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Research article

Linear generalized derivations on Banach *-algebras

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Abstract: This paper deals with some identities on Banach *-algebras that are equipped with linear generalized derivations. As an application of one of our results, we describe the structure of the underlying algebras. Precisely, we prove that for a linear generalized derivation F on a Banach *-algebra A, either we obtain the existence of a central idempotent element $e \in Q$, for which $F = 0$ on eQ and $(1-e)Q$ satisfies s_4, or the set of elements $u \in A$ such that the identity $[F(u)^n, F(u^*)^n F(u)^n] \in Z(A)$ holds for no positive integer n turns out to be dense. In addition to this we consider an identity satisfied by a semisimple Banach *-algebra and look for its commutativity. Moreover, some related results are also established.

Keywords: Banach *-algebra; linear derivation; linear generalized derivation; involution
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1. Introduction

Let A be a Banach algebra equipped with an involution *, defined over the field of complex numbers $\mathbb{C}$. A map δ on A, is said to form a linear derivation on A if δ is linear, i.e., $\delta(u+v) = \delta(u) + \delta(v)$ and $\delta(au) = a\delta(u)$ and $\delta(uv) = \delta(u)v + u\delta(v)$. The Leibniz rule is satisfied for all $u, v \in A$ and $a \in \mathbb{C}$. In a similar way, a linear generalized derivation F associated with a linear derivation δ is defined as a linear map satisfying $F(uv) = F(u)v + u\delta(v)$. A derivation δ of A is said to be X-inner if there exists some $a \in Q$ such that $\delta(u) = [a, u]$, for all $u \in A$, where Q is the symmetric Martindale quotient ring (algebra) (see [3] for more details). The center of Q is known as the extended centroid and is denoted by C. The derivations that are not X-inner are called X-outer. Also, a generalized derivation is called X-inner if its associated derivation is X-inner; otherwise it is called X-outer. A map $f : S \to A$ is said to be centralizing on a subset S of A if $[f(u), u] \in Z(A)$, and is commuting on S if this commutator turns out to be zero. Extensive study of centralizing and commuting maps was started in 1957 with

Posner's [17] remarkable results pertaining to prime rings. Then onwards there was no looking back; various algebraists ventured into this area (viz.; [1, 5, 7, 10, 18–20] and various references therein). By a prime algebra, we mean an algebra A such that whenever $uAv = 0$, then either $u = 0$ or $v = 0$, and in a semiprime algebra, whenever $uAu = 0$, it implies $u = 0$. In an analogous manner we define prime (semiprime) rings, i.e., whenever a ring T satisfies $uTv = 0$ ($uTu = 0$), it implies either $u = 0$ or $v = 0$ ($u = 0$). We recall that, for a given subset S of Banach algebra A, the right annihilator $r_A(S)$ of a set S in A is a totality of all $x \in A$ such that $Sx = 0$. Accordingly, the left annihilator $l_A(S)$ is a set of all $x \in A$ such that $xS = 0$. The intersection $ann_A(S) = r_A(S) \cap l_A(S)$ is called an annihilator of S in A. A special type of semiprime algebra is that of the semisimple algebra, which, in addition to being semiprime is Artinian as well. In the free product $Q *_C C\{X_1, X_2, \ldots\}$ of the C-algebras Q and the free C-algebra in the noncommutative indeterminates $X_1, X_2, \ldots$, an element $\phi(x_1, \ldots, x_n)$ is called a generalized polynomial identity (GPI) on A if $\phi(u_1, \ldots, u_n) = 0$ for all $u_1, \ldots, u_n \in A$ (see [3] for more details). The standard polynomial s_4 is given by $s_4(u_1, u_2, u_3, u_4) = \sum(-1)^\sigma u_{\sigma(1)} u_{\sigma(2)} u_{\sigma(3)} u_{\sigma(4)}$, where the map σ runs over the permutations of symmetric group S_4, and $(-1)^\sigma$ stands for the sign of σ. An element $u \in A$ is said to be quasi-normal if it commutes with u^*u, i.e., $[u, u^*u] = 0$. This concept was given by Brown [6] way back in 1953, where he gave structure theorems for operators on a Hilbert space satisfying the quasi-normality relation. Yood in [21] worked with the idea of quasi-normality for semisimple Banach *-algebras and was able to show the commutativity of the algebra and the denseness of certain subsets. Later in [22], he studied quasi-normality for $L(H)$, the set of all bounded linear operators on a Hilbert space H. Precisely, he established that for an algebra having a regular involution * either the algebra is commutative or the set for which $[u^n, (u^*)^n u^n] \in Z(A)$ for no positive integer n is dense in A. In [1], Alhazmi and Khan, while working with the concept of linear derivations studied the denseness of some subsets subject to certain identities. Motivated by these results, we venture in this direction to see the behavior of a semisimple Banach *-algebra equipped with linear generalized derivations. The next section will provide us with the necessary impetus and some preliminary results, followed by the main results.

