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Research article

The nonisospectral integrable hierarchies of three generalized Lie algebras

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Abstract: We construct the generalized Lie algebras $\mathfrak{sp}(4)$, $\mathfrak{so}(5)$, and $\mathfrak{so}(3,2)$, and derive three kinds of (1+1)-dimensional nonisospectral integrable hierarchies. Moreover, we obtain their Hamiltonian structures. Finally, based on Lie algebras $\mathfrak{sp}(4)$, $\mathfrak{so}(5)$, and $\mathfrak{so}(3,2)$, by using the semi-direct sum decomposition of Lie algebras, we construct three kinds of integrable coupling systems associated with these three Lie algebras.

Keywords: Lie algebra; Hamiltonian structure; nonisospectral integrable hierarchy; integrable couplings
Mathematics Subject Classification: 37K05, 37K40, 35Q53

1. Introduction

Nonlinear differential equations are important mathematical tools for describing natural phenomena. The integrability and exact solutions of nonlinear evolution equations have always been of great concern to scientists [1,2]. The exact solution of nonlinear models plays an important role in describing some complex nonlinear phenomena. Some researchers have worked on solving the exact analytic solutions of the nonlinear extended models. They proposed many methods and also obtained more new types of exact solutions [3–7].

The study of integrable systems is an important research topic in disciplines such as physics and mathematics. The trace identity proposed by Tu is a simple and powerful tool for generating integrable hierarchies of soliton equations and their corresponding Hamiltonian structures [8]. By using the trace identity, based on Lie algebras, some isospectral integrable hierarchies and the corresponding Hamiltonian structures were constructed [9–13]. Some methods had been developed in deriving (2+1)-dimensional integrable systems, such as the TAH scheme. The TAH scheme is an effective method for generating (2+1)-dimensional soliton hierarchies. Based on this method, some (2+1)-dimensional integrable hierarchies and the corresponding Hamiltonian structures were obtained [14–19].

In order to obtain more integrable sysytems, Zhang et al. proposed an approach for generating nonisospectral integrable hierarchies under the assumption $\lambda_t = \sum_{i\geq 0} k_i(t)\lambda^{-i}$ [20,21]. By using this method, [22,23] constructed some multi-component integrable hierarchies associated with multi-component non-semisimple Lie algebras. Moreover, based on Lie superalgebras, [24,25] investigated some nonisospectral super integrable hierarchies and the corresponding super Hamiltonian structures.

Matrix spectral problems and zero curvature equations play an important role in exploring the mathematical properties of associated soliton equations [26,27]. The semi-direct sum decomposition of Lie algebras provide a helpful way to construct the integrable couplings of soliton systems [28–30].

This paper is arranged as follows. In Section 2, based on Lie algebra $\mathfrak{sp}(4)$, we construct the generalized Lie algebra $G\mathfrak{sp}(4)$, and obtain the nonisospectral integrable hierarchies and their Hamiltonian structures. In Section 3, based on the semi-direct sum decomposition of Lie algebras, the nonisospectral integrable coupling hierarchies and the corresponding Hamiltonian structures are obtained. In Sections 4 and 5, we obtain the nonisospectral integrable hierarchies and their coupling systems associated with the generalized Lie algebra $G\mathfrak{so}(5)$. In Section 6, by using $\mathfrak{sp}(4) \cong \mathfrak{so}(3,2)$, we obtain that there are the same integrable hierarchies of these two generalized Lie algebras, and there are also the same integrable couplings.

2. The nonisospectral integrable hierarchy associated with the generalized Lie algebra $G\mathfrak{sp}(4)$

The compact real form $\mathfrak{sp}(4)$ of complex symplectic Lie algebra $\mathfrak{sp}(4,\mathbb{R})$ is defined as [31,32]

$$\mathfrak{sp}(4) = \{x \in \mathfrak{gl}(4,\mathbb{R}) | Hx + x^{\mathrm{T}}H = 0\},$$

where $H = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$, I_2 is the 2×2 identity matrix, and x^{T} represents the transposition of x. We introduce the generalized Lie algebra $\mathfrak{sp}(4)$, that admits a basis set as follows:

$$\begin{aligned} E_1 = e_{11} - e_{33},\ \ E_2 = e_{22} - e_{44},\ \ E_3 = e_{12} - e_{43},\ \ E_4 = \varepsilon e_{21} - \varepsilon e_{34}, \\ E_5 = \varepsilon e_{14} + \varepsilon e_{23},\ \ E_6 = e_{32} + e_{41},\ \ E_7 = \varepsilon e_{13},\ \ E_8 = e_{31},\ \ E_9 = \varepsilon e_{24},\ \ E_{10} = e_{42}, \end{aligned} \tag{2.1}$$

where $\varepsilon \in \mathbb{R}$, and e_{ij} is a 4×4 matrix with 1 in the (i,j)-th position and zero elsewhere, which satisfy the commutative relations

$$\begin{aligned} &[E_1,E_2] = 0,\ [E_1,E_3] = E_3,\ [E_1,E_4] = -E_4,\ [E_1,E_5] = E_5,\ [E_1,E_6] = -E_6, \\ &[E_1,E_7] = 2E_7,\ [E_1,E_8] = -2E_8,\ [E_1,E_9] = [E_1,E_{10}] = 0,\ [E_2,E_3] = -E_3, \\ &[E_2,E_4] = E_4,\ [E_2,E_5] = E_5,\ [E_2,E_6] = -E_6,\ [E_2,E_7] = [E_2,E_8] = 0, \\ &[E_2,E_9] = 2E_9,\ [E_2,E_{10}] = -2E_{10},\ [E_3,E_4] = \varepsilon(E_1 - E_2),\ [E_3,E_5] = 2E_7, \\ &[E_3,E_6] = -2E_{10},\ [E_3,E_7] = 0,\ [E_3,E_8] = -E_6,\ [E_3,E_9] = E_5,\ [E_3,E_{10}] = 0, \\ &[E_4,E_5] = 2E_9,\ [E_4,E_6] = -2\varepsilon E_8,\ [E_4,E_7] = E_5,\ [E_4,E_8] = [E_4,E_9] = 0, \\ &[E_4,E_{10}] = -\varepsilon E_6,\ [E_5,E_6] = \varepsilon(E_1+E_2),\ [E_5,E_7] = 0,\ [E_5,E_8] = E_4,\ [E_5,E_9] = 0, \\ &[E_5,E_{10}] = \varepsilon E_3,\ [E_6,E_7] = -\varepsilon E_3,\ [E_6,E_8] = [E_6,E_{10}] = 0,\ [E_6,E_9] = -E_4, \\ &[E_7,E_8] = \varepsilon E_1,\ [E_7,E_9] = [E_7,E_{10}] = [E_8,E_9] = [E_8,E_{10}] = 0,\ [E_9,E_{10}] = \varepsilon E_2. \end{aligned}$$

We can construct different generalized Lie algebras $G\mathfrak{sp}(4)$ by adding the real number ε to different elements of the Lie algebra $\mathfrak{sp}(4)$. Here, we will only discuss one of these cases.

Consider the linear nonisospectral problem

$$\begin{cases} \varphi_x = U_1\varphi, \\ \varphi_t = V_1\varphi, \\ \lambda_t = \sum_{i\geq 0} k_i(t)\lambda^{-i}, \end{cases}$$

where

$$U_1 = \begin{pmatrix} \lambda & 0 & \varepsilon u_3 & \varepsilon u_1 \\ 0 & \lambda & \varepsilon u_1 & \varepsilon u_5 \\ u_4 & u_2 & -\lambda & 0 \\ u_2 & u_6 & 0 & -\lambda \end{pmatrix},$$

$$V_1 = \begin{pmatrix} a & c & \varepsilon g & \varepsilon e \\ \varepsilon d & b & \varepsilon e & \varepsilon p \\ h & f & -a & -\varepsilon d \\ f & q & -c & -b \end{pmatrix} = \sum_{i\geq 0} \begin{pmatrix} a_i & c_i & \varepsilon g_i & \varepsilon e_i \\ \varepsilon d_i & b_i & \varepsilon e_i & \varepsilon p_i \\ h_i & f_i & -a_i & -\varepsilon d_i \\ f_i & q_i & -c_i & -b_i \end{pmatrix} \lambda^{-i}. \tag{2.2}$$

By solving stationary the zero curvature representation

$$V_{1x} = \frac{\partial U_1}{\partial \lambda}\lambda_t + [U_1, V_1], \tag{2.3}$$

we can obtain

$$\begin{cases} a_{ix} = \varepsilon u_1 f_i - \varepsilon u_2 e_i + \varepsilon u_3 h_i - \varepsilon u_4 g_i + k_i(t), \\ b_{ix} = \varepsilon u_1 f_i - \varepsilon u_2 e_i + \varepsilon u_5 q_i - \varepsilon u_6 p_i + k_i(t), \\ c_{ix} = \varepsilon u_1 q_i - \varepsilon u_2 g_i + \varepsilon u_3 f_i - \varepsilon u_6 e_i, \\ d_{ix} = u_1 h_i - u_2 p_i - u_4 e_i + u_5 f_i, \\ e_{ix} = 2\lambda e_i - u_1 a_i - u_1 b_i - \varepsilon u_3 d_i - u_5 c_i, \\ f_{ix} = -2\lambda f_i + u_2 a_i + u_2 b_i + u_4 c_i + \varepsilon u_6 d_i, \\ g_{ix} = 2\lambda g_i - 2u_1 c_i - 2u_3 a_i, \\ h_{ix} = -2\lambda h_i + 2\varepsilon u_2 d_i + 2u_4 a_i, \\ p_{ix} = 2\lambda p_i - 2\varepsilon u_1 d_i - 2u_5 b_i, \\ q_{ix} = -2\lambda q_i + 2u_2 c_i + 2u_6 b_i. \end{cases} \tag{2.4}$$

By taking initial values

$$a_0 = \alpha(t),\ b_0 = \beta(t),\ c_0 = d_0 = e_0 = f_0 = g_0 = h_0 = p_0 = q_0 = k_0(t) = 0,$$

one has

$$\begin{aligned} &a_1 = b_1 = k_1(t)x,\ c_1 = \frac{\varepsilon}{2}\partial^{-1}(u_1u_6 + u_2u_3)(\beta - \alpha),\ g_1 = u_3\alpha,\ p_1 = u_5\beta,\ q_1 = u_6\beta, \\ &d_1 = \frac{1}{2}\partial^{-1}(u_1u_4 + u_2u_5)(\alpha - \beta),\ e_1 = \frac{1}{2}u_1(\alpha + \beta),\ f_1 = \frac{1}{2}u_2(\alpha + \beta),\ h_1 = u_4\alpha, \\ &e_2 = \frac{1}{4}u_{1x}(\alpha + \beta) + \frac{\varepsilon}{4}\left[u_3\partial^{-1}(u_1u_4 + u_2u_5) - u_5\partial^{-1}(u_1u_6 + u_2u_3)\right](\alpha - \beta) + u_1k_1(t)x, \\ &f_2 = -\frac{1}{4}u_{2x}(\alpha + \beta) + \frac{\varepsilon}{4}\left[u_4\partial^{-1}(u_1u_6 + u_2u_3) - u_6\partial^{-1}(u_1u_4 + u_2u_5)\right](\beta - \alpha) + u_2k_1(t)x, \end{aligned}$$

$$
\begin{aligned}
g_2 &= \frac{1}{2}u_{3x}\alpha + \frac{\varepsilon}{2}u_1\partial^{-1}(u_1u_6 + u_2u_3)(\beta-\alpha) + u_3k_1(t)x,\\
h_2 &= -\frac{1}{2}u_{4x}\alpha + \frac{\varepsilon}{2}u_2\partial^{-1}(u_1u_4 + u_2u_5)(\beta-\alpha) + u_4k_1(t)x,\\
p_2 &= \frac{1}{2}u_{5x}\beta - \frac{\varepsilon}{2}u_1\partial^{-1}(u_1u_4 + u_2u_5)(\beta-\alpha) + u_5k_1(t)x,\\
q_2 &= -\frac{1}{2}u_{6x}\beta + \frac{\varepsilon}{2}u_2\partial^{-1}(u_1u_6 + u_2u_3)(\beta-\alpha) + u_6k_1(t)x,\\
&\cdots\cdots
\end{aligned}
$$

where $\alpha(t)$ is an integral constant. Noting that

$$
\begin{aligned}
&V_{1+}^{(n)} = \sum_{i=0}^{n}(a_i, b_i, c_i, d_i, e_i, f_i, g_i, h_i, p_i, q_i)^{\mathrm{T}},\quad V_{1-}^{(n)} = \sum_{i=n+1}^{\infty}(a_i, b_i, c_i, d_i, e_i, f_i, g_i, h_i, p_i, q_i)^{\mathrm{T}},\\
&\lambda_{+,x}^{(n)} = \sum_{i=0}^{n} k_i(t)\lambda^{n-i},\quad \lambda_{-,x}^{(n)} = \sum_{i=n+1}^{\infty} k_i(t)\lambda^{n-i},
\end{aligned}
$$

it follows that one has

$$
-V_{1+,x}^{(n)} + \frac{\partial U_1}{\partial\lambda}\lambda_{t,+}^{(n)} + [U_1, V_{1+}^{(n)}] = (0,0,0,0,-2e_{n+1}, 2f_{n+1}, -2g_{n+1}, 2h_{n+1}, -2p_{n+1}, 2q_{n+1})^{\mathrm{T}}.
$$

