Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(10): 27577–27592.
DOI: 10.3934/math.20241339
Received: 17 July 2024
Revised: 04 September 2024
Accepted: 11 September 2024
Published: 24 September 2024

*Research article*

# Certain new subclasses of bi-univalent function associated with bounded boundary rotation involving sălăgean derivative

**Anandan Murugan[1], Sheza M. El-Deeb[2], Mariam Redn Almutiri[2], Jong-Suk-Ro[3,4,*], Prathviraj Sharma[5] and Srikandan Sivasubramanian[5]**

[1] Department of Mathematics, College of Engineering Guindy, Anna University, Chennai 600025, Tamilnadu, India

[2] Department of Mathematics, College of Science, Qassim University, Buraydah 51452, Saudi Arabia

[3] School of Electrical and Electronics Engineering, Chung-Ang University, Dongjak-gu, Seoul 06974, Republic of Korea

[4] Department of Intelligent Energy and Industry, Chung-Ang University, Dongjak-gu, Seoul 06974, Republic of Korea

[5] Department of Mathematics, University College of Engineering Tindivanam, Anna University, Tindivanam 604001, Tamilnadu, India

* **Correspondence:** Email: jongsukro@gmail.com.

**Abstract:** In this article, using the Sălăgean operator, we introduced three new subclasses of bi-univalent functions associated with bounded boundary rotation in open unit disk $\mathbb{E}$. For these new classes, we first obtain initial Taylor-Maclaurin's coefficient bounds. Furthermore, the famous Fekete-Szegö inequality was also derived for these new subclass functions. Some improved results, when compared with those available in the literature, are also stated.

**Keywords:** analytic; bi-univalent; Sălăgean operator; bounded boundary rotation; convolution; coefficient estimates
**Mathematics Subject Classification:** Primary 30C45, 33C50, Secondary 30C80

## 1. Introduction

Indicate $\mathcal{A}$ as the class of all functions $h : \mathbb{E} \to \mathbb{C}$ defined by

$$h(u) = u + \sum_{m=2}^{\infty} h_m u^m, \tag{1.1}$$

which are analytic in open unit disk $\mathbb{E} := \{u \in \mathbb{C} : |u| < 1\}$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$, which is univalent in $\mathbb{E}$. Fix $0 \leq \delta < 1$. The well known subclasses $\mathcal{S}^*(\delta)$, $C(\delta)$ and $\mathcal{R}(\delta)$ of class $\mathcal{S}$ are the class of starlike, convex, and the class of functions whose derivatives have positive real part of order $\delta$, respectively. The analytic descriptions of the above classes are given by

$$\mathcal{S}^*(\delta) := \left\{h \in \mathcal{S} : \Re\left(\frac{uh'(u)}{h(u)}\right) > \delta\right\},$$

$$C(\delta) := \left\{h \in \mathcal{S} : \Re\left(1 + \frac{uh''(u)}{h'(u)}\right) > \delta\right\},$$

and

$$\mathcal{R}(\delta) := \{h \in \mathcal{S} : \Re\left(h'(u)\right) > \delta\}.$$

Indicate $\mathcal{V}_\vartheta$ as the class of functions $h$ given in (1.1), which maps the open unit disk $\mathbb{E}$ conformally onto an image domain $h(\mathbb{E})$ of boundary rotation at most $\vartheta\pi$. The functions belonging to the class $\mathcal{V}_\vartheta$ are known as functions of bounded boundary rotation. Pinchuk [15] introduced class $\mathcal{V}_\vartheta$. Any function $h \in \mathcal{V}_\vartheta$ is expressed as

$$\int_0^{2\pi} \left|\Re\left(\frac{(re^{i\mu}h'(re^{i\mu}))'}{h'(re^{i\mu})}\right)\right| d\mu \leq \vartheta\pi.$$

Assume $\mathcal{R}_\vartheta$ as the class of functions $h$ given in (1.1) which map open unit $\mathbb{E}$ conformally onto an image domain $h(\mathbb{E})$ of boundary radius rotation at most $\vartheta\pi$. The functions belonging to the class $\mathcal{R}_\vartheta$ are known as functions of bounded radius rotation. If a function $h \in \mathcal{R}_\vartheta$, then it can be expressed as

$$\int_0^{2\pi} \left|\Re\left(\frac{re^{i\mu}h'(re^{i\mu})}{h(re^{i\mu})}\right)\right| d\mu \leq \vartheta\pi.$$

Let $\mathcal{P}_\vartheta$ be the class of functions $t$ with $t(0) = 1$ in $\mathbb{E}$ and having an integral representation

$$t(u) = \int_0^{2\pi} \frac{1 + ue^{-i\mu}}{1 - ue^{-i\mu}} d\theta(\mu),$$

where $\theta(\mu)$ is a function of bounded variation and satisfying

$$\int_0^{2\pi} d\theta(\mu) = 2 \quad \textit{and} \quad \int_0^{2\pi} |d\theta(\mu)| \leq \vartheta.$$

Assume $\mathcal{S}_\vartheta$ be the subclass of $\mathcal{V}_\vartheta$ whose members are univalent in $\mathbb{E}$. Paatero [13] proved that $\mathcal{V}_\vartheta$ coincides with $\mathcal{S}_\vartheta$ whenever $2 \leq \vartheta \leq 4$. i.e., If $2 \leq \vartheta \leq 4$, $h \in \mathcal{V}_\vartheta$ contains only univalent functions in $\mathbb{E}$. If $\vartheta > 4$, then functions in the class $\mathcal{V}_\vartheta$ is fail to univalent conditions.

