}}(\alpha), \ddot{\vartheta}^{+}_{\ddot{\vartheta}}(\alpha)] = [1 - \ddot{\mu}^{+}_{\ddot{\mu}}(\alpha) + \ddot{\eta}^{+}_{\ddot{\eta}}(\alpha), 1 - \ddot{\mu}^{-}_{\ddot{\mu}}(\alpha) + \ddot{\eta}^{-}_{\ddot{\eta}}(\alpha)]$, where $\ddot{\mu}_{\ddot{\mu}}(\alpha)$ and $\ddot{\eta}_{\ddot{\eta}}(\alpha)$ represents the truth and falsity degrees with a neutral grade $\ddot{\vartheta}_{\ddot{\vartheta}}(\alpha) = 1 - \left(\ddot{\mu}_{\ddot{\mu}}(\alpha) + \ddot{\eta}_{\ddot{\eta}}(\alpha)\right)$. Moreover, the simple form of the CIF number (CIFN) is shown by: $°F'_{CuIF_{\omega}} = \left(\left(\ddot{\mu}_{\ddot{\mu}_{\omega}}, \ddot{\eta}_{\ddot{\eta}_{\omega}}\right), \left([\ddot{\mu}^{-}_{\ddot{\mu}_{\omega}}, \ddot{\mu}^{+}_{\ddot{\mu}_{\omega}}], [\ddot{\eta}^{-}_{\ddot{\eta}_{\omega}}, \ddot{\eta}^{+}_{\ddot{\eta}_{\omega}}]\right)\right), \omega = 1,2,\ldots,z$. Furthermore, the score function and accuracy function are given, such as

$$S_{sf}\left(°F'_{CuIF_{\omega}}\right) = \frac{1}{2}\left(\left(\ddot{\mu}_{\ddot{\mu}_{\omega}} - \ddot{\eta}_{\ddot{\eta}_{\omega}}\right) + \frac{1}{2}\left(\ddot{\mu}^{-}_{\ddot{\mu}_{\omega}} + \ddot{\mu}^{+}_{\ddot{\mu}_{\omega}} - \ddot{\eta}^{-}_{\ddot{\eta}_{\omega}} - \ddot{\eta}^{+}_{\ddot{\eta}_{\omega}}\right)\right) \in [-1,1], \tag{4}$$

$$H_{af}\left(°F'_{CuIF_{\omega}}\right) = \frac{1}{2}\left(\left(\ddot{\mu}_{\ddot{\mu}_{\omega}} + \ddot{\eta}_{\ddot{\eta}_{\omega}}\right) + \frac{1}{2}\left(\ddot{\mu}^{-}_{\ddot{\mu}_{\omega}} + \ddot{\mu}^{+}_{\ddot{\mu}_{\omega}} + \ddot{\eta}^{-}_{\ddot{\eta}_{\omega}} + \ddot{\eta}^{+}_{\ddot{\eta}_{\omega}}\right)\right) \in [-1,1]. \tag{5}$$

For the above information, we can give some characteristics, such as if $S_{sf}\left(°F'_{CuIF_1}\right) > S_{sf}\left(°F'_{CuIF_2}\right) \Rightarrow °F'_{CuIF_1} > °F'_{CuIF_2}$, if $S_{sf}\left(°F'_{CuIF_1}\right) < S_{sf}\left(°F'_{CuIF_2}\right) \Rightarrow °F'_{CuIF_1} < °F'_{CuIF_2}$, if $S_{sf}\left(°F'_{CuIF_1}\right) = S_{sf}\left(°F'_{CuIF_2}\right)$, thus $H_{af}\left(°F'_{CuIF_1}\right) > H_{af}\left(°F'_{CuIF_2}\right) \Rightarrow °F'_{CuIF_1} > °F'_{CuIF_2}$, if $H_{af}\left(°F'_{CuIF_1}\right) < H_{af}\left(°F'_{CuIF_2}\right) \Rightarrow °F'_{CuIF_1} < °F'_{CuIF_2}$.

Definition 4: [28] The Aczel-Alsina t-norm for a scaler $\Xi \geq 0$ is given below:

$$\overline{\overline{\pi_{tn}}}^{\hbar\hbar}(\widetilde{\alpha_1}, \widetilde{\alpha_2}) = \begin{cases} \overline{\overline{\pi_{tn}}}(\widetilde{\alpha_1}, \widetilde{\alpha_2}) & wh\mathfrak{E}n\ \hbar\hbar = 0, \\ min(\widetilde{\alpha_1}, \widetilde{\alpha_2}) & wh\mathfrak{E}n\ \hbar\hbar = \infty, \\ \mathfrak{E}^{-\left((-\ln(\widetilde{\alpha_1}))^{\hbar\hbar} + (-\ln(\widetilde{\alpha_2}))^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}} & oth\mathfrak{E}rwis\mathfrak{E}. \end{cases} \tag{6}$$

$$\overline{\overline{\pi_{tcn}}}^{\hbar\hbar}(\widetilde{\alpha_1}, \widetilde{\alpha_2}) = \begin{cases} \overline{\overline{\pi_{tcn}}}(\widetilde{\alpha_1}, \widetilde{\alpha_2}) & wh\mathfrak{E}n\ \hbar\hbar = 0, \\ max(\widetilde{\alpha_1}, \widetilde{\alpha_2}) & wh\mathfrak{E}n\ \hbar\hbar = \infty, \\ 1 - \mathfrak{E}^{-\left((-\ln(1-\widetilde{\alpha_1}))^{\hbar\hbar} + (-\ln(1-\widetilde{\alpha_2}))^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}} & oth\mathfrak{E}rwis\mathfrak{E}. \end{cases} \tag{7}$$

Note that $\overline{\overline{\pi_{tn}}}(\widetilde{\alpha_1}, \widetilde{\alpha_2}) = \widetilde{\alpha_1}.\widetilde{\alpha_2}$ and $\overline{\overline{\pi_{tcn}}}(\widetilde{\alpha_1}, \widetilde{\alpha_2}) = \widetilde{\alpha_1} + \widetilde{\alpha_2} - \widetilde{\alpha_1}.\widetilde{\alpha_2}$ described the algebraic norms with drastic norms $min(\widetilde{\alpha_1}, \widetilde{\alpha_2})$ and $max(\widetilde{\alpha_1}, \widetilde{\alpha_2})$.

3. Aczel-Alsina Operational laws for CIFSs

In this section, we aim to develop the Aczel-Alsina norms for CIFSs and try to derive some Aczel-Alsina operational laws. Further, we prove some important results based on these operational laws.

Definition 5: For two CIFNs $°\mathrm{F}^{/}_{CuIF_{\omega}} = \left(\left(\ddot{\mu}_{\ddot{\mu_{\omega}}}, \ddot{\eta}_{\ddot{\eta}_{\omega}}\right), \left(\left[\ddot{\mu}^{-}_{\ddot{\mu_{\omega}}}, \ddot{\mu}^{+}_{\ddot{\mu_{\omega}}}\right], \left[\ddot{\eta}^{-}_{\ddot{\eta}_{\omega}}, \ddot{\eta}^{+}_{\ddot{\eta}_{\omega}}\right]\right)\right), \omega = 1,2,$ we have Aczel-Alsina operational laws, such as

$$°\mathrm{F}^{/}_{CuIF_1} \oplus °\mathrm{F}^{/}_{CuIF_2} = \left(\begin{array}{c} \left(1-\mathfrak{E}^{-\left(\left(-\mathbb{ln}\left(1-\ddot{\mu}_{\ddot{\mu_1}}\right)\right)^{\hbar\hbar}+\left(-\mathbb{ln}\left(1-\ddot{\mu}_{\ddot{\mu_2}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, \mathfrak{E}^{-\left(\left(-\mathbb{ln}\left(\ddot{\eta}_{\ddot{\eta}_1}\right)\right)^{\hbar\hbar}+\left(-\mathbb{ln}\left(\ddot{\eta}_{\ddot{\eta}_2}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right), \\ \left(\begin{array}{c} \left[1-\mathfrak{E}^{-\left(\left(-\mathbb{ln}\left(1-\ddot{\mu}^{-}_{\ddot{\mu_1}}\right)\right)^{\hbar\hbar}+\left(-\mathbb{ln}\left(1-\ddot{\mu}^{-}_{\ddot{\mu_2}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, 1-\mathfrak{E}^{-\left(\left(-\mathbb{ln}\left(1-\ddot{\mu}^{+}_{\ddot{\mu_1}}\right)\right)^{\hbar\hbar}+\left(-\mathbb{ln}\left(1-\ddot{\mu}^{+}_{\ddot{\mu_2}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right], \\ \left[\mathfrak{E}^{-\left(\left(-\mathbb{ln}\left(\ddot{\eta}^{-}_{\ddot{\eta}_1}\right)\right)^{\hbar\hbar}+\left(-\mathbb{ln}\left(\ddot{\eta}^{-}_{\ddot{\eta}_2}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, \mathfrak{E}^{-\left(\left(-\mathbb{ln}\left(\ddot{\eta}^{+}_{\ddot{\eta}_1}\right)\right)^{\hbar\hbar}+\left(-\mathbb{ln}\left(\ddot{\eta}^{+}_{\ddot{\eta}_2}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right] \end{array}\right) \end{array}\right), \quad (8)$$

$$°\mathrm{F}^{/}_{CuIF_1} \otimes °\mathrm{F}^{/}_{CuIF_2} = \left(\begin{array}{c} \left(\mathfrak{E}^{-\left(\left(-\mathbb{ln}\left(\ddot{\mu}_{\ddot{\mu_1}}\right)\right)^{\hbar\hbar}+\left(-\mathbb{ln}\left(\ddot{\mu}_{\ddot{\mu_2}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, 1-\mathfrak{E}^{-\left(\left(-\mathbb{ln}\left(1-\ddot{\eta}_{\ddot{\eta}_1}\right)\right)^{\hbar\hbar}+\left(-\mathbb{ln}\left(1-\ddot{\eta}_{\ddot{\eta}_2}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right), \\ \left(\begin{array}{c} \left[\mathfrak{E}^{-\left(\left(-\mathbb{ln}\left(\ddot{\mu}^{-}_{\ddot{\mu_1}}\right)\right)^{\hbar\hbar}+\left(-\mathbb{ln}\left(\ddot{\mu}^{-}_{\ddot{\mu_2}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, \mathfrak{E}^{-\left(\left(-\mathbb{ln}\left(\ddot{\mu}^{+}_{\ddot{\mu_1}}\right)\right)^{\hbar\hbar}+\left(-\mathbb{ln}\left(\ddot{\mu}^{+}_{\ddot{\mu_2}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right], \\ \left[1-\mathfrak{E}^{-\left(\left(-\mathbb{ln}\left(1-\ddot{\eta}^{-}_{\ddot{\eta}_1}\right)\right)^{\hbar\hbar}+\left(-\mathbb{ln}\left(1-\ddot{\eta}^{-}_{\ddot{\eta}_2}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, 1-\mathfrak{E}^{-\left(\left(-\mathbb{ln}\left(1-\ddot{\eta}^{+}_{\ddot{\eta}_1}\right)\right)^{\hbar\hbar}+\left(-\mathbb{ln}\left(1-\ddot{\eta}^{+}_{\ddot{\eta}_2}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right] \end{array}\right) \end{array}\right), \quad (9)$$

$$\ddot{\emptyset}_s °\mathrm{F}^{/}_{CuIF_1} = \left(\begin{array}{c} \left(1-\mathfrak{E}^{-\left(\ddot{\emptyset}_s\left(-\mathbb{ln}\left(1-\ddot{\mu}_{\ddot{\mu_1}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, \mathfrak{E}^{-\left(\ddot{\emptyset}_s\left(-\mathbb{ln}\left(\ddot{\eta}_{\ddot{\eta}_1}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right), \\ \left(\begin{array}{c} \left[1-\mathfrak{E}^{-\left(\ddot{\emptyset}_s\left(-\mathbb{ln}\left(1-\ddot{\mu}^{-}_{\ddot{\mu_1}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, 1-\mathfrak{E}^{-\left(\ddot{\emptyset}_s\left(-\mathbb{ln}\left(1-\ddot{\mu}^{+}_{\ddot{\mu_1}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right], \\ \left[\mathfrak{E}^{-\left(\ddot{\emptyset}_s\left(-\mathbb{ln}\left(\ddot{\eta}^{-}_{\ddot{\eta}_1}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, \mathfrak{E}^{-\left(\ddot{\emptyset}_s\left(-\mathbb{ln}\left(\ddot{\eta}^{+}_{\ddot{\eta}_1}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right] \end{array}\right) \end{array}\right), \quad (10)$$

$$\left({}^{\circ}\mathrm{F}'_{CuIF_1}\right)^{\ddot{\emptyset}_s} = \left(\begin{array}{c} \left(\mathfrak{E}^{-\left(\ddot{\emptyset}_s\left(-\mathbb{I}\mathrm{n}\left(\ddot{\mu}_{\ddot{\mu}_1}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, 1-\mathfrak{E}^{-\left(\ddot{\emptyset}_s\left(-\mathbb{I}\mathrm{n}\left(1-\ddot{\eta}_{\ddot{\eta}_1}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right), \\ \left(\begin{array}{c} \left[\mathfrak{E}^{-\left(\ddot{\emptyset}_s\left(-\mathbb{I}\mathrm{n}\left(\ddot{\mu}^{-}_{\ddot{\mu}_1}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, \mathfrak{E}^{-\left(\ddot{\emptyset}_s\left(-\mathbb{I}\mathrm{n}\left(\ddot{\mu}^{+}_{\ddot{\mu}_1}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right], \\ \left[1-\mathfrak{E}^{-\left(\ddot{\emptyset}_s\left(-\mathbb{I}\mathrm{n}\left(1-\ddot{\eta}^{-}_{\ddot{\eta}_1}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, 1-\mathfrak{E}^{-\left(\ddot{\emptyset}_s\left(-\mathbb{I}\mathrm{n}\left(1-\ddot{\eta}^{+}_{\ddot{\eta}_1}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right] \end{array}\right) \end{array}\right). \quad (11)$$

Further, we simplify the above techniques based on some suitable examples. For this, we consider two CIF numbers, such as ${}^{\circ}\mathrm{F}'_1 = \left((0.5,0.4),([0.4,0.5],[0.1,0.2])\right)$ and ${}^{\circ}\mathrm{F}'_2 = \left((0.3,0.1),([0.3,0.5],[0.1,0.4])\right)$ with $\hbar\hbar = \ddot{\emptyset}_s = 2$, thus

$${}^{\circ}\mathrm{F}'_1 \oplus {}^{\circ}\mathrm{F}'_2 = \left((0.54138,0.08389),([0.46368,0.62479],[0.03853,0.15692])\right),$$

$${}^{\circ}\mathrm{F}'_1 \otimes {}^{\circ}\mathrm{F}'_2 = \left((0.24926,0.40642),([0.22025,0.37521],[0.13843,0.42732])\right),$$

$$2 * {}^{\circ}\mathrm{F}'_1 = \left((0.62479,0.27367),([0.51442,0.62479],[0.03853,0.10269])\right),$$

$$\left({}^{\circ}\mathrm{F}'_1\right)^2 = \left((0.37521,0.51442),([0.27367,0.37521],[0.13843,0.27063])\right).$$

Theorem 1: For any CIFNs ${}^{\circ}\mathrm{F}'_{CuIF_\omega} = \left(\left(\ddot{\mu}_{\ddot{\mu}_\omega}, \ddot{\eta}_{\ddot{\eta}_\omega}\right),\left(\left[\ddot{\mu}^{-}_{\ddot{\mu}_\omega}, \ddot{\mu}^{+}_{\ddot{\mu}_\omega}\right],\left[\ddot{\eta}^{-}_{\ddot{\eta}_\omega}, \ddot{\eta}^{+}_{\ddot{\eta}_\omega}\right]\right)\right), \omega = 1,2,\dots,z$, thus

1) ${}^{\circ}\mathrm{F}'_{CuIF_1} \oplus {}^{\circ}\mathrm{F}'_{CuIF_2} = {}^{\circ}\mathrm{F}'_{CuIF_2} \oplus {}^{\circ}\mathrm{F}'_{CuIF_1}$.
2) ${}^{\circ}\mathrm{F}'_{CuIF_1} \otimes {}^{\circ}\mathrm{F}'_{CuIF_2} = {}^{\circ}\mathrm{F}'_{CuIF_2} \otimes {}^{\circ}\mathrm{F}'_{CuIF_1}$.
3) $\ddot{\emptyset}_s {}^{\circ}\mathrm{F}'_{CuIF_1} \oplus \ddot{\emptyset}_s {}^{\circ}\mathrm{F}'_{CuIF_2} = \ddot{\emptyset}_s\left({}^{\circ}\mathrm{F}'_{CuIF_1} \oplus {}^{\circ}\mathrm{F}'_{CuIF_2}\right)$.
4) $\left({}^{\circ}\mathrm{F}'_{CuIF_1}\right)^{\ddot{\emptyset}_s} \otimes \left({}^{\circ}\mathrm{F}'_{CuIF_2}\right)^{\ddot{\emptyset}_s} = \left({}^{\circ}\mathrm{F}'_{CuIF_1} \otimes {}^{\circ}\mathrm{F}'_{CuIF_2}\right)^{\ddot{\emptyset}_s}$.

Proof.

