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Research article

On $(n_1, \cdots, n_m)$-hyponormal tuples of Hilbert space operators

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Abstract: This paper introduces a new class of multivariable operators called $(n_1, \cdots, n_m)$-hyponormal tuples, which combine joint normal and joint hyponormal operators. A tuple of operators $Q = (Q_1, \cdots, Q_m)$ is said to be an $(n_1, \cdots, n_m)$-hyponormal tuple for some $(n_1, \cdots, n_m) \in \mathbb{N}^m$ if

$$\sum_{1\leq k,\, l\leq m} \langle [Q_k^{*n_k},\; Q_l^{n_l}]\omega_k \mid \omega_l\rangle \geq 0, \quad \forall\ (\omega_k)_{1\leq k\leq m} \in \mathcal{K}^m.$$

We show several properties of this class that correspond to the properties of joint hyponormal operators.

Keywords: Hilbert space; normal operator; n-hyponormal operator; commuting tuple of operators
Mathematics Subject Classification: 47A13

1. Introduction

Throughout this work, we will denote by $\mathcal{B}(\mathcal{K})$ the algebra of bounded linear operators acting on a complex Hilbert space $\mathcal{K}$. For $Q \in \mathcal{B}(\mathcal{K})$, we denote by $\ker(Q)$ and Q^* for the null space and the operator adjoint of Q, respectively. An operator $Q \in \mathcal{B}(\mathcal{K})$ is said to be normal if $Q^*Q = QQ^*$ [10, 18, 20], hyponormal if $[Q^*, Q] := Q^*Q - QQ^* \geq 0$ ($\|Q\omega\| \geq \|Q^*\omega\|\ \forall\ \omega \in \mathcal{K}$) [7, 22]). Note that

$$[Q^*, Q] \geq 0 \Longleftrightarrow \langle [Q^*, Q]\,\omega \mid \omega\rangle \geq 0 \quad \forall\ \omega \in \mathcal{K}.$$

The authors in [14] have introduced the concept of n-hyponormality for some positive integer n as follows: an operator Q is said to be n-hyponormal if $[Q^{*n}, Q^n] \geq 0$, or equivalently ($\|Q^n\omega\| \geq \|Q^{*n}\omega\|\ \forall\ \omega \in \mathcal{K}$). Note that

$$[Q^{*n}, Q^n] \geq 0 \Longleftrightarrow \langle [Q^{*n}, Q^n]\,\omega \mid \omega\rangle \geq 0 \quad \forall\ \omega \in \mathcal{K}.$$

We invite the reader to reading [14, 15, 24] for more details on this topic.

In recent years, the study of some concepts of operators theory in several variables has been studied at several levels by many authors, based on studies carried out on the theory of operators in one variable (see [1–3, 8, 9]). We mention here the following concepts related to our study, namely, joint normality, joint hyponormality and joint quasihyponormality. A tuple $Q = (Q_1, \cdots, Q_m) \in \mathcal{B}(\mathcal{K})^m$, is said to be joint normal [4–6] if Q satisfies the following conditions:

$$\begin{cases} Q_l Q_k = Q_k Q_l \quad \forall\ (l,\ k) \in \{1, \cdots, m\}^2, \\ \\ \left[Q_k^*, Q_k\right] = 0 \quad k \in \{1, \cdots, m\}. \end{cases}$$

However, $Q = (Q_1, \cdots, Q_m)$ is said to be a joint hyponormal ([4]) if

$$\sum_{1 \le l,\ k \le m} \langle [Q_k^*,\ Q_l]\omega_k \mid \omega_l \rangle \ge 0, \quad \forall\ (\omega_k)_{1 \le k \le m} \in \mathcal{K}^m.$$

Note that $Q^* := (Q_1^*, \cdots, Q_m^*)$.

Recently, Sid Ahmed et al. [17] have introduced the concept of joint m-quasihyponormal as follows: An tuple $Q = (Q_1,\ \cdots, Q_m) \in \mathcal{B}(\mathcal{K})^m$ is said to be a joint m-quasihyponormal if Q satisfies

$$\sum_{1 \le l,\ k \le m} \langle Q_k^*[Q_k^*,\ \ Q_l]Q_l\omega_k \mid \omega_l \rangle \ge 0, \quad \forall\ (\omega_k)_{1 \le k \le m} \in \mathcal{K}^m.$$

In this work, we present natural generalizations of joint hyponormalty to $(n_1, \cdots, n_m)$-hyponrmality and joint quasihyponormality to $(q_1, \cdots, q_m)$-quasi-$(n_1, \cdots, n_m)$-hyponrmality. $Q = (Q_1, \cdots, Q_m)$ is said to be an $(n_1, \cdots, n_m)$-hyponormal if

$$\sum_{1 \le l,\ k \le m} \left\langle [Q_k^{*n_k},\ Q_l^{n_l}]\omega_k \mid \omega_l \right\rangle \ge 0, \quad \forall\ (\omega_k)_{1 \le k \le m} \in \mathcal{K}^m,$$

for some $(n_1, \cdots, n_m) \in \mathbb{N}^m$, and it is said to be $(q_1, \cdots, q_m)$ quasi-$(n_1, \cdots, n_m)$-hyponrmal if

$$\sum_{1 \le l,\ k \le m} \langle Q_k^{*q_k}[Q_k^{*n_k},\ \ Q_l^{n_l}]Q_l^{q_l}\omega_k \mid \omega_l \rangle \ge 0, \quad \forall\ (\omega_k)_{1 \le k \le m} \in \mathcal{K}^m$$

for some $(n_1, \cdots, n_m) \in \mathbb{N}^m$ and $(q_1, \cdots, q_m) \in \mathbb{N}^m$.

Additional references regarding tuples of operators are cited here [12, 16, 19, 21, 23].

2. Class of $(n_1, \cdots, n_m)$-Hyponormality of operators

In this section, the definition and properties corresponding to the $(n_1, \cdots, n_m)$-hyponormal tuples of operators are introduced.