2. The results

Throughout, A will denote a complex Banach *-algebra with * as the involution map and $Z(A)$, the center of A. Q denotes the symmetric Martindale quotient algebra of A and C its center, called the extended centroid of A. $Sym(A)$ denotes the collection of all elements of $u \in A$ satisfying $u^* = u$. We say that * forms a regular involution on A if whenever $\rho(u) = 0$ and $u^* = u$ then $u = 0$, where $\rho(u) = \lim \|u^n\|^{1/n}$ (see [9], p. 420). The following facts will come handy while proving the main results:

Fact 1. *[4, p. 91] If* $u, v \in A$ *and* $[u, [u, v]] = 0$*, then* $\rho([u, v]) = 0$.

Fact 2. *[4, Theorem 2] If A is semisimple, then the involution map * is continuous.*

Fact 3. *[4] Let A be a real or complex Banach algebra and* $p(t) = \sum_{k=0}^{m} c_k t^k$ *a polynomial in the real variable t with coefficients in A. If for an infinite set of real values of t,* $p(t) \in N$*, where N is a closed linear subspace of A, then each* c_k *lies in N.*

Fact 4. *[15, Theorem 4] Let R be a semiprime ring. Then, every generalized derivation g on a dense* *right ideal of R can be uniquely extended to U and assumes the form $g(x) = ax + d(x)$ for some $a \in U$ and a derivation d on U. Moreover, a and d are uniquely determined by the generalized derivation g.*

Fact 5. *[14] Let R be a semiprime ring with extended centroid C. The set B of idempotents of C forms a Boolean algebra with respect to the operations, $e \circ f = e + f - ef$ and $e \cdot f = ef$. Let M be a maximal ideal of B. Then, M is a prime ideal of Q and is invariant under all derivations of Q.*

Lemma 2.1. *Let $g : A \to A$ be any linear map, and n be any fixed positive integer. Suppose that $(g(h))^n \in \zeta$, for all $h = h^*$, where $\zeta \subseteq A$ is any closed linear subspace. Then $(g(u))^n \in \zeta$, for all $u \in A$.*

Proof. We are given that $(g(h))^n \in \zeta$ for all $h = h^*$, where $\zeta \subseteq A$. For any $t \in \mathbb{R}$, we obtain

$$(g(h + tk))^n = \sum_{m=0}^{n} \Psi_m t^m \in \zeta,$$

for all $h, k \in Sym(A)$. Here, Ψ_m denotes the sum of the terms in $(g(h + tk))^n$ for which the sum of exponents of $(g(k))^i$ equals m, as can be seen, $\Psi_0 = (g(h))^n$, $\Psi_1 = \sum_{l=0}^{n-1}(g(h))^l(g(k))(g(h))^{n-1-l}$ and so on. Fact 3 guarantees that each $\Psi_m \in \zeta$.

Each element $u \in A$ can be written in the form $u = h + ik$, for some $h, k \in Sym(A)$, i.e., each element of this complex Banach *-algebra can be written as the sum of two symmetric elements h, k, where h corresponds to the symmetric part and ik corresponds to the skew symmetric part of u.

Thus, $(g(u))^n = (g(h + ik))^n = \sum_{m=0}^{n} \Psi_m i^m$ for all $u \in A$. Each $\Psi_m \in \zeta$ and ζ being a subspace will contain all the linear combinations of elements from A and scalars from $\mathbb{C}$. Therefore, $\sum_{m=0}^{n} \Psi_m i^m \in \zeta$ and hence $(g(u))^n \in \zeta$ for all $u \in A$. □

Now, we state and prove the first key result of this paper.

Theorem 2.2. *Let $F : A \to A$ be a non-zero linear generalized derivation defined on a non-commutative prime Banach algebra. If $[F(u)^n, F(v)] = 0$ for all $u, v \in A$, for a fixed positive integer n. Then $dim_C AC = 4$.*

Proof. We have,

$$[F(u)^n, F(v)] = 0 \text{ for all } u, v \in A. \tag{2.1}$$

Using the Fact 4, we can write, $F(u) = au + d(u)$ for some $a \in Q$. Hence, we obtain

$$[(au + d(u))^n, av + d(v)] = 0, \tag{2.2}$$

for all $u, v \in A$. Let's first focus on a. If $a = 0$, then (2.2) becomes

$$[d(u)^n, d(v)] = 0, \tag{2.3}$$

for all $u, v \in A$. The result follows from [1, Theorem 2.2] and we obtain $dim_C AC = 4$. From now on, let us assume $a \neq 0$. Shifting our focus to the derivation d and supposing d is X-outer. Then from [13, Theorem 2], we have

$$[(au + r)^n, av + s] = 0,$$

for all $u, v, r, s \in A$. Replacing r by $r - au$ and s by $s - av$ in the last relation, we obtain $[r^n, s] = 0$ for all $r, s \in A$, that is, $r^n \in Z(A)$ for all $r \in A$. Thus, we obtain A must be commutative (see [12, Theorem 3.22]), contradicting the non-commutativity of A.