According to (2.4), it is easy to show that we have the recursion relations

$$
\begin{pmatrix} 2\varepsilon f_{n+1}\\ 2\varepsilon e_{n+1}\\ \varepsilon h_{n+1}\\ \varepsilon g_{n+1}\\ \varepsilon q_{n+1}\\ \varepsilon p_{n+1}\end{pmatrix} = L_1\begin{pmatrix} 2\varepsilon f_{n}\\ 2\varepsilon e_{n}\\ \varepsilon h_{n}\\ \varepsilon g_{n}\\ \varepsilon q_{n}\\ \varepsilon p_{n}\end{pmatrix} + \begin{pmatrix} 2\varepsilon u_2\\ 2\varepsilon u_1\\ \varepsilon u_4\\ \varepsilon u_3\\ \varepsilon u_6\\ \varepsilon u_5\end{pmatrix} k_n(t)x,
$$

where the recurrence operator L_1 is defined as

$$
L_1 = \begin{pmatrix} l_{11} & l_{12} & l_{13} & l_{14} & l_{15} & l_{16}\\ l_{21} & l_{22} & l_{23} & l_{24} & l_{25} & l_{26}\\ l_{31} & l_{32} & l_{33} & l_{34} & l_{35} & l_{36}\\ l_{41} & l_{42} & l_{43} & l_{44} & l_{45} & l_{46}\\ l_{51} & l_{52} & l_{53} & l_{54} & l_{55} & l_{56}\\ l_{61} & l_{62} & l_{63} & l_{64} & l_{65} & l_{66}\end{pmatrix},
$$

and

$$
\begin{aligned}
&l_{11} = -\frac{\partial}{2} + \frac{\varepsilon}{2}(2u_2\partial^{-1}u_1 + u_4\partial^{-1}u_3 + u_6\partial^{-1}u_5), l_{12} = -\frac{\varepsilon}{2}(2u_2\partial^{-1}u_2 + u_4\partial^{-1}u_6 + u_6\partial^{-1}u_4),\\
&l_{21} = \varepsilon u_1\partial^{-1}u_1 + \frac{\varepsilon}{2}(u_3\partial^{-1}u_5 + u_5\partial^{-1}u_3),\ l_{22} = \frac{\partial}{2} - \varepsilon u_1\partial^{-1}u_2 - \frac{\varepsilon}{2}(u_3\partial^{-1}u_4 + u_5\partial^{-1}u_6),\\
&l_{31} = \frac{1}{2}l_{15} = \frac{\varepsilon}{2}(u_2\partial^{-1}u_5 + u_4\partial^{-1}u_1),\ l_{32} = \frac{1}{2}l_{14} = -\frac{\varepsilon}{2}(u_2\partial^{-1}u_4 + u_4\partial^{-1}u_2),
\end{aligned}
$$

$$l_{41}=\frac{1}{2}l_{23}=\frac{\varepsilon}{2}(u_1\partial^{-1}u_3+u_3\partial^{-1}u_1),\ \ l_{42}=\frac{1}{2}l_{26}=-\frac{\varepsilon}{2}(u_1\partial^{-1}u_6+u_3\partial^{-1}u_2),$$
$$l_{51}=\frac{1}{2}l_{13}=\frac{\varepsilon}{2}(u_2\partial^{-1}u_3+u_6\partial^{-1}u_1),\ \ l_{52}=\frac{1}{2}l_{16}=-\frac{\varepsilon}{2}(u_2\partial^{-1}u_6+u_6\partial^{-1}u_2),$$
$$l_{61}=\frac{1}{2}l_{25}=\frac{\varepsilon}{2}(u_1\partial^{-1}u_5+u_5\partial^{-1}u_1),\ \ l_{62}=\frac{1}{2}l_{24}=-\frac{\varepsilon}{2}(u_1\partial^{-1}u_4+u_5\partial^{-1}u_2),$$
$$l_{33}=-\frac{\partial}{2}+\varepsilon u_2\partial^{-1}u_1+\varepsilon u_4\partial^{-1}u_3,\ \ l_{34}=-\varepsilon u_4\partial^{-1}u_4,\ \ l_{35}=0,\ \ l_{36}=-\varepsilon u_2\partial^{-1}u_2,$$
$$l_{43}=\varepsilon u_3\partial^{-1}u_3,\ \ l_{44}=\frac{\partial}{2}-\varepsilon u_1\partial^{-1}u_2-\varepsilon u_3\partial^{-1}u_4,\ \ l_{45}=\varepsilon u_1\partial^{-1}u_1,\ \ l_{46}=0,$$
$$l_{53}=0,\ \ l_{54}=-\varepsilon u_2\partial^{-1}u_2,\ \ l_{55}=-\frac{\partial}{2}+\varepsilon u_2\partial^{-1}u_1+\varepsilon u_6\partial^{-1}u_5,\ \ l_{56}=-\varepsilon u_6\partial^{-1}u_6,$$
$$l_{63}=\varepsilon u_1\partial^{-1}u_1,\ \ l_{64}=0,\ \ l_{65}=\varepsilon u_5\partial^{-1}u_5,\ \ l_{66}=\frac{\partial}{2}-\varepsilon u_1\partial^{-1}u_2-\varepsilon u_5\partial^{-1}u_6.$$

Taking $V_1^{(n)}=V_{1,+}^{(n)}$, then the zero curvature equation

$$-V_{1x}^{(n)}+\frac{\partial U_1}{\partial u}u_t+\frac{\partial U_1}{\partial \lambda}\lambda_{t,+}^{(n)}+[U_1,V_1^{(n)}]=0$$

leads to the following nonisospectral hierarchy

$$\begin{aligned} u_{t_n} &= \begin{pmatrix} u_1\\ u_2\\ u_3\\ u_4\\ u_5\\ u_6\end{pmatrix}_{t_n}=\begin{pmatrix} -2e_{n+1}\\ 2f_{n+1}\\ -2g_{n+1}\\ 2h_{n+1}\\ -2p_{n+1}\\ 2q_{n+1}\end{pmatrix}=J_1\begin{pmatrix} 2\varepsilon f_{n+1}\\ 2\varepsilon e_{n+1}\\ \varepsilon h_{n+1}\\ \varepsilon g_{n+1}\\ \varepsilon q_{n+1}\\ \varepsilon p_{n+1}\end{pmatrix}\\ &= J_1L_1\begin{pmatrix} 2\varepsilon f_n\\ 2\varepsilon e_n\\ \varepsilon h_n\\ \varepsilon g_n\\ \varepsilon q_n\\ \varepsilon p_n\end{pmatrix}+J_1\begin{pmatrix} 2\varepsilon u_2\\ 2\varepsilon u_1\\ \varepsilon u_4\\ \varepsilon u_3\\ \varepsilon u_6\\ \varepsilon u_5\end{pmatrix}k_n(t)x, \end{aligned} \tag{2.5}$$

where the Hamiltonian operator J_1 is

$$J_1=\frac{1}{\varepsilon}\begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -2 & 0 & 0\\ 0 & 0 & 2 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & -2\\ 0 & 0 & 0 & 0 & 2 & 0\end{pmatrix}.$$

To furnish Hamiltonian structures, we use the trace identity, and have

$$\langle V_1,\frac{\partial U_1}{\partial \lambda}\rangle=2a+2b,\ \langle V_1,\frac{\partial U_1}{\partial u_1}\rangle=2\varepsilon f,\ \langle V_1,\frac{\partial U_1}{\partial u_2}\rangle=2\varepsilon e,$$

$$\langle V_1, \frac{\partial U_1}{\partial u_3}\rangle = \varepsilon h,\ \langle V_1, \frac{\partial U_1}{\partial u_4}\rangle = \varepsilon g,\ \langle V_1, \frac{\partial U_1}{\partial u_5}\rangle = \varepsilon q,\ \langle V_1, \frac{\partial U_1}{\partial u_6}\rangle = \varepsilon p.$$

Substituting the above formulas into the trace identity yields

$$\frac{\delta}{\delta u}\int(2a+2b)dx = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\begin{pmatrix} 2\varepsilon f \\ 2\varepsilon e \\ \varepsilon h \\ \varepsilon g \\ \varepsilon q \\ \varepsilon p \end{pmatrix}.$$

Balancing coefficients of each power of λ in the above equality gives rise to

$$\frac{\delta}{\delta u}\int(2a_{n+1}+2b_{n+1})dx = (\gamma - n)\begin{pmatrix} 2\varepsilon f_n \\ 2\varepsilon e_n \\ \varepsilon h_n \\ \varepsilon g_n \\ \varepsilon q_n \\ \varepsilon p_n \end{pmatrix}.$$

Taking $n = 1$, gives $\gamma = 0$. Thus, we see

$$u_t = J_1\frac{\delta H^{(1)}_{n+1}}{\delta u} = J_1L_1\frac{\delta H^{(1)}_{n}}{\delta u} + J_1M_1k_n(t)x, \quad H^{(1)}_{n+1} = -2\int(\frac{a_{n+2}+b_{n+2}}{n+1})dx, \ \ n \geq 0,$$

where $M_1 = (2\varepsilon u_2, 2\varepsilon u_1, \varepsilon u_4, \varepsilon u_3, \varepsilon u_6, \varepsilon u_5)^{\mathrm{T}}$.

3. The nonisospectral integrable coupling hierarchy associated with Lie algebra $\mathfrak{sp}(4)$

3.1. Preliminaries

We generalize the semisimple Lie algebra g to the non-semisimple Lie algebra $\bar{g}$. This has the block matrix form [29,30]

$$M(A,B) = \begin{pmatrix} A & \varepsilon B \\ B & A \end{pmatrix}, \tag{3.1}$$

where $\varepsilon \in \mathbb{R}$, which is different from Section 2, and A, B are two arbitrary matrices with the same order. Non-semisimple Lie algebra $\bar{g}$ has two subalgebras $\widetilde{g} = \{M(A,O)\}$ and $\widetilde{g_c} = \{M(O,B)\}$, that forms the semi-direct sum of Lie algebras $\bar{g} = \widetilde{g} \oplus_s \widetilde{g_c}$. That is, $[\widetilde{g}, \widetilde{g_c}] = \{[A,B] | A \in \widetilde{g}, B \in \widetilde{g_c}\}$, $[\widetilde{g}, \widetilde{g}] \subseteq \widetilde{g}$, $[\widetilde{g_c}, \widetilde{g_c}] \subseteq \widetilde{g}$, $[\widetilde{g}, \widetilde{g_c}] \subseteq \widetilde{g_c}$.

We introduce the enlarged spectral problems

$$\begin{cases} \varphi_x = \bar{U}\varphi = \begin{pmatrix} U_1 & \varepsilon U_2 \\ U_2 & U_1 \end{pmatrix}, \\ \varphi_t = \bar{V}\varphi = \begin{pmatrix} V_1 & \varepsilon V_2 \\ V_2 & V_1 \end{pmatrix}, \\ \lambda_t = \sum_{i\geq 0} k_i(t)\lambda^{-i}. \end{cases}$$

From the corresponding enlarged stationary zero curvature equation

$$\bar{V}_x = \frac{\partial \bar{U}}{\partial \lambda}\lambda_t + [\bar{U}, \bar{V}], \tag{3.2}$$

it is easy to have

$$\begin{cases} V_{1,x} = [U_1, V_1] + \varepsilon[U_2, V_2], \\ V_{2,x} = [U_2, V_1] + [U_1, V_2]. \end{cases}$$

3.2. *The nonisospectral integrable couplings*

In this section, we will construct the nonisospectral integrable coupling hierarchies associated with Lie algebra $\mathfrak{sp}(4)$. We consider the nonisospectral problem

$$\begin{cases} \varphi_x = \bar{U}_1\varphi, \\ \varphi_t = \bar{V}_1\varphi, \\ \lambda_t = \sum_{i\geq 0} k_i(t)\lambda^{-i}, \end{cases}$$

where

$$\bar{U}_1 = \begin{pmatrix} U_1' & \varepsilon U_2' \\ U_2' & U_1' \end{pmatrix}, U_1' = \begin{pmatrix} \lambda & 0 & u_3 & u_1 \\ 0 & \lambda & u_1 & u_5 \\ u_4 & u_2 & -\lambda & 0 \\ u_2 & u_6 & 0 & -\lambda \end{pmatrix}, U_2' = \begin{pmatrix} 0 & 0 & u_3' & u_1' \\ 0 & 0 & u_1' & u_5' \\ u_4' & u_2' & 0 & 0 \\ u_2' & u_6' & 0 & 0 \end{pmatrix},$$

$$\bar{V}_1 = \begin{pmatrix} V_1' & \varepsilon V_2' \\ V_2' & V_1' \end{pmatrix}, V_1' = \begin{pmatrix} a & c & g & e \\ d & b & e & p \\ h & f & -a & -d \\ f & q & -c & -b \end{pmatrix}, V_2' = \begin{pmatrix} a' & c' & g' & e' \\ d' & b' & e' & p' \\ h' & f' & -a' & -d' \\ f' & q' & -c' & -b' \end{pmatrix}, \tag{3.3}$$

where $u_1, u_2, \cdots, u_6$ and $a, b, c, d, e, f, g, h, p, q$ are different from (2.2).

We solve the enlarged stationary zero curvature equation by means of

$$\bar{V}_{1x} = \frac{\partial \bar{U}_1}{\partial \lambda}\lambda_t + [\bar{U}_1, \bar{V}_1], \tag{3.4}$$

which yields

$$\begin{cases} a_{ix} = u_1 f_i - u_2 e_i + u_3 h_i - u_4 g_i + \varepsilon u_1' f_i' - \varepsilon u_2' e_i' + \varepsilon u_3' h_i' - \varepsilon u_4' g_i' + k_i(t), \\ b_{ix} = u_1 f_i - u_2 e_i + u_5 q_i - u_6 p_i + \varepsilon u_1' f_i' - \varepsilon u_2' e_i' + \varepsilon u_5' q_i' - \varepsilon u_6' p_i' + k_i(t), \\ c_{ix} = u_1 q_i - u_2 g_i + u_3 f_i - u_6 e_i + \varepsilon u_1' q_i' - \varepsilon u_2' g_i' + \varepsilon u_3' f_i' - \varepsilon u_6' e_i', \\ d_{ix} = u_1 h_i - u_2 p_i - u_4 e_i + u_5 f_i + \varepsilon u_1' h_i' - \varepsilon u_2' p_i' - \varepsilon u_4' e_i' + \varepsilon u_5' f_i', \\ e_{ix} = 2\lambda e_i - u_1 a_i - u_1 b_i - u_3 d_i - u_5 c_i - \varepsilon u_1' a_i' - \varepsilon u_1' b_i' - \varepsilon u_3' d_i' - \varepsilon u_5' c_i', \\ f_{ix} = -2\lambda f_i + u_2 a_i + u_2 b_i + u_4 c_i + u_6 d_i + \varepsilon u_2' a_i' + \varepsilon u_2' b_i' + \varepsilon u_4' c_i' + \varepsilon u_6' d_i', \\ g_{ix} = 2\lambda g_i - 2u_1 c_i - 2u_3 a_i - 2\varepsilon u_1' c_i' - 2\varepsilon u_3' a_i', \\ h_{ix} = -2\lambda h_i + 2u_2 d_i + 2u_4 a_i + 2\varepsilon u_2' d_i' + 2\varepsilon u_4' a_i', \\ p_{ix} = 2\lambda p_i - 2u_1 d_i - 2u_5 b_i - 2\varepsilon u_1' d_i' - 2\varepsilon u_5' b_i', \\ q_{ix} = -2\lambda q_i + 2u_2 c_i + 2u_6 b_i + 2\varepsilon u_2' c_i' + 2\varepsilon u_6' b_i', \end{cases} \tag{3.5}$$

and

$$\left\{\begin{array}{l}
a'_{ix}=u_1f'_i-u_2e'_i+u_3h'_i-u_4g'_i+u'_1f_i-u'_2e_i+u'_3h_i-u'_4g_i+k_i(t),\\
b'_{ix}=u_1f'_i-u_2e'_i+u_5q'_i-u_6p'_i+u'_1f_i-u'_2e_i+u'_5q_i-u'_6p_i+k_i(t),\\
c'_{ix}=u_1q'_i-u_2g'_i+u_3f'_i-u_6e'_i+u'_1q_i-u'_2g_i+u'_3f_i-u'_6e_i,\\
d'_{ix}=u_1h'_i-u_2p'_i-u_4e'_i+u_5f'_i+u'_1h_i-u'_2p_i-u'_4e_i+u'_5f_i,\\
e'_{ix}=2\lambda e'_i-u_1a'_i-u_1b'_i-u_3d'_i-u_5c'_i-u'_1a_i-u'_1b_i-u'_3d_i-u'_5c_i,\\
f'_{ix}=-2\lambda f'_i+u_2a'_i+u_2b'_i+u_4c'_i+u_6d'_i+u'_2a_i+u'_2b_i+u'_4c_i+u'_6d_i,\\
g'_{ix}=2\lambda g'_i-2u_1c'_i-2u_3a'_i-2u'_1c_i-2u'_3a_i,\\
h'_{ix}=-2\lambda h'_i+2u_2d'_i+2u_4a'_i+2u'_2d_i+2u'_4a_i,\\
p'_{ix}=2\lambda p'_i-2u_1d'_i-2u_5b'_i-2u'_1d_i-2u'_5b_i,\\
q'_{ix}=-2\lambda q'_i+2u_2c'_i+2u_6b'_i+2u'_2c_i+2u'_6b_i.
\end{array}\right. \tag{3.6}$$