Definition 2.1. Let $Q = (Q_1, \cdots, Q_m) \in \mathcal{B}(\mathcal{K})^m$. We say that Q is an $n = (n_1, \cdots, n_m)$-hyponormal tuple if the operator matrix $\left([Q_k^{*n_k},\ Q_l^{n_l}]\right)_{1 \le k,\ l \le m} = \left(Q_k^{*n_k}\ Q_l^{n_l} - Q_l^{n_l} Q_k^{*n_k}\right)_{1 \le k, l \le m}$ is positive on $\bigoplus_{1 \le k \le m} \mathcal{K}$ that is

$$\sum_{1 \le k,\ l \le m} \langle [Q_k^{*n_k},\ Q_l^{n_l}]\omega_k \mid \omega_l \rangle \ge 0, \qquad \text{for} \ \ (\omega_k)_{1 \le k \le m} \in \mathcal{K}^m. \tag{2.1}$$

It is clear from this definition that Q is an n-hyponormal tuple if

$$[Q^{*\mathbf{n}}, Q^{\mathbf{n}}] = \begin{pmatrix} \left[Q_1^{*n_1}, Q_1^{n_1}\right] & \left[Q_2^{*n_2}, Q_1^{n_1}\right] & \cdot\ \cdot\ \cdot & \left[Q_m^{*n_m}, Q_1^{n_1}\right] \\ \left[Q_1^{*n_1}, Q_2^{n_2}\right] & \left[Q_2^{*n_2}, Q_2^{n_2}\right] & \cdot\ \cdot\ \cdot & \left[Q_m^{*n_m}, Q_2^{n_2}\right] \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \left[Q_1^{*n_1}, Q_m^{n_m}\right] & \left[Q_2^{*n_2}, Q_m^{n_m}\right] & \cdot\ \cdot\ \cdot & \left[Q_m^{*n_m}, Q_m^{n_m}\right] \end{pmatrix}$$

is positive operator on $\oplus_{1\le i\le m}\mathcal{K} := \mathcal{K} \oplus \cdots \oplus \mathcal{K}$.

Remark 2.1. The following observations can be derived from Definition 2.1.
(i) When $m = 1$, then Q is an n-hyponormal if and only if $[Q^{*n},\ Q^n] \geq 0$. Note that

$$[Q^{*n},\ Q^n] \geq 0 \Longleftrightarrow \langle [Q^{*n},\ Q^n]\omega \mid \omega\rangle \geq 0 \Longleftrightarrow \|Q^n\omega\| \geq \|Q^{*n}\omega\| \quad \forall\ \omega \in \mathcal{K}.$$

(ii) If $m = 2$, then $Q = (Q_1, Q_2)$ is $n = (n_1, n_2)$-hyponormal pair if and only if

$$[Q^{*n}, Q^n] = \begin{pmatrix} [Q_1^{*n_1}, Q_1^{n_1}] & [Q_1^{*n_1}, Q_2^{n_2}] \\ [Q_2^{*n_2}, Q_1^{n_1}] & [Q_2^{*n_2}, Q_2^{n_2}] \end{pmatrix} \geq 0,$$

which can be expressed as:

$$\begin{aligned} & \left\langle [Q_1^{*n_1}, (Q_1^{n_1}]\omega_1 \mid \omega_1 \right\rangle + \left\langle [Q_1^{*n_1}, (Q_2^{n_2}]\omega_1 \mid \omega_2 \right\rangle \\ + \quad & \left\langle [Q_2^{*n_2}, Q_1^{n_1}]\omega_2 \mid \omega_1 \right\rangle + \left\langle [Q_2^{*n_2}, Q_2^{n_2}]\omega_2 \mid \omega_2 \right\rangle \\ \geq \quad & 0 \end{aligned}$$

for all $(\omega_1, \omega_2) \in \mathcal{K}^2$.

Example 2.1. Let $Q = (Q_1, \cdots, Q_m)$ such that each Q_k be n_k-hyponomal for $k = 1, \cdots, m$. If $[Q_l^*, Q_k] = 0$ for $k \neq l$. Then, $Q = (Q_1, \cdots, Q_m)$ is an $n = (n_1, \cdots, n_m)$-hyponormal tuple.

Taking into consideration $[Q_l^*, Q_k] = 0$ for $k \neq l$ and Q_k is an n_k-hyponprmal, we may rewrite

$$\begin{aligned} \sum_{1\le k,\ l\le m} \langle [Q_k^{*n_k},\ Q_l^{n_l}]\omega_k \mid \omega_l\rangle &= \sum_{1\le k\le m} \langle [Q_k^{*n_k},\ Q_k^{n_k}]\omega_k \mid \omega_k\rangle \\ &\geq 0, \quad \text{for } (\omega_k)_{1\le k\le m} \in \mathcal{K}^m. \end{aligned}$$

The following theorem introduces a characterization for the studied class of multivariable operators.

Theorem 2.1. *A tuple $Q = (Q_1, \cdots, Q_m) \in \mathcal{B}(\mathcal{K})^m$ is an $(n_1, \cdots, n_m)$-hyponormal tuple if and only if*

$$\sum_{1\le k,l\le m} \left\langle Q_k^{n_k}\omega_l \mid Q_l^{n_l}\omega_k \right\rangle - \left\| \sum_{1\le k\le m} Q_k^{*n_k}\omega_k \right\|^2 \geq 0 \tag{2.2}$$

for every $\omega_1, \cdots, \omega_m \in \mathcal{K}$.

Proof. We observe that

$$\sum_{1\leq k,\, l\leq m} \left\langle \left[Q_l^{*n_l},\ Q_k^{n_k} \right] \omega_l \mid \omega_k \right\rangle \geq 0, \quad \forall\ \omega_1, \cdots, \omega_m \in \mathcal{K}.$$