Now, suppose d is X-inner, i.e., there exists $b \in Q$ such that $d(u) = bu - ub$ for all $u \in A$. So, we have

$$[(au + [b, u])^n, av + [b, v]] = 0, \tag{2.4}$$

for all $u, v \in A$. If possible, suppose d is zero. Then Eq (2.2) becomes

$$[(au)^n, av] = 0, \tag{2.5}$$

for all $u, v \in A$. Replacing v by vr in (2.5), we obtain

$$av[(au)^n, r] = 0, \tag{2.6}$$

as $v \in A$, we can write

$$aA[(au)^n, r] = \{0\}, \tag{2.7}$$

for all $u, r \in A$. Since A is prime and $a \neq 0$, this implies $[(au)^n, r] = 0$, giving us $(au)^n \in Z(A)$ for all $u \in A$. Applying [12, Theorem 3.22] we obtain, either A must be commutative or A should possess a non-zero nil ideal. But A is non-commutative and prime, therefore, by [12, Lemma 2.1.1] we arrive at a contradiction.

As $d \neq 0$, it implies $b \notin C$, Eq (2.4) tells us that the GPI

$$[(aX + [b, X])^n, (ay + [b, Y])], \tag{2.8}$$

is satisfied by Q and is a non-trivial one. We know that Q forms a centrally closed prime C-algebra and Q satisfies a GPI; this implies Q contains a minimal right ideal eQ (making Q primitive), and eQe becomes a finite dimensional division algebra over C (see [16, Theorem 3]). So Q forms a primitive ring with a minimal idempotent of Q, say $e(\neq 1)$, so that $Ce = eQe$. Suppose $S = eQ$. Therefore, Q embeds in $End(S_C)$, and also Q acts densely on S_C. As $b \notin C$, there is some $s \in S$ so that s and bs are independent over C. Let us take $S = T \oplus Cs \oplus Cbs$, where $T \subseteq C_S$, a subspace. If $T = \{0\}$, then $dim_C S = 2$ making $dim_C Q = 4$. As Q is centrally closed, so $Q = AC$ and then $dim_C AC = 4$, as required. Having $T = \{0\}$, will lead us to the desired result. Now we will show that $T = \{0\}$. To the contrary, let us suppose $T \neq \{0\}$.

Suppose, we have a $t \in T$ such that t, bt, s, bs are independent over C. As Q acts densely on S_C, there exist $q_1, q_2 \in Q$ such that

$$q_1 t = 0,\ q_1 bt = t,\ q_1 s = 0,\ q_1 bs = 0,$$
$$q_2 t = 0,\ q_2 bt = -t - s.$$

With these equations, consider

$$\begin{aligned}&(aq_1 + [b, q_1])t = (aq_1 + bq_1 - q_1 b)t\\&= a(q_1 t) + b(q_1 t) - (q_1 b)t\\&= 0 + 0 - t = -t.\end{aligned}$$

Repeating the above steps, we obtain

$$(aq_1 + [b, q_1])^n t = (-1)^n t.$$

Similarly, observe that

$$\begin{aligned}(aq_2 + [b, q_2])t &= (aq_2 + bq_2 - q_2 b)t \\ &= aq_2 t + bq_2 t - q_2 bt = 0 + 0 - (-t - s) = t + s.\end{aligned}$$

By the last relation, we conclude that

$$(aq_2 + [b, q_2])t = t + s, \tag{2.9}$$

also

$$(aq_1 + [b, q_1])^n s = 0. \tag{2.10}$$

Now, consider the Eq (2.4),

$$[(aq_1 + [b, q_1])^n, (aq_2 + [b, q_2])] = 0, \tag{2.11}$$

for all $q_1, q_2 \in A$, we obtain

$$(aq_1 + [b, q_1])^n (aq_2 + [b, q_2]) - (aq_2 + [b, q_2])(aq_1 + [b, q_1])^n = 0.$$

Right multiplication in the above relation by t yields,

$$(aq_1 + [b, q_1])^n (aq_2 + [b, q_2])t - (aq_2 + [b, q_2])(aq_1 + [b, q_1])^n t = 0.$$

This implies that

$$(aq_1 + [b, q_1])^n (t + s) - (aq_2 + [b, q_2])(-1)^n t = 0,$$

which gives,

$$(aq_1 + [b, q_1])^n t + ((aq_1 + [b, q_1])^n)s + (-1)^{n+1}(aq_2 + [b, q_2])t = 0.$$

So, we have

$$(-1)^n t + (-1)^{n+1}(t + s) = 0,$$

which implies

$$(-1)^{n-1} s = 0,$$

from the last relation, we conclude

$$s = 0,$$

a contradiction. Therefore, no such $t \in T$ exists such that t, bt, s, bs are C-independent. Next, consider for any $t \in T$, t, bt, s, bs dependent over C. Choose $t \in T \backslash Cs \oplus Cbs$. As t, s, bs are C-independent, there exist $a_1, a_2, a_3 \in C$ such that $bt = a_1 t + a_2 s + a_3 bs$. Thus $(b - a_1)t = a_2 s + a_3 bs = (a_2 + a_1 a_3)s + a_3(b - a_1)s$, where s and $(b - a_1)s$ are C-independent. Observe that $ad(b - a_1) = ad(b)$. Taking b as $b - a_1$ from the beginning, we can assume that $bt = a_2 s + a_3 bs$. Let $p = a_2^{-1}(t - a_3 s)$ if $a_2 \neq 0$ and let $t - a_3 s = 0$ if $a_2 = 0$. Thus s, bs, p are C-independent, and, moreover, either $bp = s$ or $bp = 0$.

