Mathematics

AIMS Mathematics, 9(10): 27293–27307.
DOI: 10.3934/math.20241326
Received: 10 April 2024
Revised: 02 August 2024
Accepted: 20 August 2024
Published: 20 September 2024

https://www.aimspress.com/journal/Math

Research article

A newfangled isolated entropic measure in probability spaces and its applications to queueing theory

Vikramjeet Singh[1], Sunil Kumar Sharma[2,*], Om Parkash[3], Retneer Sharma[4] and Shivam Bhardwaj[5]

[1] Department of Mathematics, I. K. Gujral Punjab Technical University, Amritsar Campus, Punjab, India
[2] Department of Information Systems, College of Computer and Information Sciences, Majmaah University, Majmaah 11952, Saudi Arabia
[3] Department of Mathematics, Graphic Era Deemed to be University, Dehradun, Uttarakhand, India
[4] Research Scholar, I.K. Gujral Punjab Technical University, Jalandhar, India
[5] Master of Science in IT & Management, Mccombs School of Business, University of Texas at Austin, USA

* **Correspondence:** Email: s.sharma@mu.edu.sa.

Abstract: It is well established that a diverse range of entropic measures, while remarkably adaptable, must inevitably be complemented by innovative approaches to enhance their effectiveness across various domains. These measures play a crucial role in fields like communication and coding theory, driving researchers to develop numerous new information measures that can be applied in a wide array of disciplines. This paper introduces a pioneering isolated entropic measure and its solicitations to queueing theory the study of dissimilarities of uncertainty. By creating the newly developed discrete entropy, we have articulated an optimization principle where the space capacity is predetermined and solitary evidence accessible is around the mean size. Additionally, we have conveyed the solicitations of "maximum entropy principle" to maximize the entropy probability distributions.

Keywords: probability dissemination; queueing theory; entropic measure; continuity property; expansibility; concavity property; traffic intensity
Mathematics Subject Classification: 94A

1. Introduction

The maximum entropy principle, familiarized by Jaynes [7], is an extensively espoused methodology for allocating values to probability distributions grounded on fractional information. In most concrete circumstances, the accessible information is delivered by a set of well-maintained quantities connected with the probability dissemination, which is why this principle is principally secondhand in statistical thermodynamics. The principle warrants the exclusivity and uniformity of probability obligations resultant from numerous approaches, evidently delineating the elasticity in exhausting dissimilar procedures of previous data. Simply put, the principle advocates for maximum uncertainty, choosing a distribution that makes the fewest assumptions beyond the given prior data.

This principle is crucial for the effective goal-oriented modeling of evaluation systems. For instance, Zhang et al. [26] applied a selection measure and concluded that using the principle of maximum entropy allows for the rational distribution of limited resources, thereby enhancing overall economic efficiency. Other significant contributions to this field have been made by researchers such as Kapur [10]; Kapur, Baciu, and Keasavan [9]; Jizba and Korbel [8]; Xin et al. [14]; and Contreras-Reyes [3].

Shannon [20] established the furthermost significant perception of entropy accompanied $P=\left(p_1, p_2, ..., p_n\right)$ by means of subsequent quantitative measure:

$$H(P) = -\sum_{i=1}^{n} p_i \log p_i . \tag{1.1}$$

After Shannon's [22] entropy, numerous entropic measures were discussed and explored. Moreover, by understanding the submission region of entropic measures in probability universes, numerous researchers projected their own measures of entropy to extend their applications. Additionally, to supplement the literature, the subsequent information speculative entropic measures for the prolongation of research have been investigated. One such measure is

$$M_\alpha(P) = \frac{1}{1-\alpha}\left(\prod_{i=1}^{n} p_i^{\,p_i(\alpha-1)} - 1\right),\ \alpha > 0\ ,\ \alpha \neq 1 . \tag{1.2}$$

Equation (1.2) represents Shannon's entropy measure, a fundamental measure of uncertainty in information theory.

$$_\alpha A\left(P\right) = \frac{\sum_{i=1}^{n} p_i \alpha^{\log_D p_i} - 1}{1-\alpha},\ \alpha > 1. \tag{1.3}$$

Equation (1.3) denotes the discrete parametric entropic measure introducing a parameter for the flexible weighting of probabilities.

$$N_\alpha\left(P\right) = \frac{\alpha^{\sum_{i=1}^{n} p_i \ln p_i} - 1}{1-\alpha},\ \alpha > 1 . \tag{1.4}$$

Equation (1.4) represents the exponential measure of uncertainty.

$$J_{\alpha}^{\beta}(p)=\frac{\alpha^{\frac{1}{\beta-1}\ln\left(\sum_{i=1}^{n}p_i^{\beta}\right)}-1}{1-\alpha},\ \alpha>1,\ 0<\beta<1. \tag{1.5}$$

Equation (1.5) represents another exponential measure of uncertainty, where α and β are parameters indicating the order of entropy model, p_i represents the probabilities, and n denotes the number of observations.