And so it is

$$\begin{aligned}
& \sum_{1\leq k,\, l\leq m} \left\langle \left(Q_l^{*n_l} Q_k^{n_k} - Q_k^{n_k} Q_l^{*n_l} \right) \omega_l \ \mid\ \omega_k \right\rangle \geq 0 \\
\Leftrightarrow \quad & \sum_{1\leq k,\, l\leq m} \left\langle Q_l^{*n_l} Q_k^{n_k} \omega_l \mid \omega_k \right\rangle - \sum_{1\leq k,\, l\leq m} \left\langle Q_k^{n_k} Q_l^{*n_l} \omega_l \mid \omega_l \right\rangle \geq 0 \\
\Leftrightarrow \quad & \sum_{1\leq k,\, l\leq m} \left\langle Q_k^{n_k} \omega_l \mid Q_l^{n_l} \omega_k \right\rangle - \sum_{1\leq k,\, l\leq m} \left\langle Q_l^{*n_l} \omega_l \mid Q_k^{*n_k} \omega_k \right\rangle \geq 0 \\
\Longleftrightarrow \quad & \sum_{1\leq k,\, l\leq m} \left\langle Q_k^{n_k} \omega_l \mid Q_l^{n_l} \omega_k \right\rangle - \left\langle \sum_{1\leq l\leq m} Q_l^{*n_l} \omega_l \mid \sum_{1\leq k\leq m} Q_k^{*n_k} \omega_k \right\rangle \geq 0 \\
\Longleftrightarrow \quad & \sum_{1\leq k,\, l\leq m} \left\langle Q_k^{n_k} \omega_l \mid Q_l^{n_l} \omega_k \right\rangle - \Big\| \sum_{1\leq l\leq m} Q_l^{*n_l} \omega_l \Big\|^2 \geq 0.
\end{aligned}$$

Thus, the desired equivalence is obtained. □

Remark 2.2. When you choose $n = (1, \cdots, 1)$, Theorem 2.1 coincides with [4, Remark 1].

Corollary 2.1. *Let $Q = (Q_1, \cdots, Q_m) \in \mathcal{B}(\mathcal{K})^m$ be an $(n_1, \cdots, n_m)$-hyponormal tuple of operators. Then,*

$$\bigcap_{1\leq k\leq m} \ker(Q_k^{n_k}) \subseteq \ker\Big(\sum_{1\leq k\leq m} Q_k^{*n_k} \Big).$$

Proof. Let $\omega \in \bigcap_{1\leq k\leq m} \ker(Q_k^{n_k})$ and taking into account Theorem 2.1, we obtain

$$\underbrace{\sum_{1\leq k, l\leq m} \left\langle Q_k^{n_k} \omega \mid Q_l^{n_l} \omega \right\rangle}_{=0} - \Big\| \sum_{1\leq k\leq m} Q_k^{*n_k} \omega \Big\|^2 \geq 0.$$

Hence,

$$\Big\| \sum_{1\leq k\leq m} Q_k^{*n_k} \omega \Big\| = 0.$$

Consequently,

$$\sum_{1\leq k\leq m} Q_k^{*n_k} \omega = 0, \ \text{and so } \omega \in \ker\Big(\sum_{1\leq k\leq m} Q_k^{*n_k} \Big).$$

□

Remark 2.3. When $m = 1$, it is well known that if Q is an n-hyponormal single operator, then

$$\ker(Q^n) \subset \ker(Q^{*n}).$$

Proposition 2.1. *Let $Q \in \mathcal{B}(\mathcal{K})$ and consider $\widetilde{Q} = (Q, \cdots, Q)$. Then, $\widetilde{Q}$ is an $(n, \cdots, n)$-hyponormal tuple if and only if Q is an n-hyponormal.*

Proof. We have

$$\begin{aligned}
& Q \text{is } n\text{-hyopnromal} \\
\iff & \|Q^n\omega\| - \|Q^{*n}\omega\| \geq 0, \quad \forall \ \omega \in \mathcal{K} \\
\iff & \|Q^n(\omega_1 + \cdots + \omega_m)\|^2 - \|Q^{*n}(\omega_1 + \cdots + \omega_m)\|^2 \geq 0 \ \text{ for each collection } \omega_1, \cdots, \omega_m \in \|, \\
\iff & \langle Q(\sum_{1\leq l\leq m} \omega_l) \mid Q(\sum_{1\leq k\leq m} \omega_k)\rangle - \Big\| \sum_{1\leq k\leq m} Q^{*n}\omega_k \Big\|^2 \geq 0 \\
\iff & \sum_{1\leq k,\, l\leq m} \langle Q^n\omega_l \mid Q^n\omega_k\rangle - \Big\| \sum_{1\leq k\leq m} Q^{*n}\omega_k \Big\|^2 \geq 0 \\
\iff & \widetilde{Q} \text{ is } (n, \cdots, n) - \text{hyponormal tuple} \quad \text{(by Theorem 2.1)}.
\end{aligned}$$

□

Lemma 2.1. *Let $Q = (Q_1, \cdots, Q_m) \in \mathcal{B}(\mathcal{K})^m$, and let $\mu := (\mu_1, \cdots, \mu_m) \in \mathbb{C}^m$. If Q is an $(n_1, \cdots, n_m)$-hyponormal tuple. Then, $\mu Q := (\mu_1 Q_1, \cdots, \mu_m Q_m)$ is an $(n_1, \cdots, n_m)$-hyponormal tuple.*

Proof. Using some calculations and taking into account that $\mathcal{U}$ is an $(n_1, \cdots, n_m)$-hyponormal tuple, we have

$$\begin{aligned}
& \sum_{1\leq k,l\leq m} \langle(\mu_k Q_k)^{n_k}\omega_l \mid \mu_l Q_l)^{n_l}\omega_k\rangle - \Big\| \sum_{1\leq l\leq m} (\mu_l Q_l)^{*n_l}\omega_l \Big\|^2 \\
= & \sum_{1\leq k,l\leq m} \left\langle \overline{\mu}_l^{n_l} Q_k^{n_k}\omega_l \mid \overline{\mu_k}^{n_k} Q_l^{n_l}\omega_k \right\rangle - \Big\| \sum_{1\leq l\leq m} Q_l^{*n_l}\overline{\mu_l}^{n_l}\omega_l \Big\|^2 \\
= & \sum_{1\leq k,l\leq m} \left\langle Q_k^{n_k}(\overline{\mu}_l^{n_l}\omega_l) \mid Q_l^{n_l}(\overline{\mu}_k^{n_k}\omega_k) \right\rangle - \Big\| \sum_{1\leq l\leq m} Q_l^{*n_l}\overline{\mu_l}^{n_l}\omega_l \Big\|^2 \\
= & \sum_{1\leq k,l\leq m} \left\langle Q_k^{n_k}\psi_l \mid Q_l^{n_l}\psi_k \right\rangle - \Big\| \sum_{1\leq l\leq m} Q_l^{*n_l}\psi_l \Big\|^2 \\
\geq & \ 0.
\end{aligned}$$

□

Remark 2.4. The property of being $(n_1, \cdots, n_m)$-hyponormal for a tuple $Q = (Q_1, \cdots, Q_m)$ of operators is indeed invariant under permutations of the operators in Q.

