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Research article

Bipolar fuzzy INK-subalgebras of INK-algebras

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Abstract: This article presents a new idea for an extension of the fuzzy INK algebra called bipolar fuzzy INK subalgebra. The objective of this study is to define the features that distinguish bipolar fuzzy INK-subalgebras of INK-algebras. The algebraic operations on these sub-algebras are also studied. The thorough examination allows us to prove a number of theorems that shed light on the connections between the higher and lower-level sets related to these ideas. In addition, several related topics are thoroughly examined, and the idea of homomorphism for bipolar fuzzy INK sub-algebras is introduced.

Keywords: INK-algebra; fuzzy INK sub-algebra; bipolar fuzzy INK sub-algebra; homomorphism
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1. Introduction

Iseki and Tanaka [1,2] explored the intricate domains of BCK and BCI-algebras to learn more about their characteristics and applications. Future scholars will be able to recognize these algebraic structures and their innumerable applications because of their work. There is a large area of fuzzy sets that have practical applications in areas such as automotive systems, traffic signal control, and camera systems, and other researchers have utilized fuzzy logic to investigate further algebraic structures. The innovative INK algebra, introduced by Kaviyarasu and Indhira [3], offers a more thorough framework for algebraic analysis and is an extension of BCK/BCI-algebras. Our understanding of fuzzy logic in algebraic structures has been enhanced because of the fuzzy notions they have created inside this framework, such as fuzzy sub-algebras and fuzzy K-ideals in INK-algebras [4,5]. Expanding their scope to encompass interval-valued fuzzy sub-algebras of INK algebra, they introduced the concept of intuitionistic fuzzy INK algebra, thereby enhancing the practicality of fuzzy logic in algebraic contexts. To further illuminate the ever-changing connection between fuzzy logic and algebraic structures, they delved into the intricate realm of fuzzy p-ideals in INK-algebra [6,7]. Kaviyarasu and Indhira's work has expanded our understanding of fuzzy logic in algebra, both theoretically and practically, and has paved the way for further study in this interdisciplinary field.

Bipolar fuzzy INK sub-algebras are introduced, building on the foundational principles of INK-algebra. Based on Zadeh's groundbreaking work on fuzzy sets [8], Zhang initially proposed the concept of bipolar fuzzy sets [9–11]. No matter the nature of the polarization—positive and negative, honest and false, good and bad—these sets offer a versatile framework for describing the data. In a bipolar fuzzy (BF) set, each item in a given set is assigned two membership degrees: one for complying with the explicit property and another for conforming to the implicit counter-property. A degree of membership can be anywhere from -1 to 1, with 0 indicating that the element is completely unrelated to the linked property and has no affiliation whatsoever. Complete fulfillment of the property is indicated by a membership degree of 1, and complete fulfillment of the counter-property is also shown by a degree of -1. Uncertainties beyond those typically considered by conventional fuzzy set theory are taken into consideration by this approach. There are a lot of researchers that have explored various aspects of bipolar fuzzy configurations inside different algebraic structures; for example, Ahmad and Al-Masarwah [12,13] applied doubt bipolar fuzzy to BCK and BCI algebras. Numerous other scholars have also explored different facets of fuzzy indifferent algebras through the use of the bipolar fuzzy (BF) paradigm. Among others, Muhiuddin put up the idea of bipolar fuzzy KU-subagebras in KU-algebras [14], while Meng and Akram studied bipolar valued fuzzy ideals in BCK/BCI algebras [15]. As an extension of this, Muhiuddin et al. [16] developed new kinds of bipolar fuzzy ideals for BCK-algebras and expanded the work to encompass bipolar fuzzy implicative ideals. Further, Jana and Pal investigated fuzzy soft BCI-algebras with $(\in\alpha, \in\alpha \vee q\beta)$-dimensions [17], and then integrated the ideas of BFS and IFS sets to address decision-making challenges [18]. Similarly, Jana et al. presented the concept of BF soft algebras, which are BF point and ideal-based, inside the framework of BCI/BCK-algebras [19,20]. Jana et al. concentrated on $(\in, \in \vee q)$-bipolar fuzzy soft BCK algebras, taking into account BF soft sub-algebras and BF closed ideals [21], whereas Jana and Pal explored (α,β)-Union soft BCK/BCI-subalgebras and (α,β)-Union soft BCK/BCI-ideals [22]. Rupa et al. [23] used bipolarly valued fuzzy in d-algebras, and [24] investigated homomorphisms of anti-bipolar fuzzy ideals in d-algebras, going beyond this earlier research. Bipolar fuzzy sublattices and ideals were studied by Eswarlal and Venkata Kalyani, who also presented homomorphisms on bipolar vague normal groups and introduced the idea of bipolar vague cosets [25–27]. When it comes to reasoning and modeling with bipolar data, BF sets have been quite effective. In their work, Venkata Kalyani et al. presented BF magnified translations of groups [28], while Kawila et al. unveiled BF UP-subalgebras of UP-algebra and explored some algebraic features [29]. While Noori et al. proposed a new method to construct bipolar fuzzy sub-measure algebras [30], Mursaleen et al. investigated $(\in, \in \vee q)$-bipolar fuzzy b-ideals of BCK/BCI-algebras based on bipolar fuzzy points [31]. The unique extension of bipolar fuzziness is very important in several fields, including pattern detection, decision-making, and artificial intelligence. Bipolar fuzzy sets, for instance, are quite good at simulating the uncertainty and imprecision that people bring to decision-making when it comes to artificial intelligence. They are also good at bringing different perspectives from different parties to the table while trying to solve problems.

In addition, bipolar fuzzy sets may be used to represent ambiguous medical data in medical diagnosis. This helps to provide more accurate and reliable diagnostic findings, especially when symptoms are unclear or inconsistent and need interpretation. Due to the effectiveness of BFS in application areas as well as the development of algebraic structures in the BF environment, they play a significant role in fuzzy algebraic structures. Seeing the existing research knowledge on INK-algebras [3–7] and the effective roles of bipolar fuzzy sets [31–34], we are motivated to develop bipolar fuzzy INK-algebras, because there is no research on bipolar fuzzy INK-algebras in my knowledge. We foresee a significant improvement in the flexibility of our models. This helps us facilitate more refined decision-making and logical solutions to difficult problems. Furthermore, such integration has the capability to lead to ground-breaking developments in automated reasoning and information theory. This paper tries to fill the research gap to address bipolar fuzzy INK-subalgebras. By doing a comprehensive analysis, we are able to demonstrate several theorems that shed light on the relationships between the associated higher and lower-level sets. Furthermore, the notion of homomorphism for bipolar fuzzy INK sub-algebras is presented, and several related properties are investigated in detail.

