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Research article

Analysis of nonlinear implicit fractional differential equations with the Atangana-Baleanu derivative via measure of non-compactness

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Abstract: In this study, we proved existence results for nonlinear implicit fractional differential equations with the Caputo version of the Atangana-Baleanu derivative, subject to the boundary and nonlocal initial conditions. The Kuratowski's measure of non-compactness and its associated fixed point theorems–Darbo's fixed point theorem and Mönchh's fixed point theorem, are the foundation for the analysis in this paper. We support our results with examples of nonlinear implicit fractional differential equations involving the Caputo version of the Atangana-Baleanu derivative subject to both boundary and nonlocal initial conditions. In addition, we provide solutions to the problems we considered.

Keywords: implicit fractional differential equations; non-singular kernel; existence results; measure of non-compactness; fixed point theorem
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1. Introduction

Riemann-Liouville and Caputo fractional derivatives are most commonly used in the analysis of differential equations of non-integer order. The research monographs [1–5] present the fundamental calculus of Riemann-Liouville and Caputo fractional derivatives, along with key studies on the existence, uniqueness, and various qualitative properties of solutions to differential equations involving these operators.

According to researchers [6–9], fractional derivatives with singular kernels may not adequately describe certain natural phenomena with nonlocal characteristics. As a result, efforts are being made by mathematicians to define new, non local fractional derivatives involving a non-singular kernel. In this sense, Caputo-Fabrizio [10] and Atangana-Baleanu [11] have defined new fractional derivative operators with a non-singular kernel in the form of exponential and Mittag-Leffler functions, respectively. Similar to other fractional derivatives, the Caputo version of the Atangana-Baleanu (AB) derivative [11] also has some drawbacks, though it overcomes some of the limitations due to conventional fractional derivatives.

The AB-derivative has garnered much attention because it effectively models certain phenomena that cannot be modeled as FDEs involving other conventional fractional derivatives. Atangana and Araz used the AB derivative [12] to analyse nonlinear differential equations in order to investigate the existence and uniqueness of solutions. The approximate controllability of fractional neutral stochastic systems with indefinite delay was investigated by Dineshkumar et al. [13] using the AB derivative. Initial value problems for several classes of AB FDEs have been studied by Kucche and Sutar [14–16], who have demonstrated comparative results as well as findings on the existence and uniqueness of solutions, extremal solutions, and data dependency of solutions.

To know more about mathematical modeling utilizing the AB derivative for various outbreaks such as dengue fever, a tumor-immune surveillance mechanism, optimal control of diabetes and tuberculosis co-existence etc., one can refer to [6–9] and [17].

Byszewski [18] pioneered the investigation of differential equations with nonlocal conditions. Nonlocal conditions play a crucial role in modeling many phenomena where the state at a point depends on the state at distant points. Nonlocal conditions generalize classical initial conditions to include additional information and provide more accurate solutions. For more details on nonlocal conditions and analysis of associated differential equations, we recommend the foundational works of Byszewski [19–21] and Balachandran [22,23].

The measures of non-compactness (MNC) provides another approach to manage differential equations by leveraging fixed point theorems that are proved in the form of MNC. For a comprehensive understanding and applications of MNC in the analysis of differential equations, we refer to [24–27] and the works cited therein. Researchers working in this field are actively exploring the application of MNC to investigate a wide range of differential equations subject to local and nonlocal constraints. Here, we include a few relevant research works. Sarwar et al. [28] analyzed controllability results for semi-linear non-instantaneous impulsive neutral stochastic differential equations with the AB derivative. In another study, Thilakraj et al. [29] analyzed the sobolev-type Volterra-Fredholm functional integro-differential equation with non-local conditions.

On the other hand, implicit FDEs are essential for modeling various physical phenomena, in situations when it is not feasible to express fractional derivatives of a dependent variable explicitly. Since it might not always be possible to obtain analytical solutions for these equations, researchers are still working to develop the theory, methods, and techniques to analyze implicit FDEs. Kucche et al. [30–32] have examined different classes of implicit FDEs that involve the Caputo fractional derivative, and proved results about the existence and uniqueness of solutions as well as the interval of solution existence. The authors also investigated global existence results and Ulam-type stabilities via successive approximations for implicit AB FDEs. Sutar et al. [33] have studied implicit FDEs involving Caputo fractional derivatives through Picard and weakly Picard operator theory, along with the Pompeiu-Hausdorff functional to examine the existence, uniqueness, and dependence of the solution on the initial condition and the nonlinear-functions involved in the FDEs.

Inspired by the significance of implicit differential equations and the non-local, non-singular nature of the AB fractional derivative, we examine the boundary value problem for nonlinear implicit AB-FDEs of the following form:

$$ {}^{\mathfrak{AB}}{}_{0}\mathfrak{D}^{\mu}_{\eta}\zeta(\eta) = \mathfrak{F}\left(\eta,\ \zeta(\eta),\ {}^{\mathfrak{AB}}{}_{0}\mathfrak{D}^{\mu}_{\eta}\zeta(\eta)\right),\ \eta \in J = [0, T],\ T > 0, \tag{1.1} $$

$$ \alpha\zeta(0) + \beta\zeta(T) = \gamma, \tag{1.2} $$

and implicit AB-FDEs with non local condition of the form:

$$ {}^{\mathfrak{AB}}{}_{0}\mathfrak{D}^{\mu}_{\eta}\zeta(\eta) = \mathfrak{F}\left(\eta,\ \zeta(\eta),\ {}^{\mathfrak{AB}}{}_{0}\mathfrak{D}^{\mu}_{\eta}\zeta(\eta)\right),\ \eta \in J = [0, T],\ T > 0, \tag{1.3} $$

$$ \zeta(0) + \mathfrak{g}(\zeta) = \zeta_0, \tag{1.4} $$

in a Banach space $(E, \|\cdot\|)$ over a field $\mathbb{R}$. Here, ${}^{\mathfrak{AB}}{}_{0}\mathfrak{D}^{\mu}_{\eta}$ denotes the AB fractional derivative of order μ $(0 < \mu < 1)$, $\mathfrak{F} \in C^1(J \times E \times E, E)$ is any nonlinear function, $\mathfrak{g} : C \to E$ is a continuous function, $\alpha, \beta \in \mathbb{R}$ such that $\alpha + \beta \neq 0$ and $\gamma,\ \zeta_0 \in E$. We assume that $C = C(J, E)$ is the Banach space equipped with the supremum norm $\|\zeta\|_C = \sup\{\|\zeta(\eta)\| : \eta \in J\}$.