The following proposition describes some properties of n-hyponormal m-tuples of operators.

Proposition 2.2. *Let $Q = (Q_1, \cdots, Q_m) \in \mathcal{B}(\mathcal{K})^m$ be an $(n_1, \cdots, n_m)$-hyponormal tuple. The following properties hold:*

1) If $\mathcal{N} \in \mathcal{B}(\mathcal{H})$ is a normal operator such that $\mathcal{N}$ commutes with each Q_k, then, $\mathcal{N}Q := (\mathcal{N}Q_K, \cdots, \mathcal{N}Q_m)$ is $(n_1, \cdots, n_m)$-hyponormal tuple.
2) For any unitary operator $\mathcal{W} \in \mathcal{B}(\mathcal{K})$, the tuples $\mathcal{W}Q\mathcal{W}^* := (\mathcal{W}Q_1\mathcal{W}^*, \cdots, \mathcal{W}Q_m\mathcal{W}^*)$ is $(n_1, \cdots, n_m)$-hyponormal tuple.

Proof. 1) Given that $\mathcal{N}$ is a normal operator for which $\mathcal{N}Q_k = Q_k\mathcal{N}$ for $k = 1, \cdots, m$, and referring to Fuglede-Putnam theorem [11], we obtain $\mathcal{N}^*Q_k = Q_k\mathcal{N}^*$. Based on these statements, we can obtain the relationships

$$\begin{aligned}
& \sum_{1\leq k,l\leq m} \langle (\mathcal{N}Q_k)^{n_k}\omega_l \mid (\mathcal{N}Q_l)^{n_l}\omega_k\rangle - \Big\| \sum_{1\leq k\leq m} (\mathcal{N}Q_k)^{*n_k}\omega_k\Big\|^2 \\
= & \sum_{1\leq k,l\leq m} \left\langle \mathcal{N}^{*n_l}Q_k^{n_k}\omega_l \mid \mathcal{N}^{*n_k}Q_l^{n_l}\omega_k\right\rangle - \| \sum_{1\leq k\leq m} \mathcal{N}^{*n_k}Q_k^{*n_k}\omega_k\|^2 \\
= & \sum_{1\leq k,l\leq m} \left\langle Q_k^{n_k}(\mathcal{N}^{*n_l}\omega_l) \mid Q_l^{n_l}(\mathcal{N}^{*n_l}\omega_k)\right\rangle - \| \sum_{1\leq k\leq m} Q_k^{*n_k}\mathcal{N}^{*n_k}\omega_k\|^2 \\
= & \sum_{1\leq k,l\leq m} \left\langle Q_k^{n_k}\omega_l' \mid Q_l^{n_l}\omega_k'\right\rangle - \| \sum_{1\leq k\leq m} Q_k^{*n_k}\omega_k'\|^2 \\
\geq & \; 0.
\end{aligned}$$

Therefore, $\mathcal{N}Q := (\mathcal{N}Q_1, \cdots, \mathcal{N}Q_m)$ is n-hyponormal tuple. From it, the desired results are produced.

2) Suppose any unitary operator $\mathcal{V} \in \mathcal{B}(\mathcal{K})$ such that,

$$\begin{aligned}
[(\mathcal{V}Q_l\mathcal{V}^*)^{*n_l},\ (\mathcal{V}Q_k\mathcal{V}^*)^{n_k}] &= [\mathcal{V}Q_l^{*n_l}\mathcal{V}^*,\ \mathcal{V}Q_k^{n_k}\mathcal{V}^*] \\
&= \mathcal{V}[Q_l^{*n_l}, Q_k^{n_k}]\mathcal{V}^*.
\end{aligned}$$

Hence, for each collection $\omega_1, \cdots, \omega_m \in \mathcal{K}$, we have

$$\begin{aligned}
\sum_{1\leq k,\ l\leq m} \langle [(\mathcal{V}Q_l\mathcal{V}^*)^{*n_l},\ (\mathcal{V}Q_k\mathcal{V}^*)^{n_k}]\omega_l \mid \omega_k\rangle &= \sum_{1\leq k,\ l\leq m} \langle \mathcal{V}[Q_l^{*n_l}, Q_k^{n_k}]\mathcal{V}^*\omega_l \mid \omega_k\rangle \\
&= \sum_{1\leq i,\ j\leq m} \langle [Q_l^{*n_l}, Q_k^{n_k}]\mathcal{V}^*\omega_l \mid \mathcal{V}^*\omega_k\rangle \\
&\geq 0.
\end{aligned}$$

Which is ends of the proof □

The following theorem generalizes the statement (1) of Proposition 2.2.

Theorem 2.2. *Let $Q = (Q_1, \cdots, Q_m) \in \mathcal{B}(\mathcal{K})^m$ and $\mathcal{W} = (\mathcal{W}_1, \cdots, \mathcal{W}_m) \in \mathcal{B}(\mathcal{K})^m$ for which the following conditions are satisfied*

$$\begin{cases}
\mathcal{W}_kQ_l = Q_l\mathcal{W}_k & \text{for all } k, l \in \{1, \cdots, m\}, \\
\mathcal{W}_k^*Q_l = Q_l\mathcal{W}_k^* & \text{for all } k, l \in \{1, \cdots, m\}, \\
\mathcal{W}_k^*\mathcal{W}_l = \mathcal{W}_l^*\mathcal{W}_k & \text{for all } k, l \in \{1, \cdots, m\}.
\end{cases} \tag{2.3}$$

If Q is an $(n_1, \cdots, n_m)$-hyponormal tuple, then $\mathcal{W}Q := (\mathcal{W}_1Q_1, \cdots, \mathcal{W}_mQ_m)$ is too.