Here is how the paper is structured: In Section 2, we introduce some basic definitions of INK-algebras that are essential for the proposed study. In Section 3, we established several fundamental algebraic operations and explored the idea of bipolar fuzzy INK-subalgebras and their related properties, intersection of BF INK subalgebra and its properties are established in Section 4. In Section 5, we have proposed union of BFINK-algebras and their details. Section 6 lays out the link between bipolar fuzzy set level sets and proves some related theorems. The homomorphism, epimorphism, and endomorphism theorems in Section 7 follow. In Section 8, outlines the study's practical applications and aims to shed insight on future directions of investigation.

2. **Preliminaries**

In this area I have discussed the basic preliminaries that I have used in my work.

Definition 2.1. [4] An algebra (U, ●, 0) is defined as an INK-algebra if it has to follow the situations for any $\tilde{b}_1, \tilde{b}_2, \tilde{b}_3 \in$ U,

INK-1: $((\tilde{b}_1 ● \tilde{b}_2) ● (\tilde{b}_1 ● \tilde{b}_3)) ● (\tilde{b}_3 ● \tilde{b}_2) = 0$.

INK-2: $((\tilde{b}_1 ● \tilde{b}_3) ● (\tilde{b}_2 ● \tilde{b}_3)) ● (\tilde{b}_1 ● \tilde{b}_2) = 0$.

INK-3: $\tilde{b}_1 ● 0 = \tilde{b}_1$.

INK-4: $\tilde{b}_1 ● \tilde{b}_2 = 0$ and $\tilde{b}_2 ● \tilde{b}_1 = 0$ implies $\tilde{b}_1 = \tilde{b}_2$.

Where "0" is a constant and "●" is recognized as the binary operation of U.

Definition 2.2. [4] If Y is a non-empty subset of an INK-algebra U, then $\tilde{b} ● \tilde{c} \in$ Y, where $\tilde{b}, \tilde{c} \in$ U then Y is called a sub-algebra of INK-algebra U.

Definition 2.3. [4] If ꝥ is a fuzzy set in a INK algebra U then ꝥ $(\tilde{b} ● \tilde{c}) \geqslant$ min {ꝥ($\tilde{b}$), ꝥ($\tilde{c}$)}, is called a fuzzy sub algebra of INK algebra U for all $\tilde{b}$, $\tilde{c} \in$ U.

Definition 2.4. [19] An algebra (U, ●, 0) of form (2, 0) is a BCI-algebra if it fulfills all $\tilde{b}_1$, $\tilde{b}_2$, $\tilde{b}_3 \in$ U.

(K1) $((\tilde{b}_1 ● \tilde{b}_2) ● (\tilde{b}_1 ● \tilde{b}_3)) ● (\tilde{b}_3 ● \tilde{b}_2) = 0$.

(K2) $(\tilde{b}_1 ● (\tilde{b}_1 ● \tilde{b}_2)) ● \tilde{b}_2 = 0$.

(K3) $\tilde{b}_1 ● \tilde{b}_1 = 0$.

(K4) $\tilde{b}_1 ● \tilde{b}_2$ 0 and $\tilde{b}_2 ● \tilde{b}_1 = 0 \Rightarrow \tilde{b}_1 = \tilde{b}_2$.

Definition 2.5. [19] If $\varepsilon=(\beta_{\varepsilon}^{+}, \beta_{\varepsilon}^{-})$ and $\mathfrak{z}=(\beta_{\mathfrak{z}}^{+}, \beta_{\mathfrak{z}}^{-})$ are two BF subsets in bipolar fuzzy set U for which it has to satisfy $\beta_{\mathfrak{z}}^{+}(\tilde{b}\bullet\tilde{c})\geqslant\min\{\beta_{\varepsilon}^{+}(\tilde{b}), \beta_{\varepsilon}^{+}(\tilde{c})\}$, and $\beta_{\mathfrak{z}}^{-}(\tilde{b}\bullet\tilde{c})\leqslant\max\{\beta_{\varepsilon}^{-}(\tilde{b}),\beta_{\varepsilon}^{-}(\tilde{c})\}$, where $\beta_{\varepsilon}^{+}: U\times U\rightarrow[0,1]$, and $\beta_{\varepsilon}^{-}: U\times U\rightarrow[-1,0]$ for all $\tilde{b}\bullet\tilde{c}\in U$.

Definition 2.6. [26] Let $\varepsilon=(\beta_{\varepsilon}^{+}, \beta_{\varepsilon}^{-})$ and $\mathfrak{z}=(\beta_{\mathfrak{z}}^{+}, \beta_{\mathfrak{z}}^{-})$ be two BF sets in U, then the Union and Intersections are defined by

i) $\varepsilon\cup\mathfrak{z}(\tilde{b})=(\max(\beta_{\varepsilon}^{+}(\tilde{b}),\beta_{\mathfrak{z}}^{+}(\tilde{b}), \min(\beta_{\varepsilon}^{-}(\tilde{b}),\beta_{\mathfrak{z}}^{-}(\tilde{b}))$ for all $\tilde{b}\in U$.

ii) $\varepsilon\cap\mathfrak{z}(\tilde{b})=(\min(\beta_{\varepsilon}^{+}(\tilde{b}),\beta_{\mathfrak{z}}^{+}(\tilde{b}), \max(\beta_{\varepsilon}^{-}(\tilde{b}),\beta_{\mathfrak{z}}^{-}(\tilde{b}))$ for all $\tilde{b}\in U$.

Definition 2.7. [16] Let $\varepsilon=(U, \varepsilon^{+}, \varepsilon^{-})$ be a BF set, and $T\times S\in[-1,0]\times[0,1]$, the sets $\varepsilon_S^{+}=\{\tilde{b}\in U / \varepsilon^{+}(\tilde{b})\geqslant S\}\&\varepsilon_T^{-}=\{\tilde{b}\in U / \varepsilon^{-}(\tilde{b})\leqslant T\}$ are known as positive S-cut and negative T-cut correspondingly. For $T\times S\in[-1,0]\times[0,1]$, the set $\varepsilon_{(T,S)}=\varepsilon_S^{+}\cap\varepsilon_T^{-}$ is known as (T,S)-set of $\varepsilon=(U, \varepsilon^{+},\varepsilon^{-})$.

3. Major findings in BF INK-sub algebra

In this section, we defined bipolar fuzzy INK sub-algebra assertions and give examples that satisfy the conditions of the BF INK sub-algebra.

