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Research article

Double composed metric-like spaces via some fixed point theorems

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Abstract: The manuscript introduces the concept of a double-composed metric-like space, which is an extension of the notion of a double-composed metric space. In this new space, the self-distance is not necessarily zero, but if the distance metric equals zero, it must be for identical points of distance. Furthermore, this paper presents several results related to this novel concept in the literature, with a particular focus on Hardy–Rogers type contractions. It establishes fixed point theorems accompanied by some illustrative examples that elucidate the findings. Finally, this research provides an application to nonlinear integral equation to substantiate our theorems.

Keywords: fixed point; controlled metric-type spaces; double-controlled metric-type spaces; double-composed metric spaces; double-composed metric-like spaces
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1. Introduction

Fixed-point theorems are one of the most intriguing aspects of nonlinear analysis with wide-ranging applications, and they have grown in popularity among researchers. Initially, Banach [1] established the existence and uniqueness of a fixed point in 1922 through an empirical concept known as the "Banach contractive principle" in the metric spaces. It was the beginning point in the area, leading to further expansion of his theorem through the generalization of metric spaces or the refinement of contractions. Significant generalizations of metric space include b-metric spaces, introduced by Bakhtin [2] and Czerwik [3]. Harandi, Amini [4] extended the concept of partial metric space by defining a metric-like space. Additionally, Hitzler et al. [5] introduced the notion of dislocated metric spaces. Whereas, the most comprehensive generalization, the b-metric-like space, was introduced by Alghamdi et al. [6]. Numerous fixed-point theorems have focused on b-metric spaces (see; for extended b-metric spaces [7, 8], b-rectangular metric spaces [9] and b-metric-like spaces [10–12]). Recently, several generalizations of b-metric spaces and b-metric-like spaces have been presented, such as extended b-metric spaces introduced by Kamran et al. [13] and controlled metric-type spaces initiated by Mlaiki et al. [14]. Subsequently, in 2018, Abdeljawad et al. [15] redefined the notion under the name of double-controlled metric-type spaces. In 2020, Mlaiki [16] presented a generalization of double-controlled metric-type spaces as known by double-controlled metric-like spaces. In 2022, Karami et al. [17] proposed the concept of the fascinating generalized controlled metric-type spaces and named it an expanded b-metric spaces. In 2023, Ayoobi et al. [18, 19] proposed a new generalization of the previous types of metric spaces called double-composed metric spaces, which depend on two composed functions with triangular inequality. For more details, see [20–22].

This paper is derived from [16, 18, 21] and serves as a generalization of all various types of the previously mentioned metric spaces. It introduces a new class, known as double-composed metric-like spaces (for short, DCML-space). The primary goal is to present fixed-point results involving Banach, Kannan, Reich, and Hardy–Roger-type contractions, concentrate on the results within DCML-spaces, along with examples. Additionally, it explores certain relationships involving convergence of sequences. Finally, the paper introduces an application of a nonlinear integral equation, which supports our fixed-point theorems within these new spaces.

2. Preliminaries

This section gives definitions of controlled metric-type spaces according to Nabil Mlaiki et al. [14].

Definition 2.1. [14] Assume Γ is a nonempty set via $\omega : \Gamma \times \Gamma \to [1, \infty)$. The mapping $d_\omega : \Gamma \times \Gamma \to [0, \infty)$ is said to be a controlled metric type on Γ if for all $a, b, c \in \Gamma$ satisfied:

(C1) $d_\omega(a, b) = 0$ if and only if $a = b$,
(C2) $d_\omega(a, b) = d_\omega(b, a)$,
(C3) $d_\omega(a, b) \leq \omega(a, c)d_\omega(a, c) + \omega(c, b)d_\omega(c, b)$.

The pair (Γ, d_ω) is called a controlled metric type space. Obviously, when taking $\omega(a, c) = \omega(c, b)$, we obtain that extended b-metric space according to Kamran et al. [13], if $\omega(a, c) = \omega(b, c) = s$, we go to the b-metric space desired by Czerwik [3].

Abdeljawad et al. [15] presented an extension of controlled metric-type spaces, referred to as double-controlled type-metric spaces. Furthermore, Mlaiki et al. [16] given double-controlled metric-like spaces (DCMLS), which is a generalized to double-controlled metric-type space (for short, DCMTS).

Definition 2.2. [16] Let Γ be a nonempty set and $\omega_1, \omega_2 : \Gamma \times \Gamma \to [1, \infty)$. A function $\rho : \Gamma \times \Gamma \to [0, \infty)$ is a double-controlled metric-like if for all $a, b, c \in \Gamma$, satisfied:

(D1) $\rho(a, b) = 0$ implies $a = b$,
(D2) $\rho(a, b) = \rho(b, a)$,
(D3) $\rho(a, b) \leq \omega_1(a, c)\rho(a, c) + \omega_2(c, b)\rho(c, b)$.

The pair (Γ, ρ) is said to be a double-controlled-metric-like space (for short, DCMLS). Clearly, every DCMTS is a DCMLS, but the converse is not necessarily true (e.g., [15, 20]).

Next, Ayoob et al. [18] introduced the below-mentioned several generalizations of DCMTS, as known in that double-composed-metric space.

Definition 2.3. [18] Assume Γ is a nonempty set, and $\theta, \vartheta : [0,\infty) \to [0,\infty)$ be a non-constant function. A mapping $\rho_c : \Gamma \times \Gamma \to [0,\infty)$ is said to be a double-composed metric if for all $a, b, c \in \Gamma$, it satisfies the following conditions:

(P1) $\rho_c(a,b) = 0$ if and only if $a = b$,
(P2) $\rho_c(a,b) = \rho_c(b,a)$,
(P3) $\rho_c(a,b) \leq \theta(\rho_c(a,c)) + \vartheta(\rho_c(c,b))$.

Then pair (Γ, ρ_c) is said to be a double-composed metric space (briefly, DCMS). For more details, see [18, 21].

