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Research article

Exponential decay in a delayed wave equation with variable coefficients

Waled Al-Khulaifi[1,2,*], **Manal Alotibi**[1,3] **and Nasser-Eddine Tatar**[1,4]

[1] King Fahd University of Petroleum & Minerals, Department of Mathematics, Dhahran, 31261, Saudi Arabia

[2] Interdisciplinary Research Center for Construction and Building Materials, KFUPM, Dhahran, 31261, Saudi Arabia

[3] Center for Integrative Petroleum Research, KFUPM, Dhahran, 31261, Saudi Arabia

[4] Interdisciplinary Research Center for Intelligent Manufacturing and Robotics, KFUPM, Dhahran, 31261, Saudi Arabia

* **Correspondence:** Email: waled.alkhulaifi@kfupm.edu.sa.

Abstract: We establish an exponential stability result for a wave equation that includes weighted coefficients of structural damping and a delayed term. This study reveals cases where the delayed term may not be dominated by the damping term, yet the system is exponentially stable. Our coefficients do not obey necessarily the conditions that are usually imposed in the literature.

Keywords: wave equation; energy decay; time delay; strong damping; exponential stability
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1. Introduction

It is known that delays, in general, cause instability, chaos, and damage to structures [1]. This has brought considerable attention to many researchers who have tried possible treatments for the problem that may control these delay factors. These studies, generally speaking, among others, led to a sufficient condition on the dominance of the damping term on the delayed term to ensure the exponential stability. In this paper, we study the problem

$$\begin{cases} u_{tt} = \Delta u + \mu_1(t)\Delta u_t - \mu_2(t)u_t(t-\tau), & \text{in } \Omega \times \mathbb{R}^+, \\ u(x,t) = 0, & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & \text{in } \Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau), & \text{in } \Omega \times (0,\tau), \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^n$ ($n \geq 1$) with a smooth boundary $\partial\Omega$. The functions $\mu_1, \mu_2 : \mathbb{R}_+ \to (0, \infty)$ are bounded differentiable functions, and the constant $\tau > 0$ corresponds to the time lag. The functions u_0, u_1, and f_0 belong to appropriate spaces that will be determined later in our existence and uniqueness result. In the one-dimensional space, this problem describes the motion of a string with both ends fixed. The term Δu_t describes the structural damping of the object and is often called strong damping (which is a special case of Kelvin–Voigt damping), while the term $\mu_t(t-\tau)$ is the retarded time derivative of the state. The interaction between damping and time delay arises in many real-life models, such as signal processing [2] and vehicle suspension systems [3]. Delays are omnipresent and intrinsic in many processes and phenomena. For instance, in engineering, data are often collected by means of a sensor and then analyzed. Afterwards, a decision is sent to an actuator to apply it, involving a time lag that may not be negligible. Ignoring this time lag can lead to inaccurate results. Time retardation also exists in many fields, such as signal theory, drilling, milling, digital control, rotation, and even phenomena involving human reactions. For more applications on this topic, we refer to the books [4–6]. In the absence of the retarded term ($\mu_2(t) \equiv 0$), it is well known that the energy of the system decreases exponentially to zero as $t \to \infty$; we refer the reader to [7–9]. In the case where $\mu_1(t)$ and $\mu_2(t)$ are constants, Datko [10] showed that when $\mu_2 > 0$ and $\mu_1 = 0$, the system is unstable no matter how small μ_2 is, whether appearing in the main equation or even in the boundary feedback [11]. Xu, Yung, and Li [12] and Nicaise and Pignotti [13, 14] showed that the system could be stabilized by adding a linear frictional damping term $\mu_1 u_t$ (instead of Δu_t in (1.1)) under the condition that the weight of this term override the delay term, that is,

$$\mu_2 \leq \mu_1, \quad \forall t \geq 0. \tag{1.2}$$

This work has motivated many researchers to investigate various types of problems, for instance, viscoelastic wave equations of finite memory type [15, 20, 23] and infinite memory type [16], abstract evolution equations [17], nonlinear wave equations [18, 19, 21], and plate equations [22, 24].