Wan and Guo [24] provided significant insights into the long-standing use of information perception in exploring a vast array of physical phenomena. They elucidated the connections between observation, sudden changes in Shannon entropy, information conversion, and robust arrangement collaboration, grounded on the up-to-date intention of energy configurations.

Following the identification of this measure, researchers have conducted extensive investigations to develop various entropic measures for diverse applications. These measures have been effectively employed in disciplines such as statistics. Recently, Elgawad et al. [1] made notable advancements related to prevailing information measures, extensively studying various distinguished distributions. They highlighted the impact of their study on random order statistics, which serve as a critical classification method for ordering bivariate data within this comprehensive framework. Their work also extended the application of these measures in reliability theory.

Mondaini and Neto [16] emphasized that Khinchin-Shannon generalized inequalities, derived from entropy measures, are instrumental in exploring the synergy of probability distributions in physical systems. Segura et al. [2] discussed entropy in the context of the second law of thermodynamics, explaining that entropy increases in isolated systems, leading to greater disorder over time. In closed systems, entropy tends to increase without energy exchange with the environment, whereas in open systems, entropy can decrease due to energy exchange with the surroundings. This dissimilarity highlights that more closed structures are disposed to cumulative entropy, whereas more open structures can acclimate enhanced to fluctuating surroundings.

Rastegin [18] recently highlighted that complementary relationships among the various descriptions of a probability distribution are central to information theory. He focused on providing lower and upper bounds for the entropic function, which is essential in applied settings where certain probability constraints are known. Rastegin proposed a family of polynomials for estimating Shannon entropy from below, leading to more accurate evaluations in specific areas and generating uncertainty and inevitability relationships for positive operator-valued processes.

There is a gigantic accumulation of measures but still possibility ascends to articulate amplification in their versions. Additionally, there materializes a distinguishable strong association of interacting entropy and chi-square distribution. To carry this out, Parkash, Sharma and Singh [17] demarcated an innovative measure by the consequent manifestation:

$$H_{\alpha,\beta}(P)=\frac{1}{\beta-\alpha}\sum_{i=1}^{n}\left[1-p_i^{(\beta-\alpha)p_i}\right];\ \alpha\neq\beta,\ \beta-\alpha>0. \tag{1.6}$$

In recent studies, Zhang and Shi [25] provided remarkable insights into Shannon's entropy, emphasizing its foundational role and its indispensable application in machine learning procedures. Despite its importance, Shannon's entropy is solitary demarcated for disseminations with rapidly decomposing tails on a countable alphabet. Due to the unbounded nature of Shannon's entropy over the universal category of disseminations, this limitation restricts its full potential utility. Zhang and Shi [25] conducted an extensive study on the asymptotic properties of the plug-in estimator for countable symbols, demonstrating that these properties hold without requiring any assumptions about the underlying distribution.

Additional studies of isolated entropy measures have been completed by numerous investigators, including Kapur [11], Elgawad et al. [1], Sholehkerdar et al. [21], Fowler and Heckman [4], Gao and Deng [5], Li et al. [12], Suguro [22], and Legchenkova et al. [13].

One notable application area of entropy measures is in queueing theory, which has remained a vital and active field of research in recent years. The applications of queueing theory provides a theoretical foundation for designing and analysing numerous stochastic systems across various disciplines. These systems typically operate under random conditions, and congestion often arises from the discrepancy between variable service capacity and random service demand. Due to increasing complexities in these stochastic systems, traditional queueing theory, which was once highly effective in modelling telephone systems, has become inadequate.

While delivering solicitations in modest birth-death progression, if $p_n(t)$) is the possibility of *n* individuals at a particular time *t*, n_0 *to* be individuals at a specific time *t=0*, and we designate the probability-generating function by

$$\phi(s,t)=\sum_{n=o}^{\infty} p_n(t)s^n , \tag{1.7}$$

then, we acquire the subsequent outcome of the above appearance:

$$\phi(s,t)=\left[\frac{(\lambda-\mu)s+\mu(x-1)}{(\lambda-\lambda x)s+(\lambda x-1)}\right]^{n_0} , \lambda\neq\mu , \tag{1.8}$$

$$=\left[\frac{\lambda t-(\lambda t-1)s}{1-\lambda t-\lambda ts}\right]^{n_0} , \lambda=\mu , \tag{1.9}$$

where

$$x=\exp(\lambda-\mu)t . \tag{1.10}$$

By constructing an enlargement $\varphi(s,t)$, an individual can ascertain $p_n(t)$. The following, where λ and μ are the arrival and service rates, respectively, is prevalent in the discipline of queueing theory:

$$p_n=(1-\rho)\rho^n ,\ n=0,1,2.3,....;\ \rho=\frac{\lambda}{\mu} . \tag{1.11}$$

Our entropic measure is designed to analyze and quantify uncertainties within queueing systems, providing a novel approach to understanding how uncertainty behaves under different conditions. Specifically, the model allows us to do the following:

(1) Analyze steady-state processes: In steady-state queueing systems, the measure helps quantify how uncertainty increases with traffic intensity, offering a clearer picture of system behaviour as it approaches capacity.
(2) Examine non-steady-state processes: For non-steady-state processes, the measure captures the dynamic nature of uncertainty, which initially increases, reaches a maximum, and then decreases over time. This provides valuable insights into transient behaviours in queueing systems.
(3) Apply the maximum entropy principle: The measure uses the maximum entropy principle to derive the probability distribution that maximizes entropy under given constraints, such as system capacity and mean queue size. This application is particularly useful for predicting queue lengths and wait times in various scenarios. This paper aims to generate comprehensive knowledge of a new discrete entropic measure and its applications to queueing theory, facilitating the study of uncertainty variations.