1) Let

$${}^{\circ}\mathrm{F}'_{CuIF_1} \oplus {}^{\circ}\mathrm{F}'_{CuIF_2} = \left(\begin{array}{c} \left(1-\mathfrak{E}^{-\left(\left(-\mathbb{I}\mathrm{n}\left(1-\ddot{\mu}_{\ddot{\mu}_1}\right)\right)^{\hbar\hbar}+\left(-\mathbb{I}\mathrm{n}\left(1-\ddot{\mu}_{\ddot{\mu}_2}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, \mathfrak{E}^{-\left(\left(-\mathbb{I}\mathrm{n}\left(\ddot{\eta}_{\ddot{\eta}_1}\right)\right)^{\hbar\hbar}+\left(-\mathbb{I}\mathrm{n}\left(\ddot{\eta}_{\ddot{\eta}_2}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right), \\ \left(\begin{array}{c} \left[1-\mathfrak{E}^{-\left(\left(-\mathbb{I}\mathrm{n}\left(1-\ddot{\mu}^{-}_{\ddot{\mu}_1}\right)\right)^{\hbar\hbar}+\left(-\mathbb{I}\mathrm{n}\left(1-\ddot{\mu}^{-}_{\ddot{\mu}_2}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, 1-\mathfrak{E}^{-\left(\left(-\mathbb{I}\mathrm{n}\left(1-\ddot{\mu}^{+}_{\ddot{\mu}_1}\right)\right)^{\hbar\hbar}+\left(-\mathbb{I}\mathrm{n}\left(1-\ddot{\mu}^{+}_{\ddot{\mu}_2}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right], \\ \left[\mathfrak{E}^{-\left(\left(-\mathbb{I}\mathrm{n}\left(\ddot{\eta}^{-}_{\ddot{\eta}_1}\right)\right)^{\hbar\hbar}+\left(-\mathbb{I}\mathrm{n}\left(\ddot{\eta}^{-}_{\ddot{\eta}_2}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, \mathfrak{E}^{-\left(\left(-\mathbb{I}\mathrm{n}\left(\ddot{\eta}^{+}_{\ddot{\eta}_1}\right)\right)^{\hbar\hbar}+\left(-\mathbb{I}\mathrm{n}\left(\ddot{\eta}^{+}_{\ddot{\eta}_2}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right] \end{array}\right) \end{array}\right)$$

$$= \left(\begin{array}{c} \left(1 - \mathfrak{E}^{-\left(\left(-\ln\left(1-\ddot{\mu}_{\ddot{\mu_2}}\right)\right)^{\hbar\hbar}+\left(-\ln\left(1-\ddot{\mu}_{\ddot{\mu_1}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, \mathfrak{E}^{-\left(\left(-\ln\left(\ddot{\eta}_{\ddot{\eta}_2}\right)\right)^{\hbar\hbar}+\left(-\ln\left(\ddot{\eta}_{\ddot{\eta}_1}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right), \\ \left(\begin{array}{c} \left[1 - \mathfrak{E}^{-\left(\left(-\ln\left(1-\ddot{\mu}^{-}_{\ddot{\mu_2}}\right)\right)^{\hbar\hbar}+\left(-\ln\left(1-\ddot{\mu}^{-}_{\ddot{\mu_1}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, 1 - \mathfrak{E}^{-\left(\left(-\ln\left(1-\ddot{\mu}^{+}_{\ddot{\mu_2}}\right)\right)^{\hbar\hbar}+\left(-\ln\left(1-\ddot{\mu}^{+}_{\ddot{\mu_1}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right], \\ \left[\mathfrak{E}^{-\left(\left(-\ln\left(\ddot{\eta}^{-}_{\ddot{\eta}_2}\right)\right)^{\hbar\hbar}+\left(-\ln\left(\ddot{\eta}^{-}_{\ddot{\eta}_1}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, \mathfrak{E}^{-\left(\left(-\ln\left(\ddot{\eta}^{+}_{\ddot{\eta}_2}\right)\right)^{\hbar\hbar}+\left(-\ln\left(\ddot{\eta}^{+}_{\ddot{\eta}_1}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right] \end{array} \right) \end{array} \right)$$

$$= {}^{\circ}\mathrm{F}^{/}_{CuIF_2} \oplus {}^{\circ}\mathrm{F}^{/}_{CuIF_1}.$$

2) Let

$${}^{\circ}\mathrm{F}^{/}_{CuIF_1} \otimes {}^{\circ}\mathrm{F}^{/}_{CuIF_2}$$

$$= \left(\begin{array}{c} \left(\mathfrak{E}^{-\left(\left(-\ln\left(\ddot{\mu}_{\ddot{\mu_1}}\right)\right)^{\hbar\hbar}+\left(-\ln\left(\ddot{\mu}_{\ddot{\mu_2}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, 1 - \mathfrak{E}^{-\left(\left(-\ln\left(1-\ddot{\eta}_{\ddot{\eta}_1}\right)\right)^{\hbar\hbar}+\left(-\ln\left(1-\ddot{\eta}_{\ddot{\eta}_2}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right), \\ \left(\begin{array}{c} \left[\mathfrak{E}^{-\left(\left(-\ln\left(\ddot{\mu}^{-}_{\ddot{\mu_1}}\right)\right)^{\hbar\hbar}+\left(-\ln\left(\ddot{\mu}^{-}_{\ddot{\mu_2}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, \mathfrak{E}^{-\left(\left(-\ln\left(\ddot{\mu}^{+}_{\ddot{\mu_1}}\right)\right)^{\hbar\hbar}+\left(-\ln\left(\ddot{\mu}^{+}_{\ddot{\mu_2}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right], \\ \left[1 - \mathfrak{E}^{-\left(\left(-\ln\left(1-\ddot{\eta}^{-}_{\ddot{\eta}_1}\right)\right)^{\hbar\hbar}+\left(-\ln\left(1-\ddot{\eta}^{-}_{\ddot{\eta}_2}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, 1 - \mathfrak{E}^{-\left(\left(-\ln\left(1-\ddot{\eta}^{+}_{\ddot{\eta}_1}\right)\right)^{\hbar\hbar}+\left(-\ln\left(1-\ddot{\eta}^{+}_{\ddot{\eta}_2}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right] \end{array} \right) \end{array} \right)$$

$$= \left(\begin{array}{c} \left(\mathfrak{E}^{-\left(\left(-\ln\left(\ddot{\mu}_{\ddot{\mu_2}}\right)\right)^{\hbar\hbar}+\left(-\ln\left(\ddot{\mu}_{\ddot{\mu_1}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, 1 - \mathfrak{E}^{-\left(\left(-\ln\left(1-\ddot{\eta}_{\ddot{\eta}_2}\right)\right)^{\hbar\hbar}+\left(-\ln\left(1-\ddot{\eta}_{\ddot{\eta}_1}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right), \\ \left(\begin{array}{c} \left[\mathfrak{E}^{-\left(\left(-\ln\left(\ddot{\mu}^{-}_{\ddot{\mu_2}}\right)\right)^{\hbar\hbar}+\left(-\ln\left(\ddot{\mu}^{-}_{\ddot{\mu_1}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, \mathfrak{E}^{-\left(\left(-\ln\left(\ddot{\mu}^{+}_{\ddot{\mu_2}}\right)\right)^{\hbar\hbar}+\left(-\ln\left(\ddot{\mu}^{+}_{\ddot{\mu_1}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right], \\ \left[1 - \mathfrak{E}^{-\left(\left(-\ln\left(1-\ddot{\eta}^{-}_{\ddot{\eta}_2}\right)\right)^{\hbar\hbar}+\left(-\ln\left(1-\ddot{\eta}^{-}_{\ddot{\eta}_1}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, 1 - \mathfrak{E}^{-\left(\left(-\ln\left(1-\ddot{\eta}^{+}_{\ddot{\eta}_2}\right)\right)^{\hbar\hbar}+\left(-\ln\left(1-\ddot{\eta}^{+}_{\ddot{\eta}_1}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right] \end{array} \right) \end{array} \right)$$

$$= {}^{\circ}\mathrm{F}^{/}_{CuIF_2} \otimes {}^{\circ}\mathrm{F}^{/}_{CuIF_1}.$$

3) Consider

$$\ddot{\emptyset}_s {}^{\circ}\mathrm{F}'_{CuIF_1} \oplus \ddot{\emptyset}_s {}^{\circ}\mathrm{F}'_{CuIF_2}$$

$$= \left(\begin{array}{c} \left(1 - \mathfrak{E}^{-\left(\ddot{\emptyset}_s\left(-\mathbb{ln}\left(1-\ddot{\mu}_{\ddot{\mu}_1}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, \mathfrak{E}^{-\left(\ddot{\emptyset}_s\left(-\mathbb{ln}\left(\ddot{\eta}_{\ddot{\eta}_1}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right), \\ \left(\begin{array}{c} \left[1 - \mathfrak{E}^{-\left(\ddot{\emptyset}_s\left(-\mathbb{ln}\left(1-\ddot{\mu}^-_{\ddot{\mu}_1}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, 1 - \mathfrak{E}^{-\left(\ddot{\emptyset}_s\left(-\mathbb{ln}\left(1-\ddot{\mu}^+_{\ddot{\mu}_1}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right], \\ \left[\mathfrak{E}^{-\left(\ddot{\emptyset}_s\left(-\mathbb{ln}\left(\ddot{\eta}^-_{\ddot{\eta}_1}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, \mathfrak{E}^{-\left(\ddot{\emptyset}_s\left(-\mathbb{ln}\left(\ddot{\eta}^+_{\ddot{\eta}_1}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right] \end{array} \right) \end{array} \right)$$

$$\oplus \left(\begin{array}{c} \left(1 - \mathfrak{E}^{-\left(\ddot{\emptyset}_s\left(-\mathbb{ln}\left(1-\ddot{\mu}_{\ddot{\mu}_2}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, \mathfrak{E}^{-\left(\ddot{\emptyset}_s\left(-\mathbb{ln}\left(\ddot{\eta}_{\ddot{\eta}_2}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right), \\ \left(\begin{array}{c} \left[1 - \mathfrak{E}^{-\left(\ddot{\emptyset}_s\left(-\mathbb{ln}\left(1-\ddot{\mu}^-_{\ddot{\mu}_2}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, 1 - \mathfrak{E}^{-\left(\ddot{\emptyset}_s\left(-\mathbb{ln}\left(1-\ddot{\mu}^+_{\ddot{\mu}_2}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right], \\ \left[\mathfrak{E}^{-\left(\ddot{\emptyset}_s\left(-\mathbb{ln}\left(\ddot{\eta}^-_{\ddot{\eta}_2}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, \mathfrak{E}^{-\left(\ddot{\emptyset}_s\left(-\mathbb{ln}\left(\ddot{\eta}^+_{\ddot{\eta}_2}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right] \end{array} \right) \end{array} \right)$$

$$= \left(\begin{array}{c} \left(1 - \mathfrak{E}^{-\left(\emptyset_s\left(-\mathbb{ln}\left(1-\ddot{\mu}_{\ddot{\mu}_1}\right)\right)^{\hbar\hbar}+\left(-\mathbb{ln}\left(1-\ddot{\mu}_{\ddot{\mu}_2}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, \mathfrak{E}^{-\left(\emptyset_s\left(-\mathbb{ln}\left(\ddot{\eta}_{\ddot{\eta}_1}\right)\right)^{\hbar\hbar}+\left(-\mathbb{ln}\left(\ddot{\eta}_{\ddot{\eta}_2}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right), \\ \left(\begin{array}{c} \left[1 - \mathfrak{E}^{-\left(\emptyset_s\left(-\mathbb{ln}\left(1-\ddot{\mu}^-_{\ddot{\mu}_1}\right)\right)^{\hbar\hbar}+\left(-\mathbb{ln}\left(1-\ddot{\mu}^-_{\ddot{\mu}_2}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, 1 - \mathfrak{E}^{-\left(\emptyset_s\left(-\mathbb{ln}\left(1-\ddot{\mu}^+_{\ddot{\mu}_1}\right)\right)^{\hbar\hbar}+\left(-\mathbb{ln}\left(1-\ddot{\mu}^+_{\ddot{\mu}_2}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right], \\ \left[\mathfrak{E}^{-\left(\emptyset_s\left(-\mathbb{ln}\left(\ddot{\eta}^-_{\ddot{\eta}_1}\right)\right)^{\hbar\hbar}+\left(-\mathbb{ln}\left(\ddot{\eta}^-_{\ddot{\eta}_2}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, \mathfrak{E}^{-\left(\emptyset_s\left(-\mathbb{ln}\left(\ddot{\eta}^+_{\ddot{\eta}_1}\right)\right)^{\hbar\hbar}+\left(-\mathbb{ln}\left(\ddot{\eta}^+_{\ddot{\eta}_2}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right] \end{array} \right) \end{array} \right)$$

$$= \ddot{\emptyset}_s\left({}^{\circ}\mathrm{F}'_{CuIF_1} \oplus {}^{\circ}\mathrm{F}'_{CuIF_2}\right).$$

4) Assume that

$$\left({}^{\circ}\mathrm{F}'_{CuIF_1}\right)^{\ddot{\emptyset}_s} \otimes \left({}^{\circ}\mathrm{F}'_{CuIF_2}\right)^{\ddot{\emptyset}_s}$$

$$
=\left(\begin{array}{c}
\left(\mathfrak{E}^{-\left(\ddot{\phi}_s\left(-\mathbb{ln}\left(\ddot{\mu}_{\ddot{\mu}_1}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, 1-\mathfrak{E}^{-\left(\ddot{\phi}_s\left(-\mathbb{ln}\left(1-\ddot{\eta}_{\ddot{\eta}_1}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right), \\
\left(\begin{array}{c}
\left[\mathfrak{E}^{-\left(\ddot{\phi}_s\left(-\mathbb{ln}\left(\ddot{\mu}^{-}_{\ddot{\mu}_1}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, \mathfrak{E}^{-\left(\ddot{\phi}_s\left(-\mathbb{ln}\left(\ddot{\mu}^{+}_{\ddot{\mu}_1}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right], \\
\left[1-\mathfrak{E}^{-\left(\ddot{\phi}_s\left(-\mathbb{ln}\left(1-\ddot{\eta}^{-}_{\ddot{\eta}_1}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, 1-\mathfrak{E}^{-\left(\ddot{\phi}_s\left(-\mathbb{ln}\left(1-\ddot{\eta}^{+}_{\ddot{\eta}_1}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right]
\end{array}\right)
\end{array}\right)
$$

$$
\otimes\left(\begin{array}{c}
\left(\mathfrak{E}^{-\left(\ddot{\phi}_s\left(-\mathbb{ln}\left(\ddot{\mu}_{\ddot{\mu}_2}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, 1-\mathfrak{E}^{-\left(\ddot{\phi}_s\left(-\mathbb{ln}\left(1-\ddot{\eta}_{\ddot{\eta}_2}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right), \\
\left(\begin{array}{c}
\left[\mathfrak{E}^{-\left(\ddot{\phi}_s\left(-\mathbb{ln}\left(\ddot{\mu}^{-}_{\ddot{\mu}_2}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, \mathfrak{E}^{-\left(\ddot{\phi}_s\left(-\mathbb{ln}\left(\ddot{\mu}^{+}_{\ddot{\mu}_2}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right], \\
\left[1-\mathfrak{E}^{-\left(\ddot{\phi}_s\left(-\mathbb{ln}\left(1-\ddot{\eta}^{-}_{\ddot{\eta}_2}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, 1-\mathfrak{E}^{-\left(\ddot{\phi}_s\left(-\mathbb{ln}\left(1-\ddot{\eta}^{+}_{\ddot{\eta}_2}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right]
\end{array}\right)
\end{array}\right)
$$

$$
=\left(\begin{array}{c}
\left(\mathfrak{E}^{-\left(\ddot{\phi}_s\left(-\mathbb{ln}\left(\ddot{\mu}_{\ddot{\mu}_1}\right)\right)^{\hbar\hbar}+\left(-\mathbb{ln}\left(\ddot{\mu}_{\ddot{\mu}_2}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, 1-\mathfrak{E}^{-\left(\ddot{\phi}_s\left(-\mathbb{ln}\left(1-\ddot{\eta}_{\ddot{\eta}_1}\right)\right)^{\hbar\hbar}+\left(-\mathbb{ln}\left(1-\ddot{\eta}_{\ddot{\eta}_2}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right), \\
\left(\begin{array}{c}
\left[\mathfrak{E}^{-\left(\ddot{\phi}_s\left(-\mathbb{ln}\left(\ddot{\mu}^{-}_{\ddot{\mu}_1}\right)\right)^{\hbar\hbar}+\left(-\mathbb{ln}\left(\ddot{\mu}^{-}_{\ddot{\mu}_2}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, \mathfrak{E}^{-\left(\ddot{\phi}_s\left(-\mathbb{ln}\left(\ddot{\mu}^{+}_{\ddot{\mu}_1}\right)\right)^{\hbar\hbar}+\left(-\mathbb{ln}\left(\ddot{\mu}^{+}_{\ddot{\mu}_2}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right], \\
\left[1-\mathfrak{E}^{-\left(\ddot{\phi}_s\left(-\mathbb{ln}\left(1-\ddot{\eta}^{-}_{\ddot{\eta}_1}\right)\right)^{\hbar\hbar}+\left(-\mathbb{ln}\left(1-\ddot{\eta}^{-}_{\ddot{\eta}_2}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, 1-\mathfrak{E}^{-\left(\ddot{\phi}_s\left(-\mathbb{ln}\left(1-\ddot{\eta}^{+}_{\ddot{\eta}_1}\right)\right)^{\hbar\hbar}+\left(-\mathbb{ln}\left(1-\ddot{\eta}^{+}_{\ddot{\eta}_2}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right]
\end{array}\right)
\end{array}\right)
$$

$$
=\left({}^{\circ}\mathrm{F}'_{CuIF_1}\otimes{}^{\circ}\mathrm{F}'_{CuIF_2}\right)^{\ddot{\phi}_s}.
$$

4. Aczel-Alsina aggregation operators for CIFSs

In this section, we develop the novel CIFAAWA operator, CIFAAOWA operator, CIFAAHA operator, CIFAAWG operator, CIFAAOWG operator, and CIFAAHG operator. These operators are the combination of the Aczel-Alsina operational and CIFNs. Furthermore, we have also stated some fundamental properties for the above results.

Definition 6: For the finite collection of CIFNs ${}^{\circ}\mathrm{F}'_{CuIF_\omega}=\left(\left(\ddot{\mu}_{\ddot{\mu}_\omega},\ddot{\eta}_{\ddot{\eta}_\omega}\right),\left(\left[\ddot{\mu}^{-}_{\ddot{\mu}_\omega},\ddot{\mu}^{+}_{\ddot{\mu}_\omega}\right],\left[\ddot{\eta}^{-}_{\ddot{\eta}_\omega},\ddot{\eta}^{+}_{\ddot{\eta}_\omega}\right]\right)\right), \omega=1,2,\dots,z$, then the CIFAAWA operator is defined as:

$$CuIFAAWA: {}^{\circ}\mathrm{F}^{z}\rightarrow{}^{\circ}\mathrm{F}, by$$

$$
\begin{aligned}
CuIFAAWA\left({}^{\circ}\mathrm{F}'_{CuIF_1},{}^{\circ}\mathrm{F}'_{CuIF_2},\dots,{}^{\circ}\mathrm{F}'_{CuIF_z}\right)&=\ddot{\phi}_s^{\ 1}\,{}^{\circ}\mathrm{F}'_{CuIF_1}\oplus\ddot{\phi}_s^{\ 2}\,{}^{\circ}\mathrm{F}'_{CuIF_2}\oplus\dots\oplus\ddot{\phi}_s^{\ z}\,{}^{\circ}\mathrm{F}'_{CuIF_z} \\
&=\oplus_{\omega=1}^{z}\ddot{\phi}_s^{\ \omega}\,{}^{\circ}\mathrm{F}'_{CuIF_\omega}.
\end{aligned}
\tag{12}
$$

Note that the weighted vector is stated by: $\ddot{\emptyset}_s^{\ \omega} \in [0,1]$ with $\sum_{\omega=1}^{z} \ddot{\emptyset}_s^{\ \omega} = 1$.