Case I: First assume $bp = s$. Since Q acts densely on S_C, there exist $q_1, q_2 \in Q$ such that $q_1 s = 0, q_1(bs) = -s, q_1 p = 0, q_2 s = 0, q_2(bs) = -s - p, q_2 p = 0$. Then, we have $(aq_2 + [b, q_2])s = -q_2 bs = s + p$ and $(aq_1 + [b, q_1])s = -q_1 bs = s$, which leads to

$$(aq_1 + [b, q_1])^n s = s. \tag{2.12}$$

Furthermore, we have $(aq_1 + [b, q_1])p = -q_1 bp = -q_1 p = 0$, giving

$$(aq_1 + [b, q_1])^n p = 0. \tag{2.13}$$

Now, by right multiplication of Eq (2.4) by s and using (2.12) and (2.13) in (2.4), we obtain

$$\begin{aligned} 0 &= (aq_1 + [b, q_1])^n (aq_2 + [b, q_2])s - (aq_2 + [b, q_2])(aq_1 + [b, q_1])^n s \\ &= (aq_1 + [b, q_1])^n (s + p) - (aq_2 + [b, q_2])s = (aq_1 + [b, q_1])^n (s + p) - (aq_2 + [b, q_2])s \\ &= (aq_1 + [b, q_1])^n s + (aq_1 + [b, q_1])^n p - (aq_2 + [b, q_2])s = s + 0 - (p + s) = -p \neq 0, \end{aligned}$$

a contradiction.

Case II: Now, suppose $bp = 0$, and $q_1 s = 0, q_1(bs) = -s, q_1 p = 0, q_2 s = 0, q_2(bs) = s - p, q_2 p = 0$, then from these identities, we obtain, $(aq_1 + [b, q_1])s = -q_1 bs = s$ and $(aq_1 + [b, q_1])p = 0$, which leads to

$$(aq_1 + [b, q_1])^n p = 0, \tag{2.14}$$

therefore,

$$(aq_1 + [b, q_1])^n s = s, \tag{2.15}$$

and

$$(aq_2 + [b, q_2])s = -q_2 bp = p - s. \tag{2.16}$$

Left multiplying (2.4) by s and using (2.14)–(2.16), we obtain

$$\begin{aligned} 0 &= (aq_1 + [b, q_1])^n (aq_2 + [b, q_2])s - (aq_2 + [b, q_2])(aq_1 + [b, q_1])^n s \\ &= (aq_1 + [b, q_1])^n (p - s) - (aq_2 + [b, q_2])s \\ &= (aq_1 + [b, q_1])^n p - (aq_2 + [b, q_2])s = 0 - s - (p - s) = -s - p + s = -p \neq 0, \end{aligned}$$

a contradiction. Therefore, we conclude that $T = \{0\}$. Hence, the result follows. □

Corollary 2.3. *[1, Theorem 2.2] Let $d : A \to A$ be a non-zero linear derivation defined on a non-commutative prime Banach algebra. If $[d(u)^n, d(v)] = 0$ for all $u, v \in A$, for a fixed positive integer n. Then $dim_C AC = 4$.*

Theorem 2.4. *Let $F : A \to A$ be a linear generalized derivation defined on a semiprime Banach algebra. If $[F(u)^n, F(v)] = 0$ for all $u, v \in A$, for a fixed positive integer n. Then there is a central idempotent $e \in Q$ such that $F = 0$ on eQ and the polynomial identity s_4 is satisfied by $(1 - e)Q$.*

Proof. We know that F, a generalized derivation on A can be uniquely extended to a generalized derivation F of Q (see [15, Theorem 4]). The application of [15, Theorem 3] tells us that Q and A satisfy

$$[F(u)^n, F(v)] = 0, \tag{2.17}$$

for all $u, v \in Q$. Let M be a maximal ideal of B, where B denotes the set of idempotents of C. Define, $\overline{Q} = Q/MQ$. Since $F(u) = au + d(u)$ for some $a \in Q$ and MQ is an ideal, then we can write, $F(MQ) = a(MQ) + d(MQ)$. From Fact 5, d is invariant on MQ, i.e., $d(MQ) \subseteq MQ$, which implies $F(MQ) \subseteq MQ$, therefore MQ is F invariant as well.