Proof. Let $\omega_1, \cdots, \omega_m \in \mathcal{K}$, and taking into account (2.3), we may write

$$\langle [(\mathcal{W}_lQ_l)^{*n_l},\ (\mathcal{W}_kQ_k)^{n_k}]\omega_l \mid \omega_k\rangle = \left\langle \left((\mathcal{W}_kQ_l)^{*n_l}(\mathcal{W}_kQ_k)^{n_k} - (\mathcal{W}_kQ_k)^{n_k}(\mathcal{W}_lQ_l)^{*n_l}\right)\omega_l \mid \omega_k\right\rangle$$

$$= \left\langle \left(Q_l^{*n_l} \mathcal{W}_l^{*n_l} \mathcal{W}_k^{n_k} Q_k^{n_k} - \mathcal{W}_k^{n_k} Q_k^{n_k} Q_l^{*n_l} \mathcal{W}_L^{*n_L} \right) \omega_l \mid \omega_k \right\rangle$$
$$= \langle \mathcal{W}_k^{n_k} [Q_l^{*n_l}, Q_k^{n_k}] \mathcal{W}_l^{*n_l} \omega_l \mid \omega_k \rangle.$$

We have

$$\begin{aligned}
\sum_{1\le k,\, l\le m} \langle [(\mathcal{W}_l Q_l)^{*n_l},\ (\mathcal{W}_k Q_k)^{n_k}] \omega_l \mid \omega_k \rangle &= \sum_{1\le k,\, l\le m} \langle \mathcal{W}_k^{n_k} [Q_l^{*n_l}, Q_k^{n_k}] \mathcal{W}_l^{*n_l} \omega_l \mid \omega_k \rangle \\
&= \sum_{1\le k,\, l\le m} \langle \mathcal{W}_l^{n_l} [Q_l^{*n_l}, Q_k^{n_k}] \mathcal{W}_l^{*n_l} \omega_l \mid \omega_k \rangle \\
&= \sum_{1\le k,\, l\le m} \langle [Q_l^{*n_l}, Q_k^{n_k}] \mathcal{W}_l^{*n_l} \omega_l \mid \mathcal{W}_k^{*n_k} \omega_k \rangle \\
&= \sum_{1\le k,\, l\le m} \langle [Q_l^{*n_l}, Q_k^{n_k}] \mathcal{W}_l^{*n_l} \omega_l \mid \mathcal{W}_k^{*n_k} \omega_k \rangle \\
&= \sum_{1\le k,\, l\le m} \langle [Q_l^{*n_l}, Q_k^{n_k}] \psi_l \mid \psi_k \rangle,
\end{aligned}$$

where $\psi_k = \mathcal{W}_k^{*n_k} \omega_i$ for $i = 1, \cdots, m$ and $\omega_1, \cdots, \omega_m \in \mathcal{K}$.

In view of the fact that Q is an $(n_1, \cdots, n_m)$-hyponorml tuple, we can obtain

$$\sum_{1\le k,\, l\le m} \langle [(\mathcal{W}_l Q_l)^{*n_l},\ (\mathcal{W}_k Q_k)^{n_k}] \omega_l \mid \omega_k \rangle \ge 0.$$

This completes the proof. $\square$

Theorem 2.3. *Let $\mathcal{N} \in \mathcal{B}(\mathcal{K})$ be an invertible operator and $Q = (Q_1 \cdots, Q_m) \in \mathcal{B}(\mathcal{H})^m$ be a tuple of operators such that each Q_k commutes with $\mathcal{N}^*\mathcal{N}$ for $k = 1, \cdots, m$. Then, $Q = (Q_1 \cdots, Q_m)$ is an $(n_1, \cdots, n_m)$-hyponormal tuple if and only if*

$\mathcal{N}Q\mathcal{N}^{-1} := (\mathcal{N}Q_1\mathcal{N}^{-1}, \cdots, \mathcal{N}Q_m\mathcal{N}^{-1})$ is an $(n_1, \cdots, n_m)$-hyponormal tuple.

Proof. Assume that $Q = (Q_1, \cdots, Q_m)$ is an $(n_1, \cdots, n_m)$-hyponormal tuple. We need to show that $\mathcal{N}Q\mathcal{N}^{-1} := (\mathcal{N}Q_1\mathcal{N}^{-1}, \cdots, \mathcal{N}Q_m\mathcal{N}^{-1})$ is an $(n_1, \cdots, n_m)$-hyponormal tuple. In fact, let $\omega_1, \cdots, \omega_m \in \mathcal{H}$, we have

$$\begin{aligned}
&\sum_{1\le k,\, l\le m} \left\langle [(\mathcal{N}Q_k\mathcal{N}^{-1})^{*n_k}, (\mathcal{N}Q_l\mathcal{N}^{-1})^{n_l}] \omega_k \mid \omega_l \right\rangle \\
= &\sum_{1\le k,\, l\le m} \left\langle [\mathcal{N}Q_k^{*n_k}\mathcal{N}^{-1}, \mathcal{N}Q_l^{n_l}\mathcal{N}^{-1}] \omega_k \mid \omega_l \right\rangle \\
= &\sum_{1\le k,\, l\le m} \left\langle \mathcal{N}[Q_k^{*n_k}, Q_l^{n_l}]\mathcal{N}^{-1} \omega_k \mid \omega_l \right\rangle \\
= &\sum_{1\le k,\, l\le m} \left\langle \mathcal{N}[Q_k^{*n_k}, Q_l^{n_l}]\mathcal{N}^{-1} \omega_k \mid \mathcal{N}\mathcal{N}^{-1}\omega_l \right\rangle \\
= &\sum_{1\le k,\, l\le m} \left\langle \mathcal{N}^*\mathcal{N}[Q_k^{*n_k}, Q_l^{n_l}]\mathcal{N}^{-1} \omega_k \mid \mathcal{N}^{-1}\omega_l \right\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{1\le k,\, l\le m} \left\langle [Q_k^{*n_k}, Q_l^{n_l}]\, \sqrt{\mathcal{N}^*\mathcal{N}}\mathcal{N}^{-1}\omega_k \mid \sqrt{\mathcal{N}^*\mathcal{N}}\mathcal{N}^{-1}\omega_l \right\rangle \\
&= \sum_{1\le k,\, l\le m} \left\langle [Q_k^{*n_k}, Q_l^{n_l}]\psi_k \mid \psi_l \right\rangle \quad (\psi_j = \sqrt{\mathcal{N}^*\mathcal{N}}\mathcal{N}^{-1}\omega_j) \\
&\ge 0.
\end{aligned}
$$