Theorem 2: For any finite collection of CIFNs $°\mathrm{F}'_{CuIF_\omega} = \left((\ddot{\mu}_{\mu_{\ddot{\omega}}}, \ddot{\eta}_{\ddot{\eta}_\omega}), ([\ddot{\mu}^-_{\mu_{\ddot{\omega}}}, \ddot{\mu}^+_{\mu_{\ddot{\omega}}}], [\ddot{\eta}^-_{\ddot{\eta}_\omega}, \ddot{\eta}^+_{\ddot{\eta}_\omega}])\right), \omega = 1,2, \ldots, z$, we proved that Eq (12) is also a CIFN, such as

$$CuIFAAWA(°\mathrm{F}'_{CuIF_1}, °\mathrm{F}'_{CuIF_2}, \ldots, °\mathrm{F}'_{CuIF_z}) = \left(\begin{array}{c} \left(1 - \mathfrak{E}^{-\left(\sum_{\omega=1}^{z} \ddot{\emptyset}_s^{\ \omega}\left(-\mathbb{l}\mathrm{n}(1-\ddot{\mu}_{\mu_{\ddot{\omega}}})\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, \mathfrak{E}^{-\left(\sum_{\omega=1}^{z} \ddot{\emptyset}_s^{\ \omega}\left(-\mathbb{l}\mathrm{n}(\ddot{\eta}_{\ddot{\eta}_\omega})\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right), \\ \left(\begin{array}{c} \left[1 - \mathfrak{E}^{-\left(\sum_{\omega=1}^{z} \ddot{\emptyset}_s^{\ \omega}\left(-\mathbb{l}\mathrm{n}\left(1-\ddot{\mu}^-_{\mu_{\ddot{\omega}}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, 1 - \mathfrak{E}^{-\left(\sum_{\omega=1}^{z} \ddot{\emptyset}_s^{\ \omega}\left(-\mathbb{l}\mathrm{n}\left(1-\ddot{\mu}^+_{\mu_{\ddot{\omega}}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right], \\ \left[\mathfrak{E}^{-\left(\sum_{\omega=1}^{z} \ddot{\emptyset}_s^{\ \omega}\left(-\mathbb{l}\mathrm{n}\left(\ddot{\eta}^-_{\ddot{\eta}_\omega}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, \mathfrak{E}^{-\left(\sum_{\omega=1}^{z} \ddot{\emptyset}_s^{\ \omega}\left(-\mathbb{l}\mathrm{n}\left(\ddot{\eta}^+_{\ddot{\eta}_\omega}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right] \end{array}\right) \end{array}\right). \tag{13}$$

Proof. To prove Eq (13), we used mathematical induction. For this, first, we considered the value of $z = 2$, we have

$$\ddot{\emptyset}_s^{\ 1} °\mathrm{F}'_{CuIF_1} = \left(\begin{array}{c} \left(1 - \mathfrak{E}^{-\left(\ddot{\emptyset}_s^{\ 1}\left(-\mathbb{l}\mathrm{n}(1-\ddot{\mu}_{\mu_{\ddot{1}}})\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, \mathfrak{E}^{-\left(\ddot{\emptyset}_s^{\ 1}\left(-\mathbb{l}\mathrm{n}(\ddot{\eta}_{\ddot{\eta}_1})\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right), \\ \left(\begin{array}{c} \left[1 - \mathfrak{E}^{-\left(\ddot{\emptyset}_s^{\ 1}\left(-\mathbb{l}\mathrm{n}\left(1-\ddot{\mu}^-_{\mu_{\ddot{1}}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, 1 - \mathfrak{E}^{-\left(\ddot{\emptyset}_s^{\ 1}\left(-\mathbb{l}\mathrm{n}\left(1-\ddot{\mu}^+_{\mu_{\ddot{1}}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right], \\ \left[\mathfrak{E}^{-\left(\ddot{\emptyset}_s^{\ 1}\left(-\mathbb{l}\mathrm{n}\left(\ddot{\eta}^-_{\ddot{\eta}_1}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, \mathfrak{E}^{-\left(\ddot{\emptyset}_s^{\ 1}\left(-\mathbb{l}\mathrm{n}\left(\ddot{\eta}^+_{\ddot{\eta}_1}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right] \end{array}\right) \end{array}\right),$$

$$\ddot{\emptyset}_s^{\ 2} °\mathrm{F}'_{CuIF_2} = \left(\begin{array}{c} \left(1 - \mathfrak{E}^{-\left(\ddot{\emptyset}_s^{\ 2}\left(-\mathbb{l}\mathrm{n}(1-\ddot{\mu}_{\mu_{\ddot{2}}})\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, \mathfrak{E}^{-\left(\ddot{\emptyset}_s^{\ 2}\left(-\mathbb{l}\mathrm{n}(\ddot{\eta}_{\ddot{\eta}_2})\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right), \\ \left(\begin{array}{c} \left[1 - \mathfrak{E}^{-\left(\ddot{\emptyset}_s^{\ 2}\left(-\mathbb{l}\mathrm{n}\left(1-\ddot{\mu}^-_{\mu_{\ddot{2}}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, 1 - \mathfrak{E}^{-\left(\ddot{\emptyset}_s^{\ 2}\left(-\mathbb{l}\mathrm{n}\left(1-\ddot{\mu}^+_{\mu_{\ddot{2}}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right], \\ \left[\mathfrak{E}^{-\left(\ddot{\emptyset}_s^{\ 2}\left(-\mathbb{l}\mathrm{n}\left(\ddot{\eta}^-_{\ddot{\eta}_2}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, \mathfrak{E}^{-\left(\ddot{\emptyset}_s^{\ 2}\left(-\mathbb{l}\mathrm{n}\left(\ddot{\eta}^+_{\ddot{\eta}_2}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right] \end{array}\right) \end{array}\right),$$

$$CuIFAAWA(°\mathrm{F}'_{CuIF_1}, °\mathrm{F}'_{CuIF_2}) = \ddot{\emptyset}_s^{\ 1} °\mathrm{F}'_{CuIF_1} \oplus \ddot{\emptyset}_s^{\ 2} °\mathrm{F}'_{CuIF_2}$$

$$=\left(\begin{array}{c}\left(1-\mathfrak{E}^{-\left(\ddot{\emptyset}_s{}^{1}\left(-\mathbb{ln}\left(1-\ddot{\mu}_{\ddot{\mu}_1}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}},\mathfrak{E}^{-\left(\ddot{\emptyset}_s{}^{1}\left(-\mathbb{ln}\left(\ddot{\eta}_{\ddot{\eta}_1}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right),\\ \left(\begin{array}{c}\left[1-\mathfrak{E}^{-\left(\ddot{\emptyset}_s{}^{1}\left(-\mathbb{ln}\left(1-\ddot{\mu}^{-}_{\ddot{\mu}_1}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}},1-\mathfrak{E}^{-\left(\ddot{\emptyset}_s{}^{1}\left(-\mathbb{ln}\left(1-\ddot{\mu}^{+}_{\ddot{\mu}_1}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right],\\ \left[\mathfrak{E}^{-\left(\ddot{\emptyset}_s{}^{1}\left(-\mathbb{ln}\left(\ddot{\eta}^{-}_{\ddot{\eta}_1}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}},\mathfrak{E}^{-\left(\ddot{\emptyset}_s{}^{1}\left(-\mathbb{ln}\left(\ddot{\eta}^{+}_{\ddot{\eta}_1}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right]\end{array}\right)\end{array}\right)$$

$$\oplus\left(\begin{array}{c}\left(1-\mathfrak{E}^{-\left(\ddot{\emptyset}_s{}^{2}\left(-\mathbb{ln}\left(1-\ddot{\mu}_{\ddot{\mu}_2}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}},\mathfrak{E}^{-\left(\ddot{\emptyset}_s{}^{2}\left(-\mathbb{ln}\left(\ddot{\eta}_{\ddot{\eta}_2}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right),\\ \left(\begin{array}{c}\left[1-\mathfrak{E}^{-\left(\ddot{\emptyset}_s{}^{2}\left(-\mathbb{ln}\left(1-\ddot{\mu}^{-}_{\ddot{\mu}_2}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}},1-\mathfrak{E}^{-\left(\ddot{\emptyset}_s{}^{2}\left(-\mathbb{ln}\left(1-\ddot{\mu}^{+}_{\ddot{\mu}_2}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right],\\ \left[\mathfrak{E}^{-\left(\ddot{\emptyset}_s{}^{2}\left(-\mathbb{ln}\left(\ddot{\eta}^{-}_{\ddot{\eta}_2}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}},\mathfrak{E}^{-\left(\ddot{\emptyset}_s{}^{2}\left(-\mathbb{ln}\left(\ddot{\eta}^{+}_{\ddot{\eta}_2}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right]\end{array}\right)\end{array}\right)$$

$$=\left(\begin{array}{c}\left(1-\mathfrak{E}^{-\left(\sum_{\omega=1}^{2}\ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{ln}\left(1-\ddot{\mu}_{\ddot{\mu}_\omega}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}},\mathfrak{E}^{-\left(\sum_{\omega=1}^{2}\ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{ln}\left(\ddot{\eta}_{\ddot{\eta}_\omega}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right),\\ \left(\begin{array}{c}\left[1-\mathfrak{E}^{-\left(\sum_{\omega=1}^{2}\ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{ln}\left(1-\ddot{\mu}^{-}_{\ddot{\mu}_\omega}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}},1-\mathfrak{E}^{-\left(\sum_{\omega=1}^{2}\ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{ln}\left(1-\ddot{\mu}^{+}_{\ddot{\mu}_\omega}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right],\\ \left[\mathfrak{E}^{-\left(\sum_{\omega=1}^{2}\ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{ln}\left(\ddot{\eta}^{-}_{\ddot{\eta}_\omega}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}},\mathfrak{E}^{-\left(\sum_{\omega=1}^{2}\ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{ln}\left(\ddot{\eta}^{+}_{\ddot{\eta}_\omega}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right]\end{array}\right)\end{array}\right).$$

Equation (13) is correct for $z=2$.

We consider that it is also correct for $z=y$, thus

$$CuIFAAWA\left({}^{\circ}\mathrm{F}^{/}_{CuIF_1},{}^{\circ}\mathrm{F}^{/}_{CuIF_2},\ldots,{}^{\circ}\mathrm{F}^{/}_{CuIF_y}\right)=$$

$$\left(\begin{array}{c}\left(1-\mathfrak{E}^{-\left(\sum_{\omega=1}^{y}\ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{ln}\left(1-\ddot{\mu}_{\ddot{\mu}_\omega}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}},\mathfrak{E}^{-\left(\sum_{\omega=1}^{y}\ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{ln}\left(\ddot{\eta}_{\ddot{\eta}_\omega}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right),\\ \left(\begin{array}{c}\left[1-\mathfrak{E}^{-\left(\sum_{\omega=1}^{y}\ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{ln}\left(1-\ddot{\mu}^{-}_{\ddot{\mu}_\omega}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}},1-\mathfrak{E}^{-\left(\sum_{\omega=1}^{y}\ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{ln}\left(1-\ddot{\mu}^{+}_{\ddot{\mu}_\omega}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right],\\ \left[\mathfrak{E}^{-\left(\sum_{\omega=1}^{y}\ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{ln}\left(\ddot{\eta}^{-}_{\ddot{\eta}_\omega}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}},\mathfrak{E}^{-\left(\sum_{\omega=1}^{y}\ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{ln}\left(\ddot{\eta}^{+}_{\ddot{\eta}_\omega}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right]\end{array}\right)\end{array}\right).$$

Then, we prove that the Eq (13) is also correct for $z=y+1$, such as

$$CuIFAAWA\left({}^{\circ}F'_{CuIF_1}, {}^{\circ}F'_{CuIF_2}, \dots, {}^{\circ}F'_{CuIF_z}\right)$$

$$= \ddot{\emptyset}_s{}^{1}\,{}^{\circ}F'_{CuIF_1} \oplus \ddot{\emptyset}_s{}^{2}\,{}^{\circ}F'_{CuIF_2} \oplus \dots \oplus \ddot{\emptyset}_s{}^{y}\,{}^{\circ}F'_{CuIF_y} \oplus \ddot{\emptyset}_s{}^{y+1}\,{}^{\circ}F'_{CuIF_{y+1}}$$

$$= \bigoplus_{\omega=1}^{y} \ddot{\emptyset}_s{}^{\omega}\,{}^{\circ}F'_{CuIF_\omega} \oplus \ddot{\emptyset}_s{}^{y+1}\,{}^{\circ}F'_{CuIF_{y+1}}$$

$$= \left(\begin{array}{c} \left(1-\mathfrak{E}^{-\left(\sum_{\omega=1}^{y}\ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{In}\left(1-\ddot{\mu}_{\ddot{\mu}_\omega}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, \mathfrak{E}^{-\left(\sum_{\omega=1}^{y}\ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{In}\left(\ddot{\eta}_{\ddot{\eta}_\omega}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right), \\ \left(\begin{array}{c} \left[1-\mathfrak{E}^{-\left(\sum_{\omega=1}^{y}\ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{In}\left(1-\ddot{\mu}^{-}_{\ddot{\mu}_\omega}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, 1-\mathfrak{E}^{-\left(\sum_{\omega=1}^{y}\ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{In}\left(1-\ddot{\mu}^{+}_{\ddot{\mu}_\omega}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right], \\ \left[\mathfrak{E}^{-\left(\sum_{\omega=1}^{y}\ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{In}\left(\ddot{\eta}^{-}_{\ddot{\eta}_\omega}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, \mathfrak{E}^{-\left(\sum_{\omega=1}^{y}\ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{In}\left(\ddot{\eta}^{+}_{\ddot{\eta}_\omega}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right] \end{array} \right) \end{array} \right) \oplus \ddot{\emptyset}_s{}^{y+1}\,{}^{\circ}F'_{CuIF_{y+1}}$$

$$= \left(\begin{array}{c} \left(1-\mathfrak{E}^{-\left(\sum_{\omega=1}^{y}\ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{In}\left(1-\ddot{\mu}_{\ddot{\mu}_\omega}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, \mathfrak{E}^{-\left(\sum_{\omega=1}^{y}\ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{In}\left(\ddot{\eta}_{\ddot{\eta}_\omega}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right), \\ \left(\begin{array}{c} \left[1-\mathfrak{E}^{-\left(\sum_{\omega=1}^{y}\ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{In}\left(1-\ddot{\mu}^{-}_{\ddot{\mu}_\omega}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, 1-\mathfrak{E}^{-\left(\sum_{\omega=1}^{y}\ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{In}\left(1-\ddot{\mu}^{+}_{\ddot{\mu}_\omega}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right], \\ \left[\mathfrak{E}^{-\left(\sum_{\omega=1}^{y}\ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{In}\left(\ddot{\eta}^{-}_{\ddot{\eta}_\omega}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, \mathfrak{E}^{-\left(\sum_{\omega=1}^{y}\ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{In}\left(\ddot{\eta}^{+}_{\ddot{\eta}_\omega}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right] \end{array} \right) \end{array} \right)$$

$$\oplus \left(\begin{array}{c} \left(1-\mathfrak{E}^{-\left(\ddot{\emptyset}_s{}^{y+1}\left(-\mathbb{In}\left(1-\ddot{\mu}_{\ddot{\mu}_{y+1}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, \mathfrak{E}^{-\left(\ddot{\emptyset}_s{}^{y+1}\left(-\mathbb{In}\left(\ddot{\eta}_{\ddot{\eta}_{y+1}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right), \\ \left(\begin{array}{c} \left[1-\mathfrak{E}^{-\left(\ddot{\emptyset}_s{}^{y+1}\left(-\mathbb{In}\left(1-\ddot{\mu}^{-}_{\ddot{\mu}_{y+1}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, 1-\mathfrak{E}^{-\left(\ddot{\emptyset}_s{}^{y+1}\left(-\mathbb{In}\left(1-\ddot{\mu}^{+}_{\ddot{\mu}_{y+1}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right], \\ \left[\mathfrak{E}^{-\left(\ddot{\emptyset}_s{}^{y+1}\left(-\mathbb{In}\left(\ddot{\eta}^{-}_{\ddot{\eta}_{y+1}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, \mathfrak{E}^{-\left(\ddot{\emptyset}_s{}^{y+1}\left(-\mathbb{In}\left(\ddot{\eta}^{+}_{\ddot{\eta}_{y+1}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right] \end{array} \right) \end{array} \right)$$

$$= \left(\begin{array}{c} \left(1-\mathfrak{E}^{-\left(\sum_{\omega=1}^{y+1}\ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{In}\left(1-\ddot{\mu}_{\ddot{\mu}_\omega}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, \mathfrak{E}^{-\left(\sum_{\omega=1}^{y+1}\ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{In}\left(\ddot{\eta}_{\ddot{\eta}_\omega}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right), \\ \left(\begin{array}{c} \left[1-\mathfrak{E}^{-\left(\sum_{\omega=1}^{y+1}\ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{In}\left(1-\ddot{\mu}^{-}_{\ddot{\mu}_\omega}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, 1-\mathfrak{E}^{-\left(\sum_{\omega=1}^{y+1}\ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{In}\left(1-\ddot{\mu}^{+}_{\ddot{\mu}_\omega}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right], \\ \left[\mathfrak{E}^{-\left(\sum_{\omega=1}^{y+1}\ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{In}\left(\ddot{\eta}^{-}_{\ddot{\eta}_\omega}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, \mathfrak{E}^{-\left(\sum_{\omega=1}^{y+1}\ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{In}\left(\ddot{\eta}^{+}_{\ddot{\eta}_\omega}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right] \end{array} \right) \end{array} \right).$$

Hence, Eq (13) is correct for all positive values of z.

Property 1: If $°F'_{CuIF_\omega} = °F'_{CuIF} = \left((\ddot{\mu}_{\ddot{\mu}}, \ddot{\eta}_{\ddot{\eta}}), ([\ddot{\mu}^-_{\ddot{\mu}}, \ddot{\mu}^+_{\ddot{\mu}}], [\ddot{\eta}^-_{\ddot{\eta}}, \ddot{\eta}^+_{\ddot{\eta}}])\right), \omega = 1,2,\dots,z$, then

$$CuIFAAWA(°F'_{CuIF_1}, °F'_{CuIF_2}, \dots, °F'_{CuIF_z}) = °F'_{CuIF}. \tag{14}$$

Proof. Let $°F'_{CuIF_\omega} = °F'_{CuIF} = \left((\ddot{\mu}_{\ddot{\mu}}, \ddot{\eta}_{\ddot{\eta}}), ([\ddot{\mu}^-_{\ddot{\mu}}, \ddot{\mu}^+_{\ddot{\mu}}], [\ddot{\eta}^-_{\ddot{\eta}}, \ddot{\eta}^+_{\ddot{\eta}}])\right), \omega = 1,2,\dots,z$, thus

$$CuIFAAWA(°F'_{CuIF_1}, °F'_{CuIF_2}, \dots, °F'_{CuIF_z})$$

$$= \begin{pmatrix} \left(1 - \mathfrak{E}^{-\left(\sum_{\omega=1}^{z} \ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{I}\mathrm{n}(1-\ddot{\mu}_{\ddot{\mu}_\omega})\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, \mathfrak{E}^{-\left(\sum_{\omega=1}^{z} \ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{I}\mathrm{n}(\ddot{\eta}_{\ddot{\eta}_\omega})\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right), \\ \left(\begin{array}{c} \left[1 - \mathfrak{E}^{-\left(\sum_{\omega=1}^{z} \ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{I}\mathrm{n}\left(1-\ddot{\mu}^-_{\ddot{\mu}_\omega}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, 1 - \mathfrak{E}^{-\left(\sum_{\omega=1}^{z} \ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{I}\mathrm{n}\left(1-\ddot{\mu}^+_{\ddot{\mu}_\omega}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right], \\ \left[\mathfrak{E}^{-\left(\sum_{\omega=1}^{z} \ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{I}\mathrm{n}\left(\ddot{\eta}^-_{\ddot{\eta}_\omega}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, \mathfrak{E}^{-\left(\sum_{\omega=1}^{z} \ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{I}\mathrm{n}\left(\ddot{\eta}^+_{\ddot{\eta}_\omega}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right] \end{array}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \left(1 - \mathfrak{E}^{-\left(\sum_{\omega=1}^{z} \ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{I}\mathrm{n}(1-\ddot{\mu}_{\ddot{\mu}})\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, \mathfrak{E}^{-\left(\sum_{\omega=1}^{z} \ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{I}\mathrm{n}(\ddot{\eta}_{\ddot{\eta}})\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right), \\ \left(\begin{array}{c} \left[1 - \mathfrak{E}^{-\left(\sum_{\omega=1}^{z} \ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{I}\mathrm{n}(1-\ddot{\mu}^-_{\ddot{\mu}})\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, 1 - \mathfrak{E}^{-\left(\sum_{\omega=1}^{z} \ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{I}\mathrm{n}\left(1-\ddot{\mu}^+_{\ddot{\mu}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right], \\ \left[\mathfrak{E}^{-\left(\sum_{\omega=1}^{z} \ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{I}\mathrm{n}\left(\ddot{\eta}^-_{\ddot{\eta}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, \mathfrak{E}^{-\left(\sum_{\omega=1}^{z} \ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{I}\mathrm{n}\left(\ddot{\eta}^+_{\ddot{\eta}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right] \end{array}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \left(1 - \mathfrak{E}^{-\left(\left(-\mathbb{I}\mathrm{n}(1-\ddot{\mu}_{\ddot{\mu}})\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, \mathfrak{E}^{-\left(\left(-\mathbb{I}\mathrm{n}(\ddot{\eta}_{\ddot{\eta}})\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right), \\ \left(\begin{array}{c} \left[1 - \mathfrak{E}^{-\left(\left(-\mathbb{I}\mathrm{n}(1-\ddot{\mu}^-_{\ddot{\mu}})\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, 1 - \mathfrak{E}^{-\left(\left(-\mathbb{I}\mathrm{n}\left(1-\ddot{\mu}^+_{\ddot{\mu}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right], \\ \left[\mathfrak{E}^{-\left(\left(-\mathbb{I}\mathrm{n}\left(\ddot{\eta}^-_{\ddot{\eta}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, \mathfrak{E}^{-\left(\left(-\mathbb{I}\mathrm{n}\left(\ddot{\eta}^+_{\ddot{\eta}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right] \end{array}\right) \end{pmatrix}, \left(\sum_{\omega=1}^{z} \ddot{\emptyset}_s{}^{\omega} = 1\right)$$

$$= \begin{pmatrix} \left(1 - \mathfrak{E}^{\mathbb{I}\mathrm{n}(1-\ddot{\mu}_{\ddot{\mu}})}, \mathfrak{E}^{\mathbb{I}\mathrm{n}(\ddot{\eta}_{\ddot{\eta}})}\right), \\ \left(\begin{array}{c} \left[1 - \mathfrak{E}^{\mathbb{I}\mathrm{n}(1-\ddot{\mu}^-_{\ddot{\mu}})}, 1 - \mathfrak{E}^{\mathbb{I}\mathrm{n}\left(1-\ddot{\mu}^+_{\ddot{\mu}}\right)}\right], \\ \left[\mathfrak{E}^{\mathbb{I}\mathrm{n}\left(\ddot{\eta}^-_{\ddot{\eta}}\right)}, \mathfrak{E}^{\mathbb{I}\mathrm{n}\left(\ddot{\eta}^+_{\ddot{\eta}}\right)}\right] \end{array}\right) \end{pmatrix}$$

$$= \left((\ddot{\mu}_{\ddot{\mu}}, \ddot{\eta}_{\ddot{\eta}}), ([\ddot{\mu}^-_{\ddot{\mu}}, \ddot{\mu}^+_{\ddot{\mu}}], [\ddot{\eta}^-_{\ddot{\eta}}, \ddot{\eta}^+_{\ddot{\eta}}])\right) = °F'_{CuIF}.$$