Proving the existence results to the boundary value problem for nonlinear implicit FDEs (1.3) subject to boundary condition (1.4), and nonlinear implicit FDEs (1.1) subject to nonlocal condition (1.2) using the AB derivative, is our primary objective. Our primary findings are derived from the fixed point theorems of Darbo, Mönch, and the theory of the Kuratowski's measure of non-compactness.

We note that, the results obtained for nonlinear implicit FDEs (1.1)-(1.2) and (1.3)-(1.4) covers the study of the following implicit FDEs subject to following different types of initial and boundary conditions:

- The boundary value problem (1.1)-(1.2) reduces to an initial value problem for implicit FDEs if $\alpha = 1$ and $\beta = 0$.
- The non-local implicit FDEs (1.3)-(1.4) reduce to an initial value problem for implicit FDEs if $\mathfrak{g}(\zeta) = 0$.
- If $\mathfrak{g}(\zeta) = \sum_{k=1}^{n} c_k\,\zeta(\eta_k)$, where $\eta_k \in J$ with $0 \leq \eta_k < \eta_{k+1} \leq T$ for $k = 0, 1, \ldots n-1$, and $c_k \in E$ for $k = 1, 2, \ldots n$, then implicit FDEs (1.3)-(1.4) reduces to a non-local problem where we are excepting a solution to the problem (1.3)-(1.4) passing through n number of points.

The key highlights can be summarized as follows:

- Established several significant existence results for nonlinear implicit FDEs that incorporate AB derivative.
- The investigation of nonlinear implicit FDEs was subject to both boundary and nonlocal conditions, shedding light on the complexities of such equations.
- The foundational tools employed are Kuratowski's measure of non-compactness and associated fixed point theorems, namely Darbo's fixed point theorem and Mönch's fixed point theorem.
- The results were illustrated by an example where we also obtained the solution to the problem for boundary conditions and non local conditions.

The remaining structure of this article is as follows: In Section 2, we provide the definitions and results regarding the AB fractional derivative, Kuratowski's measure of non-compactness, and fixed point theorems. In Section 3, we prove different existence results for the boundary value problem of the implicit FDEs (1.1)-(1.2). In Section 4, we establish existence results for implicit FDE (1.3) subject to the nonlocal condition (1.4). Finally, in Section 5, we provide examples of nonlinear implicit FDEs subject to boundary and initial conditions that illustrate the existence results we obtained.

2. Preliminaries

In this section, we introduce preliminary information that will be used throughout the paper.

2.1. AB fractional derivative and integral

In this section, we provide the definitions and few properties of the AB fractional derivative and integral.

Definition 2.1. *[11] Let $\zeta \in H^1(0,1)$ and $0<\mu<1$, the left AB fractional derivative of ζ of order μ is defined by*

$$^{\mathfrak{AB}}{}_0\mathfrak{D}^{\mu}_{\eta}\zeta(\eta) = \frac{\mathfrak{A}(\mu)}{1-\mu}\int_0^{\eta}\mathbb{E}_{\mu}\left[-\frac{\mu}{1-\mu}(\eta-\xi)^{\mu}\right]\zeta'(\xi)d\xi,$$

where $\mathfrak{A}(\mu)>0$ is a normalization function satisfying $\mathfrak{A}(0)=\mathfrak{A}(1)=1$ and $\mathbb{E}_{\mu}$ is one parameter Mittag-Leffler function.

Definition 2.2. *[11] For any $\zeta \in H^1(0,1)$ and $0<\mu<1$, the fractional integral in sense of Atangana-Baleanu fractional derivative is given by*

$$^{\mathfrak{AB}}{}_0\mathfrak{I}^{\mu}_{\eta}\zeta(\eta) = \frac{1-\mu}{\mathfrak{A}(\mu)}\zeta(\eta) + \frac{\mu}{\mathfrak{A}(\mu)\Gamma(\mu)}\int_0^{\eta}(\eta-\xi)^{\mu-1}\zeta(\xi)d\xi.$$

Lemma 2.1. *[34] For $0<\mu<1$,*

(i) $^{\mathfrak{AB}}{}_0\mathfrak{I}^{\mu}_{\eta}\left[^{\mathfrak{AB}}{}_0\mathfrak{D}^{\mu}_{\eta}\zeta(\eta)\right] = \zeta(\eta)-\zeta(0)$,

(ii) $^{\mathfrak{AB}}{}_0\mathfrak{D}^{\mu}_{\eta}\left[^{\mathfrak{AB}}{}_0\mathfrak{I}^{\mu}_{\eta}\zeta(\eta)\right] = \zeta(\eta)$.

Lemma 2.2. *[35] Suppose that $\mu>0$, $c(\eta)(1-\frac{1-\mu}{\mathfrak{A}(\mu)}d(\eta))^{-1}$ is a non-negative, non decreasing and locally integrable function on $[a,b)$, $\frac{\mu d(\eta)}{\mathfrak{A}(\mu)}\left(1-\frac{1-\mu}{\mathfrak{A}(\mu)}d(\eta)\right)^{-1}$ is non negative and bounded on $[a,b)$ and $\zeta(\eta)$ is non negative and locally integrable on $[a,b)$ with $\zeta(\eta)\le c(\eta)+d(\eta)({}^{\mathfrak{AB}}_{a}\mathfrak{I}^{\mu}\upsilon)(\eta)$. Then,*

$$\zeta(\eta) \le \frac{c(\eta)\mathfrak{A}(\mu)}{\mathfrak{A}(\mu)-(1-\mu)d(\eta)}\mathbb{E}_{\mu}\left(\frac{\mu d(\eta)(\eta-a)^{\mu}}{\mathfrak{A}(\mu)-(1-\mu)d(\eta)}\right),\ \eta\in[a,b).$$

2.2. Kuratowski measure of non compactness

In this section, we provide the definition and properties of the Kuratowski measure of non compactness and the associated fixed point theorems.

Definition 2.3. *[24] (The Kuratowski measure of non compactness) Let $\mathfrak{E}$ be a Banach space and $\Omega_{\mathfrak{E}}$ the set of bounded subsets of $\mathfrak{E}$. The Kuratowski measure of non compactness is the map $\nu:\Omega_{\mathfrak{E}}\to[0,\infty]$ defined by*

$$\nu(\mathfrak{A}) = \inf\left\{\epsilon>0 : \mathfrak{E}\subseteq\cup_{i=1}^{n}\mathfrak{E}_i \text{ and } \operatorname{diam}(\mathfrak{E}_i)\le\epsilon\right\},$$

where $\mathfrak{E}_i\in\Omega_{\mathfrak{E}}$ and $diam(\mathfrak{E}_i)=\sup\{\|a-b\| : a,b\in\mathfrak{E}_i\}$.