Now, we present our generalization of the DCMS as follows.

Definition 2.4. Let Γ be a nonempty set. A mapping $L : \Gamma\times\Gamma \to [0,\infty)$ is said to be a double-composed metric-like, if for each $a, b, c \in \Gamma$, there exists two non-constant functions $f, g : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(L1) $L(a,b) = 0$, then $a = b$,
(L2) $L(a,b) = L(b,a)$,
(L3) $L(a,b) \leq f(L(a,c)) + g(L(c,b))$.

The pair (Γ, L) is called a double-composed metric-like space (DCML-space).

Notice that every DCMS is a DCML-space. But the converse is not necessarily always true.

Example 1. Suppose $\Gamma = \{0, 1, 2\}$, and let $L : \Gamma \times \Gamma \to [0,\infty)$ be defined by $L(0,0) = L(1,1) = 0, L(2,2) = \frac{1}{2}, L(0,1) = 1, L(0,2) = 4, L(1,2) = \frac{5}{4}$.

Define two functions $f, g : [0,\infty) \to [0,\infty)$ by $f(t) = 2t$ and $g(t) = 2\sqrt{t}$. Clearly, (Γ, L) is a DCML-space. In other words, we see that, $L(2,2) = \frac{1}{2} \neq 0$. Thus (Γ, L) is not a DCMS.

Example 2. Let (Γ, ρ) be a double-controlled metric-like space with two controlled functions $\omega_1, \omega_2 : \Gamma \times \Gamma \to [1,\infty)$, and let $L(a,b) = \sinh(\rho(a,b))$. We demonstrate that L is a DCML-space with two functions, $f(t) = \sinh(2\omega_1(a,c)t)$ and $g(t) = \sinh(2\omega_2(c,b)t), t \in [0,\infty)$.

First, we demonstrate (L1): Since $\sinh(0) = 0$ and by property (D1), we have $L(a,b) = 0$, then $a = b$. (L2) is evident. Since $\sinh(t)$ is an increasing function, for all $a, b \geq 0$, $\sinh(a+b) \leq \sinh(2max\{a,b\}) \leq \sinh(2a) + \sinh(2b)$. Therefore, for each $a, b, c \in \Gamma$, we obtain:

$$\begin{aligned} L(a,b) = \sinh\rho(a,b) &\leq \sinh\left(\omega_1(a,c)\rho(a,c) + \omega_2(c,b)\rho(c,b)\right) \\ &\leq \sinh\left(\omega_1(a,c)\sinh(\rho(a,c)) + \omega_2(c,b)\sinh(\rho(c,b))\right) \\ &\leq \sinh(2\omega_1(a,c)L(a,c)) + \sinh(2\omega_2(c,b)L(c,b)) \\ &= f(L(a,c)) + g(L(c,b)) \end{aligned}$$

Thus, condition (L3) of Definition 2.4 holds, and (Γ, L) is DCML-space. Moreover, note that if $a = b$, it implies that $\rho(a,a) \neq 0$ in general, so that $L(a,a) \neq 0$, which distinguishes this space from DCMS.

In the following, we proposed the topology of the DCML-space.

Definition 2.5. Let (Γ, L) be a DCML-space in respect of f and g, and let $\{\kappa_n\}_{(n\geq 0)}$ be a sequence in Γ. Then:

1) $\{a_n\}$ is said to converge to a_0 in Γ, if and only if $\lim_{n\to\infty} L(a_n, a_0) = L(a_0, a_0)$. We denote this as $\lim_{n\to\infty} a_n = a_0$.
2) $\{a_n\}$ is said to be L-Cauchy if and only if $\lim_{m,n\to\infty} L(a_m, a_n)$ exists and is finite.
3) The space (Γ, L) is said to be complete if every L-Cauchy in Γ is convergence in Γ. That is, $\lim_{n\to\infty} L(a_n, a_0) = L(a_0, a_0) = \lim_{m,n\to\infty} L(a_m, a_n)$.

Definition 2.6. Let (Γ, L) be a DCML-space via f and g. For $a_0 \in \Gamma$ and $\varepsilon > 0$. Then:

1) An open ball $\mathfrak{B}(a_0, \varepsilon) = \{\omega \in \Gamma, |L(a_0, \omega) - L(a_0, a_0)| < \varepsilon\}$.
2) The mapping $T : \Gamma \to \Gamma$ is said to be continuous at $a_0 \in \Gamma$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $T(\mathfrak{B}(a_0, \delta)) \subseteq \mathfrak{B}(T(a_0), \varepsilon)$. Thus, if T is continuous at a_0, then for all sequence $\{a_n\}$ converging to a_0, we deduce $\lim_{n\to\infty} T(a_n) = T(a_0)$. That is, $\lim_{n\to\infty} L(Ta_n, Ta_0) = L(Ta_0, Ta_0)$.

In general, the limit of a convergent sequence in DCML-space may not be unique.

Lemma 2.7. *Let (Γ, L) be a DCML-space via non-constant functions $f, g : [0, \infty) \to [0, \infty)$ satisfying $f(0) = g(0) = 0$, and let $\{a_n\}$ be a sequence in Γ such that $\lim_{n\to\infty} L(a_n, a_0) = 0$. Then every sequence convergence has a unique limit.*

Proof. The proof is omitted. □

Lemma 2.8. *Let (Γ, L) be a DCML-space with functions $f, g : [0, \infty) \to [0, \infty)$. Suppose that the inverse mappings of f and g exist.*

1) Let $\{a_n\}$ is convergent to a, and for any $w \in \Gamma$, we obtain that

$$\begin{aligned}\frac{f^{-1}(L(a, w) - \Delta_2) + g^{-1}(L(a, w) - \Delta_1)}{2} &\leq \lim_{n\to\infty} inf L(a_n, w) \leq \lim_{n\to\infty} sup L(a_n, w)\\ &\leq \Delta + \frac{f(L(a, w)) + g(L(a, w))}{2},\end{aligned}$$

where $\Delta_1 = f(L(a, a)), \Delta_2 = g(L(a, a))$ and $\Delta = \frac{\Delta_1+\Delta_2}{2}$.