Definition 3.1. A BF subset $\varepsilon=(\beta_{\varepsilon}^{+}, \beta_{\varepsilon}^{-})$ of INK-algebra U, where $\beta_{\varepsilon}^{+}: U\rightarrow[0,1]$, and $\beta_{\varepsilon}^{-}: U\rightarrow[-1,0]$, is named as BF INK sub-algebra of U if,

i) $\beta_{\varepsilon}^{+}(\tilde{b}\bullet\tilde{c})\geqslant\min\{\beta_{\varepsilon}^{+}(\tilde{b}), \beta_{\varepsilon}^{+}(\tilde{c})\}$,

ii) $\beta_{\varepsilon}^{-}(\tilde{b}\bullet\tilde{c})\leqslant\max\{\beta_{\varepsilon}^{-}(\tilde{b}), \beta_{\varepsilon}^{-}(\tilde{c})\}$, for all $\tilde{b}, \tilde{c}\in U$.

Example 3.2. Study the set U = {0, s, 1}. '●' is the binary operation, then the Cayley composition in Tables 1 and 2.

Table 1. Cayley composition table of BF INK-algebra.

●	0	s	1
0	0	1	s
1	1	s	0
s	s	0	1

Table 2. Bipolar fuzzy values (ref Table 1).

	0	s	1
β_{ε}^{+}	0.2	0.4	0.6
β_{ε}^{-}	-0.2	-0.3	-0.5

Define BF set $\varepsilon=(U;\beta_{\varepsilon}^{+}\beta\_\varepsilon\hat{}+,\beta_{\varepsilon}^{-}\beta\_\varepsilon\hat{}-)$, where $\beta_{\varepsilon}^{+}: U\rightarrow[0,1]$, and $\beta_{\varepsilon}^{-}: U\rightarrow[-1,0]$.

Now, we express the BF subset $\varepsilon=(\beta_{\varepsilon}^{+}, \beta_{\varepsilon}^{-})$ of an INK-algebra U by $\beta_{\varepsilon}^{+}(0)=0.2$, $\beta_{\varepsilon}^{+}(1)=0.4$, $\beta_{\varepsilon}^{+}(s)=0.6$, $\beta_{\varepsilon}^{-}(0)=-0.2$, $\beta_{\varepsilon}^{-}(1)=-0.3$, $\beta_{\varepsilon}^{-}(s)=-0.5$, respectively.

In conclusion, the above mentioned Cayley Table 1 satisfies the conditions of INK algebra and BF INK sub-algebra.

Lemma 3.3. If $\varepsilon=(\beta_{\varepsilon}^{+}, \beta_{\varepsilon}^{-})$ is a BF INK sub-algebra of an INK algebra U, then, $\beta_{\varepsilon}^{+}(0)\geqslant\beta_{\varepsilon}^{+}(\tilde{b})$, $\beta_{\varepsilon}^{-}(0)\leqslant\beta_{\varepsilon}^{-}(\tilde{b})$ for all $\tilde{b}\in U$.

Proof: Since $\tilde{b}\bullet\tilde{b}=0$ for all $\tilde{b}\in U$.

Then,

$$\beta^{+}_{\varepsilon}(0) = \beta^{+}_{\varepsilon}(\tilde{b} \bullet \tilde{b}) \geqslant \min\{\beta^{+}_{\varepsilon}(\tilde{b}),\ \beta^{+}_{\varepsilon}(\tilde{b})\},$$

$$\beta^{+}_{\varepsilon}(0) = \beta^{+}_{\varepsilon}(\tilde{b}).$$

Also,

$$\beta^{-}_{\varepsilon}(0) = \beta^{-}_{\varepsilon}(\tilde{b} \bullet \tilde{b}) \leqslant \max\{\beta^{-}_{\varepsilon}(\tilde{b}),\ \beta^{-}_{\varepsilon}(\tilde{b})\},$$

$$\beta^{-}_{\varepsilon}(0) = \beta^{-}_{\varepsilon}(\tilde{b}).$$

4. Intersection of BF INK sub-algebras

In this area, we proposed intersection of the BF INK sub-algebra and also studied the theorem that the arbitrary intersection of family BF INK sub-algebra is again a BF INK sub-algebra.

Definition 4.1. Let $\varepsilon = (\beta^{+}_{\varepsilon},\ \beta^{-}_{\varepsilon})$ and $\mathfrak{z} = (\beta^{+}_{\mathfrak{z}},\ \beta^{-}_{\mathfrak{z}})$ be two BF INK sub-algebras of INK algebra U. Then, $\varepsilon \cap \mathfrak{z} = (\beta^{+}_{\varepsilon\cap\mathfrak{z}}, \beta^{-}_{\varepsilon\cap\mathfrak{z}})$ are defined by

i) $\beta^{+}_{\varepsilon\cap\mathfrak{z}}(\tilde{b}) \geqslant \min\{\beta^{+}_{\varepsilon}(\tilde{b}),\ \beta^{+}_{\mathfrak{z}}(\tilde{b})\}$, and

ii) $\beta^{-}_{\varepsilon\cap\mathfrak{z}}(\tilde{c}) \leqslant \max\{\beta^{-}_{\varepsilon}(\tilde{c}),\ \beta^{-}_{\mathfrak{z}}(\tilde{c})\}$, for all $\tilde{b}, \tilde{c} \in U$.

Theorem 4.2. Let $\varepsilon = (\beta^{+}_{\varepsilon},\ \beta^{-}_{\varepsilon})$ and $\mathfrak{z} = (\beta^{+}_{\mathfrak{z}},\ \beta^{-}_{\mathfrak{z}})$ be two BFINK sub-algebras of INK algebra U. Then, $\varepsilon \cap \mathfrak{z} = (\beta^{+}_{\varepsilon\cap\mathfrak{z}}, \beta^{-}_{\varepsilon\cap\mathfrak{z}})$ is a BF sub-algebra of U.

Proof: By considering the BF INK sub-algebra, assertions are

$$\beta^{+}_{\varepsilon}(\tilde{b} \bullet \tilde{c}) \geqslant \min\{\beta^{+}_{\varepsilon}(\tilde{b}),\ \beta^{+}_{\varepsilon}(\tilde{c})\},$$

$$\beta^{-}_{\varepsilon}(\tilde{b} \bullet \tilde{c}) \leqslant \max\{\beta^{-}_{\varepsilon}(\tilde{b}),\ \beta^{-}_{\varepsilon}(\tilde{c})\}, \text{ for all } \tilde{b},\ \tilde{c} \in U.$$