In the case of non-constant coefficients with $\mu_1(t) > 0$, we mention the work of Benaissa, Benguessoum, and Messaoudi [25], who proved the well-posedness of the problem

$$u_{tt} = \Delta u - \mu_1(t) u_t - \mu_2(t) u_t(t - \tau),$$

and established an exponential decay result under the condition that there exist positive constants M, $\tilde{M}$, and $k < 1$ such that

$$\left| \frac{\mu_1'(t)}{\mu_1(t)} \right| < M, \quad |\mu_2(t)| < k\mu_1(t), \quad |\mu_2'(t)| < \tilde{M}\mu_1(t), \quad \forall t \geq 0. \tag{1.3}$$

In addition, they required $\mu_1(t)$ to be non-increasing. Barros, Nonato, and Raposo [26] extended this result to the case of time-varying delays ($\tau(t)$). Further, it has been shown that the viscoelastic damping $\int_0^t g(t-s)\Delta u(s)ds$ can also control the delay term and drive the system to equilibrium in an exponential fashion, we refer the reader to [16, 17, 27] and the references therein.

Our interest here is to study the interaction between the strong damping and the delay term and the possibility of obtaining exponential stability. A slightly different problem considered by Messouadi, Fareh, and Doudi [28], namely,

$$u_{tt} = \Delta u + \mu_1 \Delta u_t - \mu_2 \Delta u_t(t - \tau),$$

with the same initial and boundary conditions as in (1.1), and proved the existence and uniqueness of solutions. Moreover, they established that the energy decays exponentially if $|\mu_2| < \mu_1$. This result was beneficial in being implemented in different types of problems, for instance [29–31].
As far as we know, all studies suggest the dominance of the delay coefficient by one of the damping, for all time in order to reach the equilibrium state exponentially. Considering the impact of a delay factor possibly surpassing the controlling term, we shall explore in this work whether the system can still achieve stability. Our study, based on problem (1.1), reveals that, under specific conditions, exponential stability remains attainable even when the damping term does not consistently outweigh the delayed retarded term.

The paper is organized as follows: In Section 2, we outline our assumptions and provide a result on existence and uniqueness justifying our computation. Following that, in Section 3, we establish and prove our exponential stability result. Section 4 contains the numerical validation of our result.

2. Preliminaries

In this section, we prepare some material to rely on in establishing the proof of our main result in Section 3. Throughout this article, the notation $\|\cdot\|$ stands for the norm of L^2, and C_p stands for Poincaré's constant, which is the least constant such that $\|u\|^2 \le C_p\|\nabla u\|^2$ for all $u \in H_0^1(\Omega)$. Further, we impose the following two hypotheses:
(**A1**) The functions $\mu_1(t)$ and $\mu_2(t)$ are differentiable such that

$$0 < \underline{\mu_1} \le \mu_1(t) \le \overline{\mu_1} \quad \text{and} \quad 0 \le \mu_2(t) \le \overline{\mu_2}, \quad \forall t \ge 0.$$

(**A2**) The following inequality holds

$$\tau^2\overline{\mu_2}^2\overline{\mu_1} + 2\sqrt{2}\tau\overline{\mu_2}\sqrt{\tau\overline{\mu_2}C_p + C_p + \overline{\mu_2}^2C_p^2 + \overline{\mu_1}^2} < \underline{\mu_1}.$$

Remark 1. *From (**A2**) we conclude that $\tau\overline{\mu_2} < 1$.*

Remark 2. *The differentiability condition on $\mu_1(t)$ and $\mu_2(t)$ is to ensure the well-posdeness of* (1.1).

Remark 3. *Unlike [26], we do not require any monotonicity condition on $\mu_1(t)$ and $\mu_2(t)$. Moreover, the function $\mu_1(t)$ does not need to dominate $\mu_2(t)$ for all $t \ge 0$ as imposed in the literature [13, 21, 26]. An example of such functions is given in Section 4.*

Part of the main idea here is rewriting the main equation as

$$u_{tt} = \Delta u + \mu_1(t)\Delta u_t + \frac{\partial}{\partial t}\int_{t-\tau}^{t} \mu_2(s+\tau)u_t(s)\, ds - \mu_2(t+\tau)u_t(t),$$

and as a consequence, we obtain the new form

$$\frac{\partial}{\partial t}\left[u_t - \int_{t-\tau}^{t} \mu_2(s+\tau)\, u_t(s)\, ds\right] = \Delta u + \mu_1(t)\Delta u_t - \mu_2(t+\tau)u_t. \tag{2.1}$$

We define the modified energy functional

$$E(t) := \frac{1}{2}\int_\Omega \left(u_t - \int_{t-\tau}^{t} \mu_2(s+\tau)\, u_t(s)\, ds\right)^2\, dx + \frac{1}{2}\|\nabla u\|^2, \tag{2.2}$$

whereas classical energy is

$$\mathcal{E}(t) := \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla u\|^2, \quad t \geq 0. \tag{2.3}$$

We shall consider the functional $E(t)$ in (2.2). The passage from $E(t)$ to $\mathcal{E}(t)$ will be made clear at the end of the proof of our theorem.
For completeness, we state an existence and uniqueness theorem.