2) Let $\{a_n\}$ and $\{s_n\}$ be convergent to a and s, respectively. Suppose that f, g are continuous and non-decreasing functions, and g is a sub-additive. Then:

$$g^{-2}(L(a, s) - M) \leq \lim_{n\to\infty} inf L(a_n, s_n) \leq \lim_{n\to\infty} sup L(a_n, s_n) \leq M + g^2(L(a, s)),$$

where $M = \Delta_1 + \Delta_3$, and $\Delta_3 = g(f(L(s, s)))$.

Proof. First, utilizing (L3) of definition 2.4, we deduce

$$L(a, w) \leq f(L(a, a_n)) + g(L(a_n, w)).$$

Taking the lower limit as $n \to \infty$ in the above inequality, we have

$$L(a, w) \leq f(\lim_{n\to\infty} inf L(a, a_n)) + g(\lim_{n\to\infty} inf L(a_n, w))$$

$$=f(L(a,a))+g(\lim_{n\to\infty} inf L(a_n,w)).$$

Hence,

$$g^{-1}(L(a,w)-\Delta_1)\leq \lim_{n\to\infty} inf L(a_n,w). \tag{2.1}$$

Similarly, with $L(w,a)$, we obtain that

$$f^{-1}(L(w,a)-\Delta_2)\leq \lim_{n\to\infty} inf L(a_n,w). \tag{2.2}$$

By definition 2.4, (L2), which implies that (by additive (2.1) and (2.2)),

$$\frac{f^{-1}(L(a,w)-\Delta_2)+g^{-1}(L(a,w)-\Delta_1)}{2}\leq \lim_{n\to\infty} inf L(a_n,w). \tag{2.3}$$

While,

$$L(a_n,w)\leq f(L(a_n,a))+g(L(a,w)).$$

Making the upper limit as n tends to ∞, in the inequality, we reach that

$$\lim_{n\to\infty} sup L(a_n,w)\leq \Delta_1+g(L(a,w)).$$

Similarly, with $L(w,a_n)$, and by definition 2.4 of (L3), we obtain

$$\lim_{n\to\infty} sup L(a_n,w)\leq f(L(a,w))+\Delta_2.$$

This implies that,

$$\lim_{n\to\infty} sup L(a_n,w)\leq \Delta+\frac{f(L(a,w))+g(L(a,w))}{2}, \tag{2.4}$$

where $\Delta=\frac{\Delta_1+\Delta_2}{2}$. Therefore, we obtain from (2.3) and (2.4) the desired result.

Second, by means of (L3) in the DCML-space, it is easy to see that

$$\begin{aligned} L(a,s)\leq & f(L(a,a_n))+g(L(s,a_n))\\ \leq & f(L(a,a_n))+g[f(L(s,s_n))+g(L(a_n,s_n))], \end{aligned} \tag{2.5}$$

and

$$L(a_n,s_n)\leq f(L(a_n,a))+g[f(L(s_n,s))+g(L(a,s))]. \tag{2.6}$$

Taking the lower limit as $n\to\infty$ in (2.5) and using the same approach as in (2.1), given that g is a sub-additive, we obtain that

$$L(a,s)\leq f(L(a,a))+g(f(L(s,s)))+g^2(\lim_{n\to\infty} inf L(a_n,s_n)).$$

Hence,

$$g^{-2}(L(a,s)-M)\leq \lim_{n\to\infty} inf L(a_n,s_n).$$

Similarly, taking the upper limit as $n\to\infty$ in the (2.6), we obtain that

$$\lim_{n\to\infty} sup L(a_n,s_n)\leq M+g^2(L(a,s)),$$

where $M=\Delta_1+\Delta_3$, and $\Delta_3=g(f(L(s,s)))$. □

In DCMTS, the result above extends as follows, in the corollary below.

Corollary 2.9. *Let* (Γ, L) *be a DCMTS with functions* $\theta, \vartheta : [0,\infty) \to [0,\infty)$. *Suppose that the inverse mappings of* θ *and* ϑ *exist.*

1) *Let* $\{a_n\}$ *be convergent to* a, *and* $w \in \Gamma$ *is arbitrary. Then*

$$\frac{\theta^{-1}(L(a,w)) + \vartheta^{-1}(L(a,w))}{2} \leq \lim_{n\to\infty} inf L(a_n, w) \leq \lim_{n\to\infty} sup L(a_n, w) \leq \frac{\theta(L(a,w)) + \vartheta(L(a,w))}{2}.$$

2) *Let* $\{a_n\}$ *and* $\{s_n\}$ *be convergent to* a *and* s, *respectively. Suppose that* θ, ϑ *are continuous and non-decreasing functions, and* ϑ *is a sub-additive. Then*

$$\vartheta^{-2}(L(a,s)) \leq \lim_{n\to\infty} inf L(a_n, s_n) \leq \lim_{n\to\infty} sup L(a_n, s_n) \leq \vartheta^{2}(L(a,s)).$$

In particular, if $\theta(t) = \vartheta(t) = t, t \geq 0$, *then, go to* $\lim_{n\to\infty} L(a_n, s_n) = L(a, s)$.

Proof. Immediately, of Lemma 2.8, notice that the condition (P1) holds in DCMTS. □

Let Ψ be the family of all onto mappings $\psi : [0,\infty) \to [0,\infty)$ satisfying the conditions: $t \leq \psi(t)$ for each $t \in [0,\infty)$, and ψ' (the derivative of ψ) is increasing [17]. Next, in the corollary, we show that a strong formula with DCML-space is $f(t) = g(t) = \psi(t), t \geq 0$.

Corollary 2.10. *Let* (Γ, L) *be a DCML-space with auxiliary function* $\psi : [0,\infty) \to [0,\infty)$ *in* Ψ.