In Section 2, we have wrought a newfangled entropic measure for the isolated probability distributions. Section 3 provides transactions with the learning of dissimilarities of uncertainty in the steady state and non-steady state queueing processes by commissioning the newly developed generalized entropic measure. Section 4 delivers an arrangement of solicitations of the maximum entropy principle by employing our specific discovered measure.

2. An innovative discrete entropic measure in possibility universes

In the discipline of information theory, the Rényi's [19] entropy is a quantity that generalizes innumerable designs of entropy including Shannon's [20] entropy and can be looked for the most universal approach to quantify information while maintaining additivity property for independent events. On the other hand, Tsallis [23] entropy quiet identical to Havrada-Charvat's [6] entropy also participates through fundamental responsibility in deriving Tsallis distribution in statistical physics. The connection between the two (i.e., Rényi's [19] entropy and Tsallis [23] entropy) has been delivered by Mariz [15].

We now construct the consequent quantifiable isolated parametric entropic measure identified by

$$H_\gamma(P) = -\sum_{i=1}^{n} p_i \log p_i + \frac{1}{\gamma-1}\sum_{i=1}^{n}\log\left[1+(\gamma-1)p_i\right] - \frac{\log\gamma}{\gamma-1};\ \gamma>1. \tag{2.1}$$

We observe the subsequent communication in the limiting appearance:

$$\lim_{\gamma\to 1} H_\gamma(P) = -\sum_{i=1}^{n} p_i \log p_i \,.$$

We examine that measure (2.1) is a generalization of the well-accepted Shannon's [22] entropy measure.

2.1. *Numerical illustration*

Let $p_1 = 0.1$, $p_2 = 0.2$, $p_3 = 0.3$ and $p_4 = 0.4$. Thus, we have $\sum_{i=1}^{4} p_i = 1$, which confirms the soundness of the above probability distribution.

The value of Shannon entropy measure is $H(X) = 1.279854225833667$.

By taking $\gamma = 2, 3, 4, 5$, we have calculated the value of entropy measure shown in Eq (2.1) for different values of γ as shown in Table 1.

Table 1. Entropy values for different γ in comparison with Shannon entropy measure.

p_1	p_2	p_3	p_4	γ	$H_\gamma(P)$
0.1	0.2	0.3	0.4	2	1.4631752829607056
0.1	0.2	0.3	0.4	3	1.5188401252811239
0.1	0.2	0.3	0.4	4	1.5386491522103345
0.1	0.2	0.3	0.4	5	1.544551674453902

Comparing it with the Shannon entropy measure, we observe that the entropy values of our measure are greater than the Shannon entropy measure. This clearly indicates that the entropy is getting maximum in accordance with the principle of maximum entropy.

Next, to authenticate that the discrete measure (2.1) is a considerable entropy measure, we make transactions with its central properties as follows:

(i) For n degenerate disconnected allocations

$$\Delta_1 = (1, 0, 0, \ldots, 0), \Delta_2 = (0, 1, 0, \ldots, 0), \ldots, \Delta_n = (0, 0, 0, \ldots, 1),$$

we acquire

$$H_\gamma(P) = 0.$$

Meanwhile, entropy stretches the smallest assessment intended for degenerate disseminations and the smallest value is 0, so it is obligatory to have $H_\gamma(P) \geq 0$. This corroborates the non-negativity property of the projected measure.

(ii) $H_\gamma(P)$ conforms symmetry.

(iii) $H_\gamma(P)$ monitors continuity.

(iv) Concavity: To establish the concavity property of the proposed measure, we carry on with subsequent straightforward computations: We have

$$\frac{\partial H_\gamma(P)}{\partial p_i} = -1 - \log p_i + \frac{1}{1 + \gamma p_i}.$$

Also,

$$\frac{\partial^2 H_\gamma(P)}{\partial p_i^2} = -\frac{1}{p_i} - \frac{\alpha}{\{1 + \gamma p_i\}^2} < 0.$$

Thus, $H_\gamma(P)$ monitors a continuous function.

In addition, with the support of mathematical statistics displayed in the succeeding Table 1 through $n = 2$ and $\alpha = 2$, we have made manageable the anticipated isolated entropy measure $H_\gamma(P)$ counter to p by way of exposition in the succeeding Figure 1. This table is unquestionably accommodating for enlightening the information towards the concavity belongings of the anticipated measure.