Theorem 2.1. *Assume that **(A1)** and **(A2)** are satisfied. Then, given $u_0, u_1 \in H_0^1(\Omega)$, $f_0 \in L^2(\Omega\times(-\tau, 0))$ and $T > 0$, there exists a unique weak solution to the problem* (1.1) *on* $[0, T)$ *such that*

$$u \in C([0, T); H_0^1(\Omega)) \cap C^1([0, T); L^2(\Omega)) \quad \textit{and} \quad u_t \in L^2([0, T); H_0^1(\Omega)).$$

For the proof, one can use the semigroup theory approach combining the arguments in [17, 28, 32].

3. Energy decay

Within this section, we reveal and prove the exponential decay result, which reads

Theorem 3.1. *Assume that **(A1)** and **(A2)** are fulfilled. Then, there exist positive constants C and k such that the classical energy $\mathcal{E}(t)$ satisfies, along the solution of* (1.1)*, the estimate*

$$\mathcal{E}(t) \leq C\mathcal{E}(0)e^{-kt}, \quad \forall t \geq 0.$$

For the construction of the proof, we craft the following technical lemmas.

Lemma 3.2. *The modified energy functional $E(t)$ satisfies the estimate*

$$\begin{aligned} E'(t) \leq & (\mu_2(t+\tau)\varepsilon_3 - 1)\mu_2(t+\tau)\|u_t\|^2 + \varepsilon_1\|\nabla u\|^2 + (\varepsilon_2 - 1)\mu_1(t)\|\nabla u_t\|^2 \\ & + \left(\frac{1}{\varepsilon_1} + \frac{\mu_1(t)}{\varepsilon_2}\right)\frac{\tau}{4}\int_{t-\tau}^{t} \mu_2^2(s+\tau)\|\nabla u_t(s)\|^2\, ds \\ & + \frac{\tau}{4\varepsilon_3}\int_{t-\tau}^{t} \mu_2^2(s+\tau)\|u_t(s)\|^2\, ds, \quad t \geq 0, \end{aligned}$$

along the solution of (1.1) *and for every positive constant* $\{\varepsilon_i\}_{i=1}^3$.

Proof. Multiplying Eq (2.1) by $u_t - \int_{t-\tau}^t \mu_2(s+\tau)u_t(s)\, ds$ and integrating over Ω one obtains

$$\begin{aligned} E'(t) = & -\mu_1(t)\|\nabla u_t\|^2 - \mu_2(t+\tau)\|u_t\|^2 + \int_\Omega \nabla u \cdot \int_{t-\tau}^t \mu_2(s+\tau)\,\nabla u_t(s)\, ds\, dx \\ & + \mu_1(t)\int_\Omega \nabla u_t \cdot \int_{t-\tau}^t \mu_2(s+\tau)\nabla u_t(s)\, ds\, dx \\ & + \mu_2(t+\tau)\int_\Omega u_t \int_{t-\tau}^t \mu_2(s+\tau)u_t(s)\, ds\, dx, \quad t \geq 0. \end{aligned}$$

Exploiting Young's inequality, we are able to find the three estimates below, for any positive constants ε_1, ε_2 and ε_3,

$$\int_\Omega \nabla u \cdot \int_{t-\tau}^t \mu_2(s+\tau)\,\nabla u_t(s)\, ds\, dx \leq \varepsilon_1\|\nabla u\|^2 + \frac{\tau}{4\varepsilon_1}\int_{t-\tau}^t \mu_2^2(s+\tau)\|\nabla u_t(s)\|^2\, ds,$$

the second one

$$\mu_1(t)\int_\Omega \nabla u_t\cdot\int_{t-\tau}^{t}\mu_2(s+\tau)\nabla u_t(s)\,ds\,dx$$
$$\le \varepsilon_2\mu_1(t)\|\nabla u_t\|^2+\frac{\tau\mu_1(t)}{4\varepsilon_2}\int_{t-\tau}^{t}\mu_2^2(s+\tau)\|\nabla u_t(s)\|^2\,ds,$$

and the last one

$$\int_\Omega \mu_2(t+\tau)u_t\int_{t-\tau}^{t}\mu_2(s+\tau)u_t(s)\,ds\,dx$$
$$\le \varepsilon_3\mu_2^2(t+\tau)\|u_t\|^2+\frac{\tau}{4\varepsilon_3}\int_{t-\tau}^{t}\mu_2^2(s+\tau)\|u_t(s)\|^2\,ds,\, t\ge 0.$$