1) *Let* $\{a_n\}$ *be convergent to* a, *and* $w \in \Gamma$ *is arbitrary, then*

$$\psi^{-1}(L(a,w) - \hbar) \leq \lim_{n\to\infty} inf L(a_n, w) \leq \lim_{n\to\infty} sup L(a_n, w) \leq \hbar + \psi(L(a,w)),$$

   where $\hbar = \psi(L(a,a))$.
2) *Let* $\{a_n\}$ *and* $\{s_n\}$ *be convergent to* a *and* s, *respectively, then*

$$\psi^{-2}(L(a,s) - M) \leq \lim_{n\to\infty} inf L(a_n, s_n) \leq \lim_{n\to\infty} sup L(a_n, s_n) \leq M + \psi^{2}(L(a,s)),$$

   where $M = \hbar + \ell$, *and* $\ell = \psi^2(L(a,a))$.

Proof. Immediately, of Lemma 2.8, and notice that $f(t) = g(t) = \psi(t), t \geq 0$. □

Let (Γ, L) be a DCML-space. Define $\hat{L} : \Gamma^2 \to [0,\infty)$ by

$$\hat{L}(a,b) = |2L(a,b) - L(a,a) - L(b,b)|, \forall a, b \in \Gamma.$$

Obviously, $\hat{L}(a,a) = 0, \forall a \in \Gamma$.

Lemma 2.11. *Let* $\psi \in \Psi$, *then for all* $x \in [0,1]$, *we conclude:*

1) $\psi(x^p) \leq \psi^p(x)$ *and* $(\psi^{-1}(x))^p \leq \psi^{-1}(x^p)$, $p \geq 1$.
2) $\psi(x^q) \geq \psi^q(x)$ *and* $(\psi^{-1}(x))^q \geq \psi^{-1}(x^q)$, $0 < q \leq 1$.

3. Main results

This section presents some fixed point results in the framework of double-composed metric-like space. Our first theorem is corresponding to the Hardy–Roger-contraction of a fixed point theorem.

Before stating our theorems, we introduce the auxiliary functions involved in Hardy–Rogers contraction.

Let Ω be a set of all mapping $\varpi : [0,\infty) \to [0,\infty)$ satisfying,

i) ϖ is non-decreasing,

ii) $\varpi(a) < a, \forall a > 0$, and for all $a < b$ implies $\varpi(a) < \varpi(b)$,

iii) $\lim_{t\to a^+} \varpi(t) < a, \forall a > 0$, that is, $\varpi^n(t) \to 0$ as $n \to \infty, t > 0$.

Recall this function a comparison function. While if the axiom (ii.) $a < \varpi(a)$ for all $a > 0$ is known as an in-comparison function, see [12, 21].

Lemma 3.1. *[23] Let $\varpi \in \Omega$ and $\{a_n\}$ be a sequence such as $a_n \to 0$ as $n \to \infty$. Then $\varpi(a_n) \to 0$ as $n \to \infty$, and $\varpi(0) = 0$.*

Theorem 3.2. *Suppose (Γ, L) is a complete DCML-space with $f, g : [0,\infty) \to [0,\infty)$. Let $T : \Gamma \to \Gamma$ be a mapping satisfying,*

$$L(Ta, Tb) \leq \xi_1 L(a,b) + \xi_2 L(a,Ta) + \xi_3 L(b,Tb) + \xi_4 L(a,Tb) + \xi_5 L(b,Ta), \tag{3.1}$$

for all $a, b \in \Gamma, \xi_i \in [0,1), i = 1,2,\cdots,5$, and $\sum_{j=1}^{5} \xi_j < 1$. For $a_0 \in \Gamma$, take that $a_n = T^n a_0$. Suppose that,

1) f, g are continuous, non-decreasing and g is sub-additive a comparison function, and f in-comparison function.

2) $\sum_{i=m}^{n-2} g^{i-m} f\Big(\mathcal{K}^i f^i(L(a_0,a_1))\Big) + g^{n-m-1}\Big(\mathcal{K}^{n-1} f^{n-1}(L(a_0,a_1))\Big) \to 0$, (as $n, m \to \infty$), where $\mathcal{K} = \frac{\xi_1+\xi_2+\xi_4+2\xi_5}{1-\xi_3-\xi_4}$.

Then, T has fixed point. Further, if $\xi_2 = \xi_3 = 0$ (or for any fixed point a, satisfied $L(a,a) \leq L(a,w), w \in \Gamma$), then T has a unique fixed point.

Proof. Let $a_0 \in \Gamma$. By hypothesis a sequence $\{a_n\}$ in Γ form $a_n = T^n a_0$ or $a_{n+1} = Ta_n, \forall n \in \mathbb{N}$. Assuming $a = a_{n-1}$ via $b = a_n$ in the Hardy–Roger contraction, then,

$$\begin{aligned} L(a_n, a_{n+1}) = L(Ta_{n-1}, Ta_n) \leq & \xi_1 L(a_{n-1}, a_n) + \xi_2 L(a_{n-1}, Ta_{n-1}) + \xi_3 L(a_n, Ta_n) + \xi_4 L(a_{n-1}, Ta_n) \\ & + \xi_5 L(a_n, Ta_{n-1}) \\ \leq & \xi_1 L(a_{n-1}, a_n) + \xi_2 L(a_{n-1}, a_n) + \xi_3 L(a_n, a_{n+1}) + \xi_4 L(a_{n-1}, a_{n+1}) + \xi_5 L(a_n, a_n). \end{aligned} \tag{3.2}$$

In general, in DCML-space, we observe that, $L(a_n, a_n) \neq 0$. So, by (L3) in definition 2.4, we undergo

$$L(a_{n-1}, a_{n+1}) \leq f\big(L(a_{n-1}, a_n)\big) + g\big(L(a_n, a_{n+1})\big),$$

and

$$L(a_n, a_n) \leq f\big(L(a_n, a_{n-1})\big) + g\big(L(a_{n-1}, a_n)\big).$$

Therefore,

$$(1-\xi_3)L(a_n,a_{n+1})-\xi_4 g(L(a_n,a_{n+1}))\le(\xi_1+\xi_2)L(a_{n-1},a_n)+(\xi_4+\xi_5)f(L(a_{n-1},a_n))+\xi_5 g(L(a_{n-1},a_n)).$$