Noonan [11] gave the concept of order of a function for both $\mathcal{V}_\vartheta$ and $\mathcal{R}_\vartheta$ in 1971 and Padmanabhan and Parvatham [14] introduced the concept of order of a function for $\mathcal{P}_\vartheta$ in 1975. Let $\mathcal{P}_\vartheta(\delta)$ be the class of function $t$ in $\mathbb{E}$ normalized by the conditions $t(0) = 1$ and

$$\int_0^{2\pi} \left|\frac{\Re(t(u)) - \delta}{1 - \delta}\right| \leq \vartheta\pi.$$

It is well known that [5] every function $h \in \mathcal{S}$ has an inverse $h^{-1}$, defined by

$$u = h^{-1}(h(u)), \quad \forall \quad u \in \mathbb{E}$$

and

$$\omega = h(h^{-1}(\omega)), \quad \forall \quad |\omega| < r_0(h) \quad and \quad r_0(h) \geq \frac{1}{4}.$$

Hence, the inverse function $h^{-1}$ is given by

$$\gamma(\omega) = h^{-1}(\omega) = \omega - h_2\omega^2 + (2h_2^2 - h_3)\omega^2 - (5h_2^3 - 5h_2h_3 + h_4)\omega^4 + \cdots . \tag{1.2}$$

If both $h$ and $h^{-1}$ are univalent in $\mathbb{E}$, then $h$ is said to be bi-univalent in $\mathbb{E}$. Let us indicate $\Sigma$ as the class of bi-univalent functions in $\mathbb{E}$. Lewin [7] introduced the class $\Sigma$ and it was proved that $|h_2| < 1.51$. The coefficient problem for each of the following Taylor-Maclaurin coefficients:

$$|h_m|, \quad m \in \mathbb{N} \setminus \{1, 2\},$$

is an open problem. Subsequently Brannan and Clunie [3] conjectured that $|h_2| \leq \sqrt{2}$ and Netanyahu [10] showed that for $h \in \Sigma$, $\max |h_2| = \frac{4}{3}$. Several authors [6, 9, 20] introduced and investigated various subclasses of the class $\Sigma$ and obtained estimates for their coefficients $|h_2|$ and $|h_3|$ for the functions in these subclasses. Brannan and Taha [4] introduced the subclasses of bi-univalent functions $\mathcal{S}^*_\Sigma(\delta)$ and $\mathcal{K}_\Sigma(\delta)$, called bi-starlike functions of order $\delta$ and $\mathcal{K}_\Sigma(\delta)$ bi-convex functions of order $\delta$, respectively.

In geometric function theory and its related field, the study of operators plays an important role. Several authors [1, 12, 17, 18] introduced and investigated various subclasses of the class $\Sigma$ using different operators. For $h \in \mathcal{A}$, Sălăgean [16] introduced the differential operator $\mathcal{D}^\eta$, which is defined by

$$\mathcal{D}^0 h(u) = h(u);$$
$$\mathcal{D}^1 h(u) = \mathcal{D}h(u) = uh'(u);$$
$$\mathcal{D}^\eta h(u) = \mathcal{D}\left(\mathcal{D}^{\eta-1} h(u)\right), \quad \eta \in \mathbb{N},$$

then

$$\mathcal{D}^\eta h(u) = u + \sum_{m=2}^{\infty} m^\eta h_m u^m,$$

where $\eta \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

**Lemma 1.** *[2] If a function $t \in \mathcal{P}_\vartheta(\delta)$ is given in the form*

$$t(u) = 1 + t_1 u + t_2 u^2 + t_3 u^3 + \cdots , \quad u \in \mathbb{E},$$

*then for each $m \geq 1$,*

$$|t_m| \leq \vartheta(1 - \delta).$$

*This result is sharp.*

By applying the Sălăgean operator, three new subclasses of bi-univalent functions associated with bounded boundary rotations in open unit disk $\mathbb{E}$ are introduced and investigated. For these new classes, the initial coefficient estimates and the Fekete-Szegö inequality are obtained. Some of our findings improved the earlier existing results available in the literature and few of the bounds presented here generalize the result of Sharma [19].

## 2. Main results

**Definition 1.** *A function $h \in \Sigma$ given by* (1.1) *is said to be in the class $\mathcal{L}_{\Sigma}^{\eta,a}(\vartheta,\delta)$ if the following conditions*

$$(1-a)\frac{\mathcal{D}^{\eta}h(u)}{u}+a(\mathcal{D}^{\eta}h(u))'\in\mathcal{P}_{\vartheta}(\delta)$$

*and*

$$(1-a)\frac{\mathcal{D}^{\eta}\gamma(\omega)}{\omega}+a(\mathcal{D}^{\eta}\gamma(\omega))'\in\mathcal{P}_{\vartheta}(\delta),$$

*hold where $0\le a\le 1$, $2\le\vartheta\le 4$, $0\le\delta<1$ and the function $\gamma(\omega)$ is as defined by* (1.2)*.*

**Remark 1.** *If $a=1$ in Definition 1, we have $\mathcal{L}_{\Sigma}^{\eta,a}(\vartheta,\delta)\equiv\mathcal{L}_{\Sigma}^{\eta,1}(\vartheta,\delta)\equiv\mathcal{H}_{\Sigma}^{\eta}(\vartheta,\delta)$. That is, a function $h\in\Sigma$ given by* (1.1) *is said to be in the class $\mathcal{H}_{\Sigma}^{\eta}(\vartheta,\delta)$ if the following conditions*

$$(\mathcal{D}^{\eta}h(u))'\in\mathcal{P}_{\vartheta}(\delta)$$

*and*

$$(\mathcal{D}^{\eta}\gamma(\omega))'\in\mathcal{P}_{\vartheta}(\delta),$$

*hold where $2\le\vartheta\le 4$, $0\le\delta<1$ and the function $\gamma(\omega)$ is as defined by* (1.2)*.*

**Remark 2.** *If $a=0$ in Definition 1, we have $\mathcal{L}_{\Sigma}^{\eta,a}(\vartheta,\delta)\equiv\mathcal{L}_{\Sigma}^{\eta,0}(\vartheta,\delta)\equiv\mathcal{L}_{\Sigma}^{\eta}(\vartheta,\delta)$. That is a function $h\in\Sigma$ given by* (1.1) *is said to be in the class $\mathcal{L}_{\Sigma}^{\eta}(\vartheta,\delta)$ if the following conditions*

$$\frac{\mathcal{D}^{\eta}h(u)}{u}\in\mathcal{P}_{\vartheta}(\delta)$$

*and*

$$\frac{\mathcal{D}^{\eta}\gamma(\omega)}{\omega}\in\mathcal{P}_{\vartheta}(\delta),$$

*hold where $2\le\vartheta\le 4$, $0\le\delta<1$ and the function $\gamma(\omega)$ is as defined by* (1.2)*.*