Since there is a positive membership degree,
consider

$$\beta^{+}_{\varepsilon\cap\mathfrak{z}}(\tilde{b} \bullet \tilde{c}) = \min\{\beta^{+}_{\varepsilon}(\tilde{b} \bullet \tilde{c}),\ \beta^{+}_{\mathfrak{z}}(\tilde{b} \bullet \tilde{c})\}$$

$$\geqslant \min\{\{\min\{\beta^{+}_{\varepsilon}(\tilde{b}),\ \beta^{+}_{\varepsilon}(\tilde{c})\}, \min\{\beta^{+}_{\mathfrak{z}}(\tilde{b}),\ \beta^{+}_{\mathfrak{z}}(\tilde{c})\}\}$$

$$\geqslant \min\{\{\min\{\beta^{+}_{\varepsilon}(\tilde{b}),\ \beta^{+}_{\mathfrak{z}}(\tilde{b})\}, \min\{\beta^{+}_{\varepsilon}(\tilde{c}),\ \beta^{+}_{\mathfrak{z}}(\tilde{c})\}\}.$$

$$\beta^{+}_{\varepsilon\cap\mathfrak{z}}(\tilde{b} \bullet \tilde{c}) = \min\{\beta^{+}_{\varepsilon\cap\mathfrak{z}}(\tilde{b}),\ \beta^{+}_{\varepsilon\cap\mathfrak{z}}(\tilde{c})\}.$$

Since there is a negative membership degree
consider

$$\beta^{-}_{\varepsilon\cap\mathfrak{z}}(\tilde{b} \bullet \tilde{c}) = \max\{\beta^{-}_{\varepsilon}(\tilde{b} \bullet \tilde{c}),\ \beta^{-}_{\mathfrak{z}}(\tilde{b} \bullet \tilde{c})\}$$

$$\leqslant \max\{\{\max\{\beta^{-}_{\varepsilon}(\tilde{b}),\ \beta^{-}_{\varepsilon}(\tilde{c})\}, \max\{\beta^{-}_{\mathfrak{z}}(\tilde{b}),\ \beta^{-}_{\mathfrak{z}}(\tilde{c})\}\}$$

$$\leqslant \max\{\{\max\{\beta^{-}_{\varepsilon}(\tilde{b}),\ \beta^{-}_{\mathfrak{z}}(\tilde{b})\}, \max\{\beta^{-}_{\varepsilon}(\tilde{c}),\ \beta^{-}_{\mathfrak{z}}(\tilde{c})\}\}.$$

$$\beta^{-}_{\varepsilon\cap\mathfrak{z}}(\tilde{b} \bullet \tilde{c}) = \max\{\beta^{-}_{\varepsilon\cap\mathfrak{z}}(\tilde{b}),\ \beta^{-}_{\varepsilon\cap\mathfrak{z}}(\tilde{c})\}.$$

Theorem 4.3. The arbitrary intersection of the family of the BF INK sub-algebra is the BF INK sub-algebra.

Proof: Let $\{\varepsilon_i / i \in \Delta\}$ be an arbitrary intersection family of the BF INK sub-algebra of U, then the arbitrary intersection family of the BF INK sub-algebra is again BF INK sub-algebra of U, where $\bigcap_{i\in\Delta} \varepsilon_i = \{\tilde{b}, \wedge \beta^+_{\varepsilon_i}(\tilde{b}), \vee \beta^-_{\varepsilon_i}(\tilde{b})\}$.

Now, $\tilde{b}, \tilde{c} \in U$.

Consider

$$\beta^+_{\cap\varepsilon_{i\ i\in\Delta}}(\tilde{b}\bullet\tilde{c}) \geqslant \min\{\beta^+_{\cap\varepsilon_1}(\tilde{b}\bullet\tilde{c}),\ \beta^+_{\cap\varepsilon_2}(\tilde{b}\bullet\tilde{c}),\ldots\ldots\beta^+_{\cap\varepsilon_i}(\tilde{b}\bullet\tilde{c})\}$$

$$\geqslant \min\{\min\{\beta^+_{\cap\varepsilon_1}(\tilde{b})\bullet\beta^+_{\cap\varepsilon_1}(\tilde{c})\},\min\{\beta^+_{\cap\varepsilon_2}(\tilde{b})\bullet\beta^+_{\cap\varepsilon_2}(\tilde{c})\}\ldots\min\{\beta^+_{\cap\varepsilon_i}(\tilde{b})\bullet\beta^+_{\cap\varepsilon_i}(\tilde{c})\}\}$$

$$\geqslant \min\{\min\{\{\beta^+_{\cap\varepsilon_1}(\tilde{b}),\ \beta^+_{\cap\varepsilon_2}(\tilde{b})\ldots\ \beta^+_{\cap\varepsilon_i}(\tilde{b})\}\bullet\{\beta^+_{\cap\varepsilon_1}(\tilde{c}),\beta^+_{\cap\varepsilon_2}(\tilde{c})\ldots\beta^+_{\cap\varepsilon_i}(\tilde{c})\}\}\}$$

$$\geqslant \min\{\min\{\beta^+_{\cap\varepsilon_i}(\tilde{b})\bullet\beta^+_{\cap\varepsilon_i}(\tilde{c})\}\}$$

$$\geqslant \min\{\beta^+_{\cap\varepsilon_i}(\tilde{b})\bullet\beta^+_{\cap\varepsilon_i}(\tilde{c})\}.$$

5. Union of BF INK sub-algebras

In this part, we have clarified Union of BF INK sub-algebra assertions, but the following example does not satisfy the condition of Union to remark that we have anticipated the theorem that the Union of BF INK sub-algebras is again a BF INK sub-algebra by the situation of one enclosed by the other.

Definition 5.1. Let $\varepsilon=(\beta^+_{\varepsilon},\ \beta^-_{\varepsilon})$ and $ȝ=(\beta^+_{ȝ},\ \beta^-_{ȝ})$ be two BF INK sub-algebras of INK algebra U. Then, $\varepsilon \cup ȝ = (\beta^+_{\varepsilon\cup ȝ},\ \beta^-_{\varepsilon\cup ȝ})$ are defined by

i) $\beta^+_{\varepsilon\cup ȝ}(\tilde{b}) \geqslant \max\{\beta^+_{\varepsilon}(\tilde{b}),\ \beta^+_{ȝ}(\tilde{b})\}$, and

ii) $\beta^-_{\varepsilon\cup ȝ}(\tilde{c}) \leqslant \min\{\beta^-_{\varepsilon}(\tilde{c}),\ \beta^-_{ȝ}(\tilde{c})\}$, for all $\tilde{b}, \tilde{c} \in U$.

Remark 5.2. The Union of BF INK sub-algebra of the INK-algebraU need not be a BF INK sub-algebras.

Example 5.3. Consider the INK sub-algebra U = {0, 2, 4} with the following Cayley Tables 3–5.

Table 3. Cayley composition table of Union of BF INK-algebra.

●	0	2	4
0	0	4	2
2	2	0	4
4	4	2	0

Table 4. Bipolar fuzzy values of set ε (ref Table 3).