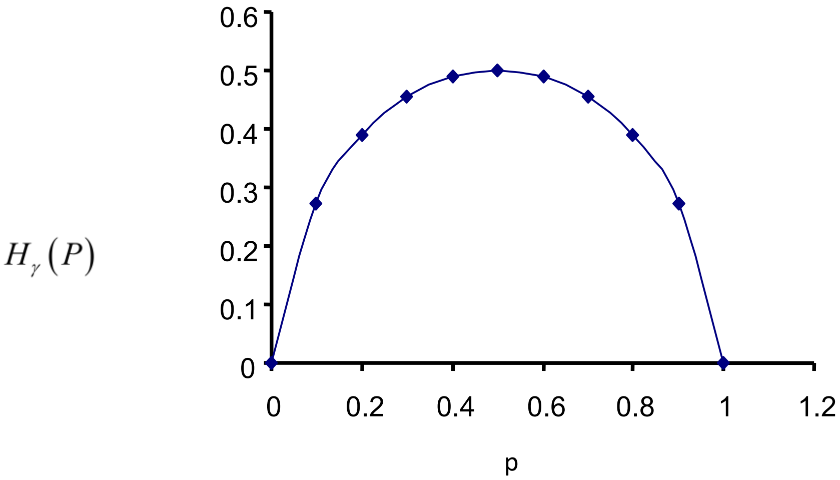

Figure 1. Concavity of $H_\gamma(P)$ with respect to P.

Figure 1 unequivocally demonstrates that the anticipated measure (2.1) maintains concavity, establishing its credibility as an entropy measure. Consequently, it can be deemed an acceptable entropy measure in terms of concavity property for various values of n and α.

(v) Maximization: We apply Lagrange's technique to exploit (2.1) subject to the acknowledged constraint conveyed by

$$\sum_{i=1}^{n} p_i = 1.$$

The necessary Lagrange's function is communicated through

$$L \equiv H_\gamma(P) - \lambda\left(\sum_{i=1}^{n} p_i - 1\right). \tag{2.2}$$

Upon differentiating (2.2) we obtain the following mathematical communications:

$$\frac{1}{1+\alpha p_1} - \log p_1 = \frac{1}{1+\alpha p_2} - \log p_2 = \ldots = \frac{1}{1+\alpha p_n} - \log p_n,$$

which is conceivable solitary if $p_1 = p_2 = \ldots = p_n$.

Further, by exhausting the prospective restriction $\sum_{i=1}^{n} p_i = 1$, we acquire the following mathematical expression in relationships of probabilities:

$$p_i = \frac{1}{n},\ \forall i = 1,2,\ldots,n.$$

Accordingly, we perceive that absolute assessment of $H_\gamma(P)$ ascends for the uniform distribution U=(1/n, 1/n,...,1/n), and this consequence is principally attractive one on the justification of the statement that there survives merely uniform distribution at which any entropy function ought to possess supreme value.

(vi) The maximum assessment $f(n)$ of the entropy function is prearranged by subsequent accurate communication:

$$f(n) = \log n + \frac{1}{\gamma-1}\left[n\log\{n+(\gamma-1)\} - n\log n\right] - \frac{\log\gamma}{\gamma-1}.$$

Thus, we arrive at the final result, f'(n) > 0, indicating that f(n) is an increasing function of n. This is a significant finding, as it supports the notion that the maximum value of entropy should consistently increase, aligning with the principle of maximum entropy.

3. Solicitations of isolated entropic measure in the theory of queues

At this stage, we have accomplished the knowledge of discrepancies of entropy in unlike states in the succeeding belongings.

3.1. Case I: Discrepancies of the entropic measure in steady state

To study such dissimilarities, we have reflected upon the entropy measure previously shaped in (2.1). This measure is prearranged by the following manifestation:

$$H_\gamma(P) = -\sum_{i=1}^{n} p_i \log p_i + \frac{1}{\gamma-1}\sum_{i=1}^{n}\log\left[1+(\gamma-1)p_i\right] - \frac{\log\gamma}{\gamma-1};\ \gamma>1.$$

In steady state, we make the following modification in the above measure as

$$S^\gamma(\lambda,\mu) = -\sum_{n=o}^{\infty} p_n \log p_n + \frac{1}{\gamma-1}\sum_{i=1}^{n}\log\left[1+(\gamma-1)p_i\right] - \frac{\log\gamma}{\gamma-1}. \tag{3.1}$$

Taking $p_n = (1-\rho)\rho^n$, we obtain the following exterior:

$$S^\gamma(\lambda,\mu) = -\sum_{n=o}^{\infty}(1-\rho)\rho^n\left\{\log((1-\rho)+\log\rho^n\right\} + \frac{1}{\gamma-1}\sum_{n=0}^{\infty}\log\left[1+(\gamma-1)(1-\rho)\rho^n\right] - \frac{\log\gamma}{\gamma-1}$$

$$= -(1-\rho)\log(1-\rho)\sum_{n=o}^{\infty}\rho^n - (1-\rho)\log\rho\sum_{n=o}^{\infty}n\rho^n + \frac{1}{\gamma-1}\sum_{n=0}^{\infty}\log\left[1+(\gamma-1)(1-\rho)\rho^n\right] - \frac{\log\gamma}{\gamma-1}$$

$$= -\frac{1}{(1-\rho)}\left[(1-\rho)\log(1-\rho)+\rho\log\rho\right] + \frac{1}{\alpha}\sum_{n=0}^{\infty}\log\left[1+\alpha(1-\rho)\rho^n\right] - \frac{\log(1+\alpha)}{\alpha}. \tag{3.2}$$