**Remark 3.** *[19] If $\eta=0$ in Definition 1, we have $\mathcal{L}_{\Sigma}^{\eta,a}(\vartheta,\delta)\equiv\mathcal{L}_{\Sigma}^{0,a}(\vartheta,\delta)\equiv\mathcal{L}_{\Sigma}^{a}(\vartheta,\delta)$. A function $h\in\Sigma$ given by* (1.1) *is said to be in the class $\mathcal{L}_{\Sigma}^{\eta}(\vartheta,\delta)$ if the following conditions*

$$(1-a)\frac{h(u)}{u}+a(h'(u))\in\mathcal{P}_{\vartheta}(\delta)$$

*and*

$$(1-a)\frac{\gamma(\omega)}{\omega}+a(\gamma'(\omega))\in\mathcal{P}_{\vartheta}(\delta),$$

*hold where $0\le a\le 1$, $2\le\vartheta\le 4$, $0\le\delta<1$ and the function $\gamma(\omega)$ is as defined by* (1.2)*.*

**Theorem 1.** *Let $h\in\mathcal{L}_{\Sigma}^{\eta,a}(\vartheta,\delta)$ be given in the form* (1.1)*. Then*

$$|h_2|\le\sqrt{\frac{\vartheta(1-\delta)}{3^{\eta}(1+2a)}}\tag{2.1}$$

*and*

$$|h_3|\le\frac{\vartheta(1-\delta)}{3^{\eta}(1+2a)}.\tag{2.2}$$

*For any* $\aleph \in \mathbb{R}$,

$$|h_3 - \aleph h_2^2| \leq \begin{cases} \dfrac{\vartheta(1-\delta)(1-\aleph)}{(1+2a)3^\eta} & \textit{for } \aleph < 0, \\ \dfrac{\vartheta(1-\delta)}{(1+2a)3^\eta} & \textit{for } 0 \leq \aleph \leq 2, \\ \dfrac{\vartheta(1-\delta)(\aleph-1)}{(1+2a)3^\eta} & \textit{for } \aleph > 2. \end{cases} \tag{2.3}$$

*Proof.* As $h \in \mathcal{L}_\Sigma^{\eta,a}(\vartheta,\delta)$, from Definition 1,

$$(1-a)\frac{\mathcal{D}^\eta h(u)}{u} + a(\mathcal{D}^\eta h(u))' = t(u) \tag{2.4}$$

and

$$(1-a)\frac{\mathcal{D}^\eta \gamma(\omega)}{\omega} + a(\mathcal{D}^\eta \gamma(\omega))' = s(\omega), \tag{2.5}$$

where $t(u)$ and $s(\omega)$ are analytic functions belonging to the class $\mathcal{P}_\vartheta(\delta)$ given by

$$t(u) = 1 + t_1 u + t_2 u^2 + t_3 u^3 + \cdots \tag{2.6}$$

and

$$s(\omega) = 1 + s_1 \omega + s_2 \omega^2 + s_3 \omega^3 + \cdots . \tag{2.7}$$

Comparing the coefficients by using (2.4)–(2.7), we have

$$(1+a)2^\eta h_2 = t_1, \tag{2.8}$$

$$(1+2a)3^\eta h_3 = t_2, \tag{2.9}$$

$$-(1+a)2^\eta h_2 = s_1, \tag{2.10}$$

and

$$2(1+2a)3^\eta h_2^2 - (1+2a)3^\eta h_3 = s_2. \tag{2.11}$$

Adding (2.9) and (2.11), we have

$$2(1+2a)3^\eta h_2^2 = t_2 + s_2. \tag{2.12}$$

Now by using Lemma 1, we have

$$|h_2|^2 \leq \frac{\vartheta(1-\delta)}{3^\eta(1+2a)},$$

gives the bound of $|h_2|$ given in (2.1). Now by using Lemma 1, in (2.9), we have

$$(1+2a)3^\eta |h_3| \leq \vartheta(1-\delta),$$

gives the bound of $|h_3|$ given in (2.2). Now fix $\aleph \in \mathbb{R}$ and by using (2.9) and (2.12), we have

$$h_3 - \aleph h_2^2 = \frac{(2-\aleph)t_2 - \aleph s_2}{2(1+2a)3^\eta}.$$

Now by using Lemma 1, we have

$$|h_3 - \aleph h_2^2| \leq \frac{\vartheta(1-\delta)[|2-\aleph| + |\aleph|]}{2(1+2a)3^\eta},$$

gives the bound of $|h_3 - \aleph h_2^2|$ given in (2.3) finishing Theorem 1. □

**Definition 2.** *A function $h \in \Sigma$ given by* (1.1) *is said to be in the class $\mathcal{B}_{\Sigma}^{\eta,b}(\vartheta,\delta)$ if the conditions*

$$\frac{u(\mathcal{D}^{\eta}h(u))'}{\mathcal{D}^{\eta}h(u)} + b\frac{u^2(\mathcal{D}^{\eta}h(u))''}{\mathcal{D}^{\eta}h(u)} \in \mathcal{P}_{\vartheta}(\delta)$$

*and*

$$\frac{\omega(\mathcal{D}^{\eta}\gamma(\omega))'}{\mathcal{D}^{\eta}\gamma(\omega)} + b\frac{\omega^2(\mathcal{D}^{\eta}\gamma(\omega))''}{\mathcal{D}^{\eta}\gamma(\omega)} \in \mathcal{P}_{\vartheta}(\delta),$$

*hold where $b \geq 0$, $2 \leq \vartheta \leq 4$, $0 \leq \delta < 1$ and the function $\gamma(\omega)$ is as defined by* (1.2).

**Remark 4.** *If $b = 0$ in Definition 2, we have $\mathcal{B}_{\Sigma}^{\eta,b}(\vartheta,\delta) \equiv \mathcal{B}_{\Sigma}^{\eta,0}(\vartheta,\delta) \equiv \mathcal{B}_{\Sigma}^{\eta}(\vartheta,\delta)$. That is, a function $h \in \Sigma$ given by* (1.1) *is said to be in the class $\mathcal{B}_{\Sigma}^{\eta}(\vartheta,\delta)$ if*

$$\frac{u\mathcal{D}^{\eta}h'(u)}{\mathcal{D}^{\eta}h(u)} \in \mathcal{P}_{\vartheta}(\delta)$$

*and*

$$\frac{\omega\mathcal{D}^{\eta}\gamma'(\omega)}{\mathcal{D}^{\eta}\gamma(\omega)} \in \mathcal{P}_{\vartheta}(\delta),$$

*where $2 \leq \vartheta \leq 4$, $0 \leq \delta < 1$ and the function $\gamma(\omega)$ is as defined by* (1.2).