By fact of condition 1, such that $g(t)<t$, for all $t\ge 0$, we obtain

$$(1-\xi_3)L(a_n,a_{n+1})-\xi_4 g(L(a_n,a_{n+1}))\le(\xi_1+\xi_2+\xi_5)L(a_{n-1},a_n)+(\xi_4+\xi_5)f(L(a_{n-1},a_n)),$$

thus, implies via condition 1, we lead to

$$\begin{aligned}(1-\xi_3)L(a_n,a_{n+1})-\xi_4 L(a_n,a_{n+1}) &\le (1-\xi_3)L(a_n,a_{n+1})-\xi_4 g(L(a_n,a_{n+1}))\\ &\le(\xi_1+\xi_2+\xi_5)L(a_{n-1},a_n)+(\xi_4+\xi_5)f(L(a_{n-1},a_n))\\ &\le(\xi_1+\xi_2+\xi_4+2\xi_5)f(L(a_{n-1},a_n)).\end{aligned}$$

Hence,

$$L(a_n,a_{n+1})\le\mathcal{K}f(L(a_{n-1},a_n)),$$

where $\mathcal{K}=\frac{\xi_1+\xi_2+\xi_4+2\xi_5}{1-\xi_3-\xi_4}$. By repeatedly this process becomes

$$L(a_n,a_{n+1})\le\mathcal{K}^n f(L(a_0,a_1)). \tag{3.3}$$

For each $n,m\in\mathbb{N}$, where $m>n$, we deduce

$$\begin{aligned}L(a_m,a_n) &\le f(L(a_m,a_{m+1}))+g(L(a_{m+1},a_n))\\ &\le f(L(a_m,a_{m+1}))+g[f(L(a_{m+1},a_{m+2}))+g(L(a_{m+2},a_n))]\\ &\le f(L(a_m,a_{m+1}))+gf(L(a_{m+1},a_{m+2}))+g^2(L(a_{m+2},a_n))\\ &\ \vdots\\ &\le\sum_{i=m}^{n-2}g^{i-m}f(L(a_i,a_{i+1}))+g^{n-m-1}(L(a_{n-1},a_n)).\end{aligned} \tag{3.4}$$

We utilize conditions 1, 2 to establish the inequality (3.3) in (3.4), we undergo

$$L(a_m,a_n)\le\sum_{i=m}^{n-2}g^{i-m}f\Big(\mathcal{K}^i f^i(L(a_0,a_1))\Big)+g^{n-m-1}\Big(\mathcal{K}^{n-1}f^{n-1}(L(a_0,a_1))\Big).$$

Letting $m,n\to\infty$ and applying condition 2 of Theorem 3.2, we obtain $L(a_m,a_n)$ for each $n,m\in\mathbb{N}$. Thus, the sequence $\{a_n\}$ is L-Cauchy in Γ. That is,

$$\lim_{n,m\to\infty}L(a_m,a_n)=0. \tag{3.5}$$

Since the sequence $\{a_n\}$ is L-Cauchy in Γ, which is a complete DCML-space, there exists an element $a\in\Gamma$ such that, $\{a_n\}\to a$. Consider

$$\lim_{n\to\infty}L(a_n,a)=L(a,a)=\lim_{n,m\to\infty}L(a_m,a_n)=0.$$

Thus, $L(a, a) = 0$. Now, we prove that $Ta = a$. With the help of (L3) in DCML-space implies that

$$L(a, a_{n+1}) \leq f\big(L(a, a_n)\big) + g\big(L(a_n, a_{n+1})\big).$$

Utilizing condition 1, and (3.5), we notice that

$$\lim_{n\to\infty} L(a, a_{n+1}) = 0. \tag{3.6}$$

By (L3) in definition 2.4, we deduce that

$$\begin{aligned} L(a, Ta) &\leq f\big(L(a, a_n)\big) + g\big(L(a_n, Ta)\big) \\ &= f\big(L(a, a_n)\big) + g\big(L(Ta_{n-1}, Ta)\big). \end{aligned} \tag{3.7}$$

Taking the limit as $n \to \infty$ in (3.7), by (3.6) and condition 1, such that $g(t) < t$, for all $t \geq 0$, we obtain $L(a, Ta) = 0$, so that $Ta = a$.

Lastly, assume that T has two fixed points, say a and s. Then

$$\begin{aligned} L(a, s) = L(Ta, Ts) &\leq \xi_1 L(a, s) + \xi_2 L(a, Ta) + \xi_3 L(s, Ts) + \xi_4 L(a, Ts) + \xi_5 L(s, Ta) \\ &= \xi_1 L(a, s) + \xi_2 L(a, a) + \xi_3 L(s, s) + \xi_4 L(a, s) + \xi_5 L(s, a). \end{aligned}$$

We further let $\xi_2 = \xi_3 = 0$, which leads that

$$L(a, s) \leq (\xi_1 + \xi_4 + \xi_5) L(a, s). \tag{3.8}$$

With $\xi_1 + \xi_4 + \xi_5 < 1$, the inequality 3.8, we obtain $L(a, s) = 0$, then $a = s$. □

Example 3. Let $\Gamma = \{1, 2, 3\}$. Define a map $L : \Gamma \times \Gamma \to [0, \infty)$ by

$$L(1, 1) = L(2, 2) = 0, L(3, 3) = 2,$$

and

$$L(1, 2) = L(2, 1) = 11, L(1, 3) = L(3, 1) = 6, L(2, 3) = L(3, 2) = 3.$$

Take $f, g : [0, \infty) \to [0, \infty)$, which is defined by $f(t) = \sinh\left(\frac{12}{11}t\right)$; and $g(t) = \left(\frac{3}{11}t\right)$.