Lemma 2.3. *[24] Let U and V bounded sets.*
(a) $\nu(U) = 0 \iff \bar{U}$ *is compact, i. e., U is relatively compact, where $\bar{U}$denotes closure of U.*
(b) *Non singularity: ν is equal to zero on every one element set.*
(c) $\nu(U) = \nu(\bar{U}) = \nu(conv\ U)$, *where conv U is the convex hull of U.*
(d) *Monotonicity:* $U \subset V \implies \nu(U) \leq \nu(V)$.
(e) *Algebraic semi-additivity:* $\nu(U + V) \leq \nu(U) + \nu(V)$, *where*

$$U + V = \{u + v : u \in U, v \in V\}.$$

(f) *Semi-homogencity:* $\nu(\lambda V) = |\lambda|\nu(V); \lambda \in \mathbb{R}$ *where* $\lambda V = \{\lambda v : v \in V\}$.
(g) *Semi-additivity:* $\nu(U \cup V) = \max\{\nu(U), \nu(V)\}$.
(h) *Invariance under translations:* $\nu(U + \zeta_0) = \nu(U)$ *for any* $\zeta_0 \in E$.

The following fixed-point theorems and subsequent lemmas that are connected to Kuratowski's measure of non-compactness serve as the foundation for our findings.

Lemma 2.4. *[25] (Darbo's fixed point theorem) Let $\mathfrak{X}$ be a Banach space and $\mathfrak{C}$ be a bounded, closed, convex and nonempty subset of $\mathfrak{X}$. Suppose a continuous mapping $\mathfrak{N} : \mathfrak{C} \to \mathfrak{C}$ is such that for all closed subsets $\mathfrak{D}$ of $\mathfrak{C}$,*

$$\nu(\mathfrak{N}(\mathfrak{D})) \leq \mathfrak{K}\nu(\mathfrak{D}),$$

where $0 < \mathfrak{K} < 1$. *Then, $\mathfrak{N}$ has a fixed point in $\mathfrak{C}$.*

Lemma 2.5. *[26] (Mönch's fixed point theorem) Let $\mathfrak{D}$ be a bounded, closed and convex subset of a Banach space such that $0 \in \mathfrak{D}$, and let $\mathfrak{N}$ be a continuous mapping of $\mathfrak{D}$ into itself. If the implication*

$$\mathfrak{V} = c\bar{o}nv\ \mathfrak{N}(\mathfrak{V}) \ or\ \mathfrak{V} = \mathfrak{N}(\mathfrak{V}) \cup \{0\} \implies \nu(\mathfrak{V}) = 0$$

holds for every subset $\mathfrak{V}$ of $\mathfrak{D}$, then, $\mathfrak{N}$ has a fixed point.

Lemma 2.6. *If $\mathfrak{V} \subset C(J, \mathbb{E})$ is a bounded and equicontinious set, then:*

(i) The function $\eta \to \nu(\mathfrak{V}(\eta))$ is continuous on J, and

$$\nu_c(\mathfrak{V}) = \sup_{\eta \in J} \nu(\mathfrak{V}(\eta)).$$

(ii)

$$\nu\left(\int_a^b \zeta(\xi)d\xi : \zeta \in \mathfrak{D}\right) \leq \int_a^b \nu(\zeta(\xi))d\xi,$$

where $\mathfrak{D}(\eta) = \{\zeta(\eta) : \zeta \in \mathfrak{D}\}$, $\eta \in J$.

3. Boundary value problem for implicit FDEs

Lemma 3.1. *Let* ζ, ${}^{\mathfrak{AB}}{}_0\mathfrak{D}^{\mu}_{\eta}\zeta \in C(J, E)$ *and* $\mathfrak{F}(0, \zeta(0), 0) = 0$. *Then,* $\zeta \in C(J, E)$ *is a solution of implicit AB-FDEs* (1.1)-(1.2) *if and only if*

$$\zeta(\eta) = \mathfrak{A}_\upsilon + {}^{\mathfrak{AB}}{}_0\mathfrak{I}^{\mu}_{\eta}\upsilon(\eta),\ \eta \in J, \tag{3.1}$$

where $\upsilon(\eta)$ is a solution of the functional equation

$$\upsilon(\eta) = \mathfrak{F}\left(\eta, \mathfrak{A}_\upsilon + {}^{\mathfrak{AB}}{}_0\mathfrak{I}^\mu_\eta \upsilon(\eta), \upsilon(\eta)\right), \ \eta \in J, \tag{3.2}$$

and

$$\mathfrak{A}_\upsilon = \frac{1}{\alpha+\beta}\left[\gamma - \beta\left({}^{\mathfrak{AB}}{}_0\mathfrak{I}^\mu_\eta \upsilon(\eta)\right)_{\eta=T}\right]. \tag{3.3}$$

Proof. Define

$${}^{\mathfrak{AB}}{}_0\mathfrak{D}^\mu_\eta \zeta(\eta) = \upsilon(\eta), \ \eta \in J. \tag{3.4}$$

Operating the AB-fractional integral operator ${}^{\mathfrak{AB}}{}_0\mathfrak{I}^\mu_\eta$ on both sides of (3.4) and utilizing Lemma 2.1, we obtain

$$\zeta(\eta) = \zeta(0) + {}^{\mathfrak{AB}}{}_0\mathfrak{I}^\mu_\eta \upsilon(\eta), \ \eta \in J. \tag{3.5}$$

Using the boundary condition given in Eq (1.2), from Eq (3.5), we get

$$\begin{aligned}\gamma &= \alpha\zeta(0) + \beta\zeta(T)\\ &= \alpha\zeta(0) + \beta\left[\zeta(0) + \left({}^{\mathfrak{AB}}{}_0\mathfrak{I}^\mu_\eta \upsilon(\eta)\right)_{\eta=T}\right]\\ &= (\alpha+\beta)\zeta(0) + \beta\left({}^{\mathfrak{AB}}{}_0\mathfrak{I}^\mu_\eta \upsilon(\eta)\right)_{\eta=T}.\end{aligned}$$

This gives

$$\zeta(0) = \frac{1}{\alpha+\beta}\left[\gamma - \beta\left({}^{\mathfrak{AB}}{}_0\mathfrak{I}^\mu_\eta \upsilon(\eta)\right)_{\eta=T}\right] := \mathfrak{A}_\upsilon. \tag{3.6}$$

From Eqs (3.5) and (3.6), we obtain Eq (3.1). Further, using Eqs (3.1) and (3.4) in the Eq (1.1) , we obtain Eq (3.2).