**Remark 5.** *[19] If $\eta = 0$ in Definition 2, we have $\mathcal{B}_{\Sigma}^{\eta,b}(\vartheta,\delta) \equiv \mathcal{B}_{\Sigma}^{0,b}(\vartheta,\delta) \equiv \mathcal{S}_{\Sigma}^{*}(b,\vartheta,\delta)$. That is, a function $h \in \Sigma$ given by* (1.1) *is said to be in the class $\mathcal{S}_{\Sigma}^{*}(b,\vartheta,\delta)$ if*

$$\frac{uh'(u)}{h(u)} + b\frac{u^2h''(u)}{h(u)} \in \mathcal{P}_{\vartheta}(\delta)$$

*and*

$$\frac{\omega\gamma'(\omega)}{\gamma(\omega)} + b\frac{\omega^2\gamma''(\omega)}{\gamma(\omega)} \in \mathcal{P}_{\vartheta}(\delta),$$

*where $b \geq 0$, $2 \leq \vartheta \leq 4$, $0 \leq \delta < 1$ and the function $\gamma(\omega)$ is as defined by* (1.2).

**Theorem 2.** *If $h \in \mathcal{B}_{\Sigma}^{\eta,b}(\vartheta,\delta)$ is of the form* (1.1)*, then*

$$|h_2| \leq \sqrt{\frac{\vartheta(1-\delta)}{2(1+3b)3^{\eta} - (1+2b)2^{2\eta}}} \tag{2.13}$$

*and*

$$|h_3| \leq \frac{\vartheta(1-\delta)}{2(1+3b)3^{\eta} - (1+2b)2^{2\eta}}. \tag{2.14}$$

*For any $\aleph \in \mathbb{R}$, then*

$$|h_3 - \aleph h_2^2| \leq \begin{cases} \dfrac{\vartheta(1-\delta)(1-\aleph)}{2(1+3b)3^{\eta} - (1+2b)2^{2\eta}} : & \aleph < \Theta, \\ \dfrac{\vartheta(1-\delta)}{2(1+3b)3^{\eta}} : & \Theta \leq \aleph \leq 2-\Theta, \\ \dfrac{\vartheta(1-\delta)(\aleph-1)}{2(1+3b)3^{\eta} - (1+2b)2^{2\eta}} : & \aleph > 2-\Theta, \end{cases} \tag{2.15}$$

*where*

$$\Theta = \frac{(1+2b)2^{2\eta}}{2(1+3b)3^{\eta}}.$$

*Proof.* As $h \in \mathcal{B}_{\Sigma}^{\eta,b}(\vartheta,\delta)$, we have

$$\frac{u(\mathcal{D}^{\eta}h(u))'}{\mathcal{D}^{\eta}h(u)} + b\frac{u^2(\mathcal{D}^{\eta}h(u))''}{\mathcal{D}^{\eta}h(u)} = t(u) \tag{2.16}$$

and

$$\frac{\omega(\mathcal{D}^{\eta}\gamma(\omega))'}{\mathcal{D}^{\eta}\gamma(\omega)} + b\frac{\omega^2(\mathcal{D}^{\eta}\gamma(\omega))''}{\mathcal{D}^{\eta}\gamma(\omega)} = s(\omega), \tag{2.17}$$

where $t(u)$ and $s(\omega)$ are analytic functions belonging to the class $\mathcal{P}_{\vartheta}(\delta)$ given by (2.6) and (2.7). Comparing the coefficients using (2.6), (2.7), (2.16), and (2.17), we have

$$(1+2b)2^{\eta}h_2 = t_1, \tag{2.18}$$

$$2(1+3b)3^{\eta}h_3 - (1+2b)2^{2\eta}h_2^2 = t_2, \tag{2.19}$$

$$-(1+2b)2^{\eta}h_2 = s_1, \tag{2.20}$$

and

$$[4(1+3b)3^{\eta} - (1+2b)2^{2\eta}]h_2^2 - 2(1+3b)3^{\eta}h_3 = s_2. \tag{2.21}$$

Adding (2.19) and (2.21), we have

$$2[2(1+3b)3^{\eta} - (1+2b)2^{2\eta}]h_2^2 = t_2 + s_2. \tag{2.22}$$

Now, using Lemma 1, in (2.22), we have

$$|h_2|^2 \leq \frac{\vartheta(1-\delta)}{2(1+3b)3^{\eta} - (1+2b)2^{2\eta}}, \tag{2.23}$$

where $b \geq 0$ and $\eta \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and (2.23) gives the bound of $|h_2|$ given in (2.13). Again from (2.19) and (2.21), we have

$$4(1+3b)3^{\eta}h_3 - 4(1+3b)h_2^2 = t_2 - s_2. \tag{2.24}$$

Now, using (2.22) in (2.24), we have

$$4(1+3b)3^{\eta}h_3 = \frac{[4(1+3b)3^{\eta} - (1+2b)2^{2\eta}]t_2 + (1+2b)2^{2\eta}s_2}{2(1+3b)3^{\eta} - (1+2b)2^{2\eta}}. \tag{2.25}$$

Now, using Lemma 1, in (2.25), we have

$$[2(1+3b)3^{\eta} - (1+2b)2^{2\eta}]|h_3| \leq \vartheta(1-\delta). \tag{2.26}$$

Equation (2.26) gives the bound of $|h_3|$ given in (2.14). Now fix $\aleph \in \mathbb{R}$ and by using (2.22) and (2.25), we have

$$h_3 - \aleph h_2^2 = \frac{[4(1+3b)3^{\eta} - (1+2b)2^{2\eta} - 2(1+3b)3^{\eta}\aleph]t_2 + [(1+2b)2^{2\eta} - 2(1+3b)3^{\eta}\aleph]s_2}{4(1+3b)3^{\eta}[2(1+3b)3^{\eta} - (1+2b)2^{2\eta}]}. \tag{2.27}$$