Combining the three estimates leads to the result. □

Lemma 3.3. *The functional*

$$V_1(t):=e^{-\beta_1 t}\int_{t-\tau}^{t}\int_{s}^{t}e^{\beta_1(\sigma+\tau)}\mu_2^2(\sigma+\tau)\|u_t(\sigma)\|^2\,d\sigma\,ds,$$

satisfies, along the solution of (1.1) *and for any positive number* β_1*,*

$$V_1'(t)\le -\beta_1V_1(t)-\int_{t-\tau}^{t}\mu_2^2(s+\tau)\|u_t(s)\|^2\,ds+\tau\mu_2^2(t+\tau)e^{\beta_1\tau}\|u_t\|^2,\ \ t\ge 0.$$

Proof. Performing differentiation of the functional $V_1(t)$ with respect to time leads to

$$\begin{aligned}V_1'(t)=&-\beta_1V_1(t)-e^{-\beta_1 t}\int_{t-\tau}^{t}e^{\beta_1(\sigma+\tau)}\mu_2^2(\sigma+\tau)\|u_t(\sigma)\|^2\,d\sigma\\&+e^{-\beta_1 t}\int_{t-\tau}^{t}e^{\beta_1(t+\tau)}\mu_2^2(t+\tau)\|u_t(t)\|^2\,ds\\=&-\beta_1V_1(t)-e^{-\beta_1 t}\int_{t-\tau}^{t}e^{\beta_1(\sigma+\tau)}\mu_2^2(\sigma+\tau)\|u_t(\sigma)\|^2\,d\sigma\\&+\tau e^{\beta_1\tau}\mu_2^2(t+\tau)\|u_t\|^2,\quad t\ge 0.\end{aligned}\tag{3.1}$$

Since

$$e^{\beta_1 t}\le e^{\beta_1(\sigma+\tau)},\quad \sigma\in[t-\tau,t],$$

the integral term in (3.1) can be estimated as

$$-e^{-\beta_1 t}\int_{t-\tau}^{t}e^{\beta_1(\sigma+\tau)}\mu_2^2(\sigma+\tau)\|u_t(\sigma)\|^2\,d\sigma\le -\int_{t-\tau}^{t}\mu_2^2(s+\tau)\|u_t(s)\|^2\,ds,$$

for all $t\ge 0$. Hence the result follows. □

If we replace u_t by ∇u_t in Lemma 3.3 and perform a similar calculation, we obtain the following conclusion.

Lemma 3.4. *The functional*

$$V_2(t) := e^{-\beta_2\tau} \int_{t-\tau}^{t} \int_{s}^{t} e^{\beta_2(\sigma+\tau)} \mu_2^2(\sigma+\tau) \|\nabla u_t(\sigma)\|^2 \, d\sigma \, ds,$$

satisfies, along the solution of (1.1), *and for any positive number* β_2,

$$V_2'(t) \le -\beta_2 V_2(t) - \int_{t-\tau}^{t} \mu_2^2(s+\tau) \|\nabla u_t(s)\|^2 \, ds + \tau\mu_2^2(t+\tau) e^{\beta_2\tau} \|\nabla u_t\|^2, \ t \ge 0.$$

Lemma 3.5. *The functional*

$$V_3(t) := \int_\Omega u \left(u_t - \int_{t-\tau}^{t} \mu_2(s+\tau) u_t(s) \, ds \right) dx$$

satisfies, along the solution of (1.1), *and for every positive constant* $\{\varepsilon_i\}_{i=4}^6$,

$$\begin{aligned} V_3'(t) \le & \left(C_p + \varepsilon_4 C_p + \varepsilon_5 \overline{\mu_1} + \varepsilon_6 C_p \overline{\mu_2} \right) \|\nabla u_t\|^2 + \left(\frac{\overline{\mu_1}}{4\varepsilon_5} + \frac{C_p \overline{\mu_2}}{4\varepsilon_6} - 1 \right) \|\nabla u\|^2 \\ & + \frac{\tau C_p}{4\varepsilon_4} \int_{t-\tau}^{t} \mu_2^2(s+\tau) \|\nabla u_t(s)\|^2 \, ds, \ t \ge 0. \end{aligned}$$