By taking initial values

$$\begin{aligned}
a_0=\alpha(t),\ b_0=\beta(t),\ c_0=d_0=e_0=f_0=g_0=h_0=p_0=q_0=k_0(t)=0,\\
a'_0=\alpha'(t),\ b'_0=\beta'(t),\ c'_0=d'_0=e'_0=f'_0=g'_0=h'_0=p'_0=q'_0=0,
\end{aligned}$$

one has

$$\begin{aligned}
a_1=&b_1=k_1(t)x,\ e_1=\frac{1}{2}u_1(\alpha+\beta)+\frac{\varepsilon}{2}u'_1(\alpha'+\beta'),\ f_1=\frac{1}{2}u_2(\alpha+\beta)+\frac{\varepsilon}{2}u'_2(\alpha'+\beta'),\\
g_1=&u_3\alpha+\varepsilon u'_3\alpha',\ h_1=u_4\alpha+\varepsilon u'_4\alpha',\ p_1=u_5\beta+\varepsilon u'_5\beta',\ q_1=u_6\beta+\varepsilon u'_6\beta',\\
c_1=&\frac{1}{2}\partial^{-1}(u_1u_6+u_2u_3+\varepsilon u'_1u'_6+\varepsilon u'_2u'_3)(\beta-\alpha)+\frac{\varepsilon}{2}\partial^{-1}(u_1u'_6+u_2u'_3+u'_1u_6+u'_2u_3)\\
&(\beta'-\alpha'),\\
d_1=&\frac{1}{2}\partial^{-1}(u_1u_4+u_2u_5+\varepsilon u'_1u'_4+\varepsilon u'_2u'_5)(\alpha-\beta)+\frac{\varepsilon}{2}\partial^{-1}(u_1u'_4+u_2u'_5+u'_1u_4+u'_2u_5)\\
&(\alpha'-\beta'),\\
a'_1=&b'_1=k_1(t)x,\ e'_1=\frac{1}{2}u'_1(\alpha+\beta)+\frac{1}{2}u_1(\alpha'+\beta'),\ f'_1=\frac{1}{2}u'_2(\alpha+\beta)+\frac{1}{2}u_2(\alpha'+\beta'),\\
g'_1=&u'_3\alpha+u_3\alpha',\ h'_1=u'_4\alpha+u_4\alpha',\ p'_1=u'_5\beta+u_5\beta',\ q'_1=u'_6\beta+u_6\beta',\\
c'_1=&\frac{1}{2}\partial^{-1}(u_1u'_6+u_2u'_3+u'_1u_6+u'_2u_3)(\beta-\alpha)+\frac{1}{2}\partial^{-1}(u_1u_6+u_2u_3+\varepsilon u'_1u'_6+\varepsilon u'_2u'_3)\\
&(\beta'-\alpha'),\\
d'_1=&\frac{1}{2}\partial^{-1}(u_1u'_4+u_2u'_5+u'_1u_4+u'_2u_5)(\alpha-\beta)+\frac{1}{2}\partial^{-1}(u_1u_4+u_2u_5+\varepsilon u'_1u'_4+\varepsilon u'_2u'_5)\\
&(\alpha'-\beta'),\\
&\cdots\cdots
\end{aligned}$$

where $\alpha(t)$, $\beta(t)$ are integral constants.

From the nonisospectral zero curvature equation

$$\frac{\partial\bar{U}_1}{\partial u}u_t+\frac{\partial\bar{U}_1}{\partial\lambda}\lambda_t-\bar{V}_{1x}^{(n)}+[\bar{U}_1,\bar{V}_1^{(n)}]=0,$$

we can obtain the following integrable couplings:

$$\bar{u}_{t_n}=\begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \end{pmatrix}_{t_n},\ \bar{u}_{1t_n}=\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix}_{t_n}=\begin{pmatrix} -2e_{n+1} \\ 2f_{n+1} \\ -2g_{n+1} \\ 2h_{n+1} \\ -2p_{n+1} \\ 2q_{n+1} \end{pmatrix},\ \bar{u}_{2t_n}=\begin{pmatrix} u'_1 \\ u'_2 \\ u'_3 \\ u'_4 \\ u'_5 \\ u'_6 \end{pmatrix}_{t_n}=\begin{pmatrix} -2e'_{n+1} \\ 2f'_{n+1} \\ -2g'_{n+1} \\ 2h'_{n+1} \\ -2p'_{n+1} \\ 2q'_{n+1} \end{pmatrix}. \tag{3.7}$$

To furnish Hamiltonian structures, we use the trace identity, and have

$$\begin{aligned}
&\langle V'_1, \frac{\partial U'_2}{\partial \lambda}\rangle + \langle V'_2, \frac{\partial U'_1}{\partial \lambda}\rangle = 2(a'+b'),\ \langle V'_1, \frac{\partial U'_2}{\partial u_1}\rangle + \langle V'_2, \frac{\partial U'_1}{\partial u_1}\rangle = 2f',\\
&\langle V'_1, \frac{\partial U'_2}{\partial u_2}\rangle + \langle V'_2, \frac{\partial U'_1}{\partial u_2}\rangle = 2e',\ \langle V'_1, \frac{\partial U'_2}{\partial u_3}\rangle + \langle V'_2, \frac{\partial U'_1}{\partial u_3}\rangle = h',\\
&\langle V'_1, \frac{\partial U'_2}{\partial u_4}\rangle + \langle V'_2, \frac{\partial U'_1}{\partial u_4}\rangle = g',\ \langle V'_1, \frac{\partial U'_2}{\partial u_5}\rangle + \langle V'_2, \frac{\partial U'_1}{\partial u_5}\rangle = q',\\
&\langle V'_1, \frac{\partial U'_2}{\partial u_6}\rangle + \langle V'_2, \frac{\partial U'_1}{\partial u_6}\rangle = p',\ \langle V'_1, \frac{\partial U'_2}{\partial u'_1}\rangle + \langle V'_2, \frac{\partial U'_1}{\partial u'_1}\rangle = 2f,\\
&\langle V'_1, \frac{\partial U'_2}{\partial u'_2}\rangle + \langle V'_2, \frac{\partial U'_1}{\partial u'_2}\rangle = 2e,\ \langle V'_1, \frac{\partial U'_2}{\partial u'_3}\rangle + \langle V'_2, \frac{\partial U'_1}{\partial u'_3}\rangle = h,\\
&\langle V'_1, \frac{\partial U'_2}{\partial u'_4}\rangle + \langle V'_2, \frac{\partial U'_1}{\partial u'_4}\rangle = g,\ \langle V'_1, \frac{\partial U'_2}{\partial u'_5}\rangle + \langle V'_2, \frac{\partial U'_1}{\partial u'_5}\rangle = q,\\
&\langle V'_1, \frac{\partial U'_2}{\partial u'_6}\rangle + \langle V'_2, \frac{\partial U'_1}{\partial u'_6}\rangle = p.
\end{aligned}$$

Substituting the above formulas into the trace identity, and balancing the coefficients of each power of λ, we give the first form

$$\frac{\delta}{\delta \bar{u}} \int 2(a'_{n+1} + b'_{n+1})dx = (\gamma - n)\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}, \tag{3.8}$$

where

$$M_1 = \begin{pmatrix} 2f'_n \\ 2e'_n \\ h'_n \\ g'_n \\ q'_n \\ p'_n \end{pmatrix},\ M_2 = \begin{pmatrix} 2f_n \\ 2e_n \\ h_n \\ g_n \\ q_n \\ p_n \end{pmatrix}.$$

At the same time, we also have

$$\langle V'_1, \frac{\partial U'_1}{\partial \lambda}\rangle + \varepsilon\langle V'_2, \frac{\partial U'_2}{\partial \lambda}\rangle = 2(a+b),\ \langle V'_1, \frac{\partial U'_1}{\partial u_1}\rangle + \varepsilon\langle V'_2, \frac{\partial U'_2}{\partial u_1}\rangle = 2f,$$

$$\begin{aligned}
&\langle V_1', \frac{\partial U_1'}{\partial u_2}\rangle + \varepsilon\langle V_2', \frac{\partial U_2'}{\partial u_2}\rangle = 2e,\ \langle V_1', \frac{\partial U_1'}{\partial u_3}\rangle + \varepsilon\langle V_2', \frac{\partial U_2'}{\partial u_3}\rangle = h,\\
&\langle V_1', \frac{\partial U_1'}{\partial u_4}\rangle + \varepsilon\langle V_2', \frac{\partial U_2'}{\partial u_4}\rangle = g,\ \langle V_1', \frac{\partial U_1'}{\partial u_5}\rangle + \varepsilon\langle V_2', \frac{\partial U_2'}{\partial u_5}\rangle = q,\\
&\langle V_1', \frac{\partial U_1'}{\partial u_6}\rangle + \varepsilon\langle V_2', \frac{\partial U_2'}{\partial u_6}\rangle = p,\ \langle V_1', \frac{\partial U_1'}{\partial u_1'}\rangle + \varepsilon\langle V_2', \frac{\partial U_2'}{\partial u_1'}\rangle = 2\varepsilon f',\\
&\langle V_1', \frac{\partial U_1'}{\partial u_2'}\rangle + \varepsilon\langle V_2', \frac{\partial U_2'}{\partial u_2'}\rangle = 2\varepsilon e',\ \langle V_1', \frac{\partial U_1'}{\partial u_3'}\rangle + \varepsilon\langle V_2', \frac{\partial U_2'}{\partial u_3'}\rangle = \varepsilon h',\\
&\langle V_1', \frac{\partial U_1'}{\partial u_4'}\rangle + \varepsilon\langle V_2', \frac{\partial U_2'}{\partial u_4'}\rangle = \varepsilon g',\ \langle V_1', \frac{\partial U_1'}{\partial u_5'}\rangle + \varepsilon\langle V_2', \frac{\partial U_2'}{\partial u_5'}\rangle = \varepsilon q',\\
&\langle V_1', \frac{\partial U_1'}{\partial u_6'}\rangle + \varepsilon\langle V_2', \frac{\partial U_2'}{\partial u_6'}\rangle = \varepsilon p'.
\end{aligned}$$

Substituting the above formulas into the trace identity, and balancing coefficients of each power of λ, we give the second form

$$\frac{\delta}{\delta u}\int 2(a_{n+1} + b_{n+1})dx = (\gamma - n)\begin{pmatrix} M_2 \\ \varepsilon M_1 \end{pmatrix},$$

where $M_1,\ M_2$ are defined as (3.8).

So, we can obtain the Hamiltonian structures of integrable couplings, which consists of the following two components. The first component has the form

$$\bar{u}_{t_n} = \begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \end{pmatrix}_{t_n} = \bar{J}_1\begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = \bar{J}_1\frac{\delta \bar{H}_{1,m}}{\delta \bar{u}},\ K_1 = \begin{pmatrix} 2f_{n+1}' \\ 2e_{n+1}' \\ h_{n+1}' \\ g_{n+1}' \\ q_{n+1}' \\ p_{n+1}' \end{pmatrix},\ K_2 = \begin{pmatrix} 2f_{n+1} \\ 2e_{n+1} \\ h_{n+1} \\ g_{n+1} \\ q_{n+1} \\ p_{n+1} \end{pmatrix},$$

where $\bar{J}_1 = \begin{pmatrix} O & J_1 \\ J_1 & O \end{pmatrix}$, and J_1 is defined as (2.5). The second component has the form

$$\bar{u}_{t_n} = \begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \end{pmatrix}_{t_n} = \bar{J}_2\begin{pmatrix} K_2 \\ \varepsilon K_1 \end{pmatrix} = \bar{J}_2\frac{\delta \bar{H}_{2,m}}{\delta \bar{u}},\quad \bar{J}_2 = \begin{pmatrix} J_1 & O \\ O & \frac{1}{\varepsilon}J_1 \end{pmatrix}.$$