Now, using Lemma 1, we have

$$|h_3 - \aleph h_2^2| \leq \frac{\vartheta(1-\delta)[|4(1+3b)3^{\eta} - (1+2b)2^{2\eta} - 2(1+3b)3^{\eta}\aleph| + |(1+2b)2^{2\eta} - 2(1+3b)3^{\eta}\aleph|]}{4(1+3b)3^{\eta}[2(1+3b)3^{\eta} - (1+2b)2^{2\eta}]},$$

gives the bound of $|h_3 - \aleph h_2^2|$ given in (2.15) finishing Theorem 2. □

**Definition 3.** *A function $h \in \Sigma$ given by* (1.1) *is said to be in the class $\mathcal{N}_{\Sigma}^{\eta,d}(\vartheta,\delta)$ if the conditions*

$$(1-d)\frac{u(\mathcal{D}^{\eta}h(u))'}{\mathcal{D}^{\eta}h(u)}+d\left(1+\frac{u(\mathcal{D}^{\eta}h(u))''}{(\mathcal{D}^{\eta}h(u))'}\right)\in\mathcal{P}_{\vartheta}(\delta)$$

*and*

$$(1-d)\frac{\omega(\mathcal{D}^{\eta}\gamma(\omega))'}{\mathcal{D}^{\eta}\gamma(\omega)}+d\left(1+\frac{\omega(\mathcal{D}^{\eta}\gamma(\omega))''}{(\mathcal{D}^{\eta}\gamma(\omega))'}\right)\in\mathcal{P}_{\vartheta}(\delta),$$

*are satisfied where $0 \le d \le 1$, $2 \le \vartheta \le 4$, $0 \le \delta < 1$ and the function $\gamma(\omega)$ is as defined by* (1.2).

**Remark 6.** *If $d = 0$ in Definition 3, we have $\mathcal{N}_{\Sigma}^{\eta,d}(\vartheta,\delta) \equiv \mathcal{N}_{\Sigma}^{\eta,0}(\vartheta,\delta) \equiv \mathcal{B}_{\Sigma}^{\eta}(\vartheta,\delta)$. A function $h \in \Sigma$ given by* (1.1) *is said to be in the class $\mathcal{B}_{\Sigma}^{\eta}(\vartheta,\delta)$ if*

$$\frac{u\mathcal{D}^{\eta}h'(u)}{\mathcal{D}^{\eta}h(u)}\in\mathcal{P}_{\vartheta}(\delta)$$

*and*

$$\frac{\omega\mathcal{D}^{\eta}\gamma'(\omega)}{\mathcal{D}^{\eta}\gamma(\omega)}\in\mathcal{P}_{\vartheta}(\delta),$$

*where $2 \le \vartheta \le 4$, $0 \le \delta < 1$ and the function $\gamma(\omega)$ is as defined by* (1.2).

**Remark 7.** *If $d = 1$ in Definition 3, we have $\mathcal{N}_{\Sigma}^{\eta,d}(\vartheta,\delta) \equiv \mathcal{N}_{\Sigma}^{\eta,1}(\vartheta,\delta) \equiv \mathcal{N}_{\Sigma}^{\eta}(\vartheta,\delta)$. A function $h \in \Sigma$ given by* (1.1) *is said to be in the class $\mathcal{N}_{\Sigma}^{\eta}(\vartheta,\delta)$ if*

$$1+\frac{u\mathcal{D}^{\eta}h''(u)}{\mathcal{D}^{\eta}h'(u)}\in\mathcal{P}_{\vartheta}(\delta)$$

*and*

$$1+\frac{\omega\mathcal{D}^{\eta}\gamma''(\omega)}{\mathcal{D}^{\eta}\gamma'(\omega)}\in\mathcal{P}_{\vartheta}(\delta),$$

*where $0 \le d \le 1$, $2 \le \vartheta \le 4$, $0 \le \delta < 1$ and the function $\gamma(\omega)$ is as defined by* (1.2).

**Remark 8.** *[19] If $\eta = 0$ in Definition 3, we have $\mathcal{N}_{\Sigma}^{\eta,d}(\vartheta,\delta) \equiv \mathcal{N}_{\Sigma}^{0,d}(\vartheta,\delta) \equiv \mathcal{M}_{\Sigma}^{d}(\vartheta,\delta)$. A function $h \in \Sigma$ given by* (1.1) *is said to be in the class $\mathcal{M}_{\Sigma}^{\eta,d}(\vartheta,\delta)$ if*

$$(1-d)\frac{uh'(u)}{h(u)}+d\left(1+\frac{uh''(u)}{h'(u)}\right)\in\mathcal{P}_{\vartheta}(\delta)$$

*and*

$$(1-d)\frac{\omega\gamma'(\omega)}{\gamma(\omega)}+d\left(1+\frac{\omega\gamma''(\omega)}{\gamma'(\omega)}\right)\in\mathcal{P}_{\vartheta}(\delta),$$

*where $0 \le d \le 1$, $2 \le \vartheta \le 4$, $0 \le \delta < 1$ and the function $\gamma(\omega)$ is as defined by* (1.2).

**Theorem 3.** *If $h \in \mathcal{N}_{\Sigma}^{\eta,d}(\vartheta,\delta)$ is given in the form* (1.1)*, then*

$$|h_2| \le \sqrt{\frac{\vartheta(1-\delta)}{2(1+2d)3^{\eta}-(1+3d)2^{2\eta}}} \tag{2.28}$$

*and*

$$|h_3| \leq \frac{\vartheta(1-\delta)}{2(1+2d)3^{\eta}-(1+3d)2^{2\eta}}. \tag{2.29}$$

*For any* $\aleph \in \mathbb{R}$, *then*

$$|h_3 - \aleph h_2^2| \leq \begin{cases} \dfrac{\vartheta(1-\delta)(1-\aleph)}{2(1+2d)3^{\eta}-(1+3d)2^{2\eta}} : & \aleph < \pounds, \\ \dfrac{\vartheta(1-\delta)}{2(1+2d)3^{\eta}} : & \pounds \leq \aleph \leq 2-\pounds, \\ \dfrac{\vartheta(1-\delta)(\aleph-1)}{2(1+2d)3^{\eta}-(1+3d)2^{2\eta}} : & \aleph > 2-\pounds, \end{cases} \tag{2.30}$$