Since $F(MQ) \subseteq MQ$, let $\overline{F}$ denote the generalized derivation on $\overline{Q}$ induced canonically by F. From (2.17), we have

$$[\overline{F}(\overline{u})^n, \overline{F}(\overline{v})] = 0, \tag{2.18}$$

for all $\overline{u}, \overline{v} \in \overline{Q}$. The application of Theorem 2.2 yields, either $\overline{F} = 0$ or $dim_{\overline{C}} \leq 4$. If $dim_{\overline{C}} \leq 4$ then $\overline{Q}$ satisfies the identity s_4. Therefore, $F(Q)Qs_4(q_1, q_2, q_3, q_4) \subseteq MQ$ for all $q_i \in Q$ and for all maximal ideals M of B. Using Fact (5), we conclude that

$$F(Q)Qs_4(q_1, q_2, q_3, q_4) = \{0\}, \tag{2.19}$$

for all $q_i \in Q$. Let $W = AF(A)A$. Define $\pi : W \oplus Ann_A(W) \to A$ as $\pi(u + v) = v$ for all $u \in W$ and $v \in Ann_A(W)$, where $Ann_A(W)$ is the annihilator set of W in the algebra A. π so defined forms an (A, A)-bimodule map. Hence, π can be realized as an element of Q that is central (say) e, i.e., the map becomes $\pi(z) = ez$ for all $z \in W \oplus Ann_A(W)$. Therefore, we have $eW = \{0\}$ and $(1 - e)Ann_A(W) = \{0\}$, which implies $(e - e^2)(W \oplus Ann_A(W)) = \{0\}$. But $W \oplus Ann_A(W)$ by definition forms an essential ideal of A. Thus $e - e^2 = 0$, implying $e \in M$. The application of [15, Theorem 3], tells us that A and Q satisfy the same type of differential identities; therefore, $eF(Q) = \{0\}$, so $F(eQ) = \{0\}$. On the other hand, it follows from Eq (2.19) that $s_4(q_1, q_2, q_3, q_4) \in Ann_A(W)$ for all $q_i \in A$ and hence for all $q_i \in Q$ (see [3, Theorem 6.4.1]). That is, $(1 - e)Q$ satisfies $s_4(q_1, q_2, q_3, q_4)$. Thus, the proof is complete. □

3. Applications

This section provides us with an application of Theorem 2.4 for the case of semisimple Banach algebras. The idea behind the following theorem comes from the study of quasi-normal elements established in [6]. The focus lies in the behavior of algebras when they possess such types of elements. In a subsequent work, documented in [22], the examination of quasi-normality was studied for the realm of $L(H)$, the set that encompasses all bounded linear operators on a Hilbert space H. Specifically, it was established that within this algebraic context, either the algebra assumes a commutative nature or a particular set, characterized by the condition $[u^n, ((u)^*)^n u^n] \in Z(A)$ for no positive integer n, emerges as dense in the algebra A. Building upon this foundation, the research presented by Alhazmi and Khan in [1] studied the concept of linear derivations, probing the denseness of specific subsets based on certain functional identities. Motivated by these significant contributions, our exploration takes a directed course into the study of the behavior of semisimple Banach *-algebras, enriched with the presence of a linear generalized derivation.

Theorem 3.1. *Let $F : A \to A$ be a linear generalized derivation defined on a semisimple Banach *-algebra over $\mathbb{C}$ with a regular involution * such that $F(u)^* = F(u^*)$ for all $u \in A$. Then, a central idempotent $e \in Q$ exists so that $F = 0$ on eQ and $(1-e)Q$ satisfies s_4, or the set of $u \in A$ such that $[F(u)^n, F(u^*)^n F(u)^n] \in Z(A)$ for no positive integer n is dense in A.*

Proof. To the contrary, suppose that the collection of all $u \in A$ such that $[F(u)^n, F(u^*)^n F(u)^n] \in Z(A)$ is not dense in A. Then we can obtain an open set G_1 so that for all $u \in G_1$, there is a positive integer $n = n(u)$ with $[F(u)^{n(u)}, F(u^*)^{n(u)} F(u)^{n(u)}] \in Z(A)$. Now, for each positive integer k, define

$$H_k = \{u \in A : [F(u)^k, F(u^*)^k F(u)^k] \notin Z(A)\}.$$

As the algebra is semisimple we can see that the involution * is regular; therefore, H_k forms an open set (see [4, p. 191]). We show that every H_k cannot be dense. If possible, suppose every H_k is dense, then by the Baire Category theorem their intersection is also dense. But this contradicts our hypothesis. So, there exists a positive l such that we acquire a non-empty set G_2 in H_l^c, such that $[F(u)^l, F(u^*)^l F(u)^l] \in Z(A)$ for all $u \in G_2$. Therefore, for some $u_0 \in G_2$, $v \in A$, and for a sufficiently small value of real t, we have $(u_0 + tv) \in G_2$. Thus,

$$[F(u_0 + tv)^l, F(u_0^* + tv^*)^l F(u_0 + tv)^l] \in Z(A).$$

Taking the coefficient of the highest power of t, we obtain

$$[F(v)^l, F(v^*)^l F(v)^l] \in Z(A),$$

for all $v \in A$. Next, we can write for any $v \in A$, $h + itk$ where $h, k \in Sym(A)$, then the expression