From (3.5) and (3.6), we can obtain the recursion relations

$$\begin{pmatrix} -2e_{n+1} \\ 2f_{n+1} \\ -2g_{n+1} \\ 2h_{n+1} \\ -2p_{n+1} \\ 2q_{n+1} \\ -2e'_{n+1} \\ 2f'_{n+1} \\ -2g'_{n+1} \\ 2h'_{n+1} \\ -2p'_{n+1} \\ 2q'_{n+1} \end{pmatrix} = \bar{L}_1 \begin{pmatrix} -2e_n \\ 2f_n \\ -2g_n \\ 2h_n \\ -2p_n \\ 2q_n \\ -2e'_n \\ 2f'_n \\ -2g'_n \\ 2h'_n \\ -2p'_n \\ 2q'_n \end{pmatrix} + \begin{pmatrix} -2u_1 - 2\varepsilon u'_1 \\ 2u_2 + 2\varepsilon u'_2 \\ -2u_3 - 2\varepsilon u'_3 \\ 2u_4 + 2\varepsilon u'_4 \\ -2u_5 - 2\varepsilon u'_5 \\ 2u_6 + 2\varepsilon u'_6 \\ -2u_1 - 2u'_1 \\ 2u_2 + 2u'_2 \\ -2u_3 - 2u'_3 \\ 2u_4 + 2u'_4 \\ -2u_5 - 2u'_5 \\ 2u_6 + 2u'_6 \end{pmatrix} k_n(t)x,$$

where the recurrence operator $\bar{L}_1$ is defined as

$$\bar{L}_1 = \begin{pmatrix} L'_1 & L'_2 \\ \frac{1}{\varepsilon}L'_2 & L'_1 \end{pmatrix},\ L'_1 = \begin{pmatrix} l_{11} & l_{12} & l_{13} & l_{14} & l_{15} & l_{16} \\ l_{21} & l_{22} & l_{23} & l_{24} & l_{25} & l_{26} \\ l_{31} & l_{32} & l_{33} & l_{34} & l_{35} & l_{36} \\ l_{41} & l_{42} & l_{43} & l_{44} & l_{45} & l_{46} \\ l_{51} & l_{52} & l_{53} & l_{54} & l_{55} & l_{56} \\ l_{61} & l_{62} & l_{63} & l_{64} & l_{65} & l_{66} \end{pmatrix},\ L'_2 = \begin{pmatrix} l'_{11} & l'_{12} & l'_{13} & l'_{14} & l'_{15} & l'_{16} \\ l'_{21} & l'_{22} & l'_{23} & l'_{24} & l'_{25} & l'_{26} \\ l'_{31} & l'_{32} & l'_{33} & l'_{34} & l'_{35} & l'_{36} \\ l'_{41} & l'_{42} & l'_{43} & l'_{44} & l'_{45} & l'_{46} \\ l'_{51} & l'_{52} & l'_{53} & l'_{54} & l'_{55} & l'_{56} \\ l'_{61} & l'_{62} & l'_{63} & l'_{64} & l'_{65} & l'_{66} \end{pmatrix},$$

and

$$\begin{aligned}
l_{11} &= \frac{1}{2}(\partial - 2u_1\partial^{-1}u_2 - u_3\partial^{-1}u_4 - u_5\partial^{-1}u_6 - 2\varepsilon u'_1\partial^{-1}u'_2 - \varepsilon u'_3\partial^{-1}u'_4 - \varepsilon u'_5\partial^{-1}u'_6),\\
l_{12} &= -\frac{1}{2}(2u_1\partial^{-1}u_1 + u_3\partial^{-1}u_5 + u_5\partial^{-1}u_3 + 2\varepsilon u'_1\partial^{-1}u'_1 + \varepsilon u'_3\partial^{-1}u'_5 + \varepsilon u'_5\partial^{-1}u'_3),\\
l_{13} &= \frac{1}{2}l_{51} = -\frac{1}{2}(u_1\partial^{-1}u_4 + u_5\partial^{-1}u_2 + \varepsilon u'_1\partial^{-1}u'_4 + \varepsilon u'_5\partial^{-1}u'_2),\\
l_{14} &= \frac{1}{2}l_{32} = -\frac{1}{2}(u_1\partial^{-1}u_3 + u_3\partial^{-1}u_1 + \varepsilon u'_1\partial^{-1}u'_3 + \varepsilon u'_3\partial^{-1}u'_1),\\
l_{15} &= \frac{1}{2}l_{31} = -\frac{1}{2}(u_1\partial^{-1}u_6 + u_3\partial^{-1}u_2 + \varepsilon u'_1\partial^{-1}u'_6 + \varepsilon u'_3\partial^{-1}u'_2),\\
l_{16} &= \frac{1}{2}l_{52} = -\frac{1}{2}(u_1\partial^{-1}u_5 + u_5\partial^{-1}u_1 + \varepsilon u'_1\partial^{-1}u'_5 + \varepsilon u'_5\partial^{-1}u'_1),\\
l'_{11} &= -\frac{\varepsilon}{2}(2u_1\partial^{-1}u'_2 + u_3\partial^{-1}u'_4 + u_5\partial^{-1}u'_6 + 2u'_1\partial^{-1}u_2 + u'_3\partial^{-1}u_4 + u'_5\partial^{-1}u_6),\\
l'_{12} &= -\frac{\varepsilon}{2}(2u_1\partial^{-1}u'_1 + u_3\partial^{-1}u'_5 + u_5\partial^{-1}u'_3 + 2u'_1\partial^{-1}u_1 + u'_3\partial^{-1}u_5 + u'_5\partial^{-1}u_3),\\
l'_{13} &= \frac{1}{2}l'_{51} = -\frac{\varepsilon}{2}(u_1\partial^{-1}u'_4 + u_5\partial^{-1}u'_2 + u'_1\partial^{-1}u_4 + u'_5\partial^{-1}u_2),\\
l'_{14} &= \frac{1}{2}l'_{32} = -\frac{\varepsilon}{2}(u_1\partial^{-1}u'_3 + u_3\partial^{-1}u'_1 + u'_1\partial^{-1}u_3 + u'_3\partial^{-1}u_1),\\
l'_{15} &= \frac{1}{2}l'_{31} = -\frac{\varepsilon}{2}(u_1\partial^{-1}u'_6 + u_3\partial^{-1}u'_2 + u'_1\partial^{-1}u_6 + u'_3\partial^{-1}u_2),
\end{aligned}$$

$$l'_{16} = \frac{1}{2}l'_{52} = -\frac{\varepsilon}{2}(u_1\partial^{-1}u'_5+u_5\partial^{-1}u'_1 + u'_1\partial^{-1}u_5+u'_5\partial^{-1}u_1),$$
$$l_{21} = \frac{1}{2}(2u_2\partial^{-1}u_2+u_4\partial^{-1}u_6+u_6\partial^{-1}u_4 + 2\varepsilon u'_2\partial^{-1}u'_2+\varepsilon u'_4\partial^{-1}u'_6+\varepsilon u'_6\partial^{-1}u'_4),$$
$$l_{22} = \frac{1}{2}(-\partial + 2u_2\partial^{-1}u_1+u_4\partial^{-1}u_3+u_6\partial^{-1}u_5 + 2\varepsilon u'_2\partial^{-1}u'_1+\varepsilon u'_4\partial^{-1}u'_3+\varepsilon u'_6\partial^{-1}u'_5),$$
$$l_{23} = \frac{1}{2}l_{41} = \frac{1}{2}(u_2\partial^{-1}u_4+u_4\partial^{-1}u_2 + \varepsilon u'_2\partial^{-1}u'_4+\varepsilon u'_4\partial^{-1}u'_2),$$
$$l_{24} = \frac{1}{2}l_{62} = \frac{1}{2}(u_2\partial^{-1}u_3+u_6\partial^{-1}u_1 + \varepsilon u'_2\partial^{-1}u'_3+\varepsilon u'_6\partial^{-1}u'_1),$$
$$l_{25} = \frac{1}{2}l_{61} = \frac{1}{2}(u_2\partial^{-1}u_6+u_6\partial^{-1}u_2 + \varepsilon u'_2\partial^{-1}u'_6+\varepsilon u'_6\partial^{-1}u'_2),$$
$$l_{26} = \frac{1}{2}l_{42} = \frac{1}{2}(u_2\partial^{-1}u_5+u_4\partial^{-1}u_1 + \varepsilon u'_2\partial^{-1}u'_5+\varepsilon u'_4\partial^{-1}u'_1),$$
$$l'_{21} = \frac{\varepsilon}{2}(2u_2\partial^{-1}u'_2+u_4\partial^{-1}u'_6+u_6\partial^{-1}u'_4 + 2u'_2\partial^{-1}u_2+u'_4\partial^{-1}u_6+u'_6\partial^{-1}u_4),$$
$$l'_{22} = \frac{\varepsilon}{2}(2u_2\partial^{-1}u'_1+u_4\partial^{-1}u'_3+u_6\partial^{-1}u'_5 + 2u'_2\partial^{-1}u_1+u'_4\partial^{-1}u_3+u'_6\partial^{-1}u_5),$$
$$l'_{23} = \frac{1}{2}l'_{41} = \frac{\varepsilon}{2}(u_2\partial^{-1}u'_4+u_4\partial^{-1}u'_2 + u'_2\partial^{-1}u_4+u'_4\partial^{-1}u_2),$$
$$l'_{24} = \frac{1}{2}l'_{62} = \frac{\varepsilon}{2}(u_2\partial^{-1}u'_3+u_6\partial^{-1}u'_1 + u'_2\partial^{-1}u_3+u'_6\partial^{-1}u_1),$$
$$l'_{25} = \frac{1}{2}l'_{61} = \frac{\varepsilon}{2}(u_2\partial^{-1}u'_6+u_6\partial^{-1}u'_2 + u'_2\partial^{-1}u_6+u'_6\partial^{-1}u_2),$$
$$l'_{26} = \frac{1}{2}l'_{42} = \frac{\varepsilon}{2}(u_2\partial^{-1}u'_5+u_4\partial^{-1}u'_1 + u'_2\partial^{-1}u_5+u'_4\partial^{-1}u_1),\ l_{35}=0,$$
$$l_{33} = \frac{\partial}{2} - u_1\partial^{-1}u_2-u_3\partial^{-1}u_4 - \varepsilon u'_1\partial^{-1}u'_2-\varepsilon u'_3\partial^{-1}u'_4,\ l_{34} = -(u_3\partial^{-1}u_3+\varepsilon u'_3\partial^{-1}u'_3),$$
$$l_{36} = -(u_1\partial^{-1}u_1+\varepsilon u'_1\partial^{-1}u'_1),\ l'_{33} = -\varepsilon(u_1\partial^{-1}u'_2+u_3\partial^{-1}u'_4+u'_1\partial^{-1}u_2+u'_3\partial^{-1}u_4),$$
$$l'_{34} = -\varepsilon(u_3\partial^{-1}u'_3+u_3\partial^{-1}u'_3),\ l'_{35}=0,\ l'_{36} = -\varepsilon(u_1\partial^{-1}u'_1+u_1\partial^{-1}u'_1),$$
$$l_{43} = u_4\partial^{-1}u_4+ \varepsilon u'_4\partial^{-1}u'_4,\ l_{44} = -\frac{\partial}{2} + u_4\partial^{-1}u_3+u_2\partial^{-1}u_1 + \varepsilon u'_4\partial^{-1}u'_3+\varepsilon u'_2\partial^{-1}u'_1,$$
$$l_{45} = u_2\partial^{-1}u_2+ \varepsilon u'_2\partial^{-1}u'_2,\ l_{46}=0,\ l'_{43} = \varepsilon(u_4\partial^{-1}u'_4 +u'_4\partial^{-1}u_4),$$
$$l'_{44} = \varepsilon(u_4\partial^{-1}u'_3+u_2\partial^{-1}u'_1+u'_4\partial^{-1}u_3 +u'_2\partial^{-1}u_1),\ l'_{45} = \varepsilon(u_2\partial^{-1}u'_2+u_2\partial^{-1}u'_2),\ l'_{46}=0,$$
$$l_{54} = -(u_1\partial^{-1}u_1+ \varepsilon u'_1\partial^{-1}u'_1),\ l_{55} = \frac{\partial}{2} - u_1\partial^{-1}u_2-u_5\partial^{-1}u_6 - \varepsilon u'_1\partial^{-1}u'_2-\varepsilon u'_5\partial^{-1}u'_6,$$
$$l_{56} = -(u_5\partial^{-1}u_5+ \varepsilon u'_5\partial^{-1}u'_5),\ l_{53} = l'_{53} = 0,\ l'_{54} = -\varepsilon(u_1\partial^{-1}u'_1 +u'_1\partial^{-1}u_1),$$
$$l'_{55} = -\varepsilon(u_1\partial^{-1}u'_2+u_5\partial^{-1}u'_6+u'_1\partial^{-1}u_2 +u'_5\partial^{-1}u_6),\ l'_{56} = -\varepsilon(u_5\partial^{-1}u'_5+u_5\partial^{-1}u'_5),$$
$$l_{63} = u_2\partial^{-1}u_2+ \varepsilon u'_2\partial^{-1}u'_2,\ l_{64} = l'_{64}=0,\ l_{65} = u_6\partial^{-1}u_6+ \varepsilon u'_6\partial^{-1}u'_6,$$
$$l_{66} = -\frac{\partial}{2} + u_2\partial^{-1}u_1+u_6\partial^{-1}u_5 + \varepsilon u'_2\partial^{-1}u'_1+\varepsilon u'_6\partial^{-1}u'_5,\ l'_{63} = \varepsilon(u_2\partial^{-1}u'_2+ u'_2\partial^{-1}u_2),$$
$$l'_{65} = \varepsilon(u_6\partial^{-1}u'_6+ u'_6\partial^{-1}u_6),\ l'_{66} = \varepsilon(u_2\partial^{-1}u'_1+u_6\partial^{-1}u'_5 + u'_2\partial^{-1}u_1+u'_6\partial^{-1}u_5).$$

4. The nonisospectral integrabl hierarchy of the generalized Lie algebra $G\mathfrak{so}(5)$

In this section, based on Lie algebra $\mathfrak{so}(5)$ [31,33], we introduce the generalized Lie algebra $G\mathfrak{so}(5)$, that admits a basis set as follows:

$$\begin{aligned} &E_1 = e_{22} - e_{44},\ \ E_2 = e_{33} - e_{55},\ \ E_3 = \varepsilon e_{32} - \varepsilon e_{45},\ \ E_4 = e_{23} - e_{54},\ E_5 = e_{52} - e_{43},\\ &E_6 = \varepsilon e_{25} - \varepsilon e_{34},\ E_7 = e_{13} + e_{51},\ E_8 = \varepsilon e_{15} + \varepsilon e_{31},\ E_9 = \varepsilon e_{12} + \varepsilon e_{41},\ E_{10} = e_{14} + e_{21}, \end{aligned} \tag{4.1}$$

where e_{ij} is a 5×5 matrix with 1 in the (i, j)-th position and zero elsewhere, which satisfy the commutative relations

$$\begin{aligned} &[E_1, E_2] = 0,\ [E_1, E_3] = -E_3,\ [E_1, E_4] = E_4,\ [E_1, E_5] = -E_5,\ [E_1, E_6] = E_6,\\ &[E_1, E_7] = [E_1, E_8] = 0,\ [E_1, E_9] = -E_9,\ [E_1, E_{10}] = E_{10},\ [E_2, E_3] = E_3,\\ &[E_2, E_4] = -E_4,\ [E_2, E_5] = -E_5,\ [E_2, E_6] = E_6,\ [E_2, E_7] = -E_7,\ [E_2, E_8] = E_8,\\ &[E_2, E_9] = [E_2, E_{10}] = 0,\ [E_3, E_4] = \varepsilon(E_2 - E_1),\ [E_3, E_5] = [E_3, E_6] = 0,\ [E_3, E_7] = -E_9,\\ &[E_3, E_8] = [E_3, E_9] = 0,\ [E_3, E_{10}] = E_8,\ [E_4, E_5] = [E_4, E_6] = [E_4, E_7] = 0,\\ &[E_4, E_8] = \varepsilon E_{10},\ [E_4, E_9] = -\varepsilon E_7,\ [E_4, E_{10}] = 0,\ [E_5, E_6] = -\varepsilon(E_1 + E_2),\ [E_5, E_7] = 0,\\ &[E_5, E_8] = -E_9,\ [E_5, E_9] = 0,\ [E_5, E_{10}] = E_7,\ [E_6, E_7] = \varepsilon E_{10},\ [E_6, E_8] = [E_6, E_{10}] = 0,\\ &[E_6, E_9] = -\varepsilon E_8,\ [E_7, E_8] = -\varepsilon E_2,\ [E_7, E_9] = \varepsilon E_5,\ [E_7, E_{10}] = -E_4,\ [E_8, E_9] = \varepsilon E_3,\\ &[E_8, E_{10}] = -E_6,\ [E_9, E_{10}] = -\varepsilon E_1. \end{aligned}$$

Consider the nonisospectral problem

$$\begin{cases} \varphi_x = U_2\varphi,\\ \varphi_t = V_2\varphi,\\ \lambda_t = \sum_{i\geq 0} k_i(t)\lambda^{-i}, \end{cases}$$

where

$$U_2 = \begin{pmatrix} 0 & \varepsilon u_5 & u_3 & u_6 & \varepsilon u_4\\ u_6 & \lambda & 0 & 0 & \varepsilon u_2\\ \varepsilon u_4 & 0 & \lambda & -\varepsilon u_2 & 0\\ \varepsilon u_5 & 0 & -u_1 & -\lambda & 0\\ u_3 & u_1 & 0 & 0 & -\lambda \end{pmatrix},$$

$$V_2 = \begin{pmatrix} 0 & \varepsilon p & g & q & \varepsilon h\\ q & a & d & 0 & \varepsilon f\\ \varepsilon h & \varepsilon c & b & -\varepsilon f & 0\\ \varepsilon p & 0 & -e & -a & -\varepsilon c\\ g & e & 0 & -d & -b \end{pmatrix} = \sum_{i\geq 0} \begin{pmatrix} 0 & \varepsilon p_i & g_i & q_i & \varepsilon h_i\\ q_i & a_i & d_i & 0 & \varepsilon f_i\\ \varepsilon h_i & \varepsilon c_i & b_i & -\varepsilon f_i & 0\\ \varepsilon p_i & 0 & -e_i & -a_i & -\varepsilon c_i\\ g_i & e_i & 0 & -d_i & -b_i \end{pmatrix} \lambda^{-i}, \tag{4.2}$$

here $u_1, u_2, \cdots, u_6$ and $a, b, c, d, e, f, g, h, p, q$ are different from (2.2) and (3.3).