*where*

$$\pounds = \frac{(1+3d)2^{2\eta}}{2(1+2d)3^{\eta}}.$$

*Proof.* As $h \in \mathcal{N}_{\Sigma}^{\eta,d}(\vartheta,\delta)$, we have,

$$(1-d)\frac{u(\mathcal{D}^{\eta}h(u))'}{\mathcal{D}^{\eta}h(u)} + d\left(1+\frac{u(\mathcal{D}^{\eta}h(u))''}{(\mathcal{D}^{\eta}h(u))'}\right) = t(u) \tag{2.31}$$

and

$$(1-d)\frac{\omega(\mathcal{D}^{\eta}\gamma(\omega))'}{\mathcal{D}^{\eta}\gamma(\omega)} + d\left(1+\frac{\omega(\mathcal{D}^{\eta}\gamma(\omega))''}{(\mathcal{D}^{\eta}\gamma(\omega))'}\right) = s(\omega), \tag{2.32}$$

where $t(u)$ and $s(\omega)$ are analytic functions belonging to the class $\mathcal{P}_{\vartheta}(\delta)$ given by (2.6) and (2.7). Comparing the coefficients using (2.6), (2.7), (2.31) and (2.32), we have

$$(1+d)2^{\eta}h_2 = t_1, \tag{2.33}$$

$$2(1+2d)3^{\eta}h_3 - (1+3d)2^{2\eta}h_2^2 = t_2, \tag{2.34}$$

$$-(1+d)2^{\eta}h_2 = s_1, \tag{2.35}$$

and

$$[4(1+2d)3^{\eta} - (1+3d)2^{2\eta}]h_2^2 - 2(1+2d)3^{\eta}h_3 = s_2. \tag{2.36}$$

Adding (2.34) and (2.36), we have

$$2[2(1+2d)3^{\eta} - (1+3d)2^{2\eta}]h_2^2 = t_2 + s_2. \tag{2.37}$$

Now, using Lemma 1, in (2.37), we have

$$|h_2|^2 \leq \frac{\vartheta(1-\delta)}{2(1+2d)3^{\eta}-(1+3d)2^{2\eta}}. \tag{2.38}$$

Equation (2.38) gives the bound of $|h_2|$ given in (2.28). Again from (2.34) and (2.36), we have

$$4(1+2d)3^{\eta}h_3 - 4(1+2d)3^{\eta}h_2^2 = t_2 - s_2. \tag{2.39}$$

Now, using (2.37) in (2.39), we have

$$4(1+2d)3^{\eta}h_3 = \frac{[4(1+2d)3^{\eta}-(1+3d)2^{2\eta}]t_2+(1+3d)2^{2\eta}s_2}{2(1+2d)3^{\eta}-(1+3d)2^{2\eta}}. \tag{2.40}$$

Now, by using Lemma 1, in (2.40), we have

$$[2(1+2d)3^{\eta}-(1+3d)2^{2\eta}]h_3 \leq \vartheta(1-\delta). \tag{2.41}$$

Equation (2.41) gives the bound of $|h_3|$ given in (2.29). Now fix $\aleph \in \mathbb{R}$ and by using (2.37) and (2.40), we have

$$h_3-\aleph h_2^2 = \frac{[4(1+2d)3^{\eta}-(1+3d)2^{2\eta}-2(1+2d)3^{\eta}\aleph]t_2+[(1+3d)2^{2\eta}-2(1+2d)3^{\eta}\aleph]s_2}{4(1+2d)3^{\eta}[2(1+2d)3^{\eta}-(1+3d)2^{2\eta}]}. \tag{2.42}$$

Now, using Lemma 1, in (2.42), we have

$$|h_3-\aleph h_2^2| \leq \frac{\vartheta(1-\delta)[|4(1+2d)3^{\eta}-(1+3d)2^{2\eta}-2(1+2d)3^{\eta}\aleph|+|(1+3d)2^{2\eta}-2(1+2d)3^{\eta}\aleph|]}{4(1+2d)3^{\eta}[2(1+2d)3^{\eta}-(1+3d)2^{2\eta}]}. \tag{2.43}$$

Equation (2.43) gives the bound of $|h_3-\aleph h_2^2|$ given in (2.30), which completes the proof of Theorem 3. □

## 3. Corollaries and remarks

For the choices of $a=1$, $a=0$ and $\eta=0$ in Theorem 1, we get the following Corollaries namely Corollary 1, Corollary 2 and Corollary 3, respectively.

**Corollary 1.** *If $h \in \mathcal{H}_{\Sigma}^{\eta}(\vartheta,\delta)$ is given in the form* (1.1)*, then*

$$|h_2| \leq \sqrt{\frac{\vartheta(1-\delta)}{3^{\eta+1}}}$$

*and*

$$|h_3| \leq \frac{\vartheta(1-\delta)}{3^{\eta+1}}.$$

*For any $\aleph \in \mathbb{R}$, then*

$$|h_3-\aleph h_2^2| \leq \begin{cases} \dfrac{\vartheta(1-\delta)(1-\aleph)}{2.3^{\eta+1}} & \textit{for } \aleph<0, \\ \dfrac{\vartheta(1-\delta)}{3^{\eta+1}} & \textit{for } 0\leq\aleph\leq 2, \\ \dfrac{\vartheta(1-\delta)(\aleph-1)}{2.3^{\eta+1}} & \textit{for } \aleph>2. \end{cases}$$

**Corollary 2.** *If $h \in \mathcal{L}_{\Sigma}^{\eta}(\vartheta,\delta)$ is of the form* (1.1)*, then*

$$|h_2| \leq \sqrt{\frac{\vartheta(1-\delta)}{3^{\eta}}}$$

*and*

$$|h_3| \leq \frac{\vartheta(1-\delta)}{3^\eta}.$$

*For any* $\aleph \in \mathbb{R}$, *then*

$$|h_3 - \aleph h_2^2| \leq \begin{cases} \dfrac{\vartheta(1-\delta)(1-\aleph)}{2.3^\eta} & \textit{for } \aleph < 0, \\ \dfrac{\vartheta(1-\delta)}{3^\eta} & \textit{for } 0 \leq \aleph \leq 2, \\ \dfrac{\vartheta(1-\delta)(\aleph-1)}{2.3^\eta} & \textit{for } \aleph > 2. \end{cases}$$

**Corollary 3.** *If* $h \in \mathcal{L}_\Sigma^a(\vartheta, \delta)$ *is given in the form* (1.1)*, then*

$$|h_2| \leq \sqrt{\frac{\vartheta(1-\delta)}{(1+2a)}}$$

*and*

$$|h_3| \leq \frac{\vartheta(1-\delta)}{(1+2a)}.$$

*For any* $\aleph \in \mathbb{R}$, *then*

$$|h_3 - \aleph h_2^2| \leq \begin{cases} \dfrac{\vartheta(1-\delta)(1-\aleph)}{2(1+2a)} & \textit{for } \aleph < 0, \\ \dfrac{\vartheta(1-\delta)}{(1+2a)} & \textit{for } 0 \leq \aleph \leq 2, \\ \dfrac{\vartheta(1-\delta)(\aleph-1)}{2(1+2a)} & \textit{for } \aleph > 2. \end{cases}$$

**Remark 9.** $\vartheta = 2$ *in Corollary 3, verifies the results obtained in [6].*

For the selection of $b = 0$, $\eta = 0$ in Theorem 2, we get the Corollaries Corollary 4, Corollary 5, respectively.