Proof. Taking the derivative of $V_3(t)$ and recalling (2.1) lead to

$$\begin{aligned} V_3'(t) = & \|u_t\|^2 - \int_\Omega u_t \int_{t-\tau}^{t} \mu_2(s+\tau) u_t(s) \, ds \, dx - \|\nabla u\|^2 \\ & - \mu_1(t) \int_\Omega \nabla u \cdot \nabla u_t \, dx - \mu_2(t+\tau) \int_\Omega u u_t \, dx, \quad t \ge 0. \end{aligned}$$

With the aid of Young's and Poincaré's inequalities, one can reach the result. □

Now we introduce the Lyapunov functional

$$L(t) := E(t) + \sum_{k=1}^{3} \lambda_k V_k, \quad t \ge 0, \tag{3.2}$$

where λ_k, $k = 1, 2, 3$, are positive constants to be determined later.

Lemma 3.6. *There exist two positive constants* M_1 *and* M_2 *such that the functional* L *satisfies the equivalence relation*

$$M_1 E(t) \le L(t) \le M_2 \left(E(t) + \int_{t-\tau}^{t} \mu_2^2(s+\tau) \|\nabla u_t\|^2 \, dx \right), \quad t \ge 0.$$

Proof. First, we reorder the integrals in $V_1(t)$ as follows:

$$V_1(t) = e^{-\beta_1 t} \int_{t-\tau}^{t} \int_{s}^{t} e^{\beta_1(\sigma+\tau)} \mu_2^2(\sigma+\tau) \|u_t(\sigma)\|^2 \, d\sigma \, ds$$

$$= e^{-\beta_1 t}\int_{t-\tau}^{t}\int_{t-\tau}^{\sigma} e^{\beta_1(\sigma+\tau)}\mu_2^2(\sigma+\tau)\|u_t(\sigma)\|^2\,ds\,d\sigma$$
$$= e^{-\beta_1 t}\int_{t-\tau}^{t}(\sigma-t+\tau)e^{\beta_1(\sigma+\tau)}\mu_2^2(\sigma+\tau)\|u_t(\sigma)\|^2\,d\sigma$$
$$\leq \tau e^{\beta_1\tau}\int_{t-\tau}^{t}\mu_2^2(s+\tau)\|u_t(s)\|^2\,ds,\quad t\geq 0.$$

Similarly

$$V_2(t)\leq \tau e^{\beta_2\tau}\int_{t-\tau}^{t}\mu_2^2(s+\tau)\|\nabla u_t(s)\|^2\,ds,\quad t\geq 0.$$

It is obvious when using Young's and Poincaré's inequalities, along with assumption (**A1**), one can reach the following

$$L(t)\leq M_2\left[E(t)+\int_{t-\tau}^{t}\mu_2^2(s+\tau)\|\nabla u_t(s)\|^2\,ds\right],\quad t\geq 0.$$

On the other hand, we exploit Young's and Poincaré's inequalities as follow:

$$L(t)\geq\left(\frac{1}{2}-\lambda_3\delta\right)\int_\Omega\left(u_t-\int_{t-\tau}^{t}\mu_2(s+\tau)u_t(s)\,ds\right)^2\,dx$$
$$+\left(\frac{1}{2}-\frac{C_p\lambda_3}{4\delta}\right)\|\nabla u\|^2,\quad t\geq 0.$$

Taking $\delta = 1/4\lambda_3$ and $\lambda_3 < 1/\sqrt{2C_p}$, we obtain the other (left) relation. □

Proof of Theorem 3.1. Differentiating the functional $L(t)$ and gathering all the estimates from Lemmas 3.2–3.5 we obtain