On the other hand, assume that ζ satisfy functional Eq (3.1). Operating the AB-fractional derivative operator ${}^{\mathfrak{AB}}{}_0\mathfrak{D}^\mu_\eta$ on both sides of Eq (3.1) and using the Lemma 2.1, provides the following equation:

$${}^{\mathfrak{AB}}{}_0\mathfrak{D}^\mu_\eta \zeta(\eta) = {}^{\mathfrak{AB}}{}_0\mathfrak{D}^\mu_\eta\left[\mathfrak{A}_\upsilon + {}^{\mathfrak{AB}}{}_0\mathfrak{I}^\mu_\eta \upsilon(\eta)\right] = {}^{\mathfrak{AB}}{}_0\mathfrak{D}^\mu_\eta\left[{}^{\mathfrak{AB}}{}_0\mathfrak{I}^\mu_\eta \upsilon(\eta)\right] = \upsilon(\eta), \ \eta \in J. \tag{3.7}$$

Equations (3.1) and (3.7) are used in Eq (3.2), to get

$${}^{\mathfrak{AB}}{}_0\mathfrak{D}^\mu_\eta \zeta(\eta) = \mathfrak{F}\left(\eta, \ \zeta(\eta), \ {}^{\mathfrak{AB}}{}_0\mathfrak{D}^\mu_\eta \zeta(\eta)\right), \ \eta \in J = [0, T], \ T > 0,$$

which is Eq (1.1).

Next, we show that ζ defined in Eq (3.1) also fulfills the boundary condition (1.2). In fact, based on Eqs (3.1) and (3.5), we have

$$\begin{aligned}\alpha\zeta(0) + \beta\zeta(T) &= \alpha\zeta(0) + \beta\left[\mathfrak{A}_\upsilon + \left({}^{\mathfrak{AB}}{}_0\mathfrak{I}^\mu_\eta \upsilon(\eta)\right)_{\eta=T}\right]\\ &= \alpha\mathfrak{A}_\upsilon + \beta\left[\mathfrak{A}_\upsilon + \left({}^{\mathfrak{AB}}{}_0\mathfrak{I}^\mu_\eta \upsilon(\eta)\right)_{\eta=T}\right]\\ &= (\alpha+\beta)\mathfrak{A}_\upsilon + \beta\left({}^{\mathfrak{AB}}{}_0\mathfrak{I}^\mu_\eta \upsilon(\eta)\right)_{\eta=T}.\end{aligned}$$

Using the value of $\mathfrak{A}_\upsilon$ defined in (3.6), from above equation, we have

$$\alpha\zeta(0) + \beta\zeta(T) = (\alpha+\beta)\left[\frac{1}{\alpha+\beta}\left[\gamma - \beta\left({}^{\mathfrak{AB}}{}_0\mathfrak{I}^\mu_\eta \upsilon(\eta)\right)_{\eta=T}\right]\right] + \beta\left({}^{\mathfrak{AB}}{}_0\mathfrak{I}^\mu_T \upsilon(\eta)\right)_{\eta=T} = \gamma.$$

Remark 3.1. *In the view of Lemma 3.1, to demonstrate the existence of the solution for the nonlinear implicit AB-FDEs* (1.1)-(1.2)*, we need to demonstrate the existence of the solution $\zeta(\eta)$ for the functional fractional integral equation Eq* (3.2)*. The function $\zeta(\eta)$ when substituted into Eq* (3.1) *provides the solution to the nonlinear implicit AB-FDEs* (1.1)-(1.2).

We require the subsequent assumption to demonstrate the existence results for functional fractional integral equation (3.2).

(H1) $\|\mathfrak{F}(\eta_1,\zeta_1,\delta_1)-\mathfrak{F}(\eta_2,\zeta_2,\delta_2)\| \le \mathfrak{L}|\eta_1-\eta_2| + \mathfrak{M}\|\zeta_1-\zeta_2\| + \mathfrak{N}\|\delta_1-\delta_2\|;\ \mathfrak{L},\mathfrak{M}\in\mathbb{R}^+, 0<\mathfrak{N}<1.$

Remark 3.2. *[27] The condition (H1) is equivalent to the following inequality:*

$$\mu(\mathfrak{F}(\eta,\mathfrak{A}_1,\mathfrak{A}_2)) \le \mathfrak{M}\mu(\mathfrak{A}_1)+\mathfrak{N}\mu(\mathfrak{A}_2),$$

for any bounded sets $\mathfrak{A}_1,\mathfrak{A}_2\subseteq E$ and for each $\eta\in J$.

The proof of the first existence result is provided through the fixed point theorem of Darbo.

Theorem 3.1. *Let the function $\mathfrak{F}\in C(J\times E\times E,E)$ satisfies Lipschitz type condition (H1). Then, the nonlinear implicit AB-FDEs* (1.1)-(1.2) *has at least one solution, provided*

$$\mathfrak{N}+\mathfrak{M}\left(1+\frac{|\beta|}{|\alpha+\beta|}\right)\left(\frac{1-\mu}{\mathfrak{A}(\mu)}+\frac{T^{\mu}}{\mathfrak{A}(\mu)\Gamma(\mu)}\right)<1. \tag{3.8}$$

Proof. We define the mapping $\mathfrak{T}:C\to C$, $C=C(J,E)$ by

$$(\mathfrak{T}\upsilon)(\eta)=\mathfrak{F}\left(\eta,\mathfrak{A}_{\upsilon}+{}^{\mathfrak{AB}}{}_{0}\mathfrak{I}^{\mu}_{\eta}\upsilon(\eta),\upsilon(\eta)\right),\ \eta\in J, \tag{3.9}$$

in the context of comments in Remark 3.1, where $\mathfrak{A}_{\upsilon}$ refers to the constant specified in Eq (3.3). We demonstrate that the mapping $\mathfrak{T}$ fulfills every condition of Lemma 2.4.

First, we establish that the mapping $\mathfrak{T}$ is continuous. Let the sequence $\{\upsilon_m\}$ of points in E convergent to $\upsilon\in E$. Then for any $\eta\in J$, from Eq (3.9), we obtain