**Corollary 4.** *If* $h \in \mathcal{B}_\Sigma^\eta(\vartheta, \delta)$ *is represented in the form* (1.1)*, then*

$$|h_2| \leq \sqrt{\frac{\vartheta(1-\delta)}{2.3^\eta - 2^{2\eta}}}$$

*and*

$$|h_3| \leq \frac{\vartheta(1-\delta)}{2.3^\eta - 2^{2\eta}}.$$

*For any* $\aleph \in \mathbb{R}$, *then*

$$|h_3 - \aleph h_2^2| \leq \begin{cases} \dfrac{\vartheta(1-\delta)(1-\aleph)}{2.3^\eta - 2^{2\eta}} : & \aleph < \Theta, \\ \dfrac{\vartheta(1-\delta)}{2.3^\eta} : & \Theta \leq \aleph \leq 2-\Theta, \\ \dfrac{\vartheta(1-\delta)(\aleph-1)}{2.3^\eta - 2^{2\eta}} : & \aleph > 2-\Theta, \end{cases}$$

*where*

$$\Theta = \frac{2^{2\eta}}{2.3^{\eta}}.$$

**Corollary 5.** *If* $h \in \mathcal{S}^*_\Sigma(b, \vartheta, \delta)$ *is given in the form* (1.1)*, then*

$$|h_2| \leq \sqrt{\frac{\vartheta(1-\delta)}{1+4b}}$$

*and*

$$|h_3| \leq \frac{\vartheta(1-\delta)}{1+4b}.$$

*For any* $\aleph \in \mathbb{R}$, *then*

$$|h_3 - \aleph h_2^2| \leq \begin{cases} \dfrac{\vartheta(1-\delta)(1-\aleph)}{1+4b} : & \aleph < \Theta, \\ \dfrac{\vartheta(1-\delta)}{2(1+3b)} : & \Theta \leq \aleph \leq 2-\Theta, \\ \dfrac{\vartheta(1-\delta)(\aleph-1)}{1+4b} : & \aleph > 2-\Theta, \end{cases}$$

*where*

$$\Theta = \frac{1+2b}{2(1+3b)}.$$

If $\eta = 0$ and $b = 0$ in Theorem 2, we get Corollary 6, which verifies the results obtained in [8, 19].

**Corollary 6.** *If* $h \in \mathcal{S}^*_\Sigma(\vartheta, \delta)$ *given in the form* (1.1)*, then*

$$|h_2| \leq \sqrt{\vartheta(1-\delta)}$$

*and*

$$|h_3| \leq \vartheta(1-\delta).$$

*For any* $\aleph \in \mathbb{R}$, *then*

$$|h_3 - \aleph h_2^2| \leq \begin{cases} \vartheta(1-\delta)(1-\aleph) : & \aleph < \dfrac{1}{2}, \\ \dfrac{\vartheta(1-\delta)}{2} : & \dfrac{1}{2} \leq \aleph \leq \dfrac{3}{2}, \\ \vartheta(1-\delta)(\aleph-1) : & \aleph > \dfrac{3}{2}. \end{cases}$$

**Remark 10.** $\vartheta = 2$ *in Corollary 6, verifies the results obtained in [4].*

For the choices $d = 1$, $\eta = 0$ in Theorem 3, we get corollaries Corollary 7 and Corollary 8, respectively.

**Corollary 7.** *If* $h \in \mathcal{N}^\eta_\Sigma(\vartheta, \delta)$ *is of the form* (1.1)*, then*

$$|h_2| \leq \sqrt{\frac{\vartheta(1-\delta)}{2.3^{\eta+1} - 2^{2\eta+2}}}$$

*and*

$$|h_3| \leq \frac{\vartheta(1-\delta)}{2.3^{\eta+1}-2^{2\eta+2}}.$$

*For any* $\aleph \in \mathbb{R}$, *then*

$$|h_3 - \aleph h_2^2| \leq \begin{cases} \frac{\vartheta(1-\delta)(1-\aleph)}{2.3^{\eta+1}-2^{2\eta+2}} : & \aleph < £, \\ \frac{\vartheta(1-\delta)}{2.3^{\eta+1}} : & £ \leq \aleph \leq 2-£, \\ \frac{\vartheta(1-\delta)(\aleph-1)}{2.3^{\eta+1}-2^{2\eta+2}} : & \aleph > 2-£, \end{cases}$$

*where*

$$£ = \frac{2^{2\eta+2}}{2.3^{\eta+1}}.$$

**Corollary 8.** *If* $h \in \mathcal{M}_{\Sigma}^{d}(\vartheta, \delta)$ *is given in the form* (1.1)*, then*

$$|h_2| \leq \sqrt{\frac{\vartheta(1-\delta)}{1+d}}$$

*and*

$$|h_3| \leq \frac{\vartheta(1-\delta)}{1+d}.$$

*For any* $\aleph \in \mathbb{R}$, *then*

$$|h_3 - \aleph h_2^2| \leq \begin{cases} \frac{\vartheta(1-\delta)(1-\aleph)}{1+d} : & \aleph < £, \\ \frac{\vartheta(1-\delta)}{2(1+2d)} : & £ \leq \aleph \leq 2-£, \\ \frac{\vartheta(1-\delta)(\aleph-1)}{1+d} : & \aleph > 2-£, \end{cases}$$

*where*

$$£ = \frac{1+3d}{2(1+2d)}.$$

**Remark 11.** *Corollary 8, verifies the results obtained in [19].*

If $\eta = 0$ and $d = 1$ in Theorem 3, we get the following corollary.