$$F(v)^l = F(h)^l + iYt + \cdots,$$

where $Y = \sum_{i=0}^{l-1} F(h)^i F(k) F(h)^{l-1-i}$. Put $w = (F(v)^l + F(v^*)^l)/2$, observe that $[w, F(v^*)^l F(v)^l] \in Z(A)$, for all $v \in A$. Thus we have

$$[F(h)^l - Y_0' t^2 + \cdots, (F(h)^l + iYt + \cdots)(F(h)^l - iYt + \cdots)] \in Z(A), \tag{3.1}$$

for all $h \in Sym(A)$, where $Y' = \sum_{i=0}^{l-1} F(h)^i F(k)^2 F(h)^{l-1-i}$.

Equation (3.1) can be written as,

$$[F(h)^l - Y_0' t^2 + \cdots, F(h)^{2l} + it[Y, F(h)^l] + \cdots] \in Z(A) \tag{3.2}$$

for all $h \in H$. Focusing on the coefficient of t, we obtain

$$[F(h)^l, [Y, F(h)^l]] \in Z(A), \tag{3.3}$$

implies

$$[F(h), [F(h)^l, [Y, F(h)^l]]] = 0 \tag{3.4}$$

for all $h \in Sym(A)$, or we can write it as

$$[F(h)^l, [F(h)^l, [F(h), Y]]] = 0, \tag{3.5}$$

for all $h \in Sym(A)$. Therefore,

$$[F(h)^l, [F(h)^l, [F(h)^l, F(k)]]] = 0, \quad (3.6)$$

for all $h, k \in Sym(A)$. By invoking Fact 1, we deduce

$$\rho([F(h)^l, [F(h)^l, F(k)]]) = 0.$$

As the element $[F(h)^l, [F(h)^l, F(k)]] \in Sym(A)$. Thus, from the regularity of the involution map, we have $[F(h)^l, [F(h)^l, F(k)]] = 0$, for all $h, k \in Sym(A)$. Applying Fact 1 again, we obtain $\rho([F(h)^l, F(k)]) = 0$ for all $h, k \in Sym(A)$, also $[F(h)^l, F(k)] \in Sym(A)$. Similarly, we conclude $[F(h)^l, F(k)] = 0$ for all $h, k \in Sym(A)$. By Lemma 2.1, we can write $[F(u)^l, F(v)] = 0$, for all $u, v \in A$. In view of Theorem 2.4, there is a central idempotent e, so that d on eQ is zero and $(1 - e)Q$ satisfies the polynomial identity s_4. This completes the proof. □

As an immediate consequence of this theorem, we obtain the result of [1] by considering a linear derivation instead of a linear generalized derivation.

Corollary 3.2. *[1, Theorem 1.1] Let $d : A \to A$ be a linear derivation defined on a semisimple Banach *-algebra over $\mathbb{C}$ with a regular involution * such that $d(u)^* = d(u^*)$ for all $u \in A$. Then a central idempotent $e \in Q$ exists so that $d = 0$ on eQ and $(1 - e)Q$ satisfies s_4, or the set of $u \in A$ such that $[d(u)^n, d(u^*)^n d(u)^n] \in Z(A)$ for no positive integer n is dense in A.*

The next result is motivated by the work of Ashraf and Wani [2], where they characterized the structure of unital prime Banach algebras with continuous generalized derivations that satisfy certain functional identities. What we try to establish is the * version of this particular result. This result is independent of the previous results but is an interesting deduction of linear generalized derivations.

Theorem 3.3. *Let A be a unital, semisimple complex Banach *-algebra, F be a nonzero linear generalized derivation with respect to the derivation d such that $d(Z(A)) \neq 0$. Suppose there exist non-empty open subsets G_1 and G_2 of A such that either*

$$F((u^*v)^m) - (u^*)^m v^m \in Z(A)$$

or

$$F((u^*v)^m) - v^m (u^*)^m \in Z(A),$$

for all $u^* \in G_1, v \in G_2$, $m = m(u^*, v) > 1$. Then A is commutative.

Proof. Step 1: We will show that for a fixed $u^* \in G_1$, we have either

$$F((u^*w)^j) - (u^*)^j w^j \in Z(A),$$

or

$$F((u^*w)^j) - w^j (u^*)^j \in Z(A),$$

for all $w \in G_2$, $j = j(u^*, w) > 1$. Let us define a set

$$D_n = \{v \in A : F((u^*v)^n) - (u^*)^n v^n \notin Z(A) \text{ and } F((u^*v)^n) - v^n (u^*)^n \notin Z(A)\}.$$