By solving the stationary zero curvature representation

$$V_{2x} = \frac{\partial U_2}{\partial \lambda}\lambda_t + [U_2, V_2], \tag{4.3}$$

we can obtain

$$\left\{\begin{array}{l} a_{ix} = -\varepsilon u_1 f_i + \varepsilon u_2 e_i - \varepsilon u_5 q_i + \varepsilon u_6 p_i + k_i(t), \\ b_{ix} = -\varepsilon u_1 f_i + \varepsilon u_2 e_i - \varepsilon u_3 h_i + \varepsilon u_4 g_i + k_i(t), \\ c_{ix} = \varepsilon u_4 p_i - \varepsilon u_5 h_i, \\ d_{ix} = -u_3 q_i + u_6 g_i, \\ e_{ix} = -2\lambda e_i + u_1 a_i + u_1 b_i + \varepsilon u_3 p_i - \varepsilon u_5 g_i, \\ f_{ix} = 2\lambda f_i - u_2 a_i - u_2 b_i - u_4 q_i + u_6 h_i, \\ g_{ix} = -\lambda g_i + u_1 q_i + u_3 b_i + \varepsilon u_5 d_i - u_6 e_i, \\ h_{ix} = \lambda h_i - \varepsilon u_2 p_i - u_4 b_i + \varepsilon u_5 f_i - u_6 c_i, \\ p_{ix} = -\lambda p_i - u_1 h_i + u_3 c_i + u_4 e_i + u_5 a_i, \\ q_{ix} = \lambda q_i + \varepsilon u_2 g_i - \varepsilon u_3 f_i - \varepsilon u_4 d_i - u_6 a_i. \end{array}\right. \tag{4.4}$$

By taking initial values

$$a_0 = \alpha(t),\ b_0 = \beta(t),\ c_0 = d_0 = e_0 = f_0 = g_0 = h_0 = p_0 = q_0 = k_0(t) = 0,$$

one has

$$\begin{aligned} & a_1 = b_1 = k_1(t)x,\ c_1 = \varepsilon\partial^{-1}u_4u_5(\alpha-\beta), d_1 = \partial^{-1}u_3u_6(\beta-\alpha),\ e_1 = \frac{1}{2}u_1(\alpha+\beta), \\ & f_1 = \frac{1}{2}u_2(\alpha+\beta),\ g_1 = u_3\beta,\ h_1 = u_4\beta,\ p_1 = u_5\alpha,\ q_1 = u_6\alpha, \\ & e_2 = -\frac{1}{4}u_{1x}(\alpha+\beta) + \frac{\varepsilon}{2}u_3u_5(\alpha-\beta) + u_1k_1(t)x, \\ & f_2 = \frac{1}{4}u_{2x}(\alpha+\beta) + \frac{1}{2}u_4u_6(\alpha-\beta) + u_2k_1(t)x, \\ & g_2 = -u_{3x}\beta + \frac{1}{2}u_1u_6(\alpha-\beta) + \varepsilon u_5\partial^{-1}u_3u_6(\beta-\alpha) + u_3k_1(t)x, \\ & h_2 = u_{4x}\beta + \frac{\varepsilon}{2}u_2u_5(\alpha-\beta) + \varepsilon u_6\partial^{-1}u_4u_5(\alpha-\beta) + u_4k_1(t)x, \\ & p_2 = -u_{5x}\alpha + \frac{1}{2}u_1u_4(\alpha-\beta) + \varepsilon u_3\partial^{-1}u_4u_5(\alpha-\beta) + u_5k_1(t)x, \\ & q_2 = u_{6x}\alpha + \frac{\varepsilon}{2}u_2u_3(\alpha-\beta) + \varepsilon u_4\partial^{-1}u_3u_6(\beta-\alpha) + u_6k_1(t)x, \\ & \cdots\cdots \end{aligned}$$

where $\alpha(t)$ is an integral constant. Noting that

$$\begin{aligned} & V_{2+}^{(n)} = \sum_{i=0}^{n}(a_i, b_i, c_i, d_i, e_i, f_i, g_i, h_i, p_i, q_i)^{\mathrm{T}},\quad V_{2-}^{(n)} = \sum_{i=n+1}^{\infty}(a_i, b_i, c_i, d_i, e_i, f_i, g_i, h_i, p_i, q_i)^{\mathrm{T}}, \\ & \lambda_{+,x}^{(n)} = \sum_{i=0}^{n} k_i(t)\lambda^{n-i},\quad \lambda_{-,x}^{(n)} = \sum_{i=n+1}^{\infty} k_i(t)\lambda^{n-i}, \end{aligned}$$

it follows that one has

$$-V_{2+,x}^{(n)} + \frac{\partial U_2}{\partial\lambda}\lambda_{t,+}^{(n)} + [U_2, V_{2+}^{(n)}] = (0,0,0,0,2e_{n+1},-2f_{n+1},g_{n+1},-h_{n+1},p_{n+1},-q_{n+1})^{\mathrm{T}}.$$

According to (4.4), it is easy to show that we have the recursion relations

$$\begin{pmatrix} 2\varepsilon f_{n+1} \\ 2\varepsilon e_{n+1} \\ 2\varepsilon h_{n+1} \\ 2\varepsilon g_{n+1} \\ 2\varepsilon q_{n+1} \\ 2\varepsilon p_{n+1} \end{pmatrix} = L_2 \begin{pmatrix} 2\varepsilon f_n \\ 2\varepsilon e_n \\ 2\varepsilon h_n \\ 2\varepsilon g_n \\ 2\varepsilon q_n \\ 2\varepsilon p_n \end{pmatrix} + \begin{pmatrix} 2\varepsilon u_2 \\ 2\varepsilon u_1 \\ 2\varepsilon u_4 \\ 2\varepsilon u_3 \\ 2\varepsilon u_6 \\ 2\varepsilon u_5 \end{pmatrix} k_n(t)x,$$

where the recurrence operator L_2 is defined as

$$L_2 = \begin{pmatrix} l_{11} & l_{12} & l_{13} & l_{14} & l_{15} & l_{16} \\ l_{21} & l_{22} & l_{23} & l_{24} & l_{25} & l_{26} \\ l_{31} & l_{32} & l_{33} & l_{34} & l_{35} & l_{36} \\ l_{41} & l_{42} & l_{43} & l_{44} & l_{45} & l_{46} \\ l_{51} & l_{52} & l_{53} & l_{54} & l_{55} & l_{56} \\ l_{61} & l_{62} & l_{63} & l_{64} & l_{65} & l_{66} \end{pmatrix},$$

and

$$\begin{aligned}
&l_{11} = \frac{\partial}{2} - \varepsilon u_2\partial^{-1}u_1,\ l_{12} = \varepsilon u_2\partial^{-1}u_2,\ l_{13} = -\frac{\varepsilon}{2}u_2\partial^{-1}u_3 - \frac{u_6}{2},\ l_{14} = \frac{\varepsilon}{2}u_2\partial^{-1}u_4,\\
&l_{15} = -\frac{\varepsilon}{2}u_2\partial^{-1}u_5 + \frac{u_4}{2},\ l_{16} = \frac{\varepsilon}{2}u_2\partial^{-1}u_6,\ l_{21} = -\varepsilon u_1\partial^{-1}u_1,\ l_{22} = -\frac{\partial}{2} + \varepsilon u_1\partial^{-1}u_2,\\
&l_{23}=-\frac{\varepsilon}{2}u_1\partial^{-1}u_3,\ l_{24}=\frac{\varepsilon}{2}u_1\partial^{-1}u_4 - \frac{\varepsilon}{2}u_5,\ l_{25}=-\frac{\varepsilon}{2}u_1\partial^{-1}u_5,\ l_{26}=\frac{\varepsilon}{2}u_1\partial^{-1}u_6 + \frac{\varepsilon}{2}u_3,\\
&l_{31} = -\varepsilon u_4\partial^{-1}u_1 - \varepsilon u_5,\ l_{32} = \varepsilon u_4\partial^{-1}u_2,\ l_{33} = \partial - \varepsilon u_4\partial^{-1}u_3 - \varepsilon u_6\partial^{-1}u_5,\ l_{35} = 0,\\
&l_{34} = \varepsilon u_4\partial^{-1}u_4,\ l_{36} = \varepsilon u_2 + \varepsilon u_6\partial^{-1}u_4,\ l_{41} = -\varepsilon u_3\partial^{-1}u_1,\ l_{42} = \varepsilon u_3\partial^{-1}u_2 - u_6,\\
&l_{43} = -\varepsilon u_3\partial^{-1}u_3,\ l_{44} = -\partial + \varepsilon u_3\partial^{-1}u_4 + \varepsilon u_5\partial^{-1}u_6,\ l_{45} = u_1 - \varepsilon u_5\partial^{-1}u_3,\ l_{46} = 0,\\
&l_{51} = \varepsilon u_3 - \varepsilon u_6\partial^{-1}u_1,\ l_{52} = \varepsilon u_6\partial^{-1}u_2,\ l_{53} = 0,\ l_{54} = -\varepsilon u_2 + \varepsilon u_4\partial^{-1}u_6,\\
&l_{55}=\partial-\varepsilon u_4\partial^{-1}u_3-\varepsilon u_6\partial^{-1}u_5,\ l_{56}=\varepsilon u_6\partial^{-1}u_6,\ l_{61}=-\varepsilon u_5\partial^{-1}u_1,\ l_{62}=u_4+\varepsilon u_5\partial^{-1}u_2,\\
&l_{63} = -u_1 - \varepsilon u_3\partial^{-1}u_5,\ l_{64} = 0,\ l_{65} = -\varepsilon u_5\partial^{-1}u_5,\ l_{66} = -\partial + \varepsilon u_3\partial^{-1}u_4 + \varepsilon u_5\partial^{-1}u_6.
\end{aligned}$$

Taking $V_2^{(n)} = V_{2,+}^{(n)}$, the zero curvature equation

$$-V_{2x}^{(n)} + \frac{\partial U_2}{\partial u}u_t + \frac{\partial U_2}{\partial \lambda}\lambda_{t,+}^{(n)} + [U_2, V_2^{(n)}] = 0$$

leads to the nonisospectral hierarchy

$$u_{t_n} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix}_{t_n} = \begin{pmatrix} 2e_{n+1} \\ -2f_{n+1} \\ g_{n+1} \\ -h_{n+1} \\ p_{n+1} \\ -q_{n+1} \end{pmatrix} = J_2 \begin{pmatrix} 2\varepsilon f_{n+1} \\ 2\varepsilon e_{n+1} \\ 2\varepsilon h_{n+1} \\ 2\varepsilon g_{n+1} \\ 2\varepsilon q_{n+1} \\ 2\varepsilon p_{n+1} \end{pmatrix}$$

$$= \quad J_2L_2\begin{pmatrix} 2\varepsilon f_n \\ 2\varepsilon e_n \\ 2\varepsilon h_n \\ 2\varepsilon g_n \\ 2\varepsilon q_n \\ 2\varepsilon p_n \end{pmatrix} + J_2\begin{pmatrix} 2\varepsilon u_2 \\ 2\varepsilon u_1 \\ 2\varepsilon u_4 \\ 2\varepsilon u_3 \\ 2\varepsilon u_6 \\ 2\varepsilon u_5 \end{pmatrix}k_n(t)x, \tag{4.5}$$

where the Hamiltonian operator J_2 is

$$J_2 = \frac{1}{\varepsilon}\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \end{pmatrix}.$$

To furnish Hamiltonian structures, we use the trace identity, and have

$$\langle V_2, \frac{\partial U_2}{\partial \lambda}\rangle = 2a + 2b,\ \langle V_2, \frac{\partial U_2}{\partial u_1}\rangle = 2\varepsilon f,\ \langle V_2, \frac{\partial U_2}{\partial u_2}\rangle = 2\varepsilon e,$$
$$\langle V_2, \frac{\partial U_2}{\partial u_3}\rangle = 2\varepsilon h,\ \langle V_2, \frac{\partial U_2}{\partial u_4}\rangle = 2\varepsilon g,\ \langle V_2, \frac{\partial U_2}{\partial u_5}\rangle = 2\varepsilon q,\ \langle V_2, \frac{\partial U_2}{\partial u_6}\rangle = 2\varepsilon p.$$

Substituting the above formulas into the trace identity and balancing coefficients of each power of λ gives rise to

$$\frac{\delta}{\delta u}\int (2a_{n+1} + 2b_{n+1})dx = (\gamma - n)\begin{pmatrix} 2\varepsilon f_n \\ 2\varepsilon e_n \\ 2\varepsilon h_n \\ 2\varepsilon g_n \\ 2\varepsilon q_n \\ 2\varepsilon p_n \end{pmatrix}.$$

Taking $n = 1$, gives $\gamma = 0$. Thus, we see

$$u_t = J_2\frac{\delta H^{(2)}_{n+1}}{\delta u} = J_2L_2\frac{\delta H^{(2)}_n}{\delta u} + J_2M_2k_n(t)x, \quad H^{(2)}_{n+1} = -2\int (\frac{a_{n+2} + b_{n+2}}{n+1})dx, \ \ n \geq 0,$$

where $M_2 = (2\varepsilon u_2, 2\varepsilon u_1, 2\varepsilon u_4, 2\varepsilon u_3, 2\varepsilon u_6, 2\varepsilon u_5)^{\mathrm{T}}$.