**Corollary 9.** *If* $h \in \mathcal{M}_{\Sigma}(\vartheta, \delta)$ *given in the form* (1.1)*, then*

$$|h_2| \leq \sqrt{\frac{\vartheta(1-\delta)}{2}}$$

*and*

$$|h_3| \leq \frac{\vartheta(1-\delta)}{2}.$$

*For any* $\aleph \in \mathbb{R}$, *then*

$$|h_3 - \aleph h_2^2| \leq \begin{cases} \dfrac{\vartheta(1-\delta)(1-\aleph)}{2} : & \aleph < \dfrac{2}{3}, \\ \dfrac{\vartheta(1-\delta)}{6} : & \dfrac{2}{3} \leq \aleph \leq \dfrac{4}{3}, \\ \dfrac{\vartheta(1-\delta)(\aleph-1)}{2} : & \aleph > \dfrac{4}{3}. \end{cases}$$

**Remark 12.** *Corollary 9, verifies the results obtained in [8]. If* $\vartheta = 2$ *in Corollary 9, verifies the results obtained in [4].*

## 4. Conclusions

By an application of the Sălăgean operator, three new subclasses of bi-univalent functions associated with bounded boundary rotation in open unit disk $\mathbb{E}$ are considered in this article. We first established initial coefficient bounds as well as the Fekete-Szegö estimates for the classes $\mathcal{L}_\Sigma^{\eta,a}(\vartheta,\delta)$, $\mathcal{B}_\Sigma^{\eta,b}(\vartheta,\delta)$, and $\mathcal{N}_\Sigma^{\eta,d}(\vartheta,\delta)$. Interesting remarks for the major results, including improvements of the earlier bounds, are also quoted. More corollaries and remarks could be reported for the selection of parameters, and those details have been omitted.

## Author contributions

All authors of this article have been contributed equally. All authors have read and approved the final version of the manuscript for publication.


## Acknowledgments

The authors would like to thank the referees for their comments and suggestions on the original manuscript.

This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government(MSIT) (No. NRF-2022R1A2C2004874). This work was supported by the Korea Institute of Energy Technology Evaluation and Planning(KETEP) and the Ministry of Trade, Industry & Energy(MOTIE) of the Republic of Korea (No. 20214000000280).


## Conflict of interest

The authors declare no conflict of interest.

## References


1. T. Al-Hawary, Sufficient conditions for analytic functions defined by Frasin differential operator, *Stud. Univ. Babeş-Bolyai Math.*, **66** (2021), 353–359. https://doi.org/10.24193/subbmath.2021.2.11

2. B. S. Alkahtani, P. Goswami, T. Bulboacă, Estimate for initial MacLaurin coefficients of certain subclasses of bi-univalent functions, *Miskolc Math.*, **17** (2016), 739–748. https://doi.org/10.18514/MMN.2017.1565
3. D. A. Brannan, J. G. Clunie, *Aspects of contemporary complex analysis*, London: Academic Press, 1980.
4. D. A. Brannan, T. S. Taha, On some classes of bi-univalent functions, *Math. Anal. Appl.*, 1985, 53–60. https://doi.org/10.1016/B978-0-08-031636-9.50012-7
5. P. L. Duren, *Univalent functions*, New York: Springer, **259** (1983).
6. B. A. Frasin, M. K. Aouf, New subclasses of bi-univalent functions, *Appl. Math. Lett.*, **24** (2011), 1569–1573. https://doi.org/10.1016/j.aml.2011.03.048
7. M. Lewin, On a coefficient problem for bi-univalent functions, *Proc. Amer. Math. Soc.*, **18** (1967), 63–68. https://doi.org/10.2307/2035225
8. Y. M. Li, K. Vijaya, G. Murugusundaramoorthy, H. Tang, On new subclasses of bi-starlike functions with bounded boundary rotation, *AIMS Math.*, **5** (2020), 3346–3356. https://doi.org/10.3934/math.2020215
9. U. H. Naik, A. B. Patil, On initial coefficient inequalities for certain new subclasses of bi-univalent functions, *J. Egyptian Math. Soc.*, **25** (2017), 291–293. https://doi.org/10.1016/j.joems.2017.04.001
10. E. Netanyahu, The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$, *Arch. Rational Mech. Anal.*, **32** (1969), 100–112. https://doi.org/10.1007/BF00247676
11. J. W. Noonan, Asymptotic behavior of functions with bounded boundary rotation, *Trans. Amer. Math. Soc.*, **164** (1972), 397–410. https://doi.org/10.2307/1995984
12. S. O. Olatunji, P. T. Ajai, On subclasses of bi-univalent functions of Bazilevic type involving linear and Salagean operator, *Int. J. Pure. Appl. Math.*, **92** (2014), 645–656. http://dx.doi.org/10.12732/ijpam.v92i5.2
13. V. Paatero, Über Gebiete von beschrankter Randdrehung, *Ann. Acad. Sci. Fenn. Ser. A.*, **37** (1933), 1–20.
14. K. S. Padmanabhan, R. Parvatham, Properties of a class of functions with bounded boundary rotation, *Ann. Pol. Math.*, **31** (1975), 311–323. https://doi.org/10.4064/ap-31-3-311-323
15. B. Pinchuk, Functions of bounded boundary rotation, *Israel J. Math.*, **10** (1971), 6–16. https://doi.org/10.1007/BF02771515
16. G. S. Sălăgean, Subclasses of univalent functions, *Complex Analysis-Fifth Romanian-Finnish Seminar*, 1981, 362–372. https://doi.org/10.1007/BFb0066543
17. T. G. Shaba, Certain new subclasses of analytic and bi-univalent functions using salagean operator, *Asia Pac. J. Math.*, **7** (2020), 29. https://doi.org/10.28924/APJM/7-29
18. T. G. Shaba, On some new subclass of bi-univalent functions associated with Opoola differential operator, *Open J. Math. Anal.*, **4** (2020), 74–79. https://doi.org/10.30538/psrp-oma2020.0064

19. P. Sharma, S. Sivasubramanian, N. E. Cho, Initial coefficient bounds for certain new subclasses of bi-univalent functions with bounded boundary rotation, *AIMS Math.*, **8** (2023), 29535–29554. https://doi.org/10.3934/math.20231512

20. H. M. Srivastava, A. K. Mishra, P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, *Appl. Math. Lett.*, **23** (2010), 1188–1192. https://doi.org/10.1016/j.aml.2010.05.009