D_n so defined forms an open subset of A. To see this, consider a sequence $\{z_k\}_{k=0}^{\infty} \in D_n^c$ such that $\lim_{k\to\infty} z_k = z$. As $\{z_k\}_{k=0}^{\infty} \in D_n^c$, it implies either

$$F((u^* z_k)^n) - (u^*)^n (z_k)^n \in Z(A),$$

or

$$F((u^* z_k)^n) - (z_k)^n (u^*)^n \in Z(A),$$

for infinitely many values of k. Taking limit $k \to \infty$ and observing that the function F is continuous and u^* is independent of variable k, we have either

$$F((u^* z)^n) - (u^*)^n (z)^n \in Z(A),$$

or

$$F((u^* z)^n) - (z)^n (u^*)^n \in Z(A).$$

Therefore $z \in D_n^c$. If each D_n is dense then by the Baire Category theorem, their intersection is also dense. But that would contradict to the non-emptiness of the set G_2. So, there should exist an integer j, such that either

$$F((u^*(a + tb))^j) - (u^*)^j (a + tb)^j \in Z(A), \tag{3.7}$$

or

$$F((u^*(a + tb))^j) - (a + tb)^j (u^*)^j \in Z(A),$$

for some $b \in A$ and sufficiently small real t, $a \in G_3$, where G_3 is an open subset of D_j^c. The expression (3.7) can be rewritten as

$$F(G_{j,0}(u^*, a, b)) - (u^*)^j H_{j,0}(a, b) + \{F(G_{j-1,1}(u^*, a, b)) - (u^*)^j H_{j-1,1}(a, b)\}t + \cdots$$
$$+\{F(G_{1,j-1}(u^*, a, b)) - (u^*)^j H_{1,j-1}(a, b)\}t^{j-1} + \{F(G_{0,j}(u^*, a, b)) - (u^*)^j H_{0,j}(a, b)\}t^j,$$

where $G_{j,k}(u^*, a, b)$ and $H_{j,k}(a, b)$ denotes the summation of all the terms in which u^*a appears j times and u^*b appears k times while expanding $F((u^*(a+tb))^j)$ and the summation of terms in which a appears j times and b appears k times in the expansion of $(a + tb)^j$ respectively. The above polynomial lies in the center $Z(A)$ using Fact 3, we deduce that each coefficient is in $Z(A)$. Therefore, the coefficient of t^j is in $Z(A)$, i.e., either

$$F((u^* b)^j) - (u^*)^j b^j \in Z(A),$$

or

$$F((u^* b)^j) - b^j (u^*)^j \in Z(A).$$

The element b was chosen arbitrarily from A, so we obtain either

$$F((u^* w)^j) - (u^*)^j w^j \in Z(A),$$

or

$$F((u^* w)^j) - w^j (u^*)^j \in Z(A),$$

for any $w \in A$.
Step 2: Now, let us fix $v \in A$ and show that either

$$F((r^* v)^k) - (r^*)^k v^k \in Z(A),$$

or

$$F((r^*v)^k) - v^k(r^*)^k \in Z(A),$$

for all $r \in A$, $k = k(r^*, v)$. For each integer n, define $E_n = \{r \in A : F((r^*v)^n) - (r^*)^n v^n \notin Z(A)$ *and* $F((r^*v)^n) - v^n(r^*)^n \notin Z(A)\}$. Since F is continuous and * is also, by invoking Fact 2, we obtain that the sets E_n are open. Following similar arguments as above, we come up with a nonempty open subset $G_4 \subseteq E_k^c$, such that for any $a \in G_4$ and $b \in A$, with a sufficiently small real t, we have, either

$$F(((a + tb)^*v)^k) - ((a + tb)^*)^k v^k \in Z(A),$$

or

$$F(((a + tb)^*v)^k) - v^k((a + tb)^*)^k \in Z(A).$$

Compairing the coefficient of t^k in the above polynomial expressions, we obtain either

$$F((b^*v)^k) - (b^*)^k v^k \in Z(A),$$

or

$$F((b^*v)^k) - v^k(b^*)^k \in Z(A),$$

for any $b \in A$.

Now, define a set K_n for each integer $n > 1$. Let K_n be the collection of all those $v \in A$ such that either

$$F(w^*v)^n - v^n(w^*)^n \in Z(A),$$

or

$$F(w^*v)^n - (w^*)^n v^n \in Z(A),$$

for each $w \in A$. Observe that the union of K_n for $k > 1$ is A, and each K_n forms a closed set. Therefore, by the Baire Category theorem, for each $p > 1$, the set K_p must contain a nonempty open subset (say) G_5 such that, for any $b \in A$, $a \in G_5$ and sufficiently small real t, we get either

$$F(w^*(a + tb))^p - (w^*)^p(a + tb)^p \in Z(A),$$

or

$$F(w^*(a + tb))^p - (a + tb)^p(w^*)^p \in Z(A).$$

By similar arguments as above, we obtain for any $w, b \in A$ either

$$F(w^*b)^p - (w^*)^p b^p \in Z(A),$$

or

$$F(w^*b)^p - b^p(w^*)^p \in Z(A).$$

Let the unity of A be denoted by e. Then, we have either

$$F(((e + tu)^*v)^p) - ((e + tu)^*)^p v^p \in Z(A),$$

or

$$F(((e + tu)^*v)^p) - v^p((e + tu)^*)^p \in Z(A),$$

for all $u, v \in A$. Considering the coefficient of t in the above expressions, either