Conversely, assume that $\mathcal{N}Q\mathcal{N}^{-1} := (\mathcal{N}Q_1\mathcal{N}^{-1}, \cdots, \mathcal{N}Q_m\mathcal{N}^{-1})$ is an $(n_1, \cdots, n_m)$-hyponormal tuple. Set $Q_k = \mathcal{N}Q_k\mathcal{N}^{-1}$ for $k = 1, \cdots, m$. We can check that each Q_k commutes with $(\mathcal{N}^{-1})^*\mathcal{N}^{-1}$, and moreover

$$
(\mathcal{N}^{-1}Q_1(\mathcal{N}^{-1})^{-1}, \cdots, \mathcal{N}^{-1}Q_m(\mathcal{N}^{-1})^{-1}) = (\mathcal{N}^{-1}Q_1\mathcal{N}, \cdots, \mathcal{N}^{-1}Q_m\mathcal{N}) = (Q_1, \cdots, Q_m).
$$

Based on the first statement, we have $(\mathcal{N}^{-1}Q_1\mathcal{N}, \cdots, \mathcal{N}^{-1}Q_m\mathcal{N})$ is an $(n_1, \cdots, n_m)$-hyponormal tuple, and so it shall be $(Q_1, \cdots, Q_m)$ is an $(n_1, \cdots, n_m)$-hyponormal tuple. □

Definition 2.2. ([5]) An operator $Q = (Q_1, \cdots, Q_m) \in \mathcal{B}(\mathcal{H})^m$ is said to be $(n_1, \cdots, n_m)$-normal tuple if

$$
\begin{cases}
[Q_l^{n_l},\ Q_k^{n_k}] = 0, \quad \forall\ k, l \in \{1, \cdots, m\}, \\[2ex]
\left[Q_k^{*n_k},\ Q_k^{n_k}\right] = 0,\ \forall\ k \in \{1, \cdots, m\}.
\end{cases}
$$

Theorem 2.4. *Let $Q = (Q_1, \cdots, Q_m) \in \mathcal{B}(\mathcal{K})^m$ and $n = (n_1, \cdots, n_m) \in \mathbb{N}^m$. The following statements hold:*

1) If Q is an $(n_1, \cdots, n_m)$-hyponormal tuple, then Q^* is an $(n_1, \cdots, n_m)$-hyponormal tuple if and only if

$$
\left\langle \left[Q_l^{*n_l},\ Q_k^{n_k}\right]\omega \mid \omega\right\rangle = iIm\left\langle \left[Q_l^{*n_l},\ Q_k^{n_k}\right]\omega \mid \omega\right\rangle = 0,\ \ \forall\ \omega \in \mathcal{K},\ k, l = 1, \cdots, m.
$$

2) Assume that $Q = (Q_1, \cdots, Q_m)$ be commuting tuple of operators. If Q and Q^* are $(n_1, \cdots, n_m)$-hyponormal tuple, then Q is an $(n_1, \cdots, n_m)$-normal tuple.

Proof. 1) Let $k, l \in \{1, 2, \cdots, m\}$, we observe that

$$
\begin{aligned}
\left[(Q_l^{*n_l})^*,\ Q_k^{*n_k}\right] &= \left[Q_l^{n_l},\ Q_k^{*n_k}\right] \\
&= Q_l^{n_l}Q_k^{*n_k} - Q_k^{*n_k}Q_l^{n_l} \\
&= -\left[Q_k^{*n_k},\ Q_l^{n_l}\right].
\end{aligned}
$$

Assume that Q and Q^* are $(n_1, \cdots, n_m)$-hyponormal tuples. It follows that for each finite collections $\omega_1,\ \cdots, \omega_m \in \mathcal{K}$, we have

$$
\begin{cases}
\displaystyle\sum_{1\le k,\, l\le m} \left\langle \left[Q_l^{*n_l},\ Q_k^{n_k}\right]\omega_l \mid \omega_k\right\rangle \ge 0, \\[3ex]
\displaystyle\sum_{1\le k,\, l\le m} \left\langle \left[(Q_l^{*n_l})^*,\ Q_k^{*n_k}\right]\omega_l \mid \omega_k\right\rangle \ge 0.
\end{cases}
$$

Or equivalently,

$$\begin{cases} \sum_{1\le k,\ l\le m} \left\langle \left[Q_l^{*n_l},\ Q_k^{n_k}\right]\omega_l \mid \omega_k \right\rangle \ge 0, \\ \sum_{1\le k,\ l\le m} \left\langle \left[Q_k^{*n_k},\ Q_l^{n_l}\right]\omega_l \mid \omega_k \right\rangle \le 0. \end{cases}$$

For fixed couple (k_0, l_0), let $\omega_1, \cdots, \omega_m \in \mathcal{K}$ be chosen so that $\omega_p = 0$ for $p \notin \{k_0, l_0\}$. Applying the first inequality and the second inequality above to the tuple $(\omega_1, \cdots, \omega_m)$, we obtain respectively,

$$\left\langle \left[Q_{k_0}^{*n_{k_0}},\ U_{k_0}^{n_{k_0}}\right]\omega_{k_0} \mid \omega_{k_0}\right\rangle + \left\langle \left[Q_{l_0}^{*n_{l_0}},\ Q_{l_0}^{n_{l_0}}\right]\omega_{l_0} \mid \omega_{l_0}\right\rangle + \left\langle \left[Q_{l_0}^{*n_{l_0}},\ Q_{k_0}^{n_{k_0}}\right]\omega_{l_0} \mid \omega_{k_0}\right\rangle + \left\langle \left[Q_{k_0}^{*n_{k_0}},\ Q_{l_0}^{n_{l_0}}\right]\omega_{l_0} \mid \omega_{k_0}\right\rangle \ge 0,$$

and

$$\left\langle \left[Q_{k_0}^{*n_{k_0}},\ Q_{k_0}^{n_{k_0}}\right]\omega_{k_0} \mid \omega_{k_0}\right\rangle + \left\langle \left[Q_{l_0}^{*n_{l_0}},\ Q_{l_0}^{n_{l_0}}\right]\omega_{l_0} \mid \omega_{l_0}\right\rangle + \left\langle \left[Q_{l_0}^{*n_{l_0}},\ Q_{K_0}^{n_{k_0}}\right]\omega_{l_0} \mid \omega_{k_0}\right\rangle + \left\langle \left[Q_{k_0}^{*n_{k_0}},\ Q_{l_0}^{n_{l_0}}\right]\omega_{l_0} \mid \omega_{k_0}\right\rangle \le 0.$$