Property 2: If $°F'_{CuIF_\omega} \leq °F^{**}_{CuIF_\omega}$, it means that $\ddot{\mu}_{\ddot{\mu}_\omega} \leq \ddot{\mu}^{**}_{\ddot{\mu}_\omega}, \ddot{\eta}_{\ddot{\eta}_\omega} \geq \ddot{\eta}^{**}_{\ddot{\eta}_\omega}$ and $\ddot{\mu}^-_{\ddot{\mu}_\omega} \leq \ddot{\mu}^-_{\ddot{\mu}_\omega}{}^{**}, \ddot{\mu}^+_{\ddot{\mu}_\omega} \leq$

$\ddot{\mu}^{+}_{\ddot{\mu_\omega}}{}^{**}, \dddot{\eta}^{-}_{\ddot{\eta}_\omega} \geq \dddot{\eta}^{-}_{\ddot{\eta}_\omega}{}^{**}, \dddot{\eta}^{+}_{\ddot{\eta}_\omega} \geq \dddot{\eta}^{+}_{\ddot{\eta}_\omega}{}^{**}$, then

$$CuIFAAWA\left({}^{\circ}\mathrm{F}^{/}_{CuIF_1}, {}^{\circ}\mathrm{F}^{/}_{CuIF_2}, \ldots, {}^{\circ}\mathrm{F}^{/}_{CuIF_z}\right) \leq CuIFAAWA\left({}^{\circ}\mathrm{F}^{**}_{CuIF_1}, {}^{\circ}\mathrm{F}^{**}_{CuIF_2}, \ldots, {}^{\circ}\mathrm{F}^{**}_{CuIF_z}\right). \tag{15}$$

Proof. Consider that ${}^{\circ}\mathrm{F}^{/}_{CuIF_\omega} \leq {}^{\circ}\mathrm{F}^{**}_{CuIF_\omega}$, which means that $\ddot{\mu}_{\ddot{\mu_\omega}} \leq \ddot{\mu}^{**}_{\ddot{\mu_\omega}}, \dddot{\eta}_{\ddot{\eta}_\omega} \geq \dddot{\eta}^{**}_{\ddot{\eta}_\omega}$ and $\ddot{\mu}^{-}_{\ddot{\mu_\omega}} \leq \ddot{\mu}^{-}_{\ddot{\mu_\omega}}{}^{**}, \ddot{\mu}^{+}_{\ddot{\mu_\omega}} \leq \ddot{\mu}^{+}_{\ddot{\mu_\omega}}{}^{**}, \dddot{\eta}^{-}_{\ddot{\eta}_\omega} \geq \dddot{\eta}^{-}_{\ddot{\eta}_\omega}{}^{**}, \dddot{\eta}^{+}_{\ddot{\eta}_\omega} \geq \dddot{\eta}^{+}_{\ddot{\eta}_\omega}{}^{**}$, thus

$$\ddot{\mu}_{\ddot{\mu_\omega}} \leq \ddot{\mu}^{**}_{\ddot{\mu_\omega}} \Rightarrow 1-\ddot{\mu}_{\ddot{\mu_\omega}} \geq 1-\ddot{\mu}^{**}_{\ddot{\mu_\omega}} \Rightarrow \mathbb{I}\mathrm{n}\left(1-\ddot{\mu}_{\ddot{\mu_\omega}}\right) \geq \mathbb{I}\mathrm{n}\left(1-\ddot{\mu}^{**}_{\ddot{\mu_\omega}}\right)$$

$$\Rightarrow -\mathbb{I}\mathrm{n}\left(1-\ddot{\mu}_{\ddot{\mu_\omega}}\right) \leq -\mathbb{I}\mathrm{n}\left(1-\ddot{\mu}^{**}_{\ddot{\mu_\omega}}\right)$$

$$\Rightarrow \left(\sum_{\omega=1}^{z} \ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{I}\mathrm{n}\left(1-\ddot{\mu}_{\ddot{\mu_\omega}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}} \leq \left(\sum_{\omega=1}^{z} \ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{I}\mathrm{n}\left(1-\ddot{\mu}^{**}_{\ddot{\mu_\omega}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}$$

$$\Rightarrow -\left(\sum_{\omega=1}^{z} \ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{I}\mathrm{n}\left(1-\ddot{\mu}_{\ddot{\mu_\omega}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}} \geq -\left(\sum_{\omega=1}^{z} \ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{I}\mathrm{n}\left(1-\ddot{\mu}^{**}_{\ddot{\mu_\omega}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}$$

$$\Rightarrow \mathfrak{E}^{-\left(\sum_{\omega=1}^{z} \ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{I}\mathrm{n}\left(1-\ddot{\mu}_{\ddot{\mu_\omega}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}} \geq \mathfrak{E}^{-\left(\sum_{\omega=1}^{z} \ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{I}\mathrm{n}\left(1-\ddot{\mu}^{**}_{\ddot{\mu_\omega}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}$$

$$\Rightarrow \mathfrak{E}^{-\left(\sum_{\omega=1}^{z} \ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{I}\mathrm{n}\left(1-\ddot{\mu}_{\ddot{\mu_\omega}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}} \leq \mathfrak{E}^{-\left(\sum_{\omega=1}^{z} \ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{I}\mathrm{n}\left(1-\ddot{\mu}^{**}_{\ddot{\mu_\omega}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}$$

$$\Rightarrow 1-\mathfrak{E}^{-\left(\sum_{\omega=1}^{z} \ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{I}\mathrm{n}\left(1-\ddot{\mu}_{\ddot{\mu_\omega}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}} \leq 1-\mathfrak{E}^{-\left(\sum_{\omega=1}^{z} \ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{I}\mathrm{n}\left(1-\ddot{\mu}^{**}_{\ddot{\mu_\omega}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}.$$

Similarly, for the lower and upper parts of the truth grade, we have

$$\Rightarrow 1-\mathfrak{E}^{-\left(\sum_{\omega=1}^{z} \ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{I}\mathrm{n}\left(1-\ddot{\mu}^{-}_{\ddot{\mu_\omega}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}} \leq 1-\mathfrak{E}^{-\left(\sum_{\omega=1}^{z} \ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{I}\mathrm{n}\left(1-\ddot{\mu}^{-}_{\ddot{\mu_\omega}}{}^{**}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}$$

$$\Rightarrow 1-\mathfrak{E}^{-\left(\sum_{\omega=1}^{z} \ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{I}\mathrm{n}\left(1-\ddot{\mu}^{+}_{\ddot{\mu_\omega}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}} \leq 1-\mathfrak{E}^{-\left(\sum_{\omega=1}^{z} \ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{I}\mathrm{n}\left(1-\ddot{\mu}^{+}_{\ddot{\mu_\omega}}{}^{**}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}.$$

Further, for the falsity of information, we have

$$\dddot{\eta}_{\ddot{\eta}_\omega} \geq \dddot{\eta}^{**}_{\ddot{\eta}_\omega} \Rightarrow \mathbb{I}\mathrm{n}\left(\dddot{\eta}_{\ddot{\eta}_\omega}\right) \geq \mathbb{I}\mathrm{n}\left(\dddot{\eta}^{**}_{\ddot{\eta}_\omega}\right)$$

$$\Rightarrow -\mathbb{I}\mathrm{n}\left(\dddot{\eta}_{\ddot{\eta}_\omega}\right) \leq -\mathbb{I}\mathrm{n}\left(\dddot{\eta}^{**}_{\ddot{\eta}_\omega}\right) \Rightarrow \sum_{\omega=1}^{z} \ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{I}\mathrm{n}\left(\dddot{\eta}_{\ddot{\eta}_\omega}\right)\right)^{\hbar\hbar} \leq \sum_{\omega=1}^{z} \ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{I}\mathrm{n}\left(\dddot{\eta}^{**}_{\ddot{\eta}_\omega}\right)\right)^{\hbar\hbar}$$

$$\Rightarrow -\left(\sum_{\omega=1}^{z} \ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{I}\mathrm{n}\left(\dddot{\eta}_{\ddot{\eta}_\omega}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}} \geq -\left(\sum_{\omega=1}^{z} \ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{I}\mathrm{n}\left(\dddot{\eta}^{**}_{\ddot{\eta}_\omega}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}$$

$$\Rightarrow \mathfrak{E}^{-\left(\sum_{\omega=1}^{z} \ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{I}\mathrm{n}\left(\dddot{\eta}_{\ddot{\eta}_\omega}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}} \geq \mathfrak{E}^{-\left(\sum_{\omega=1}^{z} \ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{I}\mathrm{n}\left(\dddot{\eta}^{**}_{\ddot{\eta}_\omega}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}.$$

Similarly, for the lower and upper parts of the falsity grade, we have

$$\Rightarrow \mathfrak{E}^{-\left(\sum_{\omega=1}^{z} \ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{I}\mathrm{n}\left(\dddot{\eta}^{-}_{\ddot{\eta}_\omega}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}} \geq \mathfrak{E}^{-\left(\sum_{\omega=1}^{z} \ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{I}\mathrm{n}\left(\dddot{\eta}^{-}_{\ddot{\eta}_\omega}{}^{**}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}$$

$$\Rightarrow \mathfrak{E}^{-\left(\sum_{\omega=1}^{z} \ddot{\phi}_s^{\ \omega}\left(-\mathbb{I}n\left(\ddot{\eta}_{\ddot{\eta}_\omega}^{+}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}} \geq \mathfrak{E}^{-\left(\sum_{\omega=1}^{z} \ddot{\phi}_s^{\ \omega}\left(-\mathbb{I}n\left(\ddot{\eta}_{\ddot{\eta}_\omega}^{+\ **}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}.$$

Finally, by the score function and accuracy function, we can easily get the following results, such as

$$CuIFAAWA\left({}^\circ\mathrm{F}'_{CuIF_1}, {}^\circ\mathrm{F}'_{CuIF_2}, \dots, {}^\circ\mathrm{F}'_{CuIF_z}\right) \leq CuIFAAWA\left({}^\circ\mathrm{F}^{**}_{CuIF_1}, {}^\circ\mathrm{F}^{**}_{CuIF_2}, \dots, {}^\circ\mathrm{F}^{**}_{CuIF_z}\right).$$

Property 3: If ${}^\circ\mathrm{F}^{-}_{CuIF_\omega} = \left(\left(\min_\omega \ddot{\mu}_{\ddot{\mu}_\omega}, \max_\omega \ddot{\eta}_{\ddot{\eta}_\omega}\right), \left(\left[\min_\omega \ddot{\mu}^{-}_{\ddot{\mu}_\omega}, \min_\omega \ddot{\mu}^{+}_{\ddot{\mu}_\omega}\right], \left[\max_\omega \ddot{\eta}^{-}_{\ddot{\eta}_\omega}, \max_\omega \ddot{\eta}^{+}_{\ddot{\eta}_\omega}\right]\right)\right)$ and

${}^\circ\mathrm{F}^{+}_{CuIF_\omega} = \left(\left(\max_\omega \ddot{\mu}_{\ddot{\mu}_\omega}, \min_\omega \ddot{\eta}_{\ddot{\eta}_\omega}\right), \left(\left[\max_\omega \ddot{\mu}^{-}_{\ddot{\mu}_\omega}, \max_\omega \ddot{\mu}^{+}_{\ddot{\mu}_\omega}\right], \left[\min_\omega \ddot{\eta}^{-}_{\ddot{\eta}_\omega}, \min_\omega \ddot{\eta}^{+}_{\ddot{\eta}_\omega}\right]\right)\right)$, then

$$ {}^\circ\mathrm{F}^{-}_{CuIF_\omega} \leq CuIFAAWA\left({}^\circ\mathrm{F}'_{CuIF_1}, {}^\circ\mathrm{F}'_{CuIF_2}, \dots, {}^\circ\mathrm{F}'_{CuIF_z}\right) \leq {}^\circ\mathrm{F}^{+}_{CuIF_\omega}. \tag{16}$$

Proof. Considering Property 1 and Property 2, we have

$$CuIFAAWA\left({}^\circ\mathrm{F}'_{CuIF_1}, {}^\circ\mathrm{F}'_{CuIF_2}, \dots, {}^\circ\mathrm{F}'_{CuIF_z}\right) \leq CuIFAAWA\left({}^\circ\mathrm{F}^{+}_{CuIF_1}, {}^\circ\mathrm{F}^{+}_{CuIF_2}, \dots, {}^\circ\mathrm{F}^{+}_{CuIF_z}\right) = {}^\circ\mathrm{F}^{+}_{CuIF_\omega},$$

$$CuIFAAWA\left({}^\circ\mathrm{F}'_{CuIF_1}, {}^\circ\mathrm{F}'_{CuIF_2}, \dots, {}^\circ\mathrm{F}'_{CuIF_z}\right) \geq CuIFAAWA\left({}^\circ\mathrm{F}^{-}_{CuIF_1}, {}^\circ\mathrm{F}^{-}_{CuIF_2}, \dots, {}^\circ\mathrm{F}^{-}_{CuIF_z}\right) = {}^\circ\mathrm{F}^{-}_{CuIF_\omega}.$$

Thus, we have

$$ {}^\circ\mathrm{F}^{-}_{CuIF_\omega} \leq CuIFAAWA\left({}^\circ\mathrm{F}'_{CuIF_1}, {}^\circ\mathrm{F}'_{CuIF_2}, \dots, {}^\circ\mathrm{F}'_{CuIF_z}\right) \leq {}^\circ\mathrm{F}^{+}_{CuIF_\omega}.$$

Definition 7: For the finite collection of CIFNs ${}^\circ\mathrm{F}'_{CuIF_\omega} = \left(\left(\ddot{\mu}_{\ddot{\mu}_\omega}, \ddot{\eta}_{\ddot{\eta}_\omega}\right), \left(\left[\ddot{\mu}^{-}_{\ddot{\mu}_\omega}, \ddot{\mu}^{+}_{\ddot{\mu}_\omega}\right], \left[\ddot{\eta}^{-}_{\ddot{\eta}_\omega}, \ddot{\eta}^{+}_{\ddot{\eta}_\omega}\right]\right)\right), \omega = 1,2,\dots,z$, then the CIFAAOWA operator is defined as

$$CuIFAAOWA: {}^\circ\mathrm{F}^{z} \to {}^\circ\mathrm{F}, by$$

$$CuIFAAOWA\left({}^\circ\mathrm{F}'_{CuIF_1}, {}^\circ\mathrm{F}'_{CuIF_2}, \dots, {}^\circ\mathrm{F}'_{CuIF_z}\right) = \ddot{\phi}_s^{\ 1}\,{}^\circ\mathrm{F}'_{CuIF_{0(1)}} \oplus \ddot{\phi}_s^{\ 2}\,{}^\circ\mathrm{F}'_{CuIF_{0(2)}} \oplus \dots \oplus \ddot{\phi}_s^{\ z}\,{}^\circ\mathrm{F}'_{CuIF_{o(z)}} = \oplus_{\omega=1}^{z} \ddot{\phi}_s^{\ \omega}\,{}^\circ\mathrm{F}'_{CuIF_{0(\omega)}}. \tag{17}$$

Note that the weighted vector is stated by: $\ddot{\phi}_s^{\ \omega} \in [0,1]$ with $\sum_{\omega=1}^{z} \ddot{\phi}_s^{\ \omega} = 1$ with $0(\omega) \leq 0(\omega - 1)$, where we can get the order of the CIFNs by the score function.

Theorem 3: For any finite collection of CIFNs ${}^\circ\mathrm{F}'_{CuIF_\omega} = \left(\left(\ddot{\mu}_{\ddot{\mu}_\omega}, \ddot{\eta}_{\ddot{\eta}_\omega}\right), \left(\left[\ddot{\mu}^{-}_{\ddot{\mu}_\omega}, \ddot{\mu}^{+}_{\ddot{\mu}_\omega}\right], \left[\ddot{\eta}^{-}_{\ddot{\eta}_\omega}, \ddot{\eta}^{+}_{\ddot{\eta}_\omega}\right]\right)\right), \omega = 1,2,\dots,z$, we proved that Eq (17) is also a CIFN, such as

$$CuIFAAOWA\left({}^{\circ}\mathrm{F}^{/}_{CuIF_1}, {}^{\circ}\mathrm{F}^{/}_{CuIF_2}, \dots, {}^{\circ}\mathrm{F}^{/}_{CuIF_z}\right) = \left(\begin{array}{c} \left(1-\mathfrak{E}^{-\left(\sum_{\omega=1}^{z}\ddot{\phi}_s{}^{\omega}\left(-\mathbb{In}\left(1-\ddot{\mu}_{\mu_{0(\omega)}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, \mathfrak{E}^{-\left(\sum_{\omega=1}^{z}\ddot{\phi}_s{}^{\omega}\left(-\mathbb{In}\left(\ddot{\eta}_{\ddot{\eta}_{0(\omega)}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right), \\ \left(\begin{array}{c} \left[1-\mathfrak{E}^{-\left(\sum_{\omega=1}^{z}\ddot{\phi}_s{}^{\omega}\left(-\mathbb{In}\left(1-\ddot{\mu}^{-}_{\mu_{0(\omega)}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, 1-\mathfrak{E}^{-\left(\sum_{\omega=1}^{z}\ddot{\phi}_s{}^{\omega}\left(-\mathbb{In}\left(1-\ddot{\mu}^{+}_{\mu_{0(\omega)}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right], \\ \left[\mathfrak{E}^{-\left(\sum_{\omega=1}^{z}\ddot{\phi}_s{}^{\omega}\left(-\mathbb{In}\left(\ddot{\eta}^{-}_{\ddot{\eta}_{0(\omega)}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, \mathfrak{E}^{-\left(\sum_{\omega=1}^{z}\ddot{\phi}_s{}^{\omega}\left(-\mathbb{In}\left(\ddot{\eta}^{+}_{\ddot{\eta}_{0(\omega)}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right] \end{array}\right) \end{array}\right). \tag{18}$$

Proof. Straightforward.