$$L'(t)\leq\left[\mu_2^2(t+\tau)\varepsilon_3-\mu_2(t+\tau)+\tau\lambda_1\mu_2^2(t+\tau)e^{\beta_1\tau}\right]\|u_t\|^2$$
$$+\left[\varepsilon_1+\left(\frac{\overline{\mu_1}}{4\varepsilon_5}+\frac{C_p\overline{\mu_2}}{4\varepsilon_6}-1\right)\lambda_3\right]\|\nabla u\|^2$$
$$+\Big[(\varepsilon_2-1)\mu_1(t)+\tau\mu_2^2(t+\tau)\lambda_2e^{\beta_2\tau}$$
$$+\left(C_p+\varepsilon_4C_p+\varepsilon_5\overline{\mu_1}+\varepsilon_6C_p\overline{\mu_2}\right)\lambda_3\Big]\|\nabla u_t\|^2$$
$$+\left(\frac{\tau}{4\varepsilon_3}-\lambda_1\right)\int_{t-\tau}^{t}\mu_2^2(s+\tau)\|u_t(s)\|^2\,ds$$
$$+\left(\frac{\tau}{4\varepsilon_1}+\frac{\tau\mu_1(t)}{4\varepsilon_2}+\frac{\tau C_p\lambda_3}{4\varepsilon_4}-\lambda_2\right)\int_{t-\tau}^{t}\mu_2^2(s+\tau)\|\nabla u_t(s)\|^2\,ds$$
$$-\beta_1\lambda_1V_1(t)-\beta_2\lambda_2V_2(t),\quad t\geq 0.$$

At this stage, we choose the positive constants $\{\varepsilon_i\}_{i=1}^4$ and $\{\lambda_i\}_{i=1}^3$ and ignore β_1 and β_2 with the aim of satisfying the following inequalities

$$\overline{\mu_2}\varepsilon_3+\tau\overline{\mu_2}\lambda_1<1\tag{3.3}$$

$$\varepsilon_1 < \left(1 - \frac{\overline{\mu_1}}{4\varepsilon_5} - \frac{C_p\overline{\mu_2}}{4\varepsilon_6}\right)\lambda_3 \tag{3.4}$$

$$\tau\overline{\mu_2}^2\lambda_2 + \left(C_p + \varepsilon_4 C_p + \varepsilon_5\overline{\mu_1} + \varepsilon_6 C_p\overline{\mu_2}\right)\lambda_3 < (1-\varepsilon_2)\underline{\mu_1} \tag{3.5}$$

$$\frac{\tau}{4\varepsilon_3} < \lambda_1 \tag{3.6}$$

$$\frac{\tau}{4\varepsilon_1} + \frac{\tau\overline{\mu_1}}{4\varepsilon_2} + \frac{\tau C_p\lambda_3}{4\varepsilon_4} < \lambda_2. \tag{3.7}$$

Inequalities (3.3) and (3.6) imply that

$$\overline{\mu_2}\left(\varepsilon_3 + \frac{\tau^2}{4\varepsilon_3}\right) < 1, \tag{3.8}$$

which is satisfied when we let $\varepsilon_3 = \tau/2$ together with the assumption $\tau\overline{\mu_2} < 1$, which follows from (**A2**). Next, we pick $\varepsilon_5 = \overline{\mu_1}$ and $\varepsilon_6 = C_p\overline{\mu_2}$, and by these choices, inequality (3.4) is fulfilled if we choose $\lambda_3 > 2\varepsilon_1$ (we will revisit this selection after fixing ε_1 in order to satisfy the equivalence relation in Lemma 3.6).
Next, we let $\varepsilon_2 = 1/2$ and combine inequalities (3.5) and (3.7), connected by λ_2, to obtain

$$\tau\overline{\mu_2}^2\left(\frac{\tau}{2\varepsilon_1} + \tau\overline{\mu_1} + \frac{\tau C_p\varepsilon_1}{\varepsilon_4}\right) + 4\varepsilon_1 C_p + 4\varepsilon_1\varepsilon_4 C_p + 4\varepsilon_1 C_p^2\overline{\mu_2}^2 + 4\varepsilon_1\overline{\mu_1}^2 < \underline{\mu_1}. \tag{3.9}$$

Now, with the choice

$$\varepsilon_4 = \frac{\tau\overline{\mu_2}}{2} \quad \text{and} \quad \varepsilon_1 = \frac{\tau\overline{\mu_2}}{\sqrt{2\left(4\tau\overline{\mu_2}C_p + 4C_p + 4\overline{\mu_2}^2C_p^2 + 4\overline{\mu_1}^2\right)}},$$

inequality (3.9) becomes

$$\tau^2\overline{\mu_2}^2\overline{\mu_1} + 2\sqrt{2}\tau\overline{\mu_2}\sqrt{\tau\overline{\mu_2}C_p + C_p + \overline{\mu_2}^2C_p^2 + \overline{\mu_1}^2} < \underline{\mu_1},$$

which follows from our assumption (**A2**).
Finally, λ_3 needs to fulfill the inequalities $2\varepsilon_1 < \lambda_3 < 1/\sqrt{2C_p}$ which necessitates

$$\frac{2\tau\overline{\mu_2}}{\sqrt{4\tau\overline{\mu_2}C_p + 4C_p + 4\overline{\mu_2}^2C_p^2 + 4\overline{\mu_1}^2}} < \frac{1}{\sqrt{C_p}}. \tag{3.10}$$