Thus, we have

$$\left\langle \left[Q_{k_0}^{*n_{k_0}},\ Q_{k_0}^{n_{k_0}}\right]\omega_{k_0} \mid \omega_{k_0}\right\rangle + \left\langle \left[Q_{l_0}^{*n_{l_0}},\ Q_{l_0}^{n_{l_0}}\right]\omega_{l_0} \mid \omega_{l_0}\right\rangle + \left\langle \left[Q_{l_0}^{*n_{l_0}},\ Q_{k_0}^{n_{k_0}}\right]\omega_{l_0} \mid \omega_{k_0}\right\rangle + \left\langle \left[Q_{k_0}^{*n_{k_0}},\ Q_{l_0}^{n_{l_0}}\right]\omega_{l_0} \mid \omega_{k_0}\right\rangle = 0$$

for each $\omega \in \mathcal{K}, k, l = 1, \cdots, m$.

Letting $\omega_{k_0} = \omega_{l_0} = \omega \in \mathcal{K}$, we obtain

$$\left\langle \left[Q_{k_0}^{*n_{k_0}},\ Q_{k_0}^{n_{k_0}}\right]\omega \mid \omega\right\rangle + \left\langle \left[Q_{l_0}^{*n_{l_0}},\ Q_{l_0}^{n_{l_0}}\right]\omega \mid \omega\right\rangle + \left\langle \left[Q_{l_0}^{*n_{l_0}},\ Q_{k_0}^{n_{k_0}}\right]\omega \mid \omega\right\rangle + \left\langle \left[Q_{k_0}^{*n_{k_0}},\ Q_{l_0}^{n_{l_0}}\right]\omega \mid \omega\right\rangle = 0 \quad (2.4)$$

for each$\omega \in \mathcal{K}$, $k, l = 1, \cdots, m$. Letting $\omega_{k_0} = \omega$ and $\omega_{l_0} = -\omega \in \mathcal{K}$, we obtain

$$\left\langle \left[Q_{k_0}^{*n_{k_0}},\ Q_{k_0}^{n_{k_0}}\right]\omega \mid \omega\right\rangle + \left\langle \left[Q_{l_0}^{*n_{l_0}},\ Q_{l_0}^{n_{l_0}}\right]\omega \mid \omega\right\rangle - \left\langle \left[Q_{l_0}^{*n_{l_0}},\ Q_{k_0}^{n_{k_0}}\right]\omega \mid \omega\right\rangle - \left\langle \left[Q_{k_0}^{*n_{k_0}},\ Q_{l_0}^{n_{l_0}}\right]\omega \mid \omega\right\rangle = 0, \quad (2.5)$$

for each $\omega \in \mathcal{K}, k, l = 1, \cdots, m$.

By combining (2.4) and (2.5), we obtain:

$$\left\langle \left[Q_{k_0}^{*n_{k_0}},\ Q_{k_0}^{n_{k_0}}\right]\omega \mid \omega\right\rangle + \left\langle \left[Q_{l_0}^{*n_{l_0}},\ Q_{l_0}^{n_{l_0}}\right]\omega \mid \omega\right\rangle = 0, \text{ for each } \omega \in \mathcal{K} \text{ and } k,\ l = 1, \cdots, m.$$

Hence,

$$\left\langle \left[Q_{l_0}^{*n_{l_0}},\ Q_{k_0}^{n_{k_0}}\right]\omega \mid \omega\right\rangle + \left\langle \left[Q_{k_0}^{*n_{k_0}},\ Q_{l_0}^{n_{l_0}}\right]\omega \mid \omega\right\rangle = 0, \text{ for each } \omega \in \mathcal{K} \text{ and } k,\ l = 1, \cdots, m.$$

Observing that

$$\left\langle \left[Q_{k_0}^{*n_{k_0}},\ Q_{l_0}^{n_{l_0}}\right]\omega \mid \omega\right\rangle = \left\langle \omega \mid \left[Q_{l_0}^{*n_{l_0}},\ Q_{k_0}^{n_{k_0}}\right]\omega\right\rangle = \overline{\left\langle \left[Q_{l_0}^{*n_{l_0}},\ Q_{k_0}^{n_{k_0}}\right]\omega \mid \omega\right\rangle},$$

this implying that

$$Re\langle \left[Q_l^{*n_l},\ Q_k^{n_k}\right]\omega \mid \omega\rangle = 0,\quad \forall\ \omega \in \mathcal{K},\ k,l = 1,\cdots,m.$$

Therefore, $\left[Q_l^{*n_l},\ Q_k^{n_k}\right]$ is purely imaginary.

Assume that $\left[Q_l^{*n_l},\ Q_k^{n_k}\right]$ is purely imaginary for all $k,l = 1.\cdots,m$. Thus,

$$Re\langle \left[Q_l^{*n_l},\ Q_k^{n_k}\right]\omega \mid \omega\rangle = 0,\quad \forall\ \omega \in \mathcal{K},\ k,l = 1,\cdots,m.$$

This means that

$$\langle \left[Q_l^{*n_l},\ Q_k^{n_k}\right]\omega \mid \omega\rangle + \langle \left[Q_k^{*n_k},\ Q_l^{n_l}\right]\omega \mid \omega\rangle = 0 \quad \text{for}\ \ k,l = 1,\cdots,m.$$

Let $\omega_1,\ \cdots,\omega_m \in \mathcal{K}$, and taking into account that

$$\sum_{1\le k,\ l\le m} \left\langle\left[Q_l^{*n_l},\ Q_k^{n_k}\right]\omega_l \mid \omega_k\right\rangle = -\sum_{1\le k,\ l\le m} \left\langle\left[Q_k^{*n_k},\ Q_l^{n_l}\right]\omega_l \mid \omega_k\right\rangle.$$

However,

$$\sum_{1\le k,\ l\le m} \left\langle\left[(Q_l^{*n_l})^*,\ Q_k^{*n_k}\right]\omega_l \mid \omega_k\right\rangle = -\sum_{1\le k,\ l\le m} \left\langle\left[Q_k^{*n_k},\ Q_l^{n_l}\right]\omega_l \mid \omega_k\right\rangle.$$

The above simplification shows:

$$\begin{aligned}\sum_{1\le k,\ l\le m} \left\langle\left[(Q_l^{*n_l})^*,\ Q_l^{*n_l}\right]\omega_l \mid \omega_k\right\rangle &= \sum_{1\le k,\ l\le m} \left\langle\left[Q_l^{*n_l},\ Q_k^{n_k}\right]\omega_l \mid \omega_k\right\rangle \\ &\ge 0.\end{aligned}$$

Which prove that Q^* is an $(n_1,\cdots,n_m)$-hyponormal tuple.