Indeed, we consider the second and third positions of (3.3) as

$$\frac{1}{\gamma-1}\sum_{n=0}^{\infty}\log\left[1+(\gamma-1)(1-\rho)\rho^{n}\right]-\frac{\log\gamma}{\gamma-1}=\frac{1}{\gamma-1}\begin{bmatrix}\log\left\{1+(\gamma-1)(1-\rho)\right\}+\log\left\{1+(\gamma-1)(1-\rho)\rho\right\}\\ \\ +\log\left\{1+(\gamma-1)(1-\rho)\rho^{2}+\ldots\right\}+\ldots\end{bmatrix}-\frac{\log\gamma}{\gamma-1}.$$

Taking the limit as $\gamma \rightarrow 1$, the above calculation gives the following:

$$lim_{\gamma\to 0}\frac{1}{\gamma-1}\sum_{n=0}^{\infty}\log\left[1+(\gamma-1)(1-\rho)\rho^{n}\right]-\frac{\log\gamma-1}{\gamma-1}$$

$$=\lim_{\gamma\to 0}\frac{1}{\gamma-1}\begin{bmatrix}\log\left\{1+(\gamma-1)(1-\rho)\right\}+\log\left\{1+(\gamma-1)(1-\rho)\rho\right\}\\ \\ +\log\left\{1+(\gamma-1)(1-\rho)\rho^{2}\}\right\}+\ldots\end{bmatrix}-\lim_{\gamma\to 0}\frac{\log\gamma}{\gamma-1}$$

$$=\left[(1-\rho)+(1-\rho)\rho+(1-\rho)\rho^{2}+(1-\rho)\rho^{3}+\ldots\right]-1$$

$$=(1-\rho)\left[1+\rho+\rho^{2}+\rho^{3}+\ldots\right]-1=0.$$

Thus, Eq (3.3) delivers the following limiting expression:

$$\lim_{\gamma\to 0}S^{\gamma}\left(\lambda,\mu\right)=-\frac{1}{(1-\rho)}\left[(1-\rho)\log(1-\rho)+\rho\log\rho\right]. \tag{3.3}$$

Upon differentiation, (3.4) explores the following consequence:

$$\lim_{\gamma\to 0}\frac{\partial}{\partial\rho}S^{\gamma}\left(\lambda,\mu\right)=-\frac{\log\rho}{(1-\rho)^{2}}>0,$$

which indicates that while dealing with the steady-state process, the uncertainty escalates from 0 to ∞ as ρ escalates from 0 to 1. Consequently, in the present case, we identify that the uncertainty increases if the traffic intensity increases.

3.2. Case II: Disparities of entropy in the non-steady state

Kapur [8] has revealed that Eqs (3.2) and (3.3) deliver the expressions for probabilities. Thus, we have the following manifestation for the possibility of n individuals at some period t:

$$p_{n}(t)=\begin{cases}\dfrac{\left(\lambda t\right)^{n-1}}{\left(1+\lambda t\right)^{n+1}},\ n\geq 1,\\ \dfrac{\lambda t}{1+\lambda t},\ n=0.\end{cases} \tag{3.4}$$

Now, we study the dissimilar discrepancies by captivating into contemplation the probabilistic entropic measure (3.1).

In the non-steady state, Eq (3.1) can be rewritten as

$$S^{\gamma}\left(\lambda,\mu\right)=-\sum_{n=o}^{\infty}p_n(t)\log p_n(t)+\frac{1}{\gamma-1}\sum_{n=0}^{\infty}\log\left[1+(\gamma-1)p_n(t)\right]-\frac{\log\gamma}{\gamma-1}. \tag{3.5}$$

To ascertain the elucidation, let us contemplate the first term of (3.5) as

$$-\sum_{n=o}^{\infty}p_n(t)\log p_n(t)$$

$$=-p_0(t)\log p_0(t)-\sum_{n=1}^{\infty}p_n(t)\log p_n(t)$$

$$=-p_0(t)\log p_0(t)-\left\{p_1(t)\log p_1(t)+p_2(t)\log p_2(t)+p_3(t)\log p_3(t)+...\right\}$$

$$=-\frac{\lambda t}{1+\lambda t}\log\frac{\lambda t}{1+\lambda t}-\left\{\frac{1}{\{1+\lambda t\}^2}\log\frac{1}{\{1+\lambda t\}^2}+\frac{1}{\{1+\lambda t\}^3}\log\frac{1}{\{1+\lambda t\}^3}+\frac{1}{\{1+\lambda t\}^4}\log\frac{1}{\{1+\lambda t\}^4}+...\right\}$$

$$=\frac{2\left[\{1+\lambda t\}\log\{1+\lambda t\}-\lambda t\log\lambda t\right]}{\{1+\lambda t\}}.$$