$$\begin{aligned}
&\|(\mathfrak{T}\upsilon_m)\eta-(\mathfrak{T}\upsilon)\eta\|\\
&=\left\|\mathfrak{F}\left(\eta,\mathfrak{A}_{\upsilon_m}+{}^{\mathfrak{AB}}{}_{0}\mathfrak{I}^{\mu}_{\eta}\upsilon_m(\eta),\upsilon_m(\eta)\right)-\mathfrak{F}\left(\eta,\mathfrak{A}_{\upsilon}+{}^{\mathfrak{AB}}{}_{0}\mathfrak{I}^{\mu}_{\eta}\upsilon(\eta),\upsilon(\eta)\right)\right\|\\
&\le\mathfrak{M}\left\{\|\mathfrak{A}_{\upsilon_m}-\mathfrak{A}_{\upsilon}\|+{}^{\mathfrak{AB}}{}_{0}\mathfrak{I}^{\mu}_{\eta}\|\upsilon_m(\eta)-\upsilon(\eta)\|\right\}+\mathfrak{N}\|\upsilon_m(\eta)-\upsilon(\eta)\|\\
&\le\mathfrak{M}\left\|\frac{1}{\alpha+\beta}\left[\gamma-\beta\left({}^{\mathfrak{AB}}{}_{0}\mathfrak{I}^{\mu}_{\eta}\upsilon_m(\eta)\right)_{\eta=T}\right]-\frac{1}{\alpha+\beta}\left[\gamma-\beta\left({}^{\mathfrak{AB}}{}_{0}\mathfrak{I}^{\mu}_{\eta}\upsilon(\eta)\right)_{\eta=T}\right]\right\|\\
&\quad+\mathfrak{M}\,{}^{\mathfrak{AB}}{}_{0}\mathfrak{I}^{\mu}_{\eta}\|\upsilon_m(\eta)-\upsilon(\eta)\|+\mathfrak{N}\|\upsilon_m(\eta)-\upsilon(\eta)\|\\
&\le\frac{\mathfrak{M}\beta}{\alpha+\beta}\left({}^{\mathfrak{AB}}{}_{0}\mathfrak{I}^{\mu}_{\eta}\|\upsilon_m(\eta)-\upsilon(\eta)\|\right)_{\eta=T}+\mathfrak{M}\,{}^{\mathfrak{AB}}{}_{0}\mathfrak{I}^{\mu}_{\eta}\|\upsilon_m(\eta)-\upsilon(\eta)\|+\mathfrak{N}\|\upsilon_m(\eta)-\upsilon(\eta)\|\\
&\le\frac{\mathfrak{M}\beta}{\alpha+\beta}\left\{\frac{1-\mu}{\mathfrak{A}(\mu)}\|\upsilon_m(T)-\upsilon(T)\|+\frac{\mu}{\mathfrak{A}(\mu)\Gamma(\mu)}\int_0^T(T-\xi)^{\mu-1}\|\upsilon_m(\xi)-\upsilon(\xi)\|d\xi\right\}\\
&\quad+\mathfrak{M}\left\{\frac{1-\mu}{\mathfrak{A}(\mu)}\|\upsilon_m(\eta)-\upsilon(\eta)\|+\frac{\mu}{\mathfrak{A}(\mu)\Gamma(\mu)}\int_0^\eta(\eta-\xi)^{\mu-1}\|\upsilon_m(\xi)-\upsilon(\xi)\|d\xi\right\}+\mathfrak{N}\|\upsilon_m(\eta)-\upsilon(\eta)\|
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{\mathfrak{M}\beta}{\alpha+\beta}\left\{\frac{1-\mu}{\mathfrak{A}(\mu)}\|\upsilon_m-\upsilon\|_C+\frac{\mu}{\mathfrak{A}(\mu)\Gamma(\mu)}\|\upsilon_m-\upsilon\|_C\int_0^T(T-\xi)^{\mu-1}d\xi\right\}\\
&\quad+\mathfrak{M}\left\{\frac{1-\mu}{\mathfrak{A}(\mu)}\|\upsilon_m-\upsilon\|_C+\frac{\mu}{\mathfrak{A}(\mu)\Gamma(\mu)}\|\upsilon_m-\upsilon\|_C\int_0^\eta(\eta-\xi)^{\mu-1}d\xi\right\}+\mathfrak{N}\|\upsilon_m-\upsilon\|_C\\
&\le \frac{\mathfrak{M}(1-\mu)}{\mathfrak{A}(\mu)}\left[\frac{\beta}{\alpha+\beta}+1\right]\|\upsilon_m-\upsilon\|_C+\frac{\mathfrak{M}\mu}{\mathfrak{A}(\mu)\Gamma(\mu)}\|\upsilon_m-\upsilon\|_C\left[\frac{\beta}{\alpha+\beta}+1\right]\frac{T^\mu}{\mu}+\mathfrak{N}\|\upsilon_m-\upsilon\|_C.
\end{aligned}
$$

This gives

$$
\|(\mathfrak{T}\upsilon_m)-(\mathfrak{T}\upsilon)\|_C\le\left(\frac{\mathfrak{M}}{\mathfrak{A}(\mu)}\frac{2\beta+\alpha}{\alpha+\beta}\left[1-\mu+\frac{T^\mu}{\Gamma(\mu)}\right]+\mathfrak{N}\right)\|\upsilon_m-\upsilon\|_C.
$$

Since

$$
\lim_{n\to\infty}\|\upsilon_m-\upsilon\|_C=0,
$$

we have

$$
\lim_{n\to\infty}\|\mathfrak{T}\upsilon_m-\mathfrak{T}\upsilon\|_C=0.
$$

We have established the mapping $\mathfrak{T}$ is continuous.

Next, consider the number R defined by

$$
R=\frac{\frac{\mathfrak{M}|\gamma|}{|\alpha+\beta|}+\mathfrak{M}_{\mathfrak{F}}}{1-\mathfrak{N}-\mathfrak{M}\left(\frac{|\beta|}{|\alpha+\beta|}+1\right)\left(\frac{1-\mu}{\mathfrak{A}(\mu)}+\frac{T^\mu}{\mathfrak{A}(\mu)\Gamma(\mu)}\right)}, \tag{3.10}
$$

where $\mathfrak{M}_{\mathfrak{F}}=\sup_{\eta\in J}\|\mathfrak{F}(\eta,0,0)\|$. Condition (3.8) implies $R>0$. Define

$$
B_R=\{\upsilon\in C:\|\upsilon\|_C\le R\}.
$$

Note that B_R is non-empty, convex, closed and bounded subset of E. Our aim is to prove that $\mathfrak{T}(B_R)\subseteq B_R$. To prove this, let's consider any $\eta\in J$ and $\upsilon\in B_r$. Then, by employing hypothesis (H1), we obtain