	0	2	4
β^+_{ε}	0.4	0.6	0.3
β^-_{ε}	-0.9	-0.8	-0.5

Define BF set $\varepsilon = (U;\beta^+_{\varepsilon},\beta^-_{\varepsilon})$, where $\beta^+_{\varepsilon}: U\rightarrow[0,1]$, and $\beta^-_{\varepsilon}: U\rightarrow[-1,0]$.

Table 5. Bipolar fuzzy values of set ζ (ref Table 3).

	0	2	4
β^{+}_{ζ}	0.5	0.4	0.4
β^{-}_{ζ}	-0.8	-0.8	-0.5

Define BF set $\zeta = (U;\beta^{+}_{\zeta},\beta^{-}_{\zeta})$, where $\beta^{+}_{\zeta}: U \to [0,1]$, and $\beta^{-}_{\zeta}: U \to [-1,0]$.

Clearly, ϵ and ζ are two BF INK sub-algebras. Here $\beta^{+}_{\epsilon\cup\zeta}(0 \bullet 2) = 0.4$, but it is not greater than or equal to, i.e., $0.5=\min\{\beta^{+}_{\epsilon\cup\zeta}(0), \beta^{+}_{\epsilon\cup\zeta}(2)\}$. Similarly, for $\beta^{-}_{\epsilon\cup\zeta}(0 \bullet 2) = -0.5$, but it is not less than or equal to, i.e., $-0.8=\max\{\beta^{-}_{\epsilon\cup\zeta}(0), \beta^{-}_{\epsilon\cup\zeta}(2)\}$. Therefore, $(\beta^{+}_{\epsilon\cup\zeta}, \beta^{-}_{\epsilon\cup\zeta})$ is not a BF INK sub-algebra. Thus, the union of BF INK sub-algebras is not a BF sub-algebra.

In particular, that follows.

Theorem 5.3. Let $\epsilon=(\beta^{+}_{\epsilon}, \beta^{-}_{\epsilon})$ and $\zeta=(\beta^{+}_{\zeta}, \beta^{-}_{\zeta})$ be two BF INK sub-algebras of INK algebra U, then $\epsilon\cup\zeta$ is a BF INK sub-algebra only if $\epsilon\subseteq\zeta$ or $\zeta\subseteq\epsilon$.

Proof: Suppose $\epsilon\subseteq\zeta$.

Let $\tilde{b}, \tilde{c}\in U$.

Now, consider

$$\begin{aligned}\beta^{+}_{\epsilon\cup\zeta}(\tilde{b}\bullet\tilde{c}) &= \max\{\beta^{+}_{\epsilon}(\tilde{b}\bullet\tilde{c}), \beta^{+}_{\zeta}(\tilde{b}\bullet\tilde{c})\}\\ &= \beta^{+}_{\zeta}(\tilde{b}\bullet\tilde{c})\\ &\geqslant \min\{\beta^{+}_{\zeta}(\tilde{b}),\beta^{+}_{\zeta}(\tilde{c})\}\\ &\geqslant \min\{\max\{\beta^{+}_{\epsilon}(\tilde{b}), \beta^{+}_{\zeta}(\tilde{b})\}\bullet \max\{\beta^{+}_{\epsilon}(\tilde{c}), \beta^{+}_{\zeta}(\tilde{c})\}\}\\ &= \max\{\beta^{+}_{\epsilon\cup\zeta}(\tilde{b}),\beta^{+}_{\epsilon\cup\zeta}(\tilde{c})\}.\end{aligned}$$

Also,

$$\begin{aligned}\beta^{-}_{\epsilon\cup\zeta}(\tilde{b}\bullet\tilde{c}) &= \min\{\beta^{-}_{\epsilon}(\tilde{b}\bullet\tilde{c}), \beta^{-}_{\zeta}(\tilde{b}\bullet\tilde{c})\}\\ &= \beta^{-}_{\zeta}(\tilde{b}\bullet\tilde{c})\\ &\leqslant \max\{\beta^{-}_{\zeta}(\tilde{b}),\beta^{-}_{\zeta}(\tilde{c})\}\\ &\leqslant \max\{\min\{\beta^{-}_{\epsilon}(\tilde{b}), \beta^{-}_{\zeta}(\tilde{b})\}\bullet\min\{\beta^{-}_{\epsilon}(\tilde{c}), \beta^{-}_{\zeta}(\tilde{c})\}\\ &= \min\{\beta^{-}_{\epsilon\cup\zeta}(\tilde{b}),\beta^{-}_{\epsilon\cup\zeta}(\tilde{c})\}.\end{aligned}$$

6. Upper and lower level sets of BF INK sub-algebra

In this section, we have worked on the level sets of the BF INK sub-algebra. Firstly, we have started with the definition of upper-s-level and lower-t-level. Later, we have proven some theorems regarding upper and lower level sets, i.e., BF subset $\epsilon=(\beta^{+}_{\epsilon}, \beta^{-}_{\epsilon})$ of an INK algebra U is a BF INK sub-algebra. Additionally, we have proposed that any INK sub-algebra U of an INK- algebra U can be recognized as a level INK sub-algebra of some BF INK sub-algebra of U.

Definition 6.1. Let $\epsilon=(\beta^{+}_{\epsilon}, \beta^{-}_{\epsilon})$ be a BF subset of an INK sub-algebra of U. The upper-s-level of β^{+}_{ϵ} is $\beta_{s}=\{\tilde{b}\in U: \beta^{+}_{\epsilon}(\tilde{b})\geqslant s\}$, and the lower-t-level of β^{-}_{ϵ} is $\beta_{t}=\{\tilde{b}\in U: \beta^{-}_{\epsilon}(\tilde{b})\leqslant t\}$, where $s\in[0,1]$, $t\in[-1,0]$.

Theorem 6.2. A BF subset ε=($\text{þ}_{\varepsilon}^{+}$, $\text{þ}_{\varepsilon}^{-}$) of an INK algebra U is a BF INK sub-algebra if for every t∈ [-1, 0], and s∈[0,1], where þ_s, $\text{þ}_t$is either empty or an INK algebra of U.

Proof: Assume that set ε=($\text{þ}_{\varepsilon}^{+}$, $\text{þ}_{\varepsilon}^{-}$) is a BF INK sub-algebra of U, and $\text{þ}_s \neq \emptyset$, $\text{þ}_t \neq \emptyset$. Then, for $\tilde{b}$, $\tilde{c}$∈ (þ_s,þ_t), we have,

i) For a positive membership degree

$$\text{þ}_{\varepsilon}^{+}(\tilde{b}\bullet\tilde{c}) \geqslant \min\{\text{þ}_{\varepsilon}^{+}(\tilde{b}),\text{þ}_{\varepsilon}^{+}(\tilde{c})\} \geqslant s.$$

Therefore,

$$\tilde{b}\bullet\tilde{c}\in\text{þ}_s,$$

þ_s which is an INK sub-algebra of U.