Thus, Eq (3.5) becomes

$$S^{\gamma}\left(\lambda,\mu\right)=\frac{2\left[\{1+\lambda t\}\log\{1+\lambda t\}-\lambda t\log\lambda t\right]}{\{1+\lambda t\}}+\frac{1}{\gamma-1}\sum_{n=0}^{\infty}\log\left[1+(\gamma-1)p_i\right]-\frac{\log\gamma}{\gamma-1}. \tag{3.6}$$

Now, the captivating limit as $\gamma\to 1$, the second term of the above Eq (3.6) stretches the subsequent limiting manifestation:

$$Lt_{\gamma\to 1}\frac{1}{\gamma-1}\sum_{n=0}^{\infty}\log\left[1+(\gamma-1)p_i\right]=\frac{\lambda t}{1+\lambda t}+\frac{1}{\{1+\lambda t\}^2}+\frac{\lambda t}{\{1+\lambda t\}^3}+\frac{\lambda^2t^2}{\{1+\lambda t\}^4}+...$$

$$=\frac{\lambda t}{1+\lambda t}+\frac{1}{\{1+\lambda t\}^2}\left[1+\frac{\lambda t}{1+\lambda t}+\frac{\lambda^2t^2}{\{1+\lambda t\}^2}+...\right]=1. \tag{3.7}$$

Likewise, the entrancing limit as $\gamma\to 1$, the third term of the above Eq (3.7) stretches $Lt_{\gamma\to 1}\frac{\log\gamma}{\gamma-1}=1$.

Consequently, in the limiting situation, (3.7) gives us the following exterior:

$$Lt_{\gamma\to 1}S^{\gamma}\left(\lambda,\mu\right)=\frac{2\left[\{1+\lambda t\}\log\{1+\lambda t\}-\lambda t\log\lambda t\right]}{\{1+\lambda t\}}. \tag{3.8}$$

Now,

$$\frac{d}{d(\lambda t)}S^{\gamma}(\lambda,t)=-\frac{2\log\lambda t}{((1-\lambda t)^2}\begin{cases}>0 & if\ \lambda t<1,\\ <0 & if\ \lambda t>1.\end{cases}$$

This manifestation suggests that improbability increases if $\lambda t < 1$ and declines if $\lambda t \geq 1$.

Also, from Eq (3.8), the maximum uncertainty occurs when $\lambda t = 1$, and in this case we obtain the following logarithmic exterior:

$$Max\ S^{\gamma}(\lambda,t) = 2\log 2 .$$

Further, once $t = 0$, the improbability is 0, and once $t \to \infty$, we obtain

$$\underset{t\to\infty}{Lt}\ S^{\gamma}(\lambda,t) = 0.$$

Consequently, in this circumstance, we analyze that improbability instigates with assessment 0 at time $t = 0$ and varnishes through assessment 0 as time $t \to \infty$, and sandwiched between, it undertakes extreme assessment at $t=\frac{1}{\lambda}$.

4. Solicitations of maximum entropy principle to the persuasion of queueing theory

To intricate this maximum entropy standard in the persuasion of queueing theory, we contemplate the succeeding measure.

4.1. Model: Optimization attitude exhausting parametric measure once the space capacity is limited and the mere evidence accessible is around the mean size of the arrangement

In this fragment, to deliver advancement headed for the optimization principle, we employ the entropy measure (3.1) previously familiarized in the above subdivision. Accordingly, our problem converts to capitalize on (3.1) under the succeeding set of surroundings:

$$\sum_{i=1}^{n} p_i = 1 \tag{4.1}$$

and

$$\sum_{i=1}^{n} i p_i = m . \tag{4.2}$$

The conforming Lagrangian function is prearranged by the consequent appearance:

$$L = -\sum_{i=1}^{n} p_i \log p_i + \frac{1}{\gamma-1}\sum_{i=1}^{n}\log\left[1+(\gamma-1)p_i\right] - \frac{\log\gamma}{\gamma-1} - \lambda\left(\sum_{i=1}^{n} p_i - 1\right) - \mu\left(\sum_{i=1}^{n} i p_i - m\right).$$

Now, in regulating cases as $\gamma \to 1$, $\frac{\partial L}{\partial p_i} = 0$ stretches the consequent communication:

$$p_i = e^{-\lambda} e^{-i\mu} = ab^i . \tag{4.3}$$

where $a = e^{-(1+\lambda)}$, $b = e^{-\mu}$.