Evidently, to verify that (Γ, L) is a complete DCML-space with respect to f, g, and $L(3, 3) \neq 0$. Therefore (Γ, L) is not a DCMS.

Also, define a map $T : \Gamma \to \Gamma$ as

$$T(a) = \begin{cases} 3 & \text{if } a = 1 \\ 2 & \text{if } a \in \{2, 3\}. \end{cases}$$

Then, T has a fixed point.

Proof. Let us choose $\xi_1 = \frac{1}{4}, \xi_2 = \frac{1}{3}, \xi_3 = \frac{1}{12}, \xi_4 = \frac{1}{13}$, and $\xi_5 = \frac{1}{20}$. Next, consider the following cases to show that the Hardy–Rogers contraction in Theorem 3.2 holds:

Case 1. $a = 1, b = 2, L(T1, T2) = L(3, 2) = 3 \leq \frac{747}{130} = \frac{1}{4}L(1, 2) + \frac{1}{3}L(1, 3) + \frac{1}{12}L(2, 2) + \frac{1}{13}L(1, 2) + \frac{1}{20}L(2, 3)$.

Case 2. $a = 1, b = 3, L(T1, T3) = L(3, 2) = 3 \leq \frac{1221}{260} = \frac{1}{4}L(1, 3) + \frac{1}{3}L(1, 3) + \frac{1}{12}L(3, 2) + \frac{1}{13}L(1, 2) +$

$\frac{1}{20}L(3,3)$.
Case 3. $a = 2, b = 3, L(T2, T3) = L(2,2) = 0 \leq \frac{23}{20} = \frac{1}{4}L(2,3) + \frac{1}{3}L(2,2) + \frac{1}{12}L(3,2) + \frac{1}{13}L(2,2) + \frac{1}{20}L(3,2)$.
Since $L(3,3) \neq 0$, we further take
Case 4. $a = 1, b = 1, L(T1, T1) = L(3,3) = 2 \leq \frac{212}{65} = \frac{1}{4}L(1,1) + \frac{1}{3}L(1,3) + \frac{1}{12}L(1,3) + \frac{1}{13}L(1,3) + \frac{1}{20}L(1,3)$.

Consider at $k_0 = 2 \in \Gamma$. Thus, $k_n = T^n k_0 = 2$ for each $n \geq 1$. For a condition 1 in Theorem 3.2, we deduce that, $t < f(t) = \sinh\left(\frac{12}{11}t\right)$, and $\left(\frac{3}{11}t\right) = g(t) < t, t > 0$. Moreover, we see that

$$\lim_{n\to\infty}\left(\frac{\xi_1 + \xi_2 + \xi_4 + 2\xi_5}{1 - \xi_3 - \xi_4}\right)^n = \lim_{n\to\infty}\left(\frac{593}{655}\right)^n = 0.$$

So, already condition 2 of Theorem 3.2 holds. Then, all the conditions of Theorem 3.2 hold, so has a fixed point given as $k = 2$. □

Afterwards, we propose some specific cases of our Theorem 3.2.

Corollary 3.3. *Suppose* (Γ, L) *is a complete DCML-space with* $f, g : [0,\infty) \to [0,\infty)$. *Let* $T : \Gamma \to \Gamma$ *be a mapping satisfying the Banach contraction,*

$$L(Ta, Tb) \leq \xi L(a,b), \forall a, b \in \Gamma, \textit{ where } \xi \in [0,1). \tag{3.9}$$

For $a_0 \in \Gamma$, *take* $a_n = T^n a_0$. *Assume that*

1) f, g *are continuous, non-decreasing functions, and g is a sub-additive function.*
2) $\sum_{i=m}^{n-2} g^{i-m} f\Big(\xi^i L(a_0, a_1)\Big) + g^{n-m-1}\Big(\xi^{n-1} L(a_0, a_1)\Big) \to 0$, *as* $n, m \to \infty$.

Then T *has a unique fixed point.*

Proof. It suffices to observe that, if $\xi = \xi_1$, and $\xi_2 = \xi_3 = \xi_4 = \xi_5 = 0$ in Theorem 3.2, we obtain the desired result. Moreover, notice that in condition 1, g does not necessarily belong to Ω. The same applies to function f. □

Corollary 3.4. *Suppose* (Γ, L) *is a complete DCML-space via* $f, g : [0,\infty) \to [0,\infty)$. *Let* $T : \Gamma \to \Gamma$ *be a mapping satisfying Kannan's contraction,*

$$L(Ta, Tb) \leq \xi_1 L(a, Ta) + \xi_2 L(b, Tb), \textit{for all } a, b \in \Gamma, \xi_1, \xi_2 \in [0,1), \textit{ and } \xi_1 + \xi_2 < 1. \tag{3.10}$$

For $a_0 \in \Gamma$, *take* $a_n = T^n a_0$. *Assume that*

1) f, g *are continuous, non-decreasing functions and g is a sub-additive comparison function.*
2) $\sum_{i=m}^{n-2} g^{i-m} f\Big(\mathcal{K}^i L(a_0, a_1)\Big) + g^{n-m-1}\Big(\mathcal{K}^{n-1} L(a_0, a_1)\Big) \to 0$, *as* $n, m \to \infty$, *where* $\mathcal{K} = \frac{\xi_1}{1-\xi_2}$.

Then T *has a fixed point. Furthermore, if for any fixed point a, such that* $L(a,a) = 0$, *then* T *has a unique fixed point.*

Proof. Whenever $\xi_1 = \xi_4 = \xi_5 = 0$ in Theorem 3.2, we obtain the desired result. In condition 1, f is not necessarily an in-comparison function. Furthermore, inspired by [16], we deduce that for any fixed point a, $L(a,a) = 0$, which implies the uniqueness of the fixed point. □

Remark 3.5. In Theorem 3.2, we can take a special case, when $\xi_4 = \xi_5 = 0$, namely Riech contraction.

In the following, we introduce the nonlinear case.