5. The nonisospectral integrable coupling hierarchy

In this section, we will construct the nonisospectral integrable coupling hierarchies associated with Lie algebra $\mathfrak{so}(5)$. We consider the nonisospectral problem

$$\begin{cases} \varphi_x = \bar{U}_2\varphi,\ \bar{U}_2 = \begin{pmatrix} U_3' & \varepsilon U_4' \\ U_4' & U_3' \end{pmatrix}, \\ \varphi_t = \bar{V}_2\varphi,\ \bar{V}_2 = \begin{pmatrix} V_3' & \varepsilon V_4' \\ V_4' & V_3' \end{pmatrix}, \\ \lambda_t = \sum_{i\geq 0} k_i(t)\lambda^{-i}, \end{cases}$$

where

$$U_3'=\begin{pmatrix}0&u_5&u_3&u_6&u_4\\u_6&\lambda&0&0&u_2\\u_4&0&\lambda&-u_2&0\\u_5&0&-u_1&-\lambda&0\\u_3&u_1&0&0&-\lambda\end{pmatrix},U_4'=\begin{pmatrix}0&u_5'&u_3'&u_6'&u_4'\\u_6'&0&0&0&u_2'\\u_4'&0&0&-u_2'&0\\u_5'&0&-u_1'&0&0\\u_3'&u_1'&0&0&0\end{pmatrix},$$

$$V_3'=\begin{pmatrix}0&p&g&q&h\\q&a&d&0&f\\h&c&b&-f&0\\p&0&-e&-a&-c\\g&e&0&-d&-b\end{pmatrix},V_4'=\begin{pmatrix}0&p'&g'&q'&h'\\q'&a'&d'&0&f'\\h'&c'&b'&-f'&0\\p'&0&-e'&-a'&-c'\\g'&e'&0&-d'&-b'\end{pmatrix},\tag{5.1}$$

where $u_1,u_2,\cdots,u_6,u_1',u_2',\cdots,u_6'$, and $a,b,c,d,e,f,g,h,p,q,a',b',c',d',e',f',g',h',p',q'$ are different from (2.2), (3.3), and (4.2).

We solve the stationary zero curvature equation by means of

$$\bar{V}_{2x}=\frac{\partial\bar{U}_2}{\partial\lambda}\lambda_t+[\bar{U}_2,\bar{V}_2]\tag{5.2}$$

which yields

$$\begin{cases}a_{ix}=-u_1f_i+u_2e_i-u_5q_i+u_6p_i-\varepsilon u_1'f_i'+\varepsilon u_2'e_i'-\varepsilon u_5'q_i'+\varepsilon u_6'p_i'+k_i(t),\\b_{ix}=-u_1f_i+u_2e_i-u_3h_i+u_4g_i-\varepsilon u_1'f_i'+\varepsilon u_2'e_i'-\varepsilon u_3'h_i'+\varepsilon u_4'g_i'+k_i(t),\\c_{ix}=u_4p_i-u_5h_i+\varepsilon u_4'p_i'-\varepsilon u_5'h_i',\\d_{ix}=-u_3q_i+u_6g_i-\varepsilon u_3'q_i'+\varepsilon u_6'g_i',\\e_{ix}=-2\lambda e_i+u_1a_i+u_1b_i+u_3p_i-u_5g_i+\varepsilon u_1'a_i'+\varepsilon u_1'b_i'+\varepsilon u_3'p_i'-\varepsilon u_5'g_i',\\f_{ix}=2\lambda f_i-u_2a_i-u_2b_i-u_4q_i+u_6h_i-\varepsilon u_2'a_i'-\varepsilon u_2'b_i'-\varepsilon u_4'q_i'+\varepsilon u_6'h_i',\\g_{ix}=-\lambda g_i+u_1q_i+u_3b_i+u_5d_i-u_6e_i+\varepsilon u_1'q_i'+\varepsilon u_3'b_i'+\varepsilon u_5'd_i'-\varepsilon u_6'e_i',\\h_{ix}=\lambda h_i-u_2p_i-u_4b_i+u_5f_i-u_6c_i-\varepsilon u_2'p_i'-\varepsilon u_4'b_i'+\varepsilon u_5'f_i'-\varepsilon u_6'c_i',\\p_{ix}=-\lambda p_i-u_1h_i+u_3c_i+u_4e_i+u_5a_i-\varepsilon u_1'h_i'+\varepsilon u_3'c_i'+\varepsilon u_4'e_i'+\varepsilon u_5'a_i',\\q_{ix}=\lambda q_i+u_2g_i-u_3f_i-u_4d_i-u_6a_i+\varepsilon u_2'g_i'-\varepsilon u_3'f_i'-\varepsilon u_4'd_i'-\varepsilon u_6'a_i',\end{cases}\tag{5.3}$$

and

$$\begin{cases}a_{ix}'=-u_1f_i'+u_2e_i'-u_5q_i'+u_6p_i'-u_1'f_i+u_2'e_i-u_5'q_i+u_6'p_i+k_i(t),\\b_{ix}'=-u_1f_i'+u_2e_i'-u_3h_i'+u_4g_i'-u_1'f_i+u_2'e_i-u_3'h_i+u_4'g_i+k_i(t),\\c_{ix}'=u_4p_i'-u_5h_i'+u_4'p_i-u_5'h_i,\\d_{ix}'=-u_3q_i'+u_6g_i'-u_3'q_i+u_6'g_i,\\e_{ix}'=-2\lambda e_i'+u_1a_i'+u_1b_i'+u_3p_i'-u_5g_i'+u_1'a_i+u_1'b_i+u_3'p_i-u_5'g_i,\\f_{ix}'=2\lambda f_i'-u_2a_i'-u_2b_i'-u_4q_i'+u_6h_i'-u_2'a_i-u_2'b_i-u_4'q_i+u_6'h_i,\\g_{ix}'=-\lambda g_i'+u_1q_i'+u_3b_i'+u_5d_i'-u_6e_i'+u_1'q_i+u_3'b_i+u_5'd_i-u_6'e_i,\\h_{ix}'=\lambda h_i'-u_2p_i'-u_4b_i'+u_5f_i'-u_6c_i'-u_2'p_i-u_4'b_i+u_5'f_i-u_6'c_i,\\p_{ix}'=-\lambda p_i'-u_1h_i'+u_3c_i'+u_4e_i'+u_5a_i'-u_1'h_i+u_3'c_i+u_4'e_i+u_5'a_i,\\q_{ix}'=\lambda q_i'+u_2g_i'-u_3f_i'-u_4d_i'-u_6a_i'+u_2'g_i-u_3'f_i-u_4'd_i-u_6'a_i.\end{cases}\tag{5.4}$$

By taking initial values

$$a_0=\alpha(t),\ b_0=\beta(t),\ c_0=d_0=e_0=f_0=g_0=h_0=p_0=q_0=k_0(t)=0,$$
$$a_0'=\alpha'(t),\ b_0'=\beta'(t),\ c_0'=d_0'=e_0'=f_0'=g_0'=h_0'=p_0'=q_0'=0,$$

one has

$$\begin{aligned}
a_1=&b_1=k_1(t)x,\ e_1=\frac{1}{2}u_1(\alpha+\beta)+\frac{\varepsilon}{2}u_1'(\alpha'+\beta'),\ f_1=\frac{1}{2}u_2(\alpha+\beta)+\frac{\varepsilon}{2}u_2'(\alpha'+\beta'),\\
g_1=&u_3\beta+\varepsilon u_3'\beta',\ h_1=u_4\beta+\varepsilon u_4'\beta',\ p_1=u_5\alpha+\varepsilon u_5'\alpha',\ q_1=u_6\alpha+\varepsilon u_6'\alpha',\\
c_1=&\partial^{-1}(u_4u_5+\varepsilon u_4'u_5')(\alpha-\beta)+\varepsilon\partial^{-1}(u_4u_5'+u_4'u_5)(\alpha'-\beta'),\\
d_1=&\partial^{-1}(u_3u_6+\varepsilon u_3'u_6')(\beta-\alpha)+\varepsilon\partial^{-1}(u_3u_6'+u_3'u_6)(\beta'-\alpha'),\\
a_1'=&b_1'=k_1(t)x,\ e_1'=\frac{1}{2}u_1'(\alpha+\beta)+\frac{1}{2}u_1(\alpha'+\beta'),\ f_1'=\frac{1}{2}u_2'(\alpha+\beta)+\frac{1}{2}u_2(\alpha'+\beta'),\\
g_1'=&u_3'\beta+u_3\beta',\ h_1'=u_4'\beta+u_4\beta',\ p_1'=u_5'\alpha+u_5\alpha',\ q_1'=u_6'\alpha+u_6\alpha',\\
c_1'=&\partial^{-1}(u_4u_5'+u_4'u_5)(\alpha-\beta)+\partial^{-1}(u_4u_5+\varepsilon u_4'u_5')(\alpha'-\beta'),\\
d_1'=&\partial^{-1}(u_3u_6'+u_3'u_6)(\beta-\alpha)+\partial^{-1}(u_3u_6+\varepsilon u_3'u_6')(\beta'-\alpha'),\\
&\cdots\cdots
\end{aligned}$$

where $\alpha(t),\ \beta(t)$ are integral constants.

From the nonisospectral zero curvature equation

$$\frac{\partial\bar{U}_2}{\partial u}u_t+\frac{\partial\bar{U}_2}{\partial\lambda}\lambda_t-\bar{V}_{2x}^{(n)}+[\bar{U}_2,\bar{V}_2^{(n)}]=0,$$

we can obtain the following integrable couplings

$$\bar{u}_{t_n}=\begin{pmatrix}\bar{u}_3\\ \bar{u}_4\end{pmatrix}_{t_n},\ \bar{u}_{3t_n}=\begin{pmatrix}u_1\\u_2\\u_3\\u_4\\u_5\\u_6\end{pmatrix}_{t_n}=\begin{pmatrix}-2e_{n+1}\\2f_{n+1}\\-2g_{n+1}\\2h_{n+1}\\-2p_{n+1}\\2q_{n+1}\end{pmatrix},\ \bar{u}_{4t_n}=\begin{pmatrix}u_1'\\u_2'\\u_3'\\u_4'\\u_5'\\u_6'\end{pmatrix}_{t_n}=\begin{pmatrix}-2e_{n+1}'\\2f_{n+1}'\\-2g_{n+1}'\\2h_{n+1}'\\-2p_{n+1}'\\2q_{n+1}'\end{pmatrix}.$$

To furnish Hamiltonian structures, we use the trace identity, and have

$$\begin{aligned}
&\langle V_3',\frac{\partial U_4'}{\partial\lambda}\rangle+\langle V_4',\frac{\partial U_3'}{\partial\lambda}\rangle=2(a'+b'),\ \langle V_3',\frac{\partial U_4'}{\partial u_1}\rangle+\langle V_4',\frac{\partial U_3'}{\partial u_1}\rangle=2f',\\
&\langle V_3',\frac{\partial U_4'}{\partial u_2}\rangle+\langle V_4',\frac{\partial U_3'}{\partial u_2}\rangle=2e',\ \langle V_3',\frac{\partial U_4'}{\partial u_3}\rangle+\langle V_4',\frac{\partial U_3'}{\partial u_3}\rangle=2h',\\
&\langle V_3',\frac{\partial U_4'}{\partial u_4}\rangle+\langle V_4',\frac{\partial U_3'}{\partial u_4}\rangle=2g',\ \langle V_3',\frac{\partial U_4'}{\partial u_5}\rangle+\langle V_4',\frac{\partial U_3'}{\partial u_5}\rangle=2q',\\
&\langle V_3',\frac{\partial U_4'}{\partial u_6}\rangle+\langle V_4',\frac{\partial U_3'}{\partial u_6}\rangle=2p',\ \langle V_3',\frac{\partial U_4'}{\partial u_1'}\rangle+\langle V_4',\frac{\partial U_3'}{\partial u_1'}\rangle=2f,
\end{aligned}$$

$$\langle V_3', \frac{\partial U_4'}{\partial u_2'}\rangle + \langle V_4', \frac{\partial U_3'}{\partial u_2'}\rangle = 2e,\ \langle V_3', \frac{\partial U_4'}{\partial u_3'}\rangle + \langle V_4', \frac{\partial U_3'}{\partial u_3'}\rangle = 2h,$$
$$\langle V_3', \frac{\partial U_4'}{\partial u_4'}\rangle + \langle V_4', \frac{\partial U_3'}{\partial u_4'}\rangle = 2g,\ \langle V_3', \frac{\partial U_4'}{\partial u_5'}\rangle + \langle V_4', \frac{\partial U_3'}{\partial u_5'}\rangle = 2q,$$
$$\langle V_3', \frac{\partial U_4'}{\partial u_6'}\rangle + \langle V_4', \frac{\partial U_3'}{\partial u_6'}\rangle = 2p.$$

Substituting the above formulas into the trace identity, and balancing coefficients of each power of λ, we give the first form

$$\frac{\delta}{\delta \bar{u}} \int 2(a_{n+1}' + b_{n+1}')dx = (\gamma - n)\begin{pmatrix} M_3 \\ M_4 \end{pmatrix}, \tag{5.5}$$

where

$$M_3 = \begin{pmatrix} 2f_n' \\ 2e_n' \\ 2h_n' \\ 2g_n' \\ 2q_n' \\ 2p_n' \end{pmatrix},\ M_4 = \begin{pmatrix} 2f_n \\ 2e_n \\ 2h_n \\ 2g_n \\ 2q_n \\ 2p_n \end{pmatrix}.$$