$$F(u^*v^p + \sum_{k=1}^{p-1} v^k u^* v^{p-k}) - pu^*v^p \in Z(A), \tag{3.8}$$

or

$$F(u^*v^p + \sum_{k=1}^{p-1} v^k u^* v^{p-k}) - pv^p u^* \in Z(A), \tag{3.9}$$

for all $u, v \in A$. On similar lines, we get either

$$F((v^*(e + tu))^p) - (v^*)^p(e + tu)^p \in Z(A),$$

or

$$F((v^*(e + tu))^p) - (e + tu)^p(v^*)^p \in Z(A),$$

for all $u, v \in A$. On replacing v by v^* and u by u^*, we obtain either

$$F(v^p u^* + \sum_{k=1}^{p-1} v^k u^* v^{p-k}) - pv^p u^* \in Z(A), \tag{3.10}$$

or

$$F(v^p u^* + \sum_{k=1}^{p-1} v^k u^* v^{p-k}) - pu^* v^p \in Z(A), \tag{3.11}$$

for all $u, v \in A$. So one of the combinations should hold, i.e., either (3.8) and (3.10) or (3.8) and (3.11) or (3.9) and (3.10) or (3.9) and (3.11). For instance, (3.8) and (3.10) hold together; therefore, their subtraction yields,

$$F[u^*, v^p] - p[u^*, v^p] \in Z(A),$$

for all $u, v \in A$. Replacing v by $e + tv$ and then considering the coefficient of t, we obtain

$$F[u^*, v] - p[u^*, v] \in Z(A), \tag{3.12}$$

for all $u, v \in A$. Similarly, if (3.8) and (3.11) hold, then

$$F[u^*, v^p] \in Z(A),$$

and again replacing v by $e + tv$, we obtain

$$F[u^*, v] \in Z(A). \tag{3.13}$$

As the involution map * is onto, we can rewrite the expressions (3.12) and (3.13) as

$$F[u^*, v] - p[u^*, v] \in Z(A), \tag{3.14}$$

and

$$F[u, v] \in Z(A).$$

Since $d(Z(A)) \neq 0$, there exists an $0 \neq \alpha \in Z(A)$, such that $d(\alpha) = 0$. Now, replacing v by $v\alpha$ in (3.14), we obtain

$$[u, v]d(\alpha) \in Z(A).$$

Hence $[u, v] \in Z(A)$ for all $u, v \in Z(A)$, making A commutative. Similarly, for $F[u, v] \in Z(A)$, replace v by $v\alpha$ to get to the desired conclusion. For the other combinations, one can easily verify using similar arguments that A is commutative. □

4. Conclusions and discussions

In this paper, we focused on certain identities on Banach *-algebras possessing maps, called the linear generalized derivations. Precisely, we characterized Banach *-algebras via linear generalized derivations. Moreover, there are various interesting open problems related to our study of linear generalized derivations on Banach *-algebras. Below, we proposed a couple of open problems that help to further understand the behavior of Banach *-algebras equipped with linear generalized derivations.

Open Questions. Suppose A is a normed algebra equipped with an involution *. Let F_1 and $F_2 : A \to A$ be linear generalized derivations associated with the derivations d_1 and d_2, respectively. In this scenario, what can be said about the structure of the algebra A and the functions F_1 and F_2, whenever:

(1) There exists no positive integer n such that the set of $u \in A$ for which $[F_1(u)^n, F_2(u^*)^n F_2(u)^n] \in Z(A)$ be dense.

(2) There exists no positive integer n such that the set of $u \in A$ for which $[F_1(u)^n, d_2(u^*)^n d_2(u)^n] \in Z(A)$ be dense.

(3) There exist non-empty open subsets G_1 and G_2 of A such that either

$$F_1((u^*v)^m) - (u^*)^m v^m \in Z(A)$$

or

$$F_1((u^*v)^m) - v^m (u^*)^m \in Z(A),$$

for all $u^* \in G_1, v \in G_2, m = m(u^*, v) > 1$.

(4) There exist non-empty open subsets G_1 and G_2 of A such that either

$$F_1((u^*v)^m) - F_1(u^*)^m F_1(v^m) \in Z(A)$$

or

$$F_1((u^*v)^m) - F_1(v^m) F_1(u^*)^m \in Z(A),$$

for all $u^* \in G_1, v \in G_2, m = m(u^*, v) > 1$.

(5) There exist non-empty open subsets G_1 and G_2 of A such that either

$$F_1((u^*v)^m) - F_2(u^*)^m F_2(v^m) \in Z(A)$$

or

$$F_1((u^*v)^m) - F_2(v^m) F_2(u^*)^m \in Z(A),$$

for all $u^* \in G_1, v \in G_2, m = m(u^*, v) > 1$.

Author contributions

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Conflict of interest

The authors declare that they have no conflicts of interest.

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