Theorem 3.6. *Suppose (Γ, L) is a complete DCML-space with respect to functions $f, g : [0, \infty) \to [0, \infty)$. Consider a mapping $T : \Gamma \to \Gamma$ for which there exists a continuous $\varpi \in \Omega$ such that*

$$L(Ta, Tb) \leq \varpi(M(a, b)), \quad M(a, b) = Max\{L(a, b), L(a, Ta), L(b, Tb)\}, \tag{3.11}$$

for all $a, b \in \Gamma$. Furthermore, assume that for each $a_0 \in \Gamma$, take $a_n = T^n a_0, n \in \mathbb{N}$, and

1) *f, g are continuous and non-decreasing functions, and g is a sub-additive function.*
2) *$\sum_{i=m}^{n-2} g^{i-m} f\Big(\varpi^i(L(a_0, a_1))\Big) + g^{n-m-1}\Big(\varpi^{n-1}(L(a_0, a_1))\Big) \to 0$, as $n, m \to \infty$.*

Then T has a unique fixed point.

Proof. Let $a_0 \in \Gamma$, and define sequence $\{a_n\}$ in Γ via $a_n = T^n a_0$ so that $a_{n+1} = Ta_n, \forall n \in \mathbb{N}$. Assume that there is $r \in \mathbb{N}$ such that $a_r = a_{r+1} = Ta_r$, hence a_r is a fixed point. Then let $a_n \neq a_{n+1}, n \geq 0$. From inequality 3.11, we conclude

$$L(a_n, a_{n+1}) = L(Ta_{n-1}, Ta_n) \leq \varpi(M(a_{n-1}, a_n)), \tag{3.12}$$

where $M(a_{n-1}, a_n) = Max\{L(a_{n-1}, a_n), L(a_n, a_{n+1})\}$. If for some n, we let $M(a_{n-1}, a_n) = L(a_n, a_{n+1})$, therefore, by 3.12 and by condition (ii) $\varpi(t) < t$ for all $t > 0$, which lead to

$$L(a_n, a_{n+1}) \leq \varpi(L(a_n, a_{n+1})) < L(a_n, a_{n+1}). \tag{3.13}$$

This is a contradiction. Thus, for all $n \in \mathbb{N}$, which implies $M(a_{n-1}, a_n) = L(a_{n-1}, a_n)$. So, $L(a_n, a_{n+1}) \leq \varpi(L(a_{n-1}, a_n))$. By induction, we deduce that, for each $n \geq 0$

$$L(a_n, a_{n+1}) \leq \varpi^n(L(a_0, a_1)).$$

By axiom of ϖ we conclude that

$$L(a_n, a_{n+1}) \to 0, \quad \text{as } n \to \infty. \tag{3.14}$$

By the same process in the proof of Theorem 3.2, for $n, m \in \mathbb{N}, m < n$, we obtain that

$$L(a_n, a_m) \leq \sum_{i=m}^{n-2} g^{i-m} f\Big(\varpi^i(L(a_0, a_1))\Big) + g^{n-m-1}\Big(\varpi^{n-1}(L(a_0, a_1))\Big).$$

With condition (2) in Theorem 3.6, as $n, m \to \infty$, we can easily deduce that $\{a_n\}$ is L-Cauchy. (Γ, L) is a complete DCML-space, so that, if $a_n \to a \in \Gamma$ as $n \to \infty$, then by condition 3.14 and utilizing the argument in the proof of Theorem 3.2, $\lim_{n\to\infty} L(a_n, a) = 0$, we obtain $Ta = a$. Lastly, assuming that a and s are two fixed, distinct fixed points of T, by inequality 3.11, we consider

$$L(a, s) = L(Ta, Ts) \leq \varpi(M(a, s)) = \varpi(L(a, s)) < L(a, s).$$

Thereby $a = s$, as required. □

Remark 3.7. Clearly, if $\varpi(t) = \xi t, 0 < \xi < 1$, then the inequality 3.11 in Theorem 3.6 becomes

$$L(Ta, Tb) \leq \xi Max\{L(a, b), L(a, Ta), L(b, Tb)\}.$$

4. Applications

The concepts of existence and uniqueness have become attractive for researchers in nonlinear analysis, particularly for solving differential equations, integral equations, and fractional differential equations, etc. This has led to improvements in the applications of fixed-point techniques.

Let us examine the existence of solutions for the nonlinear integral equation:

$$\mathfrak{a}(\tau) = \int_0^1 \mathfrak{S}(\tau, \mu, \mathfrak{a}(\mu))d\mu, \quad \text{for } \tau \in [0,1], \tag{4.1}$$

where $\mathfrak{S}(\tau, \mu, \mathfrak{a}(\mu))$ is a continuous function from $[0,1]^3$ into $\mathbb{R}$.

Let $\Gamma = C([0,1])$ be the set of all continuous functions defined on $[0,1]$. We endow Γ with DCML-space as follows:

$$L(\mathfrak{a}, \mathfrak{b}) = sup_{\tau\in[0,1]} \sinh\left(\frac{(|\mathfrak{a}(\tau)| + |\mathfrak{b}(\tau)|)^q}{2}\right),$$

for each $\mathfrak{a}, \mathfrak{b} \in \Gamma$, and $q \geq 1$.

Clearly, (Γ, L) is a complete DCML-space, where $f(\tau) = g(\tau) = \sinh(2^{q-1}\tau)$.

Theorem 4.1. *Suppose that for each* $\mathfrak{a}, \mathfrak{b} \in \Gamma$,

1) There is a continuous $\varrho : [0,1]^2 \to \mathbb{R}$, *and* $0 < \xi < 1$, *such that,*

$$|\mathfrak{S}(\tau, \mu, \mathfrak{a}(\mu))| + |\mathfrak{S}(\tau, \mu, \mathfrak{b}(\mu))| < \left(\frac{\xi}{2}\right)^{1/q} \varrho(\tau, \mu)(|\mathfrak{a}(\mu)| + |\mathfrak{b}(\mu)|). \tag{4.2}$$

2) $sup_{\tau\in[0,1]} \int_0^1 \varrho(\tau, \mu)d\mu \leq 1.$

Then the integral equation (4.1) has a unique solution.