Property 4: If ${}^{\circ}\mathrm{F}^{/}_{CuIF_\omega} = {}^{\circ}\mathrm{F}^{/}_{CuIF} = \left(\left(\ddot{\mu}_{\ddot{\mu}}, \ddot{\eta}_{\ddot{\eta}}\right), \left(\left[\ddot{\mu}^{-}_{\ddot{\mu}}, \ddot{\mu}^{+}_{\ddot{\mu}}\right], \left[\ddot{\eta}^{-}_{\ddot{\eta}}, \ddot{\eta}^{+}_{\ddot{\eta}}\right]\right)\right), \omega = 1,2,\dots,z$, then

$$CuIFAAOWA\left({}^{\circ}\mathrm{F}^{/}_{CuIF_1}, {}^{\circ}\mathrm{F}^{/}_{CuIF_2}, \dots, {}^{\circ}\mathrm{F}^{/}_{CuIF_z}\right) = {}^{\circ}\mathrm{F}^{/}_{CuIF}. \tag{19}$$

Proof. Straightforward.

Property 5: If ${}^{\circ}\mathrm{F}^{/}_{CuIF_\omega} \le {}^{\circ}\mathrm{F}^{**}_{CuIF_\omega}$, it means that $\ddot{\mu}_{\ddot{\mu}_\omega} \le \ddot{\mu}^{**}_{\ddot{\mu}_\omega}, \ddot{\eta}_{\ddot{\eta}_\omega} \ge \ddot{\eta}^{**}_{\ddot{\eta}_\omega}$ and $\ddot{\mu}^{-}_{\ddot{\mu}_\omega} \le \ddot{\mu}^{-}_{\ddot{\mu}_\omega}{}^{**}, \ddot{\mu}^{+}_{\ddot{\mu}_\omega} \le \ddot{\mu}^{+}_{\ddot{\mu}_\omega}{}^{**}, \ddot{\eta}^{-}_{\ddot{\eta}_\omega} \ge \ddot{\eta}^{-}_{\ddot{\eta}_\omega}{}^{**}, \ddot{\eta}^{+}_{\ddot{\eta}_\omega} \ge \ddot{\eta}^{+}_{\ddot{\eta}_\omega}{}^{**}$, then

$$CuIFAAOWA\left({}^{\circ}\mathrm{F}^{/}_{CuIF_1}, {}^{\circ}\mathrm{F}^{/}_{CuIF_2}, \dots, {}^{\circ}\mathrm{F}^{/}_{CuIF_z}\right) \le CuIFAAOWA\left({}^{\circ}\mathrm{F}^{**}_{CuIF_1}, {}^{\circ}\mathrm{F}^{**}_{CuIF_2}, \dots, {}^{\circ}\mathrm{F}^{**}_{CuIF_z}\right). \tag{20}$$

Proof. Straightforward.

Property 6: If ${}^{\circ}\mathrm{F}^{-}_{CuIF_\omega} = \left(\left(\min_\omega \ddot{\mu}_{\ddot{\mu}_\omega}, \max_\omega \ddot{\eta}_{\ddot{\eta}_\omega}\right), \left(\left[\min_\omega \ddot{\mu}^{-}_{\ddot{\mu}_\omega}, \min_\omega \ddot{\mu}^{+}_{\ddot{\mu}_\omega}\right], \left[\max_\omega \ddot{\eta}^{-}_{\ddot{\eta}_\omega}, \max_\omega \ddot{\eta}^{+}_{\ddot{\eta}_\omega}\right]\right)\right)$ and ${}^{\circ}\mathrm{F}^{+}_{CuIF_\omega} = \left(\left(\max_\omega \ddot{\mu}_{\ddot{\mu}_\omega}, \min_\omega \ddot{\eta}_{\ddot{\eta}_\omega}\right), \left(\left[\max_\omega \ddot{\mu}^{-}_{\ddot{\mu}_\omega}, \max_\omega \ddot{\mu}^{+}_{\ddot{\mu}_\omega}\right], \left[\min_\omega \ddot{\eta}^{-}_{\ddot{\eta}_\omega}, \min_\omega \ddot{\eta}^{+}_{\ddot{\eta}_\omega}\right]\right)\right)$, then

$${}^{\circ}\mathrm{F}^{-}_{CuIF_\omega} \le CuIFAAOWA\left({}^{\circ}\mathrm{F}^{/}_{CuIF_1}, {}^{\circ}\mathrm{F}^{/}_{CuIF_2}, \dots, {}^{\circ}\mathrm{F}^{/}_{CuIF_z}\right) \le {}^{\circ}\mathrm{F}^{+}_{CuIF_\omega}. \tag{21}$$

Proof. Straightforward.

Definition 8: For the finite collection of CIFNs ${}^{\circ}\mathrm{F}^{/}_{CuIF_\omega} = \left(\left(\ddot{\mu}_{\ddot{\mu}_\omega}, \ddot{\eta}_{\ddot{\eta}_\omega}\right), \left(\left[\ddot{\mu}^{-}_{\ddot{\mu}_\omega}, \ddot{\mu}^{+}_{\ddot{\mu}_\omega}\right], \left[\ddot{\eta}^{-}_{\ddot{\eta}_\omega}, \ddot{\eta}^{+}_{\ddot{\eta}_\omega}\right]\right)\right), \omega = 1,2,\dots,z$, then the CIFAAHA operator is defined as

$$CuIFAAHA: {}^{\circ}\mathrm{F}^{z} \to {}^{\circ}\mathrm{F}, by$$

$$CuIFAAHA\left({}^{\circ}\mathrm{F}^{/}_{CuIF_1}, {}^{\circ}\mathrm{F}^{/}_{CuIF_2}, \dots, {}^{\circ}\mathrm{F}^{/}_{CuIF_z}\right) = \ddot{\phi}_s{}^{1}\,{}^{\circ}\mathrm{F}^{*}_{CuIF_{0(1)}} \oplus \ddot{\phi}_s{}^{2}\,{}^{\circ}\mathrm{F}^{*}_{CuIF_{0(2)}} \oplus \dots \oplus \ddot{\phi}_s{}^{z}\,{}^{\circ}\mathrm{F}^{*}_{CuIF_{o(z)}} = \oplus_{\omega=1}^{z}\ddot{\phi}_s{}^{\omega}\,{}^{\circ}\mathrm{F}^{*}_{CuIF_{0(\omega)}}. \tag{22}$$

Note that the weighted vector is stated by: $\ddot{\emptyset}_s^{\ \omega} \in [0,1]$ with $\sum_{\omega=1}^{z} \ddot{\emptyset}_s^{\ \omega} = 1$ with $0(\omega) \leq 0(\omega-1)$, where we can get the order of the CIFN by the score function and $°\mathrm{F}^{*}_{CuIF_{0(\omega)}} = z\ddot{\emptyset}_w^{\ \omega}\, °\mathrm{F}^{/}_{CuIF_\omega}, \omega = 1,2,\dots,z$ with another weight vector $\ddot{\emptyset}_w^{\ \omega} \in [0,1]$ with $\sum_{\omega=1}^{z} \ddot{\emptyset}_w^{\ \omega} = 1$.

Theorem 4: For any finite collection of CIFNs $°\mathrm{F}^{/}_{CuIF_\omega} = \left((\ddot{\mu}_{\ddot{\mu}_\omega}, \ddot{\eta}_{\ddot{\eta}_\omega}), ([\ddot{\mu}^{-}_{\ddot{\mu}_\omega}, \ddot{\mu}^{+}_{\ddot{\mu}_\omega}], [\ddot{\eta}^{-}_{\ddot{\eta}_\omega}, \ddot{\eta}^{+}_{\ddot{\eta}_\omega}])\right), \omega = 1,2,\dots,z$, we proved that Eq (22) is also a CIFN, such as

$$CuIFAAHA\left(°\mathrm{F}^{/}_{CuIF_1}, °\mathrm{F}^{/}_{CuIF_2}, \dots, °\mathrm{F}^{/}_{CuIF_z}\right) = \left(\begin{array}{c}\left(1-\mathfrak{E}^{-\left(\sum_{\omega=1}^{z}\ddot{\emptyset}_s^{\ \omega}\left(-\mathbb{ln}\left(1-\ddot{\mu}^{*}_{\mu_{0(\omega)}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, \mathfrak{E}^{-\left(\sum_{\omega=1}^{z}\ddot{\emptyset}_s^{\ \omega}\left(-\mathbb{ln}\left(\ddot{\eta}^{*}_{\ddot{\eta}_{0(\omega)}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right), \\ \left(\begin{array}{c}\left[1-\mathfrak{E}^{-\left(\sum_{\omega=1}^{z}\ddot{\emptyset}_s^{\ \omega}\left(-\mathbb{ln}\left(1-\ddot{\mu}^{*}_{\mu_{0(\omega)}}{}^{-}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, 1-\mathfrak{E}^{-\left(\sum_{\omega=1}^{z}\ddot{\emptyset}_s^{\ \omega}\left(-\mathbb{ln}\left(1-\ddot{\mu}^{*}_{\mu_{0(\omega)}}{}^{+}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right], \\ \left[\mathfrak{E}^{-\left(\sum_{\omega=1}^{z}\ddot{\emptyset}_s^{\ \omega}\left(-\mathbb{ln}\left(\ddot{\eta}^{*}_{\ddot{\eta}_{0(\omega)}}{}^{-}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, \mathfrak{E}^{-\left(\sum_{\omega=1}^{z}\ddot{\emptyset}_s^{\ \omega}\left(-\mathbb{ln}\left(\ddot{\eta}^{*}_{\ddot{\eta}_{0(\omega)}}{}^{+}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right]\end{array}\right)\end{array}\right). \quad (23)$$

Proof. Straightforward.

Property 7: If $°\mathrm{F}^{/}_{CuIF_\omega} = °\mathrm{F}^{/}_{CuIF} = \left((\ddot{\mu}_{\ddot{\mu}}, \ddot{\eta}_{\ddot{\eta}}), ([\ddot{\mu}^{-}_{\ddot{\mu}}, \ddot{\mu}^{+}_{\ddot{\mu}}], [\ddot{\eta}^{-}_{\ddot{\eta}}, \ddot{\eta}^{+}_{\ddot{\eta}}])\right), \omega = 1,2,\dots,z$, then

$$CuIFAAHA\left(°\mathrm{F}^{/}_{CuIF_1}, °\mathrm{F}^{/}_{CuIF_2}, \dots, °\mathrm{F}^{/}_{CuIF_z}\right) = °\mathrm{F}^{/}_{CuIF}. \quad (24)$$

Proof. Straightforward.

Property 8: If $°\mathrm{F}^{/}_{CuIF_\omega} \leq °\mathrm{F}^{**}_{CuIF_\omega}$, it means that $\ddot{\mu}_{\ddot{\mu}_\omega} \leq \ddot{\mu}^{**}_{\ddot{\mu}_\omega}, \ddot{\eta}_{\ddot{\eta}_\omega} \geq \ddot{\eta}^{**}_{\ddot{\eta}_\omega}$ and $\ddot{\mu}^{-}_{\ddot{\mu}_\omega} \leq \ddot{\mu}^{-}_{\ddot{\mu}_\omega}{}^{**}, \ddot{\mu}^{+}_{\ddot{\mu}_\omega} \leq \ddot{\mu}^{+}_{\ddot{\mu}_\omega}{}^{**}, \ddot{\eta}^{-}_{\ddot{\eta}_\omega} \geq \ddot{\eta}^{-}_{\ddot{\eta}_\omega}{}^{**}, \ddot{\eta}^{+}_{\ddot{\eta}_\omega} \geq \ddot{\eta}^{+}_{\ddot{\eta}_\omega}{}^{**}$, then

$$CuIFAAHA\left(°\mathrm{F}^{/}_{CuIF_1}, °\mathrm{F}^{/}_{CuIF_2}, \dots, °\mathrm{F}^{/}_{CuIF_z}\right) \leq CuIFAAHA\left(°\mathrm{F}^{**}_{CuIF_1}, °\mathrm{F}^{**}_{CuIF_2}, \dots, °\mathrm{F}^{**}_{CuIF_z}\right). \quad (25)$$

Proof. Straightforward.

Property 9: If $°\mathrm{F}^{-}_{CuIF_\omega} = \left(\left(\min_\omega \ddot{\mu}_{\ddot{\mu}_\omega}, \max_\omega \ddot{\eta}_{\ddot{\eta}_\omega}\right), \left(\left[\min_\omega \ddot{\mu}^{-}_{\ddot{\mu}_\omega}, \min_\omega \ddot{\mu}^{+}_{\ddot{\mu}_\omega}\right], \left[\max_\omega \ddot{\eta}^{-}_{\ddot{\eta}_\omega}, \max_\omega \ddot{\eta}^{+}_{\ddot{\eta}_\omega}\right]\right)\right)$ and $°\mathrm{F}^{+}_{CuIF_\omega} = \left(\left(\max_\omega \ddot{\mu}_{\ddot{\mu}_\omega}, \min_\omega \ddot{\eta}_{\ddot{\eta}_\omega}\right), \left(\left[\max_\omega \ddot{\mu}^{-}_{\ddot{\mu}_\omega}, \max_\omega \ddot{\mu}^{+}_{\ddot{\mu}_\omega}\right], \left[\min_\omega \ddot{\eta}^{-}_{\ddot{\eta}_\omega}, \min_\omega \ddot{\eta}^{+}_{\ddot{\eta}_\omega}\right]\right)\right)$, then

$$°\mathrm{F}^{-}_{CuIF_\omega} \leq CuIFAAHA\left(°\mathrm{F}^{/}_{CuIF_1}, °\mathrm{F}^{/}_{CuIF_2}, \dots, °\mathrm{F}^{/}_{CuIF_z}\right) \leq °\mathrm{F}^{+}_{CuIF_\omega}. \quad (26)$$

Proof. Straightforward.

Definition 9: For the finite collection of CIFNs $°\text{F}^{/}_{CuIF_\omega} = \left(\left(\ddot{\mu}_{\ddot{\mu}_\omega}, \ddot{\eta}_{\ddot{\eta}_\omega}\right), \left(\left[\ddot{\mu}^{-}_{\ddot{\mu}_\omega}, \ddot{\mu}^{+}_{\ddot{\mu}_\omega}\right], \left[\ddot{\eta}^{-}_{\ddot{\eta}_\omega}, \ddot{\eta}^{+}_{\ddot{\eta}_\omega}\right]\right)\right), \omega = 1,2,\dots,z$, then the CIFAAWG operator is defined as:

$$CuIFAAWG: °\text{F}^{z} \rightarrow °\text{F}, by$$

$$CuIFAAWG\left(°\text{F}^{/}_{CuIF_1}, °\text{F}^{/}_{CuIF_2}, \dots, °\text{F}^{/}_{CuIF_z}\right) = \left(°\text{F}^{/}_{CuIF_1}\right)^{\ddot{\emptyset}_s^{\ 1}} \otimes \left(°\text{F}^{/}_{CuIF_2}\right)^{\ddot{\emptyset}_s^{\ 2}} \otimes \dots \otimes \left(°\text{F}^{/}_{CuIF_z}\right)^{\ddot{\emptyset}_s^{\ z}} = \otimes_{\omega=1}^{z} \left(°\text{F}^{/}_{CuIF_\omega}\right)^{\ddot{\emptyset}_s^{\ \omega}}. \quad (27)$$

Note that the weighted vector is stated by: $\ddot{\emptyset}_s^{\ \omega} \in [0,1]$ with $\sum_{\omega=1}^{z} \ddot{\emptyset}_s^{\ \omega} = 1$.

Theorem 5: For any finite collection of CIFNs $°\text{F}^{/}_{CuIF_\omega} = \left(\left(\ddot{\mu}_{\ddot{\mu}_\omega}, \ddot{\eta}_{\ddot{\eta}_\omega}\right), \left(\left[\ddot{\mu}^{-}_{\ddot{\mu}_\omega}, \ddot{\mu}^{+}_{\ddot{\mu}_\omega}\right], \left[\ddot{\eta}^{-}_{\ddot{\eta}_\omega}, \ddot{\eta}^{+}_{\ddot{\eta}_\omega}\right]\right)\right), \omega = 1,2,\dots,z$, we proved that the Eq (27) is also a CIFN, such as

$$CuIFAAWG\left(°\text{F}^{/}_{CuIF_1}, °\text{F}^{/}_{CuIF_2}, \dots, °\text{F}^{/}_{CuIF_z}\right) = \left(\begin{array}{c} \left(\mathfrak{E}^{-\left(\sum_{\omega=1}^{z} \ddot{\emptyset}_s^{\ \omega}\left(-\mathbb{I}\mathrm{n}\left(\ddot{\mu}_{\ddot{\mu}_\omega}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, 1-\mathfrak{E}^{-\left(\sum_{\omega=1}^{z} \ddot{\emptyset}_s^{\ \omega}\left(-\mathbb{I}\mathrm{n}\left(1-\ddot{\eta}_{\ddot{\eta}_\omega}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right), \\ \left(\begin{array}{c} \left[\mathfrak{E}^{-\left(\sum_{\omega=1}^{z} \ddot{\emptyset}_s^{\ \omega}\left(-\mathbb{I}\mathrm{n}\left(\ddot{\mu}^{-}_{\ddot{\mu}_\omega}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, \mathfrak{E}^{-\left(\sum_{\omega=1}^{z} \ddot{\emptyset}_s^{\ \omega}\left(-\mathbb{I}\mathrm{n}\left(\ddot{\mu}^{+}_{\ddot{\mu}_\omega}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right], \\ \left[1-\mathfrak{E}^{-\left(\sum_{\omega=1}^{z} \ddot{\emptyset}_s^{\ \omega}\left(-\mathbb{I}\mathrm{n}\left(1-\ddot{\eta}^{-}_{\ddot{\eta}_\omega}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, 1-\mathfrak{E}^{-\left(\sum_{\omega=1}^{z} \ddot{\emptyset}_s^{\ \omega}\left(-\mathbb{I}\mathrm{n}\left(1-\ddot{\eta}^{+}_{\ddot{\eta}_\omega}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right] \end{array}\right) \end{array}\right). \quad (28)$$

Proof. Straightforward.

Property 10: If $°\text{F}^{/}_{CuIF_\omega} = °\text{F}^{/}_{CuIF} = \left(\left(\ddot{\mu}_{\ddot{\mu}}, \ddot{\eta}_{\ddot{\eta}}\right), \left(\left[\ddot{\mu}^{-}_{\ddot{\mu}}, \ddot{\mu}^{+}_{\ddot{\mu}}\right], \left[\ddot{\eta}^{-}_{\ddot{\eta}}, \ddot{\eta}^{+}_{\ddot{\eta}}\right]\right)\right), \omega = 1,2,\dots,z$, then

$$CuIFAAWG\left(°\text{F}^{/}_{CuIF_1}, °\text{F}^{/}_{CuIF_2}, \dots, °\text{F}^{/}_{CuIF_z}\right) = °\text{F}^{/}_{CuIF}. \quad (29)$$

Proof. Straightforward.