At the same time, we also have

$$\langle V_3', \frac{\partial U_3'}{\partial \lambda}\rangle + \varepsilon\langle V_4', \frac{\partial U_4'}{\partial \lambda}\rangle = 2(a+b),\ \langle V_3', \frac{\partial U_3'}{\partial u_1}\rangle + \varepsilon\langle V_4', \frac{\partial U_4'}{\partial u_1}\rangle = 2f,$$
$$\langle V_3', \frac{\partial U_3'}{\partial u_2}\rangle + \varepsilon\langle V_4', \frac{\partial U_4'}{\partial u_2}\rangle = 2e,\ \langle V_3', \frac{\partial U_3'}{\partial u_3}\rangle + \varepsilon\langle V_4', \frac{\partial U_4'}{\partial u_3}\rangle = 2h,$$
$$\langle V_3', \frac{\partial U_3'}{\partial u_4}\rangle + \varepsilon\langle V_4', \frac{\partial U_4'}{\partial u_4}\rangle = 2g,\ \langle V_3', \frac{\partial U_3'}{\partial u_5}\rangle + \varepsilon\langle V_4', \frac{\partial U_4'}{\partial u_5}\rangle = 2q,$$
$$\langle V_3', \frac{\partial U_3'}{\partial u_6}\rangle + \varepsilon\langle V_4', \frac{\partial U_4'}{\partial u_6}\rangle = 2p,\ \langle V_3', \frac{\partial U_3'}{\partial u_1'}\rangle + \varepsilon\langle V_4', \frac{\partial U_4'}{\partial u_1'}\rangle = 2\varepsilon f',$$
$$\langle V_3', \frac{\partial U_3'}{\partial u_2'}\rangle + \varepsilon\langle V_4', \frac{\partial U_4'}{\partial u_2'}\rangle = 2\varepsilon e',\ \langle V_3', \frac{\partial U_3'}{\partial u_3'}\rangle + \varepsilon\langle V_4', \frac{\partial U_4'}{\partial u_3'}\rangle = 2\varepsilon h',$$
$$\langle V_3', \frac{\partial U_3'}{\partial u_4'}\rangle + \varepsilon\langle V_4', \frac{\partial U_4'}{\partial u_4'}\rangle = 2\varepsilon g',\ \langle V_3', \frac{\partial U_3'}{\partial u_5'}\rangle + \varepsilon\langle V_4', \frac{\partial U_4'}{\partial u_5'}\rangle = 2\varepsilon q',$$
$$\langle V_3', \frac{\partial U_3'}{\partial u_6'}\rangle + \varepsilon\langle V_4', \frac{\partial U_4'}{\partial u_6'}\rangle = 2\varepsilon p'.$$

Substituting the above formulas into the trace identity, and balancing coefficients of each power of λ, we give the second form

$$\frac{\delta}{\delta u} \int 2(a_{n+1} + b_{n+1})dx = (\gamma - n)\begin{pmatrix} M_4 \\ \varepsilon M_3 \end{pmatrix},$$

where $M_3,\ M_4$ are defined as (5.5).

So, we can obtain the Hamiltonian structures of integrable couplings, that consists of the following two components. The first component has the form

$$\bar{u}_{t_n}=\begin{pmatrix}\bar{u}_3\\ \bar{u}_4\end{pmatrix}_{t_n}=\bar{J}_3\begin{pmatrix}K_3\\ K_4\end{pmatrix}=\bar{J}_3\frac{\delta\bar{H}_{3,m}}{\delta\bar{u}},\ K_3=\begin{pmatrix}2f'_{n+1}\\ 2e'_{n+1}\\ 2h'_{n+1}\\ 2g'_{n+1}\\ 2q'_{n+1}\\ 2p'_{n+1}\end{pmatrix},\ K_4=\begin{pmatrix}2f_{n+1}\\ 2e_{n+1}\\ 2h_{n+1}\\ 2g_{n+1}\\ 2q_{n+1}\\ 2p_{n+1}\end{pmatrix},$$

where $\bar{J}_3=\begin{pmatrix}O & J_2\\ J_2 & O\end{pmatrix}$, and J_2 is defined as (4.5). The second component has the form

$$\bar{u}_{t_n}=\begin{pmatrix}\bar{u}_3\\ \bar{u}_4\end{pmatrix}_{t_n}=\bar{J}_4\begin{pmatrix}K_4\\ \varepsilon K_3\end{pmatrix}=\bar{J}_4\frac{\delta\bar{H}_{4,m}}{\delta\bar{u}},\ \ \bar{J}_4=\begin{pmatrix}J_2 & O\\ O & \frac{1}{\varepsilon}J_2\end{pmatrix}.$$

From (5.3) and (5.4), we obtain the following recursion relations

$$\begin{pmatrix}-2e_{n+1}\\ 2f_{n+1}\\ -2g_{n+1}\\ 2h_{n+1}\\ -2p_{n+1}\\ 2q_{n+1}\\ -2e'_{n+1}\\ 2f'_{n+1}\\ -2g'_{n+1}\\ 2h'_{n+1}\\ -2p'_{n+1}\\ 2q'_{n+1}\end{pmatrix}=\bar{L}_2\begin{pmatrix}-2e_n\\ 2f_n\\ -2g_n\\ 2h_n\\ -2p_n\\ 2q_n\\ -2e'_n\\ 2f'_n\\ -2g'_n\\ 2h'_n\\ -2p'_n\\ 2q'_n\end{pmatrix}+\begin{pmatrix}2u_2+2u'_2\\ 2u_1+2u'_1\\ 2u_4+2u'_4\\ 2u_3+2u'_3\\ 2u_6+2u'_6\\ 2u_5+2u'_5\\ 2u_2+2\varepsilon u'_2\\ 2u_1+2\varepsilon u'_1\\ 2u_4+2\varepsilon u'_4\\ 2u_3+2\varepsilon u'_3\\ 2u_6+2\varepsilon u'_6\\ 2u_5+2\varepsilon u'_5\end{pmatrix}k_n(t)x,$$

where the recurrence operator $\bar{L}_2$ is defined as

$$\bar{L}_2=\begin{pmatrix}L'_3 & L'_4\\ \frac{1}{\varepsilon}L'_4 & L'_3\end{pmatrix},\ L'_3=\begin{pmatrix}l_{11}&l_{12}&l_{13}&l_{14}&l_{15}&l_{16}\\ l_{21}&l_{22}&l_{23}&l_{24}&l_{25}&l_{26}\\ l_{31}&l_{32}&l_{33}&l_{34}&l_{35}&l_{36}\\ l_{41}&l_{42}&l_{43}&l_{44}&l_{45}&l_{46}\\ l_{51}&l_{52}&l_{53}&l_{54}&l_{55}&l_{56}\\ l_{61}&l_{62}&l_{63}&l_{64}&l_{65}&l_{66}\end{pmatrix},\ L'_4=\begin{pmatrix}l'_{11}&l'_{12}&l'_{13}&l'_{14}&l'_{15}&l'_{16}\\ l'_{21}&l'_{22}&l'_{23}&l'_{24}&l'_{25}&l'_{26}\\ l'_{31}&l'_{32}&l'_{33}&l'_{34}&l'_{35}&l'_{36}\\ l'_{41}&l'_{42}&l'_{43}&l'_{44}&l'_{45}&l'_{46}\\ l'_{51}&l'_{52}&l'_{53}&l'_{54}&l'_{55}&l'_{56}\\ l'_{61}&l'_{62}&l'_{63}&l'_{64}&l'_{65}&l'_{66}\end{pmatrix},$$

and

$$\begin{aligned}&l_{11}=\frac{\partial}{2}-u_2\partial^{-1}u_1-\varepsilon u'_2\partial^{-1}u'_1,\ l_{12}=u_2\partial^{-1}u_2+\varepsilon u'_2\partial^{-1}u'_2,\ l_{13}=-\frac{1}{2}(u_2\partial^{-1}u_3+\varepsilon u'_2\partial^{-1}u'_3+u_6),\\ &l_{14}=\frac{1}{2}(u_2\partial^{-1}u_4+\varepsilon u'_2\partial^{-1}u'_4),\ l_{15}=-\frac{1}{2}(u_2\partial^{-1}u_5+\varepsilon u'_2\partial^{-1}u'_5-u_4),\ l_{16}=\frac{1}{2}(u_2\partial^{-1}u_6+\varepsilon u'_2\partial^{-1}u'_6),\end{aligned}$$

$$l'_{11}=-u_2\partial^{-1}u'_1-u'_2\partial^{-1}u_1,\ l'_{12}=u_2\partial^{-1}u'_2+u'_2\partial^{-1}u_2,\ l'_{13}=-\frac{1}{2}(u_2\partial^{-1}u'_3+u'_2\partial^{-1}u_3+u'_6),$$
$$l'_{14}=\frac{1}{2}(u_2\partial^{-1}u'_4+u'_2\partial^{-1}u_4),\ l'_{15}=-\frac{1}{2}(u_2\partial^{-1}u'_5+u'_2\partial^{-1}u_5-u'_4),\ l'_{16}=\frac{1}{2}(u_2\partial^{-1}u'_6+u'_2\partial^{-1}u_6),$$
$$l_{21}=-u_1\partial^{-1}u_1-\varepsilon u'_1\partial^{-1}u'_1,\ l_{22}=-\frac{\partial}{2}+u_1\partial^{-1}u_2+\varepsilon u'_1\partial^{-1}u'_2,\ l_{23}=-\frac{1}{2}(u_1\partial^{-1}u_3+\varepsilon u'_1\partial^{-1}u'_3),$$
$$l_{24}=\frac{1}{2}(u_1\partial^{-1}u_4+\varepsilon u'_1\partial^{-1}u'_4-u_5),l_{25}=-\frac{1}{2}(u_1\partial^{-1}u_5+\varepsilon u'_1\partial^{-1}u'_5),l_{26}=\frac{1}{2}(u_1\partial^{-1}u_6+\varepsilon u'_1\partial^{-1}u'_6+u_3),$$
$$l'_{21}=-u_1\partial^{-1}u'_1-u'_1\partial^{-1}u_1,\ l'_{22}=u_1\partial^{-1}u'_2+u'_1\partial^{-1}u_2,\ l'_{23}=-\frac{1}{2}(u_1\partial^{-1}u'_3+u'_1\partial^{-1}u_3),$$
$$l'_{24}=\frac{1}{2}(u_1\partial^{-1}u'_4+u'_1\partial^{-1}u_4-u'_5),l'_{25}=-\frac{1}{2}(u_1\partial^{-1}u'_5+u'_1\partial^{-1}u_5),l'_{26}=\frac{1}{2}(u_1\partial^{-1}u'_6+u'_1\partial^{-1}u_6+u'_3),$$
$$l_{31}=-u_4\partial^{-1}u_1-\varepsilon u'_4\partial^{-1}u'_1-u_5,\ l_{32}=u_4\partial^{-1}u_2+\varepsilon u'_4\partial^{-1}u'_2,\ l_{35}=l'_{35}=0,$$
$$l_{33}=\partial-u_4\partial^{-1}u_3-u_6\partial^{-1}u_5-\varepsilon u'_4\partial^{-1}u'_3-\varepsilon u'_6\partial^{-1}u'_5,\ l_{34}=u_4\partial^{-1}u_4+\varepsilon u'_4\partial^{-1}u'_4,$$
$$l_{36}=u_6\partial^{-1}u_4+\varepsilon u'_6\partial^{-1}u'_4+u_2,l'_{31}=-u_4\partial^{-1}u'_1-u'_4\partial^{-1}u_1-u'_5,l'_{32}=u_4\partial^{-1}u'_2+u'_4\partial^{-1}u_2,$$
$$l'_{33}=-u_4\partial^{-1}u'_3-u_6\partial^{-1}u'_5-u'_4\partial^{-1}u_3-u'_6\partial^{-1}u_5,\ l'_{34}=u_4\partial^{-1}u'_4+u'_4\partial^{-1}u_4,$$
$$l'_{36}=u_6\partial^{-1}u'_4+u'_6\partial^{-1}u_4+u'_2,l_{41}=-u_3\partial^{-1}u_1-\varepsilon u'_3\partial^{-1}u'_1,l_{42}=u_3\partial^{-1}u_2+\varepsilon u'_3\partial^{-1}u'_2-u_6,$$
$$l_{43}=-u_3\partial^{-1}u_3-\varepsilon u'_3\partial^{-1}u'_3,\ l_{44}=-\partial+u_3\partial^{-1}u_4+u_5\partial^{-1}u_6+\varepsilon u'_3\partial^{-1}u'_4+\varepsilon u'_5\partial^{-1}u'_6,$$
$$l_{45}=-u_5\partial^{-1}u_3+\varepsilon u'_5\partial^{-1}u'_3+u_1,\ l'_{41}=-u_3\partial^{-1}u'_1-u'_3\partial^{-1}u_1,\ l_{46}=l'_{46}=0,$$
$$l'_{42}=u_3\partial^{-1}u'_2+u'_3\partial^{-1}u_2-u'_6,\ l'_{43}=-u_3\partial^{-1}u'_3-u'_3\partial^{-1}u_3,\ l_{53}=l'_{53}=0,$$
$$l'_{44}=u_3\partial^{-1}u'_4+u_5\partial^{-1}u'_6+u'_3\partial^{-1}u_4+u'_5\partial^{-1}u_6,\ l'_{45}=-u_5\partial^{-1}u'_3+u'_5\partial^{-1}u_3+u'_1,$$
$$l_{51}=-u_6\partial^{-1}u_1-\varepsilon u'_6\partial^{-1}u'_1+u_3,l_{52}=u_6\partial^{-1}u_2+\varepsilon u_6\partial^{-1}u'_2,l_{54}=u_4\partial^{-1}u_6+\varepsilon u'_4\partial^{-1}u'_6-u_2,$$
$$l_{55}=\partial-u_4\partial^{-1}u_3-u_6\partial^{-1}u_5-\varepsilon u'_4\partial^{-1}u'_3-\varepsilon u'_6\partial^{-1}u'_5,\ l_{56}=u_6\partial^{-1}u_6+\varepsilon u'_6\partial^{-1}u'_6,$$
$$l'_{51}=-u_6\partial^{-1}u'_1-u'_6\partial^{-1}u_1+u'_3,\ l'_{52}=u_6\partial^{-1}u'_2+u'_6\partial^{-1}u_2,\ l'_{54}=u_4\partial^{-1}u'_6+u'_4\partial^{-1}u_6-u'_2,$$
$$l'_{55}=-u_4\partial^{-1}u'_3-u_6\partial^{-1}u'_5-u'_4\partial^{-1}u_3-u'_6\partial^{-1}u_5,\ l'_{56}=u_6\partial^{-1}u'_6+u'_6\partial^{-1}u_6,$$
$$l_{61}=-u_5\partial^{-1}u_1-\varepsilon u'_5\partial^{-1}u'_1,l_{62}=u_5\partial^{-1}u_2+\varepsilon u_5\partial^{-1}u'_2+u_4,l_{63}=-u_3\partial^{-1}u_5-\varepsilon u'_3\partial^{-1}u'_5-u_1,$$
$$l_{64}=0,\ l_{65}=-u_5\partial^{-1}u_5-\varepsilon u'_5\partial^{-1}u'_5,\ l_{66}=-\partial+u_3\partial^{-1}u_4+u_5\partial^{-1}u_6+\varepsilon u'_3\partial^{-1}u'_4+\varepsilon u'_5\partial^{-1}u'_6,$$
$$l'_{61}=-u_5\partial^{-1}u'_1-u'_5\partial^{-1}u_1,\ l'_{62}=u_5\partial^{-1}u'_2+u'_5\partial^{-1}u_2+u'_4,\ l'_{63}=-u_3\partial^{-1}u'_5-u'_3\partial^{-1}u_5-u'_1,$$
$$l'_{64}=0,\ l'_{65}=-u_5\partial^{-1}u'_5-u'_5\partial^{-1}u_5,\ l'_{66}=u_3\partial^{-1}u'_4+u_5\partial^{-1}u'_6+u'_3\partial^{-1}u_4+u'_5\partial^{-1}u_6.$$

6. The nonisospectral integrable hierarchy associated with Lie algebra $\mathfrak{so}(3,2)$

6.1. The generalized Lie algebra $G\mathfrak{so}(3,2)$

Lie algebra $\mathfrak{so}(3,2)$ is defined as [31]

$$\mathfrak{so}(3,2)=\{x\in\mathfrak{gl}(5,\mathbb{R}\mid x=-I_{32}x^{\mathrm{T}}I_{32},tr(x)=0\},$$

where $I_{32}=\begin{pmatrix}-I_3&0\\0&I_2\end{pmatrix}$.