2) Obviously that

$$[Q_l, Q_k] = 0 \Longrightarrow \left[Q_l^{n_l}, Q_k^{n_k}\right] = 0 \quad \forall\ k,l \in \{1,\cdots,m\}.$$

Given that both Q and Q^* are $(n_1,\cdots,n_m)$-hyponormal, it follows that

$$\sum_{1\le k,\ l\le m} \langle [Q_k^{*n_k},\ Q_l^{n_l}]\omega_k \mid \omega_l\rangle = 0.$$

In particular,

$$\langle [Q_k^{*n_k},\ Q_k^{n_k}]\omega_k \mid \omega_k\rangle = 0 \quad \text{for}\ k = 1.\cdots,m.$$

□

Lemma 2.2. *([5]) Let $\mathcal{T},\mathcal{S} \in \mathcal{B}(\mathcal{K})$ and $p,q \in \mathbb{N}$. Then,*

$$[\mathcal{T}^p,\ \mathcal{S}^q] = \sum_{\substack{\alpha+\alpha' = p-1\\ \beta+\beta' = q-1}} \mathcal{T}^\alpha \mathcal{S}^\beta\, [\mathcal{T},\ \mathcal{S}]\, \mathcal{S}^{\beta'}\mathcal{T}^{\alpha'}.$$

Proposition 2.3. *Let $Q = (Q_1, \cdots, Q_m) \in \mathcal{B}(\mathcal{K})^m$ and $n = (n_1, \cdots, n_m) \in \mathbb{N}^m$. $Q = (Q_1, \cdots, Q_m)$ is an $(n_1, \cdots, n_m)$-hyponormal tuple, then $Q^r = (Q_1^{r_1}, \cdots, Q_m^{r_m})$ is too.*

Proof. If $q_r \in \{0, 1\}$ for all $k \in \{1, \cdots, m\}$, then $[(Q_k^{n_k})^{r_k}, (Q_l^{n_l})^{r_l}] = 0$ for all $k, l = 1, \cdots, m$.
Assume that $r_k > 1$ for all $k \in \{1, \cdots, m\}$, and taking into account Lemma 2.2

$$[(Q_l^{*n_l})^{r_l}, (Q_k^{n_k})^{r_k}] = \sum_{\substack{\alpha+\alpha'=r_l-1 \\ \beta+\beta'=r_k-1}} (Q_l^{*n_l})^{\alpha}(Q_k^{n_k})^{\beta}[Q_l^{*n_l},\ (Q_k^{n_k})](Q_k^{n_k})^{\alpha'}(Q_l^{*n_l})^{\beta'}.$$

We infer that

$$\begin{aligned}
&\sum_{1\le k,\ l\le m} \left\langle \left[(Q_l^{*n_l})^{r_l}, (Q_k^{n_k})^{r_k}\right] \omega_l \mid \omega_k \right\rangle \\
=\ &\sum_{1\le k,\ l\le m} \left\langle \left(\sum_{\substack{\alpha+\alpha'=q_l-1 \\ \beta+\beta'=q_i-1}} (Q_l^{*n_l})^{\alpha}(Q_k^{n_k})^{\beta}[Q_l^{*n_l},\ Q_l^{n_l}](Q_k^{n_k})^{\alpha'}(Q_l^{*n_l})^{\beta'} \right) \omega_l \mid \omega_k \right\rangle \\
=\ &\sum_{1\le k,\ l\le m} \sum_{\substack{\alpha+\alpha'=r_l-1 \\ \beta+\beta'=r_k-1}} \left(\left\langle \left([(Q_l^{*n_l}),\ (Q_l^{n_l})](Q_k^{n_k})^{\alpha'}(Q_l^{*n_l})^{\beta'} \right) \omega_l \mid (Q_k^{n_k})^{\beta}(Q_l^{n_l})^{\alpha}\omega_k \right\rangle \right).
\end{aligned}$$

Using the $(n_1, \cdots, n_m)$-hyponormality of Q, we obtain for all $(\omega_k)_{1\le k\le m} \in \mathcal{K}^m$

$$\sum_{1\le k,\ l\le m} \left\langle \left[Q_l^{*n_l},\ Q_k^{n_k}\right] \omega_l \mid \omega_k \right\rangle \ge 0,$$

which implies that

$$\sum_{1\le k,\ l\le m} \left\langle \left[(Q_l^{*n_l})^{r_l}),\ (Q_k^{n_k})^{r_k}\right] \omega_l \mid \omega_k \right\rangle \ge 0$$

for all $(\omega_k)_{1\le k\le m} \in \mathcal{K}^m$. Therefore, Q^r is an $(n_1, \cdots, n_m)$-hyponormal tuple. □

3. Conclusions

This paper introduces and explores the concept of $(n_1, \cdots, n_m)$-hyponormal tuples, a new class of multivariable operators that integrates the notions of joint normal and joint hyponormal operators. The study demonstrates that $(n_1, \cdots, n_m)$-hyponormal tuples inherit several important properties from joint hyponormal operators. This new class of operators not only enriches the theory of multivariable operators but also provides a framework for further exploration and analysis of operator tuples.

Author contributions

Sid Ahmed Ould Beinane and Sid Ahmed Ould Ahmed Mahmoud: conceptualization, validation, formal analysis, supervision, writing-review and editing. All authors contributed equally to the writing of this article. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

The authors affirm that they have no conflicts of interest to disclose.

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