$$
\begin{aligned}
\|(\mathfrak{T}\upsilon)(\eta)\|&=\|\mathfrak{F}\left(\eta,\mathfrak{A}_\upsilon+{}^{\mathfrak{AB}}{}_0\mathfrak{I}^\mu_\eta\upsilon(\eta),\upsilon(\eta)\right)\|\\
&\le\|\mathfrak{F}\left(\eta,\mathfrak{A}_\upsilon+{}^{\mathfrak{AB}}{}_0\mathfrak{I}^\mu_\eta\upsilon(\eta),\upsilon(\eta)\right)-\mathfrak{F}(\eta,0,0)\|+\|\mathfrak{F}(\eta,0,0)\|\\
&\le\mathfrak{M}\|\mathfrak{A}_\upsilon+{}^{\mathfrak{AB}}{}_0\mathfrak{I}^\mu_\eta\upsilon(\eta)\|+\mathfrak{N}\|\upsilon(\eta)\|+\mathfrak{M}_{\mathfrak{F}}\\
&\le\mathfrak{M}\|\mathfrak{A}_\upsilon\|+\mathfrak{M}\,\|{}^{\mathfrak{AB}}{}_0\mathfrak{I}^\mu_\eta\upsilon(\eta)\|+\mathfrak{N}\|\upsilon(\eta)\|+\mathfrak{M}_{\mathfrak{F}}\\
&\le\mathfrak{M}\left(\frac{1}{|\alpha+\beta|}\left[|\gamma|+|\beta|\left({}^{\mathfrak{AB}}{}_0\mathfrak{I}^\mu_\eta\|\upsilon(\eta)\|\right)_{\eta=T}\right]\right)+\mathfrak{M}\,{}^{\mathfrak{AB}}{}_0\mathfrak{I}^\mu_\eta\|\upsilon(\eta)\|+\mathfrak{N}\|\upsilon(\eta)\|+\mathfrak{M}_{\mathfrak{F}}\\
&\le\mathfrak{M}\left(\frac{1}{|\alpha+\beta|}\left[|\gamma|+|\beta|R\left({}^{\mathfrak{AB}}{}_0\mathfrak{I}^\mu_\eta(1)\right)_{\eta=T}\right]\right)+\mathfrak{M}R\,{}^{\mathfrak{AB}}{}_0\mathfrak{I}^\mu_\eta(1)+\mathfrak{N}R+\mathfrak{M}_{\mathfrak{F}}\\
&\le\mathfrak{M}\left(\frac{1}{|\alpha+\beta|}\left[|\gamma|+|\beta|R\left(\frac{1-\mu}{\mathfrak{A}(\mu)}+\frac{\eta^\mu}{\mathfrak{A}(\mu)\Gamma(\mu)}\right)_{\eta=T}\right]\right)+\mathfrak{M}R\left(\frac{1-\mu}{\mathfrak{A}(\mu)}+\frac{\eta^\mu}{\mathfrak{A}(\mu)\Gamma(\mu)}\right)+\mathfrak{N}R+\mathfrak{M}_{\mathfrak{F}}\\
&\le\left(\frac{\mathfrak{M}|\gamma|}{|\alpha+\beta|}+\mathfrak{M}_{\mathfrak{F}}\right)+R\left\{\mathfrak{M}\left(\frac{|\beta|}{|\alpha+\beta|}+1\right)\left[\frac{1-\mu}{\mathfrak{A}(\mu)}+\frac{T^\mu}{\mathfrak{A}(\mu)\Gamma(\mu)}\right]+\mathfrak{N}\right\}.
\end{aligned}
$$

Using Eq (3.10), we have

$$\left(\frac{\mathfrak{M}|\gamma|}{|\alpha+\beta|}+\mathfrak{M}_{\mathfrak{F}}\right)=R\left(1-\mathfrak{M}\left(\frac{|\beta|}{|\alpha+\beta|}+1\right)\left[\frac{1-\mu}{\mathfrak{A}(\mu)}+\frac{T^{\mu}}{\mathfrak{A}(\mu)\Gamma(\mu)}\right]-\mathfrak{N}\right).$$

Applying it in the aforementioned inequality leads us to

$$\begin{aligned}\|(\mathfrak{T}\upsilon)(\eta)\| &= R\left(1-\mathfrak{M}\left(\frac{|\beta|}{|\alpha+\beta|}+1\right)\left[\frac{1-\mu}{\mathfrak{A}(\mu)}+\frac{T^{\mu}}{\mathfrak{A}(\mu)\Gamma(\mu)}\right]-\mathfrak{N}\right)\\ &\quad + R\left\{\mathfrak{M}\left(\frac{|\beta|}{|\alpha+\beta|}+1\right)\left[\frac{1-\mu}{\mathfrak{A}(\mu)}+\frac{T^{\mu}}{\mathfrak{A}(\mu)\Gamma(\mu)}\right]+\mathfrak{N}\right\}=R.\end{aligned}$$

Consequently, $\|\mathfrak{T}(\upsilon)\|_C \le R$, implying $\mathfrak{T}\upsilon \in B_R$, and the proof of $\mathfrak{T}(B_R) \subseteq B_R$ is completed. Moreover, for each $\upsilon \in B_R$, $\|\mathfrak{T}\upsilon\|_C \le R$ indicates that $\mathfrak{T}(B_R)$ is bounded.

To demonstrate $\mathfrak{T}(B_R)$ is equicontinuous, consider any $\eta_1, \eta_2 \in J$ with $\eta_1 \le \eta_2$. Then,