On the contrary, suppose that $\text{þ}_s$is an INK sub-algebra of U.

Let $\tilde{b}$,$\tilde{c}$∈U, take s = min {$\text{þ}_{\varepsilon}^{+}(\tilde{b})$,$\text{þ}_{\varepsilon}^{+}(\tilde{c})$}.

Then, the statement $\text{þ}_s$is an INK sub-algebra of U which suggests $\tilde{b}\bullet\tilde{c}\in\text{þ}_s$.

Therefore, $\text{þ}_{\varepsilon}^{+}(\tilde{b}\bullet\tilde{c}) \geqslant$ s = min {$\text{þ}_{\varepsilon}^{+}(\tilde{b})$,$\text{þ}_{\varepsilon}^{+}(\tilde{c})$}.

Hence, $\text{þ}_{\varepsilon}^{+}$ it is an INK sub-algebra of U.

ii) For a negative membership degree

$$\text{þ}_{\varepsilon}^{-}(\tilde{b}\bullet\tilde{c}) \leqslant \max\{\text{þ}_{\varepsilon}^{-}(\tilde{b}),\text{þ}_{\varepsilon}^{-}(\tilde{c})\} \leqslant t.$$

Hence,

$$\tilde{b}\bullet\tilde{c}\in\text{þ}_t,$$

þ_t which is an INK sub-algebra of U.

On the contrary, suppose that $\text{þ}_t$is an INK sub-algebra of U.

Let $\tilde{b}$, $\tilde{c}$∈U, take t = max {$\text{þ}_{\varepsilon}^{-}(\tilde{b})$,$\text{þ}_{\varepsilon}^{-}(\tilde{c})$}.

Then, the statement $\text{þ}_t$is an INK sub-algebra of U which suggests $\tilde{b}\bullet\tilde{c}\in\text{þ}_t$.

Therefore, $\text{þ}_{\varepsilon}^{-}(\tilde{b}\bullet\tilde{c}) \leqslant$ t= max {$\text{þ}_{\varepsilon}^{-}(\tilde{b})$, $\text{þ}_{\varepsilon}^{-}(\tilde{c})$}.

Hence, $\text{þ}_{\varepsilon}^{-}$it is an INK sub-algebra of U.

Theorem 6.3. Any INK sub-algebra U of an INK- algebra U be able to be recognized as a level INK sub-algebra of some BF INK sub-algebra of U.

Proof: Let $\hat{U}$ be an INK sub-algebra taken from INK-algebra U. Let ε=(þ_{s}^{+}, þ_{t}^{-}) be a fuzzy set in U defined by

$$\text{þ}_{s}^{+}= \begin{cases} s, \text{ if } \tilde{b} \in U; \\ 0, \text{ if } \tilde{b} \in U. \end{cases} \qquad \text{þ}_{t}^{-}= \begin{cases} t, \text{ if } \tilde{b} \in U; \\ 0, \text{ if } \tilde{b} \in U. \end{cases}$$

Where s∈[0,1] and t∈[-1,0] are fixed.

It is clear that þ_{s}^{+}=þ_{t}^{-}=U.

Now, we will verify that(þ_{s}^{+}, þ_{t}^{-}) is the BF INK sub-algebra of U.

If $\tilde{b}$,$\tilde{c}$ ∈U, then also $\tilde{b}\bullet\tilde{c}$∈U.

i) For s∈[0,1], then

we have

$$\text{þ}_{s}^{+}(\tilde{b}) = \text{þ}_{s}^{+}(\tilde{c}) = \text{þ}_{s}^{+}\left(\tilde{b}\bullet\tilde{c}\right) = s,$$

and

$$\text{þ}_s^+(\tilde{b}\bullet\tilde{c})) \geqslant \min\{\text{þ}_s^+(\tilde{b}),\text{þ}_s^+(\tilde{c})\}.$$

If, $\tilde{b}, \tilde{c} \notin \widehat{U}$,
Then,

$$\text{þ}_s^+(\tilde{b}) = \text{þ}_s^+(\tilde{c}) = 0,$$

and in consequence,

$$\text{þ}_s^+(\tilde{b}\bullet\tilde{c})) \geqslant \min\{\text{þ}_s^+(\tilde{b}),\text{þ}_s^+(\tilde{c})\} = 0.$$

If at most one of $\tilde{b},\tilde{c} \in \widehat{U}$, then at least any of $\text{þ}_s^+(\tilde{b})$ and $\text{þ}_s^+(\tilde{c})$ is equal to zero.
Hence,

$$\min\{\text{þ}_s^+(\tilde{b}),\text{þ}_s^+(\tilde{c})\} = 0,$$

so that

$$\text{þ}(\tilde{b}\bullet\tilde{c})) \geqslant 0,$$

which concludes the proof for s∈[0,1].
ii) For t∈[-1,0], then,
we have,

$$\text{þ}_t^-(\tilde{b}) = \text{þ}_t^-(\tilde{c}) = \text{þ}_t^-\left(\tilde{b}\bullet\tilde{c}\right) = t,$$

and

$$\mu_t^-(\tilde{b}\bullet\tilde{c})) \leqslant \max\{\text{þ}_t^-(\tilde{b}), \text{þ}_t^-(\tilde{c})\}.$$

If $\tilde{b}, \tilde{c} \notin \hat{\chi}$,
then,

$$\text{þ}_t^-(\tilde{b}) = \text{þ}_t^-(\tilde{c}) = 0,$$

and in consequence

$$\text{þ}_t^-(\tilde{b}\bullet\tilde{c})) \leqslant \max\{\text{þ}_t^-(\tilde{b}), \text{þ}_t^-(\tilde{c})\} = 0.$$

If at most one of $\tilde{b}, \tilde{c} \in \widehat{U}$, then at least any of $\text{þ}_t^-(\tilde{b})$ and $\text{þ}_t^-(\tilde{c})$ is equal to zero.
Hence,

$$\max\{\text{þ}_t^-(\tilde{b}),\text{þ}_t^-(\tilde{c})\} = 0,$$

so that

$$\text{þ}_t^-(\tilde{b}\bullet\tilde{c})) \leqslant 0,$$

which concludes the proof for t∈[-1,0].
Theorem 6.4. Two-level INK sub-algebras $\text{þ}_S = (\text{þ}_{s_1}^+, \text{þ}_{s_2}^-)$ and $\text{þ}_T = (\text{þ}_{t_1}^+, \text{þ}_{t_2}^-)$, i.e., $s_1 < t_1$ and $s_2 > t_2$ of INK sub-algebra $\widehat{U}$ of an INK algebra U are equal if and only if there is no $\tilde{b} \in U$ such that $s_1 \leq \text{þ}^+(\tilde{b}) < t_1$, and $s_2 \geq \text{þ}^-(\tilde{b}) > t_2$, where $\text{þ}^+(\tilde{b}) = (\text{þ}_{s_1}^+, \text{þ}_{t_1}^+)$, and $\text{þ}^-(\tilde{b}) = (\text{þ}_{s_2}^-, \text{þ}_{t_2}^-)$.
Proof: Let $\text{þ}_S = \text{þ}_T$ for some $s_1 < t_1$ and $s_2 > t_2$.
If there exists $\tilde{b} \in U, \exists$ $s1 \leq \text{þ}^+(\tilde{b}) < t_1$, and $s_2 \geq \text{þ}^-(\tilde{b}) > t_2$, then þ_T is proper subset of þ_S, which is contradiction.