The subsequent set of equalities will determine the parameters a and b.

$$a\sum_{i=1}^{n} b^i = 1 \text{ and } a\sum_{i=1}^{n} ib^i = m \ . \tag{4.4}$$

Now,

$$a\sum_{i=1}^{n} b^i = a\left[b + b^2 + b^3 + b^4 + ... + b^n\right] = ab\left\{\frac{1-b^n}{1-b}\right\}.$$

Also,

$$a\sum_{i=1}^{n} ib^i = a\left[b + 2b^2 + 3b^3 + 4b^4 + ... + nb^n\right] = ab\left[\frac{1-b^n}{(1-b)^2} - \frac{nb^n}{1-b}\right].$$

Engaging these standards the equivalences (4.4) obtain the following:

$$ab\frac{1-b^n}{1-b} = 1 \text{ and } \frac{1}{1-b} - \frac{nb^n}{1-b^n} = m \ . \tag{4.5}$$

Through identified standards m and n, equivalences (4.5) stretch standards of a and b and henceforth equivalence (4.3) decide a mandatory set of possibilities. Accordingly, we witness that exploiting entropy likelihood distribution is a “geometric distribution”.

The exceeding technique has been embodied through the assistance of succeeding mathematical sketch.

4.2. Numerical illustration

We maximize (3.1) satisfying the constraints (4.1) and (4.2) for $n = 11$ and for diverse values of m predominantly, for $n = 11$ and $m = 1.5$, (4.5) and (4.3) provide the subsequent probabilities:

$$p_1 = 0.5843, \quad p_2 = 0.1265, \quad p_3 = 0.0709, \quad p_4 = 0.0493, \quad p_5 = 0.0377, \quad p_6 = 0.0306,$$

$$p_7 = 0.0257, \quad p_8 = 0.0223, \quad p_9 = 0.0195, \quad p_{10} = 0.0174, \quad p_{11} = 0.0157 .$$

We have $\sum_{i=1}^{11} p_i = 1$, which confirms the soundness of the above probability distribution.

The above- modus operandi is made continual for diverse standards of m when $n = 11$. The significance of functioning has been uncovered in Table 2.

Table 2. Probabilities for different values of m fixed $n=11$.

m	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}	p_{11}
1.5	0.5843	0.1265	0.0709	0.0493	0.0377	0.0306	0.0257	0.0223	0.0195	0.0174	0.0157
2.5	0.3579	0.1646	0.1068	0.0791	0.0628	0.0521	0.0445	0.0388	0.0344	0.0309	0.0289
3.5	0.2039	0.1504	0.1192	0.0987	0.0843	0.0735	0.0653	0.0585	0.0531	0.0486	0.0448
4.5	0.1175	0.1103	0.1040	0.0983	0.0933	0.0887	0.0845	0.0807	0.0773	0.0742	0.0712
5.5	0.0000	0.0000	0.1511	0.1371	0.1253	0.1155	0.1071	0.0998	0.0934	0.0879	0.08210
6.5	0.0000	0.0000	0.0000	0.0000	0.2093	0.1765	0.1525	0.1343	0.1110	0.1084	0.0989
7.5	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.32811	0.2299	0.1769	0.1438	0.1212
8.5	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.6372	0.2257	0.1372
9.5	0.0000	0.0000	0.00000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.5000	0.5000

5. Conclusions

In the literature of entropic measures, there are many well approved representations with their individual rewards and limitations but every model may fit in every condition. This statement suggests and subsequently induces the researchers to outline numerous newfangled measures from applicability argument of understanding. With the growth of advanced entropy, we have reflected learning of dissimilarities of improbability and witnessed that improbability continually increases in the steady state while in the non-steady state, improbability originally increases and collects supreme value and then reduces reduction and obtains its minima. The significance of our discoveries lies in their inherent authenticity, making our conclusions both compelling and credible.

We have also employed the maximum entropy principle using our own entropy measure and concluded that such maximizing entropy probability distributions can be originated with the assistance of other entropic measures. Inspecting the solicitation extents, the present work can be made to produce multiplicity of isolated and uninterrupted innovative information measures for their communication with coding theory.

Author contributions

Vikramjeet Singh, Sunil Sharma and Om Parkash: Conceptualization, Formal analysis, Validation, Methodology, Data interpretation, Visualization, Resources, Writing–original draft, Writing–review and editing, Supervision; Retneer Sharma and Shivam Bhardwaj: Formal analysis, Investigation, Validation, Data interpretation, Visualization. All authors have read and agreed to the published version of the article.

Acknowledgments

The author extends the appreciation to the Deanship of Postgraduate Studies and Scientific Research at Majmaah University for funding this research work through the project number (ICR-2024-1260).

Conflict of interest

The authors affirm the absence of any potential conflicts of interest concerning the research, authorship, and publication of this paper.