Property 11: If $°\text{F}^{/}_{CuIF_\omega} \leq °\text{F}^{**}_{CuIF_\omega}$, it means that $\ddot{\mu}_{\ddot{\mu}_\omega} \leq \ddot{\mu}^{**}_{\ddot{\mu}_\omega}, \ddot{\eta}_{\ddot{\eta}_\omega} \geq \ddot{\eta}^{**}_{\ddot{\eta}_\omega}$ and $\ddot{\mu}^{-}_{\ddot{\mu}_\omega} \leq \ddot{\mu}^{-\ \ **}_{\ddot{\mu}_\omega}, \ddot{\mu}^{+}_{\ddot{\mu}_\omega} \leq \ddot{\mu}^{+\ \ **}_{\ddot{\mu}_\omega}, \ddot{\eta}^{-}_{\ddot{\eta}_\omega} \geq \ddot{\eta}^{-\ \ **}_{\ddot{\eta}_\omega}, \ddot{\eta}^{+}_{\ddot{\eta}_\omega} \geq \ddot{\eta}^{+\ \ **}_{\ddot{\eta}_\omega}$, then

$$CuIFAAWG\left(°\text{F}^{/}_{CuIF_1}, °\text{F}^{/}_{CuIF_2}, \dots, °\text{F}^{/}_{CuIF_z}\right) \leq CuIFAAWG\left(°\text{F}^{**}_{CuIF_1}, °\text{F}^{**}_{CuIF_2}, \dots, °\text{F}^{**}_{CuIF_z}\right). \quad (30)$$

Proof. Straightforward.

Property 12: If $°\text{F}^{-}_{CuIF_\omega} = \left(\left(\min_{\omega} \ddot{\mu}_{\ddot{\mu}_\omega}, \max_{\omega} \ddot{\eta}_{\ddot{\eta}_\omega}\right), \left(\left[\min_{\omega} \ddot{\mu}^{-}_{\ddot{\mu}_\omega}, \min_{\omega} \ddot{\mu}^{+}_{\ddot{\mu}_\omega}\right], \left[\max_{\omega} \ddot{\eta}^{-}_{\ddot{\eta}_\omega}, \max_{\omega} \ddot{\eta}^{+}_{\ddot{\eta}_\omega}\right]\right)\right)$ and

$$°F^{+}_{CuIF_{\omega}} = \left(\left(\max_{\omega} \ddot{\mu}_{\ddot{\mu_{\omega}}}, \min_{\omega} \ddot{\eta}_{\ddot{\eta}_{\omega}}\right), \left(\left[\max_{\omega} \ddot{\mu}^{-}_{\ddot{\mu_{\omega}}}, \max_{\omega} \ddot{\mu}^{+}_{\ddot{\mu_{\omega}}}\right], \left[\min_{\omega} \ddot{\eta}^{-}_{\ddot{\eta}_{\omega}}, \min_{\omega} \ddot{\eta}^{+}_{\ddot{\eta}_{\omega}}\right]\right)\right)$$, then

$$°F^{-}_{CuIF_{\omega}} \leq CuIFAAWG\left(°F^{/}_{CuIF_{1}}, °F^{/}_{CuIF_{2}}, \ldots, °F^{/}_{CuIF_{z}}\right) \leq °F^{+}_{CuIF_{\omega}}. \tag{31}$$

Proof. Straightforward.

Definition 10: For the finite collection of CIFNs $°F^{/}_{CuIF_{\omega}} = \left(\left(\ddot{\mu}_{\ddot{\mu_{\omega}}}, \ddot{\eta}_{\ddot{\eta}_{\omega}}\right), \left(\left[\ddot{\mu}^{-}_{\ddot{\mu_{\omega}}}, \ddot{\mu}^{+}_{\ddot{\mu_{\omega}}}\right], \left[\ddot{\eta}^{-}_{\ddot{\eta}_{\omega}}, \ddot{\eta}^{+}_{\ddot{\eta}_{\omega}}\right]\right)\right), \omega = 1,2, \ldots, z$, then the CIFAAOWG operator is defined as:
$CuIFAAOWG: °F^{z} \rightarrow °F, by$

$$\begin{aligned} &CuIFAAOWG\left(°F^{/}_{CuIF_{1}}, °F^{/}_{CuIF_{2}}, \ldots, °F^{/}_{CuIF_{z}}\right) \\ &= \left(°F^{/}_{CuIF_{0(1)}}\right)^{\ddot{\varnothing}_{s}^{\;1}} \otimes \left(°F^{/}_{CuIF_{0(2)}}\right)^{\ddot{\varnothing}_{s}^{\;2}} \otimes \ldots \otimes \left(°F^{/}_{CuIF_{0(z)}}\right)^{\ddot{\varnothing}_{s}^{\;z}} \\ &= \otimes_{\omega=1}^{z} \left(°F^{/}_{CuIF_{0(\omega)}}\right)^{\ddot{\varnothing}_{s}^{\;\omega}}. \end{aligned} \tag{32}$$

Note that the weighted vector is stated by: $\ddot{\varnothing}_{s}^{\;\omega} \in [0,1]$ with $\sum_{\omega=1}^{z} \ddot{\varnothing}_{s}^{\;\omega} = 1$ with $0(\omega) \leq 0(\omega - 1)$, where we can get the order of the CIFN by the score function.

Theorem 6: For any finite collection of CIFNs $°F^{/}_{CuIF_{\omega}} = \left(\left(\ddot{\mu}_{\ddot{\mu_{\omega}}}, \ddot{\eta}_{\ddot{\eta}_{\omega}}\right), \left(\left[\ddot{\mu}^{-}_{\ddot{\mu_{\omega}}}, \ddot{\mu}^{+}_{\ddot{\mu_{\omega}}}\right], \left[\ddot{\eta}^{-}_{\ddot{\eta}_{\omega}}, \ddot{\eta}^{+}_{\ddot{\eta}_{\omega}}\right]\right)\right), \omega = 1,2, \ldots, z$, we proved that the Eq (32) is also a CIFN, such as

$$CuIFAAOWG\left(°F^{/}_{CuIF_{1}}, °F^{/}_{CuIF_{2}}, \ldots, °F^{/}_{CuIF_{z}}\right)$$

$$= \left(\begin{array}{c} \left(\mathfrak{E}^{-\left(\sum_{\omega=1}^{z} \ddot{\varnothing}_{s}^{\;\omega}\left(-\mathbb{I}\mathrm{n}\left(\ddot{\mu}_{\mu_{0(\ddot{\omega})}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, 1 - \mathfrak{E}^{-\left(\sum_{\omega=1}^{z} \ddot{\varnothing}_{s}^{\;\omega}\left(-\mathbb{I}\mathrm{n}\left(1-\ddot{\eta}_{\ddot{\eta}_{0(\omega)}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right), \\ \left(\begin{array}{c} \left[\mathfrak{E}^{-\left(\sum_{\omega=1}^{z} \ddot{\varnothing}_{s}^{\;\omega}\left(-\mathbb{I}\mathrm{n}\left(\ddot{\mu}^{-}_{\mu_{0(\ddot{\omega})}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, \mathfrak{E}^{-\left(\sum_{\omega=1}^{z} \ddot{\varnothing}_{s}^{\;\omega}\left(-\mathbb{I}\mathrm{n}\left(\ddot{\mu}^{+}_{\mu_{0(\ddot{\omega})}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right], \\ \left[1 - \mathfrak{E}^{-\left(\sum_{\omega=1}^{z} \ddot{\varnothing}_{s}^{\;\omega}\left(-\mathbb{I}\mathrm{n}\left(1-\ddot{\eta}^{-}_{\ddot{\eta}_{0(\omega)}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, 1 - \mathfrak{E}^{-\left(\sum_{\omega=1}^{z} \ddot{\varnothing}_{s}^{\;\omega}\left(-\mathbb{I}\mathrm{n}\left(1-\ddot{\eta}^{+}_{\ddot{\eta}_{0(\omega)}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right] \end{array}\right) \end{array}\right). \tag{33}$$

Proof. Straightforward.

Property 13: If $°F^{/}_{CuIF_{\omega}} = °F^{/}_{CuIF} = \left(\left(\ddot{\mu}_{\ddot{\mu}}, \ddot{\eta}_{\ddot{\eta}}\right), \left(\left[\ddot{\mu}^{-}_{\ddot{\mu}}, \ddot{\mu}^{+}_{\ddot{\mu}}\right], \left[\ddot{\eta}^{-}_{\ddot{\eta}}, \ddot{\eta}^{+}_{\ddot{\eta}}\right]\right)\right), \omega = 1,2, \ldots, z$, then

$$CuIFAAOWG\left(°F^{/}_{CuIF_{1}}, °F^{/}_{CuIF_{2}}, \ldots, °F^{/}_{CuIF_{z}}\right) = °F^{/}_{CuIF}. \tag{34}$$

Proof. Straightforward.

Property 14: If $°F^{/}_{CuIF_\omega} \leq °F^{**}_{CuIF_\omega}$, it means that $\ddot{\mu}_{\ddot{\mu_\omega}} \leq \ddot{\mu}^{**}_{\ddot{\mu_\omega}}, \ddot{\eta}_{\ddot{\eta}_\omega} \geq \ddot{\eta}^{**}_{\ddot{\eta}_\omega}$ and $\ddot{\mu}^{-}_{\ddot{\mu_\omega}} \leq \ddot{\mu}^{-}_{\ddot{\mu_\omega}}{}^{**}, \ddot{\mu}^{+}_{\ddot{\mu_\omega}} \leq \ddot{\mu}^{+}_{\ddot{\mu_\omega}}{}^{**}, \ddot{\eta}^{-}_{\ddot{\eta}_\omega} \geq \ddot{\eta}^{-}_{\ddot{\eta}_\omega}{}^{**}, \ddot{\eta}^{+}_{\ddot{\eta}_\omega} \geq \ddot{\eta}^{+}_{\ddot{\eta}_\omega}{}^{**}$, then

$$CuIFAAOWG\left(°F^{/}_{CuIF_1}, °F^{/}_{CuIF_2}, \dots, °F^{/}_{CuIF_z}\right) \leq CuIFAAOWG\left(°F^{**}_{CuIF_1}, °F^{**}_{CuIF_2}, \dots, °F^{**}_{CuIF_z}\right). \tag{35}$$

Proof. Straightforward.

Property 15: If $°F^{-}_{CuIF_\omega} = \left(\left(\min_\omega \ddot{\mu}_{\ddot{\mu_\omega}}, \max_\omega \ddot{\eta}_{\ddot{\eta}_\omega}\right), \left(\left[\min_\omega \ddot{\mu}^{-}_{\ddot{\mu_\omega}}, \min_\omega \ddot{\mu}^{+}_{\ddot{\mu_\omega}}\right], \left[\max_\omega \ddot{\eta}^{-}_{\ddot{\eta}_\omega}, \max_\omega \ddot{\eta}^{+}_{\ddot{\eta}_\omega}\right]\right)\right)$ and

$°F^{+}_{CuIF_\omega} = \left(\left(\max_\omega \ddot{\mu}_{\ddot{\mu_\omega}}, \min_\omega \ddot{\eta}_{\ddot{\eta}_\omega}\right), \left(\left[\max_\omega \ddot{\mu}^{-}_{\ddot{\mu_\omega}}, \max_\omega \ddot{\mu}^{+}_{\ddot{\mu_\omega}}\right], \left[\min_\omega \ddot{\eta}^{-}_{\ddot{\eta}_\omega}, \min_\omega \ddot{\eta}^{+}_{\ddot{\eta}_\omega}\right]\right)\right)$, then

$$°F^{-}_{CuIF_\omega} \leq CuIFAAOWG\left(°F^{/}_{CuIF_1}, °F^{/}_{CuIF_2}, \dots, °F^{/}_{CuIF_z}\right) \leq °F^{+}_{CuIF_\omega}. \tag{36}$$

Proof. Straightforward.

Definition 11: For the finite collection of CIFNs $°F^{/}_{CuIF_\omega} = \left(\left(\ddot{\mu}_{\ddot{\mu_\omega}}, \ddot{\eta}_{\ddot{\eta}_\omega}\right), \left(\left[\ddot{\mu}^{-}_{\ddot{\mu_\omega}}, \ddot{\mu}^{+}_{\ddot{\mu_\omega}}\right], \left[\ddot{\eta}^{-}_{\ddot{\eta}_\omega}, \ddot{\eta}^{+}_{\ddot{\eta}_\omega}\right]\right)\right), \omega = 1,2, \dots, z$, then the CIFAAHG operators are defined as: $CuIFAAHG: °F^{z} \to °F, by$

$$\begin{aligned} &CuIFAAHG\left(°F^{/}_{CuIF_1}, °F^{/}_{CuIF_2}, \dots, °F^{/}_{CuIF_z}\right) \\ &= \left(°F^{*}_{CuIF_{0(1)}}\right)^{\ddot{\emptyset}_s^{\,1}} \otimes \left(°F^{*}_{CuIF_{0(2)}}\right)^{\ddot{\emptyset}_s^{\,2}} \otimes \dots \otimes \left(°F^{*}_{CuIF_{o(z)}}\right)^{\ddot{\emptyset}_s^{\,z}} \\ &= \otimes_{\omega=1}^{z} \left(°F^{*}_{CuIF_{0(\omega)}}\right)^{\ddot{\emptyset}_s^{\,\omega}}. \end{aligned} \tag{37}$$

Note that the weighted vector is stated by: $\ddot{\emptyset}_s^{\,\omega} \in [0,1]$ with $\sum_{\omega=1}^{z} \ddot{\emptyset}_s^{\,\omega} = 1$ with $0(\omega) \leq 0(\omega - 1)$, where we can get the order of the CIFN by the score function and $°F^{*}_{CuIF_{0(\omega)}} = z\ddot{\emptyset}_w^{\,\omega} °F^{/}_{CuIF_\omega}, \omega = 1,2, \dots, z$ with another weight vector $\ddot{\emptyset}_w^{\,\omega} \in [0,1]$ with $\sum_{\omega=1}^{z} \ddot{\emptyset}_w^{\,\omega} = 1$.

Theorem 7: For any finite collection of CIFNs $°F^{/}_{CuIF_\omega} = \left(\left(\ddot{\mu}_{\ddot{\mu_\omega}}, \ddot{\eta}_{\ddot{\eta}_\omega}\right), \left(\left[\ddot{\mu}^{-}_{\ddot{\mu_\omega}}, \ddot{\mu}^{+}_{\ddot{\mu_\omega}}\right], \left[\ddot{\eta}^{-}_{\ddot{\eta}_\omega}, \ddot{\eta}^{+}_{\ddot{\eta}_\omega}\right]\right)\right), \omega = 1,2, \dots, z$, we proved that Eq (37) is also a CIFN, such as

$$CuIFAAHG\left({}^{\circ}\mathrm{F}^{/}_{CuIF_1}, {}^{\circ}\mathrm{F}^{/}_{CuIF_2}, \dots, {}^{\circ}\mathrm{F}^{/}_{CuIF_z}\right) = \left(\begin{array}{c}\left(\mathfrak{E}^{-\left(\sum_{\omega=1}^{z}\ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{ln}\left(\ddot{\mu}^{*}_{\mu_{0(\omega)}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, 1-\mathfrak{E}^{-\left(\sum_{\omega=1}^{z}\ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{ln}\left(1-\ddot{\eta}^{*}_{\eta_{0(\omega)}}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right), \\ \left(\begin{array}{c}\left[\mathfrak{E}^{-\left(\sum_{\omega=1}^{z}\ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{ln}\left(\ddot{\mu}^{*}_{\mu_{0(\omega)}}{}^{-}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, \mathfrak{E}^{-\left(\sum_{\omega=1}^{z}\ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{ln}\left(\ddot{\mu}^{*}_{\mu_{0(\omega)}}{}^{+}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right], \\ \left[1-\mathfrak{E}^{-\left(\sum_{\omega=1}^{z}\ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{ln}\left(1-\ddot{\eta}^{*}_{\eta_{0(\omega)}}{}^{-}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}, 1-\mathfrak{E}^{-\left(\sum_{\omega=1}^{z}\ddot{\emptyset}_s{}^{\omega}\left(-\mathbb{ln}\left(1-\ddot{\eta}^{*}_{\eta_{0(\omega)}}{}^{+}\right)\right)^{\hbar\hbar}\right)^{\frac{1}{\hbar\hbar}}}\right]\end{array}\right)\end{array}\right). \tag{38}$$

Proof. Straightforward.

Property 16: If ${}^{\circ}\mathrm{F}^{/}_{CuIF_\omega} = {}^{\circ}\mathrm{F}^{/}_{CuIF} = \left(\left(\ddot{\mu}_{\ddot{\mu}}, \ddot{\eta}_{\ddot{\eta}}\right), \left(\left[\ddot{\mu}^{-}_{\ddot{\mu}}, \ddot{\mu}^{+}_{\ddot{\mu}}\right], \left[\ddot{\eta}^{-}_{\ddot{\eta}}, \ddot{\eta}^{+}_{\ddot{\eta}}\right]\right)\right), \omega = 1,2, \dots, z$, then

$$CuIFAAHG\left({}^{\circ}\mathrm{F}^{/}_{CuIF_1}, {}^{\circ}\mathrm{F}^{/}_{CuIF_2}, \dots, {}^{\circ}\mathrm{F}^{/}_{CuIF_z}\right) = {}^{\circ}\mathrm{F}^{/}_{CuIF}. \tag{39}$$

Proof. Straightforward.

Property 17: If ${}^{\circ}\mathrm{F}^{/}_{CuIF_\omega} \leq {}^{\circ}\mathrm{F}^{**}_{CuIF_\omega}$, it means that $\ddot{\mu}_{\ddot{\mu}_\omega} \leq \ddot{\mu}^{**}_{\ddot{\mu}_\omega}, \ddot{\eta}_{\ddot{\eta}_\omega} \geq \ddot{\eta}^{**}_{\ddot{\eta}_\omega}$ and $\ddot{\mu}^{-}_{\ddot{\mu}_\omega} \leq \ddot{\mu}^{-}_{\ddot{\mu}_\omega}{}^{**}, \ddot{\mu}^{+}_{\ddot{\mu}_\omega} \leq \ddot{\mu}^{+}_{\ddot{\mu}_\omega}{}^{**}, \ddot{\eta}^{-}_{\ddot{\eta}_\omega} \geq \ddot{\eta}^{-}_{\ddot{\eta}_\omega}{}^{**}, \ddot{\eta}^{+}_{\ddot{\eta}_\omega} \geq \ddot{\eta}^{+}_{\ddot{\eta}_\omega}{}^{**}$, then

$$CuIFAAHG\left({}^{\circ}\mathrm{F}^{/}_{CuIF_1}, {}^{\circ}\mathrm{F}^{/}_{CuIF_2}, \dots, {}^{\circ}\mathrm{F}^{/}_{CuIF_z}\right) \leq CuIFAAHG\left({}^{\circ}\mathrm{F}^{**}_{CuIF_1}, {}^{\circ}\mathrm{F}^{**}_{CuIF_2}, \dots, {}^{\circ}\mathrm{F}^{**}_{CuIF_z}\right). \tag{40}$$

Proof. Straightforward.

Property 18: If ${}^{\circ}\mathrm{F}^{-}_{CuIF_\omega} = \left(\left(\min_{\omega} \ddot{\mu}_{\ddot{\mu}_\omega}, \max_{\omega} \ddot{\eta}_{\ddot{\eta}_\omega}\right), \left(\left[\min_{\omega} \ddot{\mu}^{-}_{\ddot{\mu}_\omega}, \min_{\omega} \ddot{\mu}^{+}_{\ddot{\mu}_\omega}\right], \left[\max_{\omega} \ddot{\eta}^{-}_{\ddot{\eta}_\omega}, \max_{\omega} \ddot{\eta}^{+}_{\ddot{\eta}_\omega}\right]\right)\right)$ and

${}^{\circ}\mathrm{F}^{+}_{CuIF_\omega} = \left(\left(\max_{\omega} \ddot{\mu}_{\ddot{\mu}_\omega}, \min_{\omega} \ddot{\eta}_{\ddot{\eta}_\omega}\right), \left(\left[\max_{\omega} \ddot{\mu}^{-}_{\ddot{\mu}_\omega}, \max_{\omega} \ddot{\mu}^{+}_{\ddot{\mu}_\omega}\right], \left[\min_{\omega} \ddot{\eta}^{-}_{\ddot{\eta}_\omega}, \min_{\omega} \ddot{\eta}^{+}_{\ddot{\eta}_\omega}\right]\right)\right)$, then

$${}^{\circ}\mathrm{F}^{-}_{CuIF_\omega} \leq CuIFAAHG\left({}^{\circ}\mathrm{F}^{/}_{CuIF_1}, {}^{\circ}\mathrm{F}^{/}_{CuIF_2}, \dots, {}^{\circ}\mathrm{F}^{/}_{CuIF_z}\right) \leq {}^{\circ}\mathrm{F}^{+}_{CuIF_\omega}. \tag{41}$$

Proof. Straightforward.