$$\begin{aligned}&\|(\mathfrak{T}\upsilon)(\eta_1)-(\mathfrak{T}\upsilon)(\eta_2)\|\\ =\ &\left\|\mathfrak{F}\left(\eta_1,\mathfrak{A}_{\upsilon}+{}^{\mathfrak{AB}}{}_0\mathfrak{I}^{\mu}_{\eta_1}\upsilon(\eta),\upsilon(\eta_1)\right)-\mathfrak{F}\left(\eta_2,\mathfrak{A}_{\upsilon}+{}^{\mathfrak{AB}}{}_0\mathfrak{I}^{\mu}_{\eta_2}\upsilon(\eta),\upsilon(\eta_2)\right)\right\|\\ \le\ &\mathfrak{L}|\eta_1-\eta_2|+\mathfrak{M}\left\|{}^{\mathfrak{AB}}{}_0\mathfrak{I}^{\mu}_{\eta_1}\upsilon(\eta)-{}^{\mathfrak{AB}}{}_0\mathfrak{I}^{\mu}_{\eta_2}\upsilon(\eta)\right\|+\mathfrak{N}\|\upsilon(\eta_1)-\upsilon(\eta_2)\|\\ \le\ &\mathfrak{L}|\eta_1-\eta_2|+\mathfrak{M}\left\|\left\{\frac{1-\mu}{\mathfrak{A}(\mu)}\upsilon(\eta_1)+\frac{\mu}{\mathfrak{A}(\mu)\Gamma(\mu)}\int_0^{\eta_1}(\eta_1-\xi)^{\mu-1}\upsilon(\xi)d\xi\right\}\right.\\ &\left.-\left\{\frac{1-\mu}{\mathfrak{A}(\mu)}\upsilon(\eta_2)+\frac{\mu}{\mathfrak{A}(\mu)\Gamma(\mu)}\int_0^{\eta_2}(\eta_2-\xi)^{\mu-1}\upsilon(\xi)d\xi\right\}\right\|+\mathfrak{N}\|\upsilon(\eta_1)-\upsilon(\eta_2)\|\\ \le\ &\mathfrak{L}|\eta_1-\eta_2|+\frac{\mathfrak{M}(1-\mu)}{\mathfrak{A}(\mu)}\|\upsilon(\eta_1)-\upsilon(\eta_2)\|+\frac{\mathfrak{M}\mu}{\mathfrak{A}(\mu)\Gamma(\mu)}\int_0^{\eta_1}\left\{(\eta_1-\xi)^{\mu-1}-(\eta_2-\xi)^{\mu-1}\right\}\|\upsilon\|_C d\xi\\ &+\frac{\mathfrak{M}\mu}{\mathfrak{A}(\mu)\Gamma(\mu)}\int_{\eta_1}^{\eta_2}(\eta_2-\xi)^{\mu-1}\|\upsilon\|_C d\xi+\mathfrak{N}\|\upsilon(\eta_1)-\upsilon(\eta_2)\|\\ \le\ &\mathfrak{L}|\eta_1-\eta_2|+\left(\frac{\mathfrak{M}(1-\mu)}{\mathfrak{A}(\mu)}+\mathfrak{N}\right)\|\upsilon(\eta_1)-\upsilon(\eta_2)\|+\frac{\mathfrak{M}R\mu}{\mathfrak{A}(\mu)\Gamma(\mu)}\int_0^{\eta_1}\left\{(\eta_1-\xi)^{\mu-1}-(\eta_2-\xi)^{\mu-1}\right\}d\xi\\ &+\frac{\mathfrak{M}R\mu}{\mathfrak{A}(\mu)\Gamma(\mu)}\int_{\eta_1}^{\eta_2}(\eta_2-\xi)^{\mu-1}d\xi\\ \le\ &\mathfrak{L}|\eta_1-\eta_2|+\left(\frac{\mathfrak{M}(1-\mu)}{\mathfrak{A}(\mu)}+\mathfrak{N}\right)\|\upsilon(\eta_1)-\upsilon(\eta_2)\|+\frac{\mathfrak{M}R\mu}{\mathfrak{A}(\mu)\Gamma(\mu+1)}\left\{-(\eta_2-\eta_1)^{\mu}-\eta_1^{\mu}+\eta_2^{\mu}\right\}\\ &+\frac{\mathfrak{M}R\mu}{\mathfrak{A}(\mu)\Gamma(\mu+1)}(\eta_2-\eta_1)^{\mu}.\end{aligned}$$

From the above inequality, it can be concluded that

$$\|(\mathfrak{T}\upsilon)\eta_1-(\mathfrak{T}\upsilon)\eta_2\|\to 0 \text{ as } |\eta_1-\eta_2|\to 0.$$

Therefore, the demonstration of the equicontinuity of $\mathfrak{T}(B_R)$ is complete. To finalize the proof, it is necessary to confirm that the mapping $\mathfrak{T}: B_R \to B_R$ satisfies the following condition:

$$\nu_c(\mathfrak{T}(\mathfrak{B})) \le k\nu_c(\mathfrak{B}),$$

for any closed subset $\mathfrak{V}$ of B_R and $0 < k < 1$. Employing the properties ν as described in Lemma 2.3, we have

$$
\begin{aligned}
&\nu(\mathfrak{T}(\upsilon)(\eta) : \upsilon \in \mathfrak{V}) \\
&= \nu\left(\mathfrak{F}\left(\eta, \mathfrak{A}_\upsilon + {}^{\mathfrak{AB}}{}_0\mathfrak{I}^\mu_\eta \upsilon(\eta), \upsilon(\eta)\right) : \upsilon \in \mathfrak{V}\right) \\
&\leq \mathfrak{M}\nu\left(\mathfrak{A}_\upsilon + {}^{\mathfrak{AB}}{}_0\mathfrak{I}^\mu_\eta \upsilon(\eta)\right) + \mathfrak{N}\nu(\upsilon(\eta)) \\
&\leq \mathfrak{M}\nu\left(\frac{1-\mu}{\mathfrak{A}(\mu)}\upsilon(\eta) + \frac{\mu}{\mathfrak{A}(\mu)\Gamma(\mu)}\int_0^\eta (\eta-\xi)^{\mu-1}\upsilon(\xi)d\xi\right) + \mathfrak{N}\nu(\upsilon(\eta)) \\
&\leq \mathfrak{M}\left(\frac{1-\mu}{\mathfrak{A}(\mu)}\nu(\upsilon(\eta)) + \frac{\mu}{\mathfrak{A}(\mu)\Gamma(\mu)}\int_0^\eta (\eta-\xi)^{\mu-1}\nu(\upsilon(\xi))d\xi\right) + \mathfrak{N}\nu(\upsilon(\eta)) \\
&\leq \mathfrak{M}\left(\frac{1-\mu}{\mathfrak{A}(\mu)}\nu_c(\mathfrak{V}) + \frac{\mu}{\mathfrak{A}(\mu)\Gamma(\mu)}\int_0^\eta (\eta-\xi)^{\mu-1}\nu_c(\mathfrak{V})d\xi\right) + \mathfrak{N}\nu_c(\mathfrak{V}) \\
&\leq \left\{\frac{\mathfrak{M}}{\mathfrak{A}(\mu)}\left(1-\mu+\frac{T^\mu}{\Gamma(\mu)}\right) + \mathfrak{N}\right\}\nu_c(\mathfrak{V}).
\end{aligned}
$$

This gives

$$\nu_c(\mathfrak{T}(\mathfrak{V})) \leq \left\{\frac{\mathfrak{M}}{\mathfrak{A}(\mu)}\left(1-\mu+\frac{T^\mu}{\Gamma(\mu)}\right) + \mathfrak{N}\right\}\nu_c(\mathfrak{V}). \tag{3.11}$$

Since

$$1 < \left(1 + \frac{|\beta|}{|\alpha+\beta|}\right),$$

we get

$$\mathfrak{N} + \frac{\mathfrak{M}}{\mathfrak{A}(\mu)}\left(1-\mu+\frac{T^\mu}{\Gamma(\mu)}\right) < \mathfrak{N} + \frac{\mathfrak{M}}{\mathfrak{A}(\mu)}\left(1 + \frac{|\beta|}{|\alpha+\beta|}\right)\left(1-\mu+\frac{T^\mu}{\Gamma(\mu)}\right).$$

The condition (3.8) allows us write

$$\mathfrak{N} + \frac{\mathfrak{M}}{\mathfrak{A}(\mu)}\left(1-\mu+\frac{T^\mu}{\Gamma(\mu)}\right) < 1.$$

Inequality (3.11) leads us to conclude that ν is a set contraction. In light of Lemma 2.4, the mapping T possesses a fixed point, which is the solution to the functional fractional integral equation (3.2). After adding this to Eq (3.1) the solution of problem (1.1)-(1.2) is provided.