References

1. M. A. Abd Elgawad, H. M. Barakat, S. W. Xiong, S. A. Alyami, Information measures for generalized order statistics and their concomitants under general framework from Huang-Kotz FGM bivariate distribution, *Entropy*, **23** (2021), 1–17. https://doi.org/10.3390/e23030335
2. E. Abad-Segura, M. D. González-Zamar, M. Squillante, Examining the research on business information-entropy correlation in the accounting process of organizations, *Entropy*, **23** (2021), 1–25. https://doi.org/10.3390/e23111493
3. J. E. Contreras-Reyes, Lerch distribution based on maximum nonsymmetric entropy principle: Application to Conway's game of life cellular automaton, *Chaos Solitons Fract.*, **151** (2021), 111272. https://doi.org/10.1016/j.chaos.2021.111272
4. R. Fowler, J. J. Heckman, Misanthropic entropy and renormalization as a communication channel, *Int. J. Modern Phys. A*, **37** (2022), 2250109. https://doi.org/10.1142/s0217751x22501093
5. X. Z. Gao, Y. Deng, The pseudo-Pascal triangle of maximum Deng entropy, *Int. J. Comput. Commun. Control*, **15** (2020), 1–10. https://doi.org/10.15837/ijccc.2020.1.3735
6. J. Havrada, F. Charvat, Quantification methods of classification process. Concept of structural α-entropy, *Kybernetika*, **3** (1967), 30–35.
7. E. T. Jaynes, Information theory and statistical mechanics, *Phys. Rev.*, **106** (1957), 620. https://doi.org/10.1103/physrev.106.620
8. P. Jizba, J. Korbel, Maximum entropy principle in statistical inference: case for non-Shannonian entropies, *Phys. Rev. Lett.*, **122** (2019), 120601. https://doi.org/10.1103/physrevlett.122.120601
9. J. N. Kapur, G. Baciu, H. K. Kesavan, The MinMax information measure, *Int. J. Syst. Sci.*, **26** (1995), 1–12. https://doi.org/10.1080/00207729508929020
10. J. N. Kapur, Review of maximum-entropy models in science and engineering, *Biometrics*, **48** (1992), 333–334. https://doi.org/10.2307/2532770
11. J. N. Kapur, Review of measures of information and their applications, *Biometrics*, **52** (1996), 379. https://doi.org/10.2307/2533186
12. Y. X. Li, B. Geng, S. B. Jiao, Dispersion entropy-based Lempel-Ziv complexity: a new metric for signal analysis, *Chaos Solitons Fract.*, **161** (2022), 112400. https://doi.org/10.1016/j.chaos.2022.112400
13. I. Legchenkova, M. Frenkel, N. Shvalb, S. Shoval, O. V. Gendelman, E. Bormashenko, From chaos to ordering: New studies in the Shannon entropy of 2D patterns, *Entropy*, **24** (2022), 1–16. https://doi.org/10.3390/e24060802
14. X. Liu, X. Y. Wang, J. Xie, B. T. Li, Construction of probability box model based on maximum entropy principle and corresponding hybrid reliability analysis approach, *Struct. Multidiscip. Optim.*, **61** (2019), 599–617. https://doi.org/10.1007/s00158-019-02382-9
15. A. M. Mariz, On the irreversible nature of the Tsallis and Rényi entropies, *Phys. Lett. A*, **165** (1992), 409–411. https://doi.org/10.1016/0375-9601(92)90339-N

16. R. P. Mondaini, S. C. de Albuquerque Neto, Alternative entropy measures and generalized Khinchin-Shannon inequalities, *Entropy*, **23** (2021), 1–12. https://doi.org/10.3390/e23121618
17. O. Parkash, V. Singh, R. Sharma, A new discrete information model and its applications for the study of contingency tables, *J. Discrete Math. Sci. Cryptogr.*, **25** (2022), 785–792. https://doi.org/10.1080/09720529.2021.2014135
18. A. Rastegin, Estimating the Shannon entropy and (un)certainty relations for design-structured POVMs, *SIAM J. Appl. Math.*, **82** (2022), 1001–1019. https://doi.org/10.1137/21m1408105
19. A. Renyi, On measures of entropy and information, In: *Proceedings 4th Berkeley symposium on mathematical statistics and probability*, **1** (1961), 547–561.
20. C. E. Shannon, A mathematical theory of communication, *Bell Syst. Tech. J.*, **27** (1948), 379–423. https://doi.org/10.1002/j.1538-7305.1948.tb01338.x
21. A. Sholehkerdar, J. Tavakoli, Z. Liu, Theoretical analysis of Tsallis entropy-based quality measure for weighted averaging image fusion, *Inform. Fusion*, **58** (2020), 69–81. https://doi.org/10.1016/j.inffus.2019.12.010
22. T. Suguro, Shannon's inequality for the Rényi entropy and an application to the uncertainty principle, *J. Funct. Anal.*, **283** (2022), 109566. https://doi.org/10.1016/j.jfa.2022.109566
23. C. Tsallis, Possible generalization of Boltzmann-Gibbs statistics, *J. Statist. Phys.*, **52** (1988), 479–487. https://doi.org/10.1007/BF01016429
24. J. J. Wan, N. Guo, Shannon entropy in configuration space for Ni-like isoelectronic sequence, *Entropy*, **22** (2020), 1–23. https://doi.org/10.3390/e22010033
25. J. L. Zhang, J. Y. Shi, Asymptotic normality for plug-in estimators of generalized Shannon's entropy, *Entropy*, **24** (2022), 1–10. https://doi.org/10.3390/e24050683
26. Z. Zhang, S. Cheng, H. S. Xu, R. J. Pan, M. Kuang, Research on airport leading goods selection based on maximum entropy principle, In: *Proceedings of the 2018 international workshop on education reform and social sciences (ERSS 2018)*, 2019. https://doi.org/10.2991/erss-18.2019.97