5. Classifications of artificial neutral networks based on proposed operators

In this section, we select the best type of artificial neural network among the five artificial neural networks based on the proposed method for CIFS. Furthermore, according to the internet, ANNs mean "artificial neural networks", which are the collection of computational techniques motivated by the shape and specification of biological neural networks discovered in the human brain. Based on the CIFAAWA operator and CIFAAWG operator, we select the best one among five artificial neural networks.

For this, we collect a finite collection of alternatives $°F^{**}_{CuIF_1}, °F^{**}_{CuIF_2}, \dots, °F^{**}_{CuIF_z}$ and for each alternative, we have the collection of finite attributes $°F^{atribut\mathfrak{E}}_{CuIF_1}, °F^{atribut\mathfrak{E}}_{CuIF_2}, \dots, °F^{atribut\mathfrak{E}}_{CuIF_n}$ with well-known weight vectors $\ddot{\emptyset}_s{}^{\omega} \in [0,1]$ with $\sum_{\omega=1}^{z} \ddot{\emptyset}_s{}^{\omega} = 1$. Further, we get a matrix with the CIF values, where $0 \leq \ddot{\mu}_{\ddot{\mu}}(\alpha) + \ddot{\eta}_{\ddot{\eta}}(\alpha) \leq 1$ and $0 \leq \ddot{\mu}^{+}_{\ddot{\mu}}(\alpha) + \ddot{\eta}^{+}_{\ddot{\eta}}(\alpha) \leq 1$, and $[\ddot{\mu}^{-}_{\ddot{\mu}}(\alpha), \ddot{\mu}^{+}_{\ddot{\mu}}(\alpha)]$ and $[\ddot{\eta}^{-}_{\ddot{\eta}}(\alpha), \ddot{\eta}^{+}_{\ddot{\eta}}(\alpha)]$ represent the interval-valued truth and interval-valued falsity degrees with a neutral grade $\ddot{\vartheta}_{\ddot{\vartheta}}(\alpha) = [\ddot{\vartheta}^{-}_{\ddot{\vartheta}}(\alpha), \ddot{\vartheta}^{+}_{\ddot{\vartheta}}(\alpha)] = [1 - \ddot{\mu}^{+}_{\ddot{\mu}}(\alpha) + \ddot{\eta}^{+}_{\ddot{\eta}}(\alpha), 1 - \ddot{\mu}^{-}_{\ddot{\mu}}(\alpha) + \ddot{\eta}^{-}_{\ddot{\eta}}(\alpha)]$, where $\ddot{\mu}_{\ddot{\mu}}(\alpha)$ and $\ddot{\eta}_{\ddot{\eta}}(\alpha)$ represent the truth and falsity degrees with a neutral grade $\ddot{\vartheta}_{\ddot{\vartheta}}(\alpha) = 1 - \left(\ddot{\mu}_{\ddot{\mu}}(\alpha) + \ddot{\eta}_{\ddot{\eta}}(\alpha)\right)$. Moreover, the simple form of the CIF number (CIFN) is shown by: $°F^{/}_{CuIF_{\omega}} = \left((\ddot{\mu}_{\ddot{\mu}_{\omega}}, \ddot{\eta}_{\ddot{\eta}_{\omega}}), ([\ddot{\mu}^{-}_{\ddot{\mu}_{\omega}}, \ddot{\mu}^{+}_{\ddot{\mu}_{\omega}}], [\ddot{\eta}^{-}_{\ddot{\eta}_{\omega}}, \ddot{\eta}^{+}_{\ddot{\eta}_{\omega}}])\right), \omega = 1,2,\dots,z$. After getting the matrix, we will use the following procedure for evaluating the best decision between five decisions. The geometrical representation of the proposed algorithm is mentioned in Figure 3.

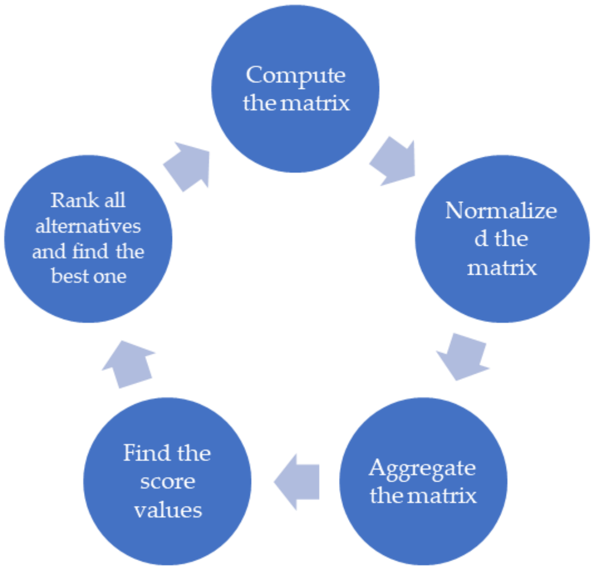

Figure 3. The geometrical shape of the proposed algorithm.

Step 1: During the collection of CIF values, we have two possibilities, such as benefit or cost type of information, if we have cost type of data in the decision matrix, then we aim to normalize the matrix, such as

$$N = \begin{cases} \left((\ddot{\mu}_{\ddot{\mu}_{\omega}}, \ddot{\eta}_{\ddot{\eta}_{\omega}}), ([\ddot{\mu}^{-}_{\ddot{\mu}_{\omega}}, \ddot{\mu}^{+}_{\ddot{\mu}_{\omega}}], [\ddot{\eta}^{-}_{\ddot{\eta}_{\omega}}, \ddot{\eta}^{+}_{\ddot{\eta}_{\omega}}])\right) & for\ benefit, \\ \left((\ddot{\eta}_{\ddot{\eta}_{\omega}}, \ddot{\mu}_{\ddot{\mu}_{\omega}}), ([\ddot{\eta}^{-}_{\ddot{\eta}_{\omega}}, \ddot{\eta}^{+}_{\ddot{\eta}_{\omega}}], [\ddot{\mu}^{-}_{\ddot{\mu}_{\omega}}, \ddot{\mu}^{+}_{\ddot{\mu}_{\omega}}])\right) & for\ cost. \end{cases}$$

However, if we have a benefit type of data, we do not need to normalize the data.

Step 2: For aggregating the normalized data into a singleton one, we use the CIFAAWA operator and CIFAAWG operator.
Step 3: For getting the score values, we use Eq (4) or Eq (5).
Step 4: Ranking the order of the alternatives based on their score values to examine the best optimal among the five ones.

Further, we simplify the above procedure with the help of some practical examples, which are related to artificial neural networks. For this, we consider five artificial neural networks and select the best one.

5.1. Numerical example

The ANN technique is used in many fields because of their features and dominancy. In this example, we aim to consider five alternatives, and for each alternative, we have four attributes with weight vectors $(0.25,0.25,0.25,0.25)^T$. Furthermore, each alternative can be stated below:

1) Feedforward Neural Networks (FNNs) "$°F^{**}_{CuIF_1}$": FNNs are the valuable and simple kind of ANNs, containing input, hidden, and output layers.
2) Recurrent Neural Networks (RNNs) "$°F^{**}_{CuIF_2}$": RNNs are specially constructed for coping with sequential information like time series or natural language.
3) Long Short-Term Memory Networks (LSTMs) "$°F^{**}_{CuIF_3}$": LSTMs are a valuable and dominant type of RNN that evaluates the vanishing gradient problems.
4) Convolutional Neural Networks (CNNs) "$°F^{**}_{CuIF_4}$": CNNs are specifically designed for the primary computer vision tasks.
5) Generative Adversarial Networks (GANs) "$°F^{**}_{CuIF_5}$": GANs consist of two ANNs, a generator, and a discriminator, that are trained together in a game-like setting.

To choose the best one, we use the following features which are stated as the main attribute or criteria, such as: $°F^{atribut\mathfrak{E}}_{CuIF_1}$: Risk analysis, $°F^{atribut\mathfrak{E}}_{CuIF_2}$: Growth analysis, $°F^{atribut\mathfrak{E}}_{CuIF_3}$: Enviromental impact, and $°F^{atribut\mathfrak{E}}_{CuIF_4}$: Social and political impact. Then, we get the data in Table 2.

After getting the matrix, we will use the following procedure to get the best decision, such as:

Step 1: Because $°F^{Atribut\mathfrak{E}}_{CuIF_1}$ is the cost type, we aim to normalize the matrix in Table 2, such as

$$N=\begin{cases}\left(\left(\ddot{\mu}_{\ddot{\mu}_{\omega}},\ddot{\eta}_{\ddot{\eta}_{\omega}}\right),\left(\left[\ddot{\mu}^{-}_{\ddot{\mu}_{\omega}},\ddot{\mu}^{+}_{\ddot{\mu}_{\omega}}\right],\left[\ddot{\eta}^{-}_{\ddot{\eta}_{\omega}},\ddot{\eta}^{+}_{\ddot{\eta}_{\omega}}\right]\right)\right) & for\ benefit,\\ \left(\left(\ddot{\eta}_{\ddot{\eta}_{\omega}},\ddot{\mu}_{\ddot{\mu}_{\omega}}\right),\left(\left[\ddot{\eta}^{-}_{\ddot{\eta}_{\omega}},\ddot{\eta}^{+}_{\ddot{\eta}_{\omega}}\right],\left[\ddot{\mu}^{-}_{\ddot{\mu}_{\omega}},\ddot{\mu}^{+}_{\ddot{\mu}_{\omega}}\right]\right)\right) & for\ cost.\end{cases}$$

then the normalized matrix is given in Table 3.

Table 2. Cubic intuitionistic fuzzy decision matrix.

	$^{\circ}F^{Atribut\mathfrak{E}}_{CuIF_1}$	$^{\circ}F^{Atribut\mathfrak{E}}_{CuIF_2}$
$^{\circ}F^{**}_{CuIF_1}$	$\begin{pmatrix}(0.2,0.7),\\([0.2,0.3],[0.4,0.5])\end{pmatrix}$	$\begin{pmatrix}(0.71,0.21),\\([0.41,0.51],[0.21,0.31])\end{pmatrix}$
$^{\circ}F^{**}_{CuIF_2}$	$\begin{pmatrix}(0.1,0.5),\\([0.1,0.2],[0.2,0.4])\end{pmatrix}$	$\begin{pmatrix}(0.51,0.11),\\([0.21,0.41],[0.11,0.21])\end{pmatrix}$
$^{\circ}F^{**}_{CuIF_3}$	$\begin{pmatrix}(0.2,0.4),\\([0.3,0.4],[0.3,0.5])\end{pmatrix}$	$\begin{pmatrix}(0.41,0.21),\\([0.31,0.51],[0.31,0.41])\end{pmatrix}$
$^{\circ}F^{**}_{CuIF_4}$	$\begin{pmatrix}(0.2,0.3),\\([0.1,0.2],[0.1,0.2])\end{pmatrix}$	$\begin{pmatrix}(0.31,0.21),\\([0.11,0.21],[0.11,0.21])\end{pmatrix}$
$^{\circ}F^{**}_{CuIF_5}$	$\begin{pmatrix}(0.1,0.8),\\([0.1,0.2],[0.5,0.6])\end{pmatrix}$	$\begin{pmatrix}(0.81,0.11),\\([0.51,0.61],[0.11,0.21])\end{pmatrix}$

	$^{\circ}F^{Atribut\mathfrak{E}}_{CuIF_3}$	$^{\circ}F^{Atribut\mathfrak{E}}_{CuIF_4}$
$^{\circ}F^{**}_{CuIF_1}$	$\begin{pmatrix}(0.72,0.22),\\([0.42,0.52],[0.22,0.32])\end{pmatrix}$	$\begin{pmatrix}(0.73,0.23),\\([0.43,0.53],[0.23,0.33])\end{pmatrix}$
$^{\circ}F^{**}_{CuIF_2}$	$\begin{pmatrix}(0.52,0.12),\\([0.22,0.42],[0.12,0.22])\end{pmatrix}$	$\begin{pmatrix}(0.53,0.13),\\([0.23,0.43],[0.13,0.23])\end{pmatrix}$
$^{\circ}F^{**}_{CuIF_3}$	$\begin{pmatrix}(0.42,0.22),\\([0.32,0.52],[0.32,0.42])\end{pmatrix}$	$\begin{pmatrix}(0.43,0.23),\\([0.33,0.53],[0.33,0.43])\end{pmatrix}$
$^{\circ}F^{**}_{CuIF_4}$	$\begin{pmatrix}(0.32,0.22),\\([0.12,0.22],[0.12,0.22])\end{pmatrix}$	$\begin{pmatrix}(0.33,0.23),\\([0.13,0.23],[0.13,0.23])\end{pmatrix}$
$^{\circ}F^{**}_{CuIF_5}$	$\begin{pmatrix}(0.82,0.12),\\([0.52,0.62],[0.12,0.22])\end{pmatrix}$	$\begin{pmatrix}(0.83,0.13),\\([0.53,0.63],[0.13,0.23])\end{pmatrix}$

Table 3. Normalized Cubic intuitionistic fuzzy decision matrix.

	$^{\circ}F^{\boldsymbol{Atribut\mathfrak{E}}}_{\boldsymbol{CuIF_1}}$	$^{\circ}F^{\boldsymbol{Atribut\mathfrak{E}}}_{\boldsymbol{CuIF_2}}$
$^{\circ}F^{**}_{\boldsymbol{CuIF_1}}$	$\begin{pmatrix}(0.7,0.2),\\([0.4,0.5],[0.2,0.3])\end{pmatrix}$	$\begin{pmatrix}(0.71,0.21),\\([0.41,0.51],[0.21,0.31])\end{pmatrix}$
$^{\circ}F^{**}_{\boldsymbol{CuIF_2}}$	$\begin{pmatrix}(0.5,0.1),\\([0.2,0.4],[0.1,0.2])\end{pmatrix}$	$\begin{pmatrix}(0.51,0.11),\\([0.21,0.41],[0.11,0.21])\end{pmatrix}$
$^{\circ}F^{**}_{\boldsymbol{CuIF_3}}$	$\begin{pmatrix}(0.4,0.2),\\([0.3,0.5],[0.3,0.4])\end{pmatrix}$	$\begin{pmatrix}(0.41,0.21),\\([0.31,0.51],[0.31,0.41])\end{pmatrix}$
$^{\circ}F^{**}_{\boldsymbol{CuIF_4}}$	$\begin{pmatrix}(0.3,0.2),\\([0.1,0.2],[0.1,0.2])\end{pmatrix}$	$\begin{pmatrix}(0.31,0.21),\\([0.11,0.21],[0.11,0.21])\end{pmatrix}$
$^{\circ}F^{**}_{\boldsymbol{CuIF_5}}$	$\begin{pmatrix}(0.8,0.1),\\([0.5,0.6],[0.1,0.2])\end{pmatrix}$	$\begin{pmatrix}(0.81,0.11),\\([0.51,0.61],[0.11,0.21])\end{pmatrix}$

	$^{\circ}F^{Atribut\mathfrak{E}}_{CuIF_3}$	$^{\circ}F^{Atribut\mathfrak{E}}_{CuIF_4}$
$^{\circ}F^{**}_{\boldsymbol{CuIF_1}}$	$\begin{pmatrix}(0.72,0.22),\\([0.42,0.52],[0.22,0.32])\end{pmatrix}$	$\begin{pmatrix}(0.73,0.23),\\([0.43,0.53],[0.23,0.33])\end{pmatrix}$
$^{\circ}F^{**}_{\boldsymbol{CuIF_2}}$	$\begin{pmatrix}(0.52,0.12),\\([0.22,0.42],[0.12,0.22])\end{pmatrix}$	$\begin{pmatrix}(0.53,0.13),\\([0.23,0.43],[0.13,0.23])\end{pmatrix}$
$^{\circ}F^{**}_{\boldsymbol{CuIF_3}}$	$\begin{pmatrix}(0.42,0.22),\\([0.32,0.52],[0.32,0.42])\end{pmatrix}$	$\begin{pmatrix}(0.43,0.23),\\([0.33,0.53],[0.33,0.43])\end{pmatrix}$
$^{\circ}F^{**}_{\boldsymbol{CuIF_4}}$	$\begin{pmatrix}(0.32,0.22),\\([0.12,0.22],[0.12,0.22])\end{pmatrix}$	$\begin{pmatrix}(0.33,0.23),\\([0.13,0.23],[0.13,0.23])\end{pmatrix}$
$^{\circ}F^{**}_{\boldsymbol{CuIF_5}}$	$\begin{pmatrix}(0.82,0.12),\\([0.52,0.62],[0.12,0.22])\end{pmatrix}$	$\begin{pmatrix}(0.83,0.13),\\([0.53,0.63],[0.13,0.23])\end{pmatrix}$

Step 2: For aggregating the normalized data into a singleton one, we use the CIFAAWA operator and CIFAAWG operator to get the results, shown in Table 4.

Table 4. Aggregated decision matrix.

	CIFAAWA Operator	CIFAAWG Operator
$°F^{**}_{CuIF_1}$	$\begin{pmatrix}(0.4204,0.5126),\\([0.2077,0.2697],[0.5126,0.6053])\end{pmatrix}$	$\begin{pmatrix}(0.8643,0.0998),\\([0.6824,0.7495],[0.0998,0.1515])\end{pmatrix}$
$°F^{**}_{CuIF_2}$	$\begin{pmatrix}(0.2697,0.3900),\\([0.0998,0.2077],[0.3900,0.5126])\end{pmatrix}$	$\begin{pmatrix}(0.7495,0.0517),\\([0.5126,0.6824],[0.0517,0.0998])\end{pmatrix}$
$°F^{**}_{CuIF_3}$	$\begin{pmatrix}(0.2077,0.5126),\\([0.1515,0.2697],[0.6053,0.6824])\end{pmatrix}$	$\begin{pmatrix}(0.6824,0.0998),\\([0.6053,0.7495],[0.1515,0.2077])\end{pmatrix}$
$°F^{**}_{CuIF_4}$	$\begin{pmatrix}(0.1515,0.5126),\\([0.0517,0.0998],[0.3900,0.5126])\end{pmatrix}$	$\begin{pmatrix}(0.6053,0.0998),\\([0.3900,0.5126],[0.0517,0.0998])\end{pmatrix}$
$°F^{**}_{CuIF_5}$	$\begin{pmatrix}(0.5198,0.3900),\\([0.2697,0.3394],[0.3900,0.5126])\end{pmatrix}$	$\begin{pmatrix}(0.5149,0.0517),\\([0.7495,0.8096],[0.0517,0.0998])\end{pmatrix}$

Step 3: For getting the score values, we use Eq (4) to calculate it, as shown in Table 5.

Table 5. The score values of the aggregated values.

	CIFAAWA Operator	CIFAAWG Operator
$°F^{**}_{CuIF_1}$	-0.20622	0.67741
$°F^{**}_{CuIF_2}$	-0.20895	0.6098
$°F^{**}_{CuIF_3}$	-0.36904	0.54018
$°F^{**}_{CuIF_4}$	-0.36835	0.44056
$°F^{**}_{CuIF_5}$	-0.00851	0.78353

Step 4: We can get the ranking order of the alternatives based on their score values, and get the best optimal among the five ones, see Table 6.