Squaring both sides of (3.10) yields

$$4(\tau\overline{\mu_2})^2 - 4(\tau\overline{\mu_2}) - 4 - 4\overline{\mu_2}^2C_p - \frac{4\overline{\mu_1}^2}{C_p} < 0,$$

which is valid when $\tau\overline{\mu_2} < 1$.
By virtue of the right relation in Lemma 3.6, we may write

$$L'(t) < -\gamma L(t), \quad t \geq 0,$$

for some positive constant γ and therefore by integration over the interval $(0, t)$ yields

$$L(t) < L(0)e^{-\gamma t}, \quad t \geq 0.$$

This property is immediately inherited by $E(t)$ through the equivalence with $L(t)$, that is, $E(t) \leq C_1 e^{-\gamma t}$, $t \geq 0$. Now we need to pass to the classical energy $\mathcal{E}(t)$. To this end, we employ Minkowski inequality and the left-hand side relation in Lemma 3.6 to find

$$\begin{aligned} \|u_t\| &= \left\| u_t - \int_{t-\tau}^{t} \mu_2(s+\tau)u_t(s)\,ds + \int_{t-\tau}^{t} \mu_2(s+\tau)u_t(s)\,ds \right\| \\ &\leq \left\| u_t - \int_{t-\tau}^{t} \mu_2(s+\tau)u_t(s)\,ds \right\| + \left\| \int_{t-\tau}^{t} \mu_2(s+\tau)u_t(s)\,ds \right\| \\ &\leq \sqrt{2C_2}e^{-\gamma t/2} + \sqrt{\tau}\left(\int_{t-\tau}^{t} \mu_2^2(s+\tau)\|u_t(s)\|^2\,ds \right)^{1/2}, \text{ where } C_2 = \frac{C_1}{M_1}. \end{aligned}$$

Squaring both sides and Young's inequality with $\eta > 0$ leads to

$$\|u_t\|^2 \leq \left(1 + \frac{1}{\eta}\right) 2C_2 e^{-\gamma t} + (\eta + 1)\,\tau\overline{\mu_2}^2 \int_{t-\tau}^{t} \|u_t(s)\|^2\,ds,$$

or

$$\begin{aligned} e^{\gamma t}\|u_t\|^2 &\leq 2C_2\left(1 + \frac{1}{\eta}\right) + (\eta + 1)\,\tau\overline{\mu_2}^2 \int_{t-\tau}^{t} e^{\gamma(t-s)}e^{\gamma s}\|u_t(s)\|^2\,ds \\ &\leq 2C_2\left(1 + \frac{1}{\eta}\right) + (\eta + 1)\,\tau\overline{\mu_2}^2 \left(\frac{e^{\gamma\tau} - 1}{\gamma}\right) \sup_{t-\tau\leq s\leq t} e^{\gamma s}\|u_t(s)\|^2 \\ &\leq 2C_2\left(1 + \frac{1}{\eta}\right) + (\eta + 1)\,\tau\overline{\mu_2}^2 \left(\frac{e^{\gamma\tau} - 1}{\gamma}\right) \sup_{0\leq s\leq t} e^{\gamma s}\|u_t(s)\|^2, \; t \geq \tau. \end{aligned} \tag{3.11}$$

By replacing t by s in relation (3.11) and then taking the supremum of both sides, we claim that

$$\sup_{0\leq s\leq t} e^{\gamma s}\|u_t(s)\|^2 \leq \frac{2C_2\left(1 + \frac{1}{\eta}\right)}{1 - (\eta + 1)\,\tau\overline{\mu_2}^2 \left(\frac{e^{\gamma\tau}-1}{\gamma}\right)}.$$

Indeed, for small values of γ, the expression $\dfrac{e^{\gamma\tau} - 1}{\gamma\tau}$ is close to 1. Therefore, the relation

$$(\eta + 1)\,\tau\overline{\mu_2}^2 \left(\frac{e^{\gamma\tau} - 1}{\gamma}\right) = (\eta + 1)\,\tau^2\overline{\mu_2}^2 \left(\frac{e^{\gamma\tau} - 1}{\gamma\tau}\right) < 1$$

is true when $\tau^2\overline{\mu_2}^2 < 1$, which is guaranteed by assumption (**A2**) and for small η. Hence, we can conclude that

$$\|u_t\|^2 \leq C^* e^{-\gamma t}, \quad \forall t \geq \tau,$$

for some $C^* > 0$. Obviously, a similar estimation holds on $[0, \tau]$ as well. The proof is complete. □