On the contrary, assume that there is no $\tilde{b} \in U$, ∃.

For $s_1 \leq \mu^+(\tilde{b}) < t_1$.

If $\tilde{b} \in \mathfrak{p}_S$, then $\mathfrak{p}^+(\tilde{b}) \geqslant s_1$ and $\mathfrak{p}^+(\tilde{b}) \geqslant t_1$, subsequently, $\mathfrak{p}^+(\tilde{b})$ does not lie among s_1 and t_1.
Thus, $\tilde{b} \in \tilde{b}_T$ which gives $\mathfrak{p}_S \subset \mathfrak{p}_T$. Also, $\mathfrak{p}_T \subset \mathfrak{p}_S$.
Therefore, $\mathfrak{p}_S = \mathfrak{p}_T$.

For $s_2 \geq \mathfrak{p}^-(\tilde{b}) > t_2$.

If $\tilde{b} \in \mathfrak{p}_S$, then $\mathfrak{p}^-(\tilde{b}) \leq s_2$ and $\mathfrak{p}^-(\tilde{b}) \leq t_2$, subsequently, $\mathfrak{p}^-(\tilde{b})$ does not lie among s_1 and t_1.
Thus, $\tilde{b} \in \mathfrak{p}_T$, which gives $\mathfrak{p}_S \subset \mathfrak{p}_T$. Also, $\mathfrak{p}_T \subset \mathfrak{p}_S$.
Therefore, $\mathfrak{p}_S = \mathfrak{p}_T$.

7. Homomorphism of BF INK sub-algebra

In this section, we have discussed homomorphism, epimorphism, and endomorphism in INK algebra. Sequentially, we have proposed some theorems of homomorpism, endomorphism, and epimorphism of the BF INK sub-algebra.

Definition 7.1. Let $\not{c}$:U→ U be a homomorphism of INK algebra and ε=($\mathfrak{p}^+_{\varepsilon}$, $\mathfrak{p}^-_{\varepsilon}$) be a bipolar fuzzy subset in U, then the BF subset $\varepsilon_{\not{c}} = (\mathfrak{p}^+_{\varepsilon_{\not{c}}}, \mathfrak{p}^-_{\varepsilon_{\not{c}}})$ in U is defined by BF subset $\mathfrak{p}^+_{\varepsilon_{\not{c}}}= \mathfrak{p}^+_{\varepsilon}(\not{c}(\tilde{b}))$, $\mathfrak{p}^-_{\varepsilon_{\not{c}}}= \mathfrak{p}^-_{\varepsilon}(\not{c}(\tilde{b}))$ for all $\tilde{b} \in U$ is called pre-image of ε under $\not{c}$.

Theorem 7.2. A homomorphic pre-image of a bipolar fuzzy INK sub algebra is a bipolar fuzzy INK sub-algebra.

Proof: Let $\not{c}$:U → U be a homomorphism of INK algebra. If a BF subset ε=($\mathfrak{p}^+_{\varepsilon}$, $\mathfrak{p}^-_{\varepsilon}$) is a BF INK sub-algebra of U and $\varepsilon_{\not{c}} = (\mathfrak{p}^+_{\varepsilon_{\not{c}}}, \mathfrak{p}^-_{\varepsilon_{\not{c}}})$ be the pre-image of ε under $\not{c}$ is defined by $\mathfrak{p}^+_{\varepsilon_{\not{c}}}= \mathfrak{p}^+_{\varepsilon}(\not{c}(\tilde{b}))$, $\mathfrak{p}^-_{\varepsilon_{\not{c}}}= \mathfrak{p}^-_{\varepsilon}(\not{c}(\tilde{b}))$ for all $\tilde{b} \in U$.

Consider

$$\mathfrak{p}^+_{\varepsilon_{\not{c}}}(\tilde{b}\bullet\tilde{c}) = \min\{\mathfrak{p}^+_{\varepsilon_{\not{c}}}(\tilde{b}), \mathfrak{p}^+_{\varepsilon_{\not{c}}}(\tilde{c})\}$$

$$\geqslant \min\{\mathfrak{p}^+_{\varepsilon}\left(¢(\tilde{b})\right), \mathfrak{p}^+_{\varepsilon}\left(¢(\tilde{c})\right)\}$$

$$\geqslant \min\{¢(\mathfrak{p}^+_{\varepsilon}(\tilde{b})),\ ¢(\mathfrak{p}^+_{\varepsilon}(\tilde{c}))\}$$

$$\geqslant \min\{¢(\mathfrak{p}^+_{\varepsilon}(\tilde{b}),\mathfrak{p}^+_{\varepsilon}(\tilde{c}))\}.$$

$$\mathfrak{p}^+_{\varepsilon_{\not{c}}}(\tilde{b}\bullet\tilde{c}) = ¢(\mathfrak{p}^+_{\varepsilon}(\tilde{b}\bullet\tilde{c})).$$

Also,

$$\mathfrak{p}^-_{\varepsilon_{\not{c}}}(\tilde{b}\bullet\tilde{c}) = \max\{\mathfrak{p}^-_{\varepsilon_{\not{c}}}(\tilde{b}), \mathfrak{p}^-_{\varepsilon_{\not{c}}}(\tilde{c})\}$$

$$\leq \max\{\mathfrak{p}^-_{\varepsilon}\left(¢(\tilde{b})\right), \mathfrak{p}^-_{\varepsilon}\left(¢(\tilde{c})\right)\}$$

$$\leq \max\{¢(\mathfrak{p}^-_{\varepsilon}(\tilde{b})),\ ¢(\mathfrak{p}^-_{\varepsilon}(\tilde{c}))\}$$

$$\leq \max\{¢(\mathfrak{p}^-_{\varepsilon}(\tilde{b}),\mathfrak{p}^-_{\varepsilon}(\tilde{c}))\}.$$

$$\mathfrak{p}^-_{\gamma\varepsilon_{\not{c}}}(\tilde{b}\bullet\tilde{c}) = ¢(\mathfrak{p}^-_{\varepsilon}(\tilde{b}\bullet\tilde{c})).$$

Hence the proof.