Table 6. Ranking values.

Methods	Ranking values	Best decision
CIFAAWA Operator	$°F^{**}_{CuIF_5} > °F^{**}_{CuIF_1} > °F^{**}_{CuIF_2} > °F^{**}_{CuIF_4} > °F^{**}_{CuIF_3}$	$°F^{**}_{CuIF_5}$
CIFAAWG Operator	$°F^{**}_{CuIF_5} > °F^{**}_{CuIF_1} > °F^{**}_{CuIF_2} > °F^{**}_{CuIF_3} > °F^{**}_{CuIF_4}$	$°F^{**}_{CuIF_5}$

From the Table 6, we observed that the best optimal is $°F^{**}_{CuIF_5}$ (Generative Adversarial Networks (GANs)) by the two different techniques based on CIFSs. Further, we try to discuss different types of cases using the data in Table 2. For instance, if we exclude the interval-valued information from the data in Table 2, then the aggregated values are shown in Table 7.

Moreover, we derive the ranking order of the alternatives based on their score values and get the best optimal among the five (see Table 8).

Table 7. Score values of IFSs.

	IFAAWA Operator	IFAA WG Operator
$°F^{**}_{CuIF_1}$	-0.0922	0.7645
$°F^{**}_{CuIF_2}$	-0.1203	-0.2975
$°F^{**}_{CuIF_3}$	-0.3048	0.5825
$°F^{**}_{CuIF_4}$	-0.3610	0.5055
$°F^{**}_{CuIF_5}$	0.1297	0.8632

Table 8. The ranking results are based on Table 7.

Methods	Ranking values	Best decision
IFAAWA Operator	$°F^{**}_{CuIF_5} > °F^{**}_{CuIF_1} > °F^{**}_{CuIF_2} > °F^{**}_{CuIF_3} > °F^{**}_{CuIF_4}$	$°F^{**}_{CuIF_5}$
IFAAWG Operator	$°F^{**}_{CuIF_5} > °F^{**}_{CuIF_1} > °F^{**}_{CuIF_3} > °F^{**}_{CuIF_4} > °F^{**}_{CuIF_2}$	$°F^{**}_{CuIF_5}$

From Table 8, we observed that the best optimal is $°F^{**}_{CuIF_5}$ (Generative Adversarial Networks (GANs)) by the two different techniques based on CIFSs.

Further, we excluded the intuitionistic information from the data in Table 2. The aggregated values are shown in Table 9.

Table 9. Score values for IVIFSs.

	IVIFAAWA Operator (0.25,0.25,0.25,0.25)	IVIFAAWG Operator (0.25,0.25,0.25,0.25)	IVIFAAWA Operator (0.2,0.3,0.2,0.3)	IVIFAAWG Operator (0.2,0.3,0.2,0.3)
$°F^{**}_{CuIF_1}$	-0.3202	0.5902	0.9575	0.9739
$°F^{**}_{CuIF_2}$	-0.2975	0.5217	0.944	0.9673
$°F^{**}_{CuIF_3}$	-0.4332	0.4977	0.9476	0.9558
$°F^{**}_{CuIF_4}$	-0.3756	0.3756	0.914	0.9452
$°F^{**}_{CuIF_5}$	-0.1467	0.7038	0.9682	0.9818

Moreover, we derive the ranking order of the alternatives based on their score values and get the best optimal among the five (see Table 10).

Table 10. The ranking results from Table 9.

Methods	Ranking values	Best decision
CIFAAWA Operator (0.25,0.25,0.25,0.25)	$°F^{**}_{CuIF_5} > °F^{**}_{CuIF_2} > °F^{**}_{CuIF_1} > °F^{**}_{CuIF_4} > °F^{**}_{CuIF_3}$	$°F^{**}_{CuIF_5}$
CIFAAWG Operator (0.25,0.25,0.25,0.25)	$°F^{**}_{CuIF_5} > °F^{**}_{CuIF_1} > °F^{**}_{CuIF_2} > °F^{**}_{CuIF_3} > °F^{**}_{CuIF_4}$	$°F^{**}_{CuIF_5}$
CIFAAWA Operator (0.2,0.3,0.2,0.3)	$°F^{**}_{CuIF_5} > °F^{**}_{CuIF_1} > °F^{**}_{CuIF_3} > °F^{**}_{CuIF_2} > °F^{**}_{CuIF_1}$	$°F^{**}_{CuIF_5}$
CIFAAWG Operator (0.2,0.3,0.2,0.3)	$°F^{**}_{CuIF_5} > °F^{**}_{CuIF_1} > °F^{**}_{CuIF_2} > °F^{**}_{CuIF_3} > °F^{**}_{CuIF_4}$	$°F^{**}_{CuIF_5}$

From Table 10, we observe that the best optimal is $°F^{**}_{CuIF_5}$ (Generative Adversarial Networks (GANs)) by the two different techniques based on CIFSs. This means that when we use only IFSs, IVIFSs, and CIF types of data, the proposed technique could easily get the same results. Moreover, we check the influence of the parameters and do a comparative analysis of the proposed method with some methods.

5.2. Influence of parameters

In this subsection, we verify the stability or influences of the proposed work with the help of the different values of parameters $\hbar\hbar \geq 1$.

For this, we consider the data in Table 2, and then based on the CIFAAWA operator and CIFAAWG operators, we check the ranking results. For the CIFAAWA operator, the influence of the possible values of $\hbar\hbar$ is shown in Table 11.

Table 11. Influence of the parameter based on the CIFAAWA operator.

Parameter	Score values	Ranking results
$\hbar\hbar = 1$	-0.2062,-0.2089,-0.3690,-0.3683,-0.0085	$°F^{**}_{CuIF_5} > °F^{**}_{CuIF_1} > °F^{**}_{CuIF_2} > °F^{**}_{CuIF_4} > °F^{**}_{CuIF_3}$
$\hbar\hbar = 3$	-0.2056,-0.208,-0.3685,-0.3675,-0.0075	$°F^{**}_{CuIF_5} > °F^{**}_{CuIF_1} > °F^{**}_{CuIF_2} > °F^{**}_{CuIF_4} > °F^{**}_{CuIF_3}$
$\hbar\hbar = 5$	-0.2049,-0.2071,-0.3679,-0.3667,-0.0065	$°F^{**}_{CuIF_5} > °F^{**}_{CuIF_1} > °F^{**}_{CuIF_2} > °F^{**}_{CuIF_4} > °F^{**}_{CuIF_3}$
$\hbar\hbar = 7$	-0.2043,-0.2062,-0.3673,-0.3659,-0.0055	$°F^{**}_{CuIF_5} > °F^{**}_{CuIF_1} > °F^{**}_{CuIF_2} > °F^{**}_{CuIF_4} > °F^{**}_{CuIF_3}$
$\hbar\hbar = 9$	-0.2037,-0.2053,-0.3668,-0.3651,-0.0045	$°F^{**}_{CuIF_5} > °F^{**}_{CuIF_1} > °F^{**}_{CuIF_2} > °F^{**}_{CuIF_4} > °F^{**}_{CuIF_3}$
$\hbar\hbar = 11$	-0.2031,-0.2045,-0.3662,-0.3644,-0.0036	$°F^{**}_{CuIF_5} > °F^{**}_{CuIF_1} > °F^{**}_{CuIF_2} > °F^{**}_{CuIF_4} > °F^{**}_{CuIF_3}$

From Table 11, we observe that the best optimal is $°F^{**}_{CuIF_5}$ (Generative Adversarial Networks (GANs)) by the Aczel-Alsina weighted averaging based on CIFSs for all possible values of the parameter. Furthermore, the influence of the possible values of $\hbar\hbar$ for the CIFAAWG operator is shown in Table 12.

From Table 12, we observe that the best optimal is $°F^{**}_{CuIF_5}$ (Generative Adversarial Networks (GANs)) by the Aczel-Alsina weighted geometric based on CIFSs for all possible values of the parameter. Furthermore, we use the data in Table 2 to perform a comparative analysis of the proposed method with some existing methods.

Table 12. Influence of the parameter based on the CIFAAWG operator.

Parameter	Score values	Ranking values
$\hbar\hbar = 1$	0.6774,0.6098,0.5401,0.4405,0.7835	$°F^{**}_{CuIF_5} > °F^{**}_{CuIF_1} > °F^{**}_{CuIF_2} > °F^{**}_{CuIF_3} > °F^{**}_{CuIF_4}$
$\hbar\hbar = 3$	0.6768,0.609,0.5396,0.4398,0.7827	$°F^{**}_{CuIF_5} > °F^{**}_{CuIF_1} > °F^{**}_{CuIF_2} > °F^{**}_{CuIF_3} > °F^{**}_{CuIF_4}$
$\hbar\hbar = 5$	0.6763,0.6083,0.5391,0.439,0.782	$°F^{**}_{CuIF_5} > °F^{**}_{CuIF_1} > °F^{**}_{CuIF_2} > °F^{**}_{CuIF_3} > °F^{**}_{CuIF_4}$
$\hbar\hbar = 7$	0.6757,0.6076,0.5386,0.4382,0.7813	$°F^{**}_{CuIF_5} > °F^{**}_{CuIF_1} > °F^{**}_{CuIF_2} > °F^{**}_{CuIF_3} > °F^{**}_{CuIF_4}$
$\hbar\hbar = 9$	0.6752,0.6069,0.5381,0.4374,0.7806	$°F^{**}_{CuIF_5} > °F^{**}_{CuIF_1} > °F^{**}_{CuIF_2} > °F^{**}_{CuIF_3} > °F^{**}_{CuIF_4}$
$\hbar\hbar = 11$	0.6747,0.6063,0.5375,0.4367,0.7799	$°F^{**}_{CuIF_5} > °F^{**}_{CuIF_1} > °F^{**}_{CuIF_2} > °F^{**}_{CuIF_3} > °F^{**}_{CuIF_4}$

5.3. Comparative analysis

In this subsection, our main target is to compare the proposed method with some existing methods to show the effectiveness of the derived method. For this, we consider the following existing methods, for instance, Xu [14] addressed the AOs for IFSs. Moreover, Xu and Yager [16] developed the geometric AOs for IFSs. Wang et al. [18] presented the AOs for IVIFSs. Further, Senapati et al. [19] derived the geometric AOs for IVIFSs. Wei and Wang [21] addressed the geometric AOs for IVIFSs. Moreover, Xu and Chen [22] presented the geometric AOs for IVIFSs. Kaur and Garg [25] developed the AOs for the CIF set. Finally, Kaur and Garg [26] proposed the generalized AOs for CIF values. Based on the data in Table 2, the comparative analysis is shown in Table 13.

Table 13. Comparative analysis for the CIF values.

Methods	Score values	Ranking values
Xu [14]	Failed	Failed
Xu and Yager [16]	Failed	Failed
Wang et al. [18]	Failed	Failed
Senapati et al. [19]	Failed	Failed
Wei and Wang [21]	Failed	Failed
Xu and Chen [22]	Failed	Failed
Kaur and Garg [25]	0.3504,0.2756,0.1253,0.0504,0.5507	$°F^{**}_{CuIF_5} > °F^{**}_{CuIF_1} > °F^{**}_{CuIF_2} > °F^{**}_{CuIF_3} > °F^{**}_{CuIF_4}$
Kaur and Garg [26]	0.3498,0.2748,0.1248,0.0496,0.5498	$°F^{**}_{CuIF_5} > °F^{**}_{CuIF_1} > °F^{**}_{CuIF_2} > °F^{**}_{CuIF_3} > °F^{**}_{CuIF_4}$
CIFAAWA operator	-0.2062,-0.209,-0.369,-0.3683,-0.0085	$°F^{**}_{CuIF_5} > °F^{**}_{CuIF_1} > °F^{**}_{CuIF_2} > °F^{**}_{CuIF_4} > °F^{**}_{CuIF_3}$
CIFAAWG operator	0.6774,0.6098,0.5402,0.4406,0.7835	$°F^{**}_{CuIF_5} > °F^{**}_{CuIF_1} > °F^{**}_{CuIF_2} > °F^{**}_{CuIF_3} > °F^{**}_{CuIF_4}$

From Table 13, we see that the best decision is $°F^{**}_{CuIF_5}$ (Generative Adversarial Networks (GANs)) according to the proposed CIFAAWA, CIFAAWG, and the methods proposed by Kaur and Garg [25,26] because these operators are based on CIF information, but the other existing techniques failed to solve this problem because they are based on IFS or IVIFSs. If we use only the IFS from the data in Table 2, then the comparison is stated in Table 14.

Table 14. Comparative analysis for the IFSs.

Methods	Score values	Ranking values
Xu [14]	0.5005,0.4007,0.2004,0.1004,0.7009	$°F^{**}_{CuIF_5} > °F^{**}_{CuIF_1} > °F^{**}_{CuIF_3} > °F^{**}_{CuIF_4} > °F^{**}_{CuIF_2}$
Xu and Yager [16]	0.4998,0.3998,0.1998,0.0997,0.6999	$°F^{**}_{CuIF_5} > °F^{**}_{CuIF_1} > °F^{**}_{CuIF_3} > °F^{**}_{CuIF_4} > °F^{**}_{CuIF_2}$
Wang et al. [18]	Failed	Failed
Senapati et al. [19]	Failed	Failed
Wei and Wang [21]	Failed	Failed
Xu and Chen [22]	Failed	Failed
Kaur and Garg [25]	0.5005,0.4007,0.2004,0.1004,0.7009	$°F^{**}_{CuIF_5} > °F^{**}_{CuIF_1} > °F^{**}_{CuIF_3} > °F^{**}_{CuIF_4} > °F^{**}_{CuIF_2}$
Kaur and Garg [26]	0.4998,0.3998,0.1998,0.0997,0.6999	$°F^{**}_{CuIF_5} > °F^{**}_{CuIF_1} > °F^{**}_{CuIF_3} > °F^{**}_{CuIF_4} > °F^{**}_{CuIF_2}$
CIFAAWA operator	-0.0922,-0.1203,-0.3048,-0.3610,0.1297	$°F^{**}_{CuIF_5} > °F^{**}_{CuIF_1} > °F^{**}_{CuIF_2} > °F^{**}_{CuIF_3} > °F^{**}_{CuIF_4}$
CIFAAWG operator	0.7645,-0.2975,0.5825,0.5055,0.8632	$°F^{**}_{CuIF_5} > °F^{**}_{CuIF_1} > °F^{**}_{CuIF_3} > °F^{**}_{CuIF_4} > °F^{**}_{CuIF_2}$

From Table 14, we see that the best decision is $°F^{**}_{CuIF_5}$ (Generative Adversarial Networks (GANs)) according to the proposed CIFAAWA, and CIFAAWG, the methods proposed by Kaur and Garg [25,26], Xu [14], and Xu and Yager [16], because these operators are based on CIF information or IFS, but the other techniques failed to solve this problem because they are based on IVIFSs. If we used only the IVIFS from the data in Table 2, then the comparison is shown in Table 15.

From Table 15, we see that the best decision is $°F^{**}_{CuIF_5}$ (Generative Adversarial Networks (GANs)) according to the proposed CIFAAWA, CIFAAWG, the methods proposed by Kaur and Garg [25,26], Wang et al. [18], Senapati et al. [19], Wei and Wang [21], and Xu and Chen [22], because these operators are based on CIF information and IVIFS.However, the other techniques failed to solve this problem because they are based on IFSs. Hence the proposed method is massively powerful and dominant compared to the existing techniques.

Table 15. Comparative analysis for the IVIFSs.

Methods	Score values	Ranking values
Xu [14]	Failed	Failed
Xu and Yager [16]	Failed	Failed
Wang et al. [18]	0.2004,0.1505,0.0503,0.0005,0.4006	$°F^{**}_{CuIF_5} > °F^{**}_{CuIF_1} > °F^{**}_{CuIF_2} > °F^{**}_{CuIF_3} > °F^{**}_{CuIF_4}$
Senapati et al. [19]	0.9575,0.9437,0.9475,0.913,0.9682	$°F^{**}_{CuIF_5} > °F^{**}_{CuIF_1} > °F^{**}_{CuIF_3} > °F^{**}_{CuIF_2} > °F^{**}_{CuIF_4}$
Wei and Wang [21]	0.1998,0.1497,0.0497,-0.0005,0.3998	$°F^{**}_{CuIF_5} > °F^{**}_{CuIF_1} > °F^{**}_{CuIF_2} > °F^{**}_{CuIF_3} > °F^{**}_{CuIF_4}$
Xu and Chen [22]	0.1998,0.1497,0.0497,-0.0005,0.3998	$°F^{**}_{CuIF_5} > °F^{**}_{CuIF_1} > °F^{**}_{CuIF_2} > °F^{**}_{CuIF_3} > °F^{**}_{CuIF_4}$
Kaur and Garg [25]	0.2004,0.1505,0.0503,0.0005,0.4006	$°F^{**}_{CuIF_5} > °F^{**}_{CuIF_1} > °F^{**}_{CuIF_2} > °F^{**}_{CuIF_3} > °F^{**}_{CuIF_4}$
Kaur and Garg [26]	0.1998,0.1497,0.0497,-0.0005,0.3998	$°F^{**}_{CuIF_5} > °F^{**}_{CuIF_1} > °F^{**}_{CuIF_2} > °F^{**}_{CuIF_3} > °F^{**}_{CuIF_4}$
CIFAAWA operator	-0.3202,-0.2975,-0.4332,-0.3756,-0.1467	$°F^{**}_{CuIF_5} > °F^{**}_{CuIF_2} > °F^{**}_{CuIF_1} > °F^{**}_{CuIF_4} > °F^{**}_{CuIF_3}$
CIFAAWG operator	0.5902,0.5217,0.4977,0.3756,0.7038	$°F^{**}_{CuIF_5} > °F^{**}_{CuIF_1} > °F^{**}_{CuIF_2} > °F^{**}_{CuIF_3} > °F^{**}_{CuIF_4}$

6. Conclusions

The model of cubic intuitionistic fuzzy sets is the combination of two different techniques, called cubic and intuitionistic fuzzy sets, and is a reliable technique to cope with vague and uncertain information. The major influences of this article are listed below:

1) We addressed or computed the model of Aczel-Alsina operational laws under the consideration of the CIF set as well as AATN and AATCN, where the model of Algebraic norms and Drastic norms are the special parts of the Aczel-Alsina norms.
2) Using the above invented operational laws, we aimed to develop the model of Aczel-Alsina average/geometric aggregation operators, called CIFAAWA, CIFAAOWA, CIFAAHA, CIFAAWG, CIFAAOWG, and CIFAAHG operators with some well-known and desirable properties.
3) A procedure of decision-making technique is presented for finding the best type of artificial neural networks with the help of MADM problems based on CIF aggregation information.
4) We determined a numerical example for showing the rationality and advantages of the developed method by comparing their ranking values with the ranking values of many prevailing tools.

6.1. Limitations of the proposed model

The model of cubic intuitionistic fuzzy sets is very flexible but due to ambiguity and problems, they are not working in many places. For instance, when a person provides information in the form of yes, no, and abstinence, then the model of the CIF set has been failed. For this, we aim to compute the model of cubic picture fuzzy sets and their extensions.

6.2. Future directions

In the future, we will extend the Aczel-Alsina operators to complex cubic intuitionistic fuzzy, Pythagorean fuzzy, q-rung orthopair fuzzy, and their extensions. Further, we will also concentrate on their application in green supply chain management, artificial intelligence, road signals, and decision-making problems.

Author contributions

Chunxiao Lu: Conceptualization, methodology, investigation, validation, writing–review and editing; Zeeshan Ali: methodology, formal analysis, validation, software, writing–original draft preparation; Peide Liu: Conceptualization, software, supervision, fund, writing–review and editing.

Conflict of interest

About the publication of this manuscript, the authors declare that they have no conflict of interest.

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