Proof. Let $T : \Gamma \to \Gamma$ be a continuous defined by $T\mathfrak{a}(\tau) = \sinh^{-1}\left(\int_0^1 \mathfrak{S}(\tau, \mu, \mathfrak{a}(\mu))d\mu\right)$. Then

$$L(T\mathfrak{a}, T\mathfrak{b}) = sup_{\tau\in[0,1]} \sinh\left(\frac{(|T\mathfrak{a}(\tau)| + |T\mathfrak{b}(\tau)|)^q}{2}\right).$$

We have

$$\begin{aligned}\sinh\left(\frac{(|T\mathfrak{a}(\tau)| + |T\mathfrak{b}(\tau)|)^q}{2}\right) &= \sinh\frac{1}{2}\left(\left|\sinh^{-1}\left(\int_0^1 \mathfrak{S}(\tau, \mu, \mathfrak{a}(\mu))d\mu\right)\right| + \left|\sinh^{-1}\left(\int_0^1 \mathfrak{S}(\tau, \mu, \mathfrak{a}(\mu))d\mu\right)\right|\right)^q \\ &\leq \sinh\frac{1}{2}\left(\sinh^{-1}\left(\left|\int_0^1 \mathfrak{S}(\tau, \mu, \mathfrak{a}(\mu))d\mu\right|\right) + \sinh^{-1}\left(\left|\int_0^1 \mathfrak{S}(\tau, \mu, \mathfrak{a}(\mu))d\mu\right|\right)\right)^q,\end{aligned}$$

utilizing Lemma 2.11, we conclude that

$$\leq \left(\sinh\left(\sinh^{-1}\left(\left|\int_0^1 \mathfrak{S}(\tau, \mu, \mathfrak{a}(\mu))d\mu\right|\right)\right) + \sinh\left(\sinh^{-1}\left(\left|\int_0^1 \mathfrak{S}(\tau, \mu, \mathfrak{a}(\mu))d\mu\right|\right)\right)\right)^q$$

$$\leq \left(\int_0^1 \left|\mathfrak{S}(\tau, \mu, \mathfrak{a}(\mu))\right| d\mu + \int_0^1 \left|\mathfrak{S}(\tau, \mu, \mathfrak{a}(\mu))\right| d\mu \right)^q$$
$$= \left(\int_0^1 \left|\mathfrak{S}(\tau, \mu, \mathfrak{a}(\mu))\right| + \left|\mathfrak{S}(\tau, \mu, \mathfrak{a}(\mu))\right| d\mu \right)^q$$
$$\leq \left(\int_0^1 \left(\frac{\xi}{2}\right)^{1/q} \varrho(\tau, \mu)(|\mathfrak{a}(\mu)| + |\mathfrak{b}(\mu)|) d\mu \right)^q$$
$$= \xi \left(\int_0^1 \varrho(\tau, \mu) \left(\frac{(|\mathfrak{a}(\mu)| + |\mathfrak{b}(\mu)|)^q}{2} \right)^{1/q} d\mu \right)^q$$
$$\leq \xi \left(\int_0^1 \varrho(\tau, \mu) \left(\sinh\left(\frac{(|\mathfrak{a}(\mu)| + |\mathfrak{b}(\mu)|)^q}{2} \right) \right)^{1/q} d\mu \right)^q$$
$$\leq \xi \left(\int_0^1 \varrho(\tau, \mu) L^{1/q}(\mathfrak{a}(\tau), \mathfrak{b}(\tau)) d\mu \right)^q$$
$$= \xi L(\mathfrak{a}(\tau), \mathfrak{b}(\tau)) \left(\int_0^1 \varrho(\tau, \mu) d\mu \right)^q \leq \xi L(\mathfrak{a}(\tau), \mathfrak{b}(\tau)),$$

yields that $L(T\mathfrak{a}(\tau), T\mathfrak{b}(\tau)) \leq \varpi\big(L(\mathfrak{a}(\tau), \mathfrak{b}(\tau))\big)$. Moreover, all the hypotheses of Theorem 3.6 are satisfied with $\varpi(t) = \xi t, 0 < \xi < 1$. Therefore, Eq (4.1) has a unique solution. □

5. Conclusions

This research establishes a novel concept of generalized metric spaces, called double-composed metric-like spaces, illustrated through several examples. We derived a Hardy–Rogers type contraction theorem for various contraction mappings in double-composed metric-like spaces and provided several related results to our theorems. Furthermore, it presents numerous examples to support the main theorems. The study demonstrates an application of nonlinear integral equations, proving the existence of solutions. This particular new generalization provides valuable tools for studying fixed-point theorems.

The following points are suggestions for open problems and future research directions:

- Explore new open generalizations of double-composed metric-like spaces, such as double-composed cone-metric-like spaces, fuzzy double -composed metric-like spaces, and neutrosophic double-composed metric-like spaces.
- Establish new fixed-point results in double-composed metric-like spaces for various types of contractions, including nonlinear rational contractions, weak contractions, almost-contraction, and (ϕ, F)-contraction, etc.
- Investigate the application of our results to expansive mappings (see [6]).
- Develop deep and non-trivial applications of our main results.

Author contributions

A.A.H: Conceptualization, Methodology, Validation, Formal analysis, Investigation, Writing-original draft, Writing–review & editing; L.K.S: Conceptualization, Methodology, Validation, Formal analysis, Investigation, Writing-original draft, Writing–review & editing; S.A: Conceptualization, Methodology, Validation, Formal analysis, Investigation, Writing-original draft, Writing–review & editing; N.M: Methodology, Validation, Formal analysis, Investigation, Supervision, Writing – review & editing. All authors reviewed the manuscript.

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Conflict of interest

The authors declare no conflict of interest.

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