Theorem 7.3. Let $¢$: U→ U be an epimorphism of INK algebra. If $ɛ_{¢} = (þ^{+}_{ɛ_{¢}}, þ^{-}_{ɛ_{¢}})$ is an bipolar fuzzy INK sub-algebra of U, then,$(þ^{+}_{ɛ},\ þ^{-}_{ɛ})$ is BF INK sub-algebra of U.

Proof: Let $þ^{+}_{ɛ_{¢}}$ be an BF INK sub-algebra of U.

Then, there exist $\tilde{b}$, $\tilde{c}, \tilde{d} \in U$ such that $¢(\tilde{b}) = \tilde{b}$, $¢(\tilde{c}) = \tilde{c}$, $¢(\tilde{d}) = \tilde{d}$.

It follows that

$$þ^{+}_{ɛ}(\tilde{b} \bullet \tilde{c}) = þ^{+}_{ɛ}(¢(\tilde{b} \bullet \tilde{c}))$$

$$= þ^{+}_{ɛ_{¢}}(\tilde{b} \bullet \tilde{c})$$

$$\geqslant \min\{þ^{+}_{ɛ_{¢}}(\tilde{b}), þ^{+}_{ɛ_{¢}}(\tilde{c})\}$$

$$\geqslant \min\{þ^{+}_{ɛ}\left(¢(\tilde{b})\right), þ^{+}_{ɛ}\left(¢(\tilde{c})\right)\}$$

$$= \min\{þ^{+}_{ɛ}(\tilde{b}), þ^{+}_{ɛ}(\tilde{c})\}.$$

$$þ^{-}_{ɛ}(\tilde{b} \bullet \tilde{c}) = þ^{-}_{ɛ}(¢(\tilde{b} \bullet \tilde{c}))$$

$$\leq þ^{-}_{ɛ_{¢}}(\tilde{b} \bullet \tilde{c})$$

$$\leq \max\{þ^{-}_{ɛ_{¢}}(\tilde{b}), þ^{-}_{ɛ_{¢}}(\tilde{c})\}$$

$$\leq \max\{þ^{-}_{ɛ}\left(¢(\tilde{b})\right), þ^{-}_{ɛ}\left(¢(\tilde{c})\right)\}$$

$$= \max\{þ^{-}_{ɛ}(\tilde{b}), þ^{-}_{ɛ}(\tilde{c})\}.$$

Therefore, $(þ^{+}_{ɛ},\ þ^{-}_{ɛ})$ is a BF INK sub-algebra of U.

Definition 7.4. Let $¢$:U→U be an endomorphism, and $ɛ = (þ^{+}_{ɛ},\ þ^{-}_{ɛ})$ be an BF subset in U. Then, $ɛ_{¢} = (þ^{+}_{ɛ_{¢}}, þ^{-}_{ɛ_{¢}})$ is new BF subset in U under $¢$ by $þ^{+}_{ɛ_{¢}} = þ^{+}_{ɛ}(¢(\tilde{b}))$, $þ^{-}_{ɛ_{¢}} = þ^{-}_{ɛ}(¢(\tilde{b}))$ for all $\tilde{b} \in U$.

Theorem 7.5. Let $¢$ be an endomorphism of an INK algebra U. If $ɛ = (þ^{+}_{ɛ},\ þ^{-}_{ɛ})$ is a BF INK sub-algebra of U, formerly so $þ^{+}_{ɛ_{¢}} = þ^{+}_{ɛ}(¢(\tilde{b}))$, $þ^{-}_{ɛ_{¢}} = þ^{-}_{ɛ}(¢(\tilde{b}))$ for all $\tilde{b} \in U$.

Proof: **Case i)**

$$þ^{+}_{ɛ_{¢}}\left(\tilde{b} \bullet \tilde{c}\right) = þ^{+}_{ɛ}(¢(\tilde{b} \bullet \tilde{c}))$$

$$\geq \min\{þ^{+}_{ɛ}(¢(\tilde{b})),\ þ^{+}_{ɛ}(¢(\tilde{c}))$$

$$= \min\ \{þ^{+}_{ɛ_{¢}}(\tilde{b}), þ^{+}_{ɛ_{¢}}(\tilde{c})\}.$$

Case ii)

$$þ^{-}_{ɛ_{¢}}\left(\tilde{b} \bullet \tilde{c}\right) = þ^{-}_{ɛ}(¢(\tilde{b} \bullet \tilde{c}))$$

$$\leq \max\{þ^{-}_{ɛ}(¢(\tilde{b})),\ þ^{-}_{ɛ}(¢(\tilde{c}))$$

$$= \max\{þ^{-}_{ɛ_{¢}}(\tilde{b}), þ^{-}_{ɛ_{¢}}(\tilde{c})\}.$$

Hence, $ɛ_{¢}$ is a bipolar fuzzy INK sub-algebra of U.

8. Conclusions

We have explored the structure, properties, and linkages of BF INK sub-algebras through various algebraic theorems. Our research has centered on BF INK subalgebra intersections, where we have demonstrated specific needs for overlapping components. This theory also includes the union of BF INK sub-algebras, which is based on their confinement link. In addition, we have analyzed and detailed the structure of the higher and lower-level sets of BF INK sub-algebras. Throughout our conversation, we have proven that these mappings preserve algebraic structures and defined homomorphism in BF INK sub-algebras. This research led to the formulation of theorems on endomorphism and epimorphism in this INK algebraic structure. The limitations of this study are that INK-algebras and INK-subalgebras,—all these kinds of algebras—are generalizations of BCK/BCI-algebras from combinatorial logic and networks, which have no immediate applications in real life. In the future, we will use this proposed study to develop metric space [35], different algebraic equations [36], fuzzy k-clustering [37], computer science [38], and Fermatean fuzzy INK-algebra [39]. Ambiguous bipolar INK Expanding beyond BF INK sub-algebras, our upcoming work encompasses bi-ideals and bipolar fuzzy INK hyper-ideals. These additions seek to enhance our knowledge of algebraic structures in bipolar fuzzy sets, which is expected to significantly promote both theoretical and practical breakthroughs in mathematics.

Author contributions

Remala Mounikalakshmi: writing-original draft; Tamma Eswarlal and Chiranjibe Jana: reviewing and editing. All authors have read and approved the final version of the manuscript for publication.

Conflict of interest

The authors declare that they have no conflicts of interest.

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