So, elements of Lie algebra $\mathfrak{so}(3,2)$ have the form

$$\begin{pmatrix} X_1 & X_2 \\ X_2^{\mathrm{T}} & X_3 \end{pmatrix},$$

where $X_1^{\mathrm{T}} = -X_1,\ X_3^{\mathrm{T}} = -X_3$, and X_1, X_2, X_3 are 3×3, 3×2, and 2×2 real matrices, respectively. It is easy to get the elements of Lie algebra $\mathfrak{so}(3,2)$ with the form

$$\begin{pmatrix} 0 & \lambda_1 & \lambda_2 & \lambda_5 & \lambda_6 \\ -\lambda_1 & 0 & \lambda_3 & \lambda_7 & \lambda_8 \\ -\lambda_2 & -\lambda_3 & 0 & \lambda_9 & \lambda_{10} \\ \lambda_5 & \lambda_7 & \lambda_9 & 0 & \lambda_4 \\ \lambda_6 & \lambda_8 & \lambda_{10} & -\lambda_4 & 0 \end{pmatrix}.$$

We can obtain the bases of Lie algebra $\mathfrak{so}(3,2)$ as

$$\begin{aligned} &E_1 = e_{12} - e_{21},\ E_2 = e_{13} - e_{31},\ E_3 = e_{23} - e_{32},\ E_4 = e_{45} - e_{54},\ E_5 = e_{14} + e_{41}, \\ &E_6 = e_{15} + e_{51},\ E_7 = e_{24} + e_{42},\ E_8 = e_{25} + e_{52},\ E_9 = e_{34} + e_{43},\ E_{10} = e_{35} + e_{53}, \end{aligned} \tag{6.1}$$

where e_{ij} is a 5×5 matrix with 1 in the (i,j)-th position and zero elsewhere. Next, we consider the generalized Lie algebra $G\mathfrak{so}(3,2)$, that admits a basis set as follows:

$$\begin{aligned} &E_1' = E_7 + E_{10},\ E_2' = -E_7 + E_{10},\ E_3' = -E_1 + E_5,\ E_4' = \varepsilon E_1 + \varepsilon E_5,\ E_5' = \varepsilon E_2 - \varepsilon E_6, \\ &E_6' = -E_2 - E_6,\ E_7' = \varepsilon E_3 - \varepsilon E_4 - \varepsilon E_8 + \varepsilon E_9,\ E_8' = -E_3 + E_4 - E_8 + E_9, \\ &E_9' = -\varepsilon E_3 - \varepsilon E_4 + \varepsilon E_8 + \varepsilon E_9,\ E_{10}' = E_3 + E_4 + E_8 + E_9, \end{aligned} \tag{6.2}$$

where $E_i, i = 1, 2, \cdots, 10$ are defined as (6.1), and satisfy the following commutative relations:

$$\begin{aligned} &[E_1', E_2'] = 0,\ [E_1', E_3'] = E_3',\ [E_1', E_4'] = -E_4',\ [E_1', E_5'] = E_5',\ [E_1', E_6'] = -E_6', \\ &[E_1', E_7'] = 2E_7',\ [E_1', E_8'] = -2E_8',\ [E_1', E_9'] = [E_1', E_{10}'] = 0,\ [E_2', E_3'] = -E_3', \\ &[E_2', E_4'] = E_4',\ [E_2', E_5'] = E_5', [E_2', E_6'] = -E_6', [E_2', E_7'] = [E_2', E_8'] = 0, [E_2', E_9'] = 2E_9', \\ &[E_2', E_{10}'] = -2E_{10}',\ [E_3', E_4'] = \varepsilon E_1' - \varepsilon E_2',\ [E_3', E_5'] = E_7',\ [E_3', E_6'] = -E_{10}',\ [E_3', E_7'] = 0, \\ &[E_3', E_8'] = -2E_6',\ [E_3', E_9'] = 2E_5',\ [E_3', E_{10}'] = 0,\ [E_4', E_5'] = \varepsilon E_9',\ [E_4', E_6'] = -\varepsilon E_8', \\ &[E_4', E_7'] = 2\varepsilon E_5',\ [E_4', E_8'] = [E_4', E_9'] = 0,\ [E_4', E_{10}'] = -2\varepsilon E_6',\ [E_5', E_6'] = \varepsilon E_1' + \varepsilon E_2', \\ &[E_5', E_7'] = 0,\ [E_5', E_8'] = 2E_4',\ [E_5', E_9'] = 0,\ [E_5', E_{10}'] = 2\varepsilon E_3',\ [E_6', E_7'] = -2\varepsilon E_3', \\ &[E_6', E_8'] = [E_6', E_{10}'] = 0,\ [E_6', E_9'] = -2E_4',\ [E_7', E_8'] = 4\varepsilon E_1',\ [E_7', E_9'] = [E_7', E_{10}'] = 0, \\ &[E_8', E_9'] = [E_8', E_{10}'] = 0,\ [E_9', E_{10}'] = 4\varepsilon E_2'. \end{aligned}$$

Consider the nonisospectral problem

$$\begin{cases} \varphi_x = U_3\varphi, \\ \varphi_t = V_3\varphi, \\ \lambda_t = \sum_{i\geq 0} k_i(t)\lambda^{-i}, \end{cases}$$

where

$$U_3 = \lambda(E_1' + E_2') + \sqrt{2}u_1E_5' + \frac{\sqrt{2}}{2}u_2E_6' + u_3E_7' + \frac{u_4}{4}E_8' + \frac{u_5}{2}E_9' + \frac{u_6}{2}E_{10}', \tag{6.3}$$
$$V_3 = aE_1' + bE_2' + \sqrt{2}cE_3' + \frac{\sqrt{2}}{2}dE_4' + \sqrt{2}eE_5' + \frac{\sqrt{2}}{2}fE_6' + gE_7' + \frac{h}{4}E_8' + \frac{p}{2}E_9' + \frac{q}{2}E_{10}',$$

where $u_1, u_2, \cdots, u_6$ and $a, b, c, d, e, f, g, h, p, q$ are the same as (2.2).

By solving the stationary zero curvature representation

$$V_{3x} = \frac{\partial U_3}{\partial \lambda}\lambda_t + [U_3, V_3], \tag{6.4}$$

we can obtain the same equation as (2.4). This means that integrable hierarchies obtained from the linear nonisospectral problems $\begin{cases} \varphi_x = U_3\varphi, \\ \varphi_t = V_3\varphi, \\ \lambda_t = \sum_{i\geq 0} k_i(t)\lambda^{-i} \end{cases}$ are the same as (2.5).

If we consider the spectral matrix U_1 and time spectral matrix V_1 in Lie algebra $G\mathfrak{sp}(4)$, and choose spectral matrices U_3 and V_3 in Lie algebra $G\mathfrak{so}(3,2)$, then from zero curvature equations $V_{1x} = \frac{\partial U_1}{\partial \lambda}\lambda_t + [U_1, V_1]$ and $V_{3x} = \frac{\partial U_3}{\partial \lambda}\lambda_t + [U_3, V_3]$, we can obtain the same nonisospectral integrable hierarchies. So, based on $\mathfrak{sp}(4) \cong \mathfrak{so}(3,2)$, as long as we select the corresponding spectral problem between Lie algebras $G\mathfrak{sp}(4)$ and $G\mathfrak{so}(3,2)$, and we can obtain the same hierarchies.

6.2. *The nonisospectral integrable couplings*

In this section, we will construct the nonisospectral integrable coupling hierarchies associated with Lie algebra $\mathfrak{so}(3,2)$. We consider the nonisospectral problem

$$\begin{cases} \varphi_x = \bar{U}_3\varphi, \\ \varphi_t = \bar{V}_3\varphi, \\ \lambda_t = \sum_{i\geq 0} k_i(t)\lambda^{-i}, \end{cases}$$

where

$$\bar{U}_3 = \begin{pmatrix} U_5' & \varepsilon U_6' \\ U_6' & U_5' \end{pmatrix}, \quad \bar{V}_3 = \begin{pmatrix} V_5' & \varepsilon V_6' \\ V_6' & V_5' \end{pmatrix}, \tag{6.5}$$

and

$$\begin{aligned} U_5' &= 2\lambda E_{10} + \sqrt{2}(u_1 - \frac{u_2}{2})E_2 - \sqrt{2}(u_1 + \frac{u_2}{2})E_6 + (u_3 - \frac{u_4}{4} - \frac{u_5}{2} + \frac{u_6}{2})E_3 \\ &\quad +(-u_3 + \frac{u_4}{4} - \frac{u_5}{2} + \frac{u_6}{2})E_4 + (-u_3 - \frac{u_4}{4} + \frac{u_5}{2} + \frac{u_6}{2})E_8 + (u_3 + \frac{u_4}{4} + \frac{u_5}{2} + \frac{u_6}{2})E_9, \\ U_6' &= \sqrt{2}(u_1' - \frac{u_2'}{2})E_2 - \sqrt{2}(u_1' + \frac{u_2'}{2})E_6 + (u_3' - \frac{u_4'}{4} - \frac{u_5'}{2} + \frac{u_6'}{2})E_3 \\ &\quad +(-u_3' + \frac{u_4'}{4} - \frac{u_5'}{2} + \frac{u_6'}{2})E_4 + (-u_3' - \frac{u_4'}{4} + \frac{u_5'}{2} + \frac{u_6'}{2})E_8 + (u_3' + \frac{u_4'}{4} + \frac{u_5'}{2} + \frac{u_6'}{2})E_9, \\ V_5' &= \sqrt{2}(\frac{d}{2} - c)E_1 + \sqrt{2}(e - \frac{f}{2})E_2 + (g - \frac{h}{4} - \frac{p}{2} + \frac{q}{2})E_3 + (-g + \frac{h}{4} - \frac{p}{2} + \frac{q}{2})E_4 \end{aligned}$$

$$
\begin{aligned}
& +\sqrt{2}(c+\frac{d}{2})E_5-\sqrt{2}(e+\frac{f}{2})E_6+(a-b)E_7+(-g-\frac{h}{4}+\frac{p}{2}+\frac{q}{2})E_8 \\
& +(g+\frac{h}{4}+\frac{p}{2}+\frac{q}{2})E_9+(a+b)E_{10}, \\
V_6' \;=\; & \sqrt{2}(\frac{d'}{2}-c')E_1+\sqrt{2}(e'-\frac{f'}{2})E_2+(g'-\frac{h'}{4}-\frac{p'}{2}+\frac{q'}{2})E_3+(-g'+\frac{h'}{4}-\frac{p'}{2}+\frac{q'}{2})E_4 \\
& +\sqrt{2}(c'+\frac{d'}{2})E_5-\sqrt{2}(e'+\frac{f'}{2})E_6+(a'-b')E_7+(-g'-\frac{h'}{4}+\frac{p'}{2}+\frac{q'}{2})E_8 \\
& +(g'+\frac{h'}{4}+\frac{p'}{2}+\frac{q'}{2})E_9+(a'+b')E_{10},
\end{aligned}
$$

where $u_1, u_2, \cdots, u_6, u_1', u_2', \cdots, u_6'$, and $a, b, c, d, e, f, g, h, p, q, a', b', c', d', e', f', g', h', p', q'$ are the same as (3.3).

Solving the stationary zero curvature equation

$$
\bar{V}_{3x} = \frac{\partial \bar{U}_3}{\partial \lambda}\lambda_t + [\bar{U}_3, \bar{V}_3], \tag{6.6}
$$

we can obtain the same equations as (3.5) and (3.6). This means that integrable coupling systems obtained from the linear nonisospectral problems $\begin{cases} \varphi_x = \bar{U}_3\varphi, \\ \varphi_t = \bar{V}_3\varphi, \\ \lambda_t = \sum_{i\geq 0} k_i(t)\lambda^{-i} \end{cases}$ are the same as (3.7).

If we consider the spectral matrix $\bar{U}_1$ and time spectral matrix $\bar{V}_1$ in Lie algebra $\mathfrak{sp}(4)$, and choose spectral matrices $\bar{U}_3$ and $\bar{V}_3$ in Lie algebra $\mathfrak{so}(3,2)$, then from zero curvature equations $\bar{V}_{1x} = \frac{\partial \bar{U}_1}{\partial \lambda}\lambda_t + [\bar{U}_1, \bar{V}_1]$ and $\bar{V}_{3x} = \frac{\partial \bar{U}_3}{\partial \lambda}\lambda_t + [\bar{U}_3, \bar{V}_3]$, we can obtain the same nonisospectral integrable coupling systems. So, based on $\mathfrak{sp}(4) \cong \mathfrak{so}(3,2)$, as long as we select the corresponding spectral problem between Lie algebras $\mathfrak{sp}(4)$ and $\mathfrak{so}(3,2)$, we can obtain the same integrable couplings.

7. Conclusions

By adding any real number ε, we construct the generalized Lie algebras $G\mathfrak{sp}(4)$, $G\mathfrak{so}(5)$, and $G\mathfrak{so}(3,2)$. Based on these three Lie algebras, we introduce the spectral parameter $\lambda_t = \sum_{i\geq 0} k_i(t)\lambda^{-i}$, and obtain the nonisospectral integrable hierarchies and their Hamiltonian structures of these three Lie algebras. Additionally, based on the semi-direct sum decomposition of Lie algebras, we derive the integrable coupling systems associated with Lie algebras $\mathfrak{sp}(4)$, $\mathfrak{so}(5)$, and $\mathfrak{so}(3,2)$. At the same time, we use $\mathfrak{sp}(4) \cong \mathfrak{so}(3,2)$, and further discuss the relationship between the integrable couplings systems corresponding to these two Lie algebras.

Author contributions

Baiying He: Conceived of the study, Completed the computations, Writing-original draft; Siyu Gao: Writing review and editing, Writing-original draft. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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