4. Numerical analysis

In this section, we illustrate the exponential decay result stated in Theorem 3.1 through a numerical example. We approximate the solution of problem (1.1) using the finite difference method in time and spatial variables. In our numerical example, we let $\Omega = [0, 1]$ (1-dimension case), and the time interval is $[0, 10]$. The initial functions are $u_0(x) = \sin(x)\cos(\frac{\pi x}{2})$ and $u_1(x) = 0$. Moreover, based on the assumptions of Theorem 3.1, we choose $\mu_1(t) = 2 + \sin t$ and $\mu_2(t) = 2 + \cos t$ with $\tau = \frac{1}{72}, C_p = \frac{1}{4}$, (see Figure 1). We point out that the functions and constants chosen do not comply with (1.3).

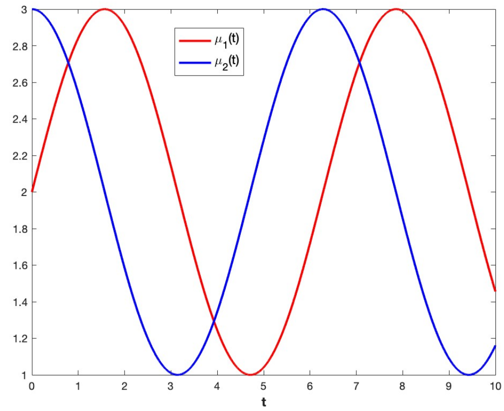

Figure 1. The weighted coefficients $\mu_1(t)$ and $\mu_2(t)$.

We plot the energy norm in Figure 2. Our approximate solution shows an exponential decay in the energy norm under the assumptions of Theorem 3.1. The solution evolving to the equilibrium state is shown in Figure 3.

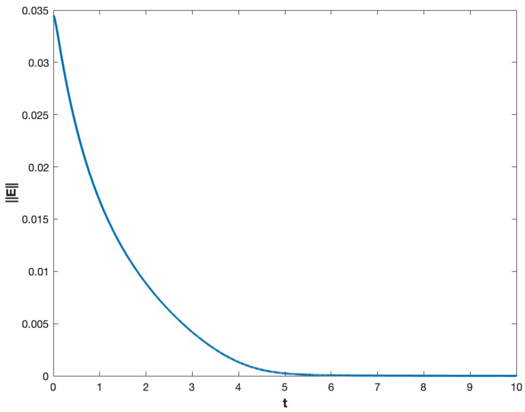

Figure 2. The exponential decay of the energy norm.

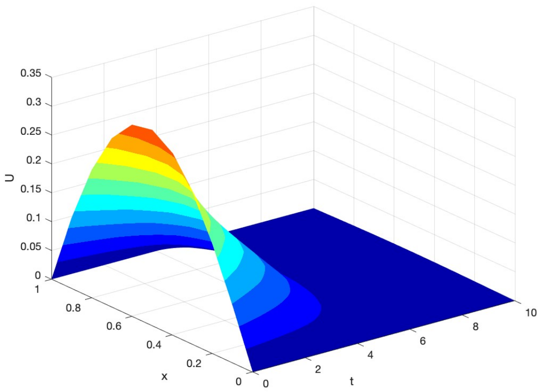

Figure 3. Solution's progression to equilibrium state.

Remark 4. *Considering the example provided, it becomes evident that the class of functions specified by condition* (1.3) *is not optimal; however, we are able to extend this in our assumptions. The question of finding the optimal range is very interesting to explore.*

5. Conclusions

In this paper, we establish that the system incorporating a delay term and weighted coefficient can achieve exponential stabilization by introducing a strong damping. Importantly, our approach eases the restrictive conditions found in existing literature, enabling us to include a larger class of functions. This study opens avenues for exploring the optimal relationship between the damping (whether it is strong or linear) and the delay term.

Author contributions

Waled Al-Khulaifi: Conceptualization, methodology, formal analysis, writing-original draft; Manal Alotibi: Formal analysis, validation, software, visualization; Nasser-Eddine Tatar: Conceptualization, methodology, formal analysis, writing-review and editing. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

The authors declare no conflicts of interest.

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