In the subsequent theorem, we demonstrated that, with the same assumptions as given in Theorem 3.1, the existence result for nonlinear implicit FDEs (1.1)-(1.2) can be proved using the fixed point theorem of Mönch.

Theorem 3.2. *Assume that the hypothesis (H1) hold. Then, the nonlinear implicit AB-FDEs* (1.1)-(1.2) *has a solution.*

Proof. Consider the set B_R we used in the Theorem 3.1. Clearly B_R is a convex, closed and bounded subset of a Banach space E containing 0. Consider the operator $\mathfrak{T} : B_R \to B_R$ defined as in Theorem 3.1. We show that $\mathfrak{T}$ satisfies the requirement of Mönach's fixed point theorem. As established in the proof of Theorem 3.1, $\mathfrak{T}$ is bounded and continuous. Hence, it suffices to prove that the implication

$$\mathfrak{V} = c\bar{o}nvT(\mathfrak{V}) \text{ or } \mathfrak{V} = T(\mathfrak{V}) \cup \{0\} \implies \nu(\mathfrak{V}) = 0$$

holds for every subset $\mathfrak{V}$ of B_R.

Let $\mathfrak{V}$ be an equicontinuous subset of B_R such that $\mathfrak{V} \subset conv(\mathfrak{T}(\mathfrak{V}) \cup \{0\})$. Define

$$x : J \to [0, \infty) \text{ by } x(\eta) = \nu(\mathfrak{V}(\eta)).$$

Using Lemma 2.6, it follows that x is continuous. Further, utilizing Remark 3.2, Lemma 2.5 and the properties of the measure ν, for each $\eta \in J$, we obtain

$$\begin{aligned}
x(\eta) &= \nu(\mathfrak{V}(\eta)) = \nu(\mathfrak{T}(\mathfrak{V})(\eta) \cup \{0\}) \leq \nu(\mathfrak{T}(\mathfrak{V})(\eta)) \\
&= \nu\left(\mathfrak{F}\left(\eta, \mathfrak{A}_{\mathfrak{V}} + {}^{\mathfrak{AB}}{}_0\mathfrak{I}_\eta^\mu \mathfrak{V}(\eta), \mathfrak{V}(\eta)\right)\right) \\
&\leq \mathfrak{M}\nu\left(\mathfrak{A}_{\mathfrak{V}} + {}^{\mathfrak{AB}}{}_0\mathfrak{I}_\eta^\mu \mathfrak{V}(\eta)\right) + \mathfrak{N}\nu(\mathfrak{V}(\eta)) \leq \mathfrak{M}\nu\left({}^{\mathfrak{AB}}{}_0\mathfrak{I}_\eta^\nu \mathfrak{V}(\eta)\right) + \mathfrak{N}\nu(\mathfrak{V}(\eta)) \\
&\leq \mathfrak{M}\nu\left(\frac{1-\mu}{\mathfrak{A}(\mu)}\mathfrak{V}(\eta) + \frac{\mu}{\mathfrak{A}(\mu)\Gamma(\mu)}\int_0^\eta (\eta-\xi)^{\mu-1}\mathfrak{V}(\xi)d\xi\right) + \mathfrak{N}\nu(\mathfrak{V}(\eta)) \\
&\leq \mathfrak{M}\left(\frac{1-\mu}{\mathfrak{A}(\mu)}\nu(\mathfrak{V}(\eta)) + \frac{\mu}{\mathfrak{A}(\mu)\Gamma(\mu)}\int_0^\eta (\eta-\xi)^{\mu-1}\nu(\mathfrak{V}(\xi))d\xi\right) + \mathfrak{N}\nu(\mathfrak{V}(\eta)) \\
&= \mathfrak{M}\left(\frac{1-\mu}{\mathfrak{A}(\mu)}x(\eta) + \frac{\mu}{\mathfrak{A}(\mu)\Gamma(\mu)}\int_0^\eta (\eta-\xi)^{\mu-1}x(\xi)d\xi\right) + \mathfrak{N}x(\eta).
\end{aligned}$$

This gives

$$x(\eta) \leq \frac{\mathfrak{M}}{1-\mathfrak{N}}\, {}^{\mathfrak{AB}}{}_0\mathfrak{I}_\eta^\mu(x)(\eta).$$

By an application of Lemma 2.2 with $c(\eta) = 0$ and $d(\eta) = \dfrac{\mathfrak{M}}{1-\mathfrak{N}}$, we get $x(\eta) = 0$ for all $\eta \in J$. This gives $\mathfrak{V}(\eta) = 0$ for all $\eta \in J$. Consequently, this gives $\nu_c(\mathfrak{V}) = 0$. Applying Theorem 2.5, we conclude that $\mathfrak{T}$ has a fixed point $\upsilon \in B_R$, which act as a solution of functional equation (3.1). Utilizing this υ, we are able to obtain the required solution of nonlinear implicit AB-FDEs (1.1)-(1.2).

4. Implicit FDEs with non-local condition

Essential conditions are established in this section for the solution to exist about the following implicit FDE:

$$ {}^{\mathfrak{AB}}{}_0\mathfrak{D}_\eta^\mu\zeta(\eta) = \mathfrak{F}\left(\eta,\ \zeta(\eta),\ {}^{\mathfrak{AB}}{}_0\mathfrak{D}_\eta^\mu\zeta(\eta)\right),\ \eta \in J = [0, T],\ T > 0, \tag{4.1}$$

subject to nonlocal initial condition

$$\zeta(0) + \mathfrak{g}(\zeta) = \zeta_0. \tag{4.2}$$

Lemma 4.1. *Let ζ, ${}^{\mathfrak{AB}}{}_0\mathfrak{D}_\eta^\mu\zeta \in E$ and $\mathfrak{F}(0, \zeta(0), 0) = 0$. Then, $\zeta \in E$ is a solution of nonlinear implicit AB-FDEs* (4.1)-(4.2) *if and only if*

$$\zeta(\eta) = \zeta_0 - \mathfrak{g}(\zeta) + {}^{\mathfrak{AB}}{}_0\mathfrak{I}_\eta^\mu \upsilon_{\mathfrak{g}}(\eta),\ \eta \in J, \tag{4.3}$$

where υ_ζ is a solution of functional equation

$$\upsilon_\zeta(\eta) = \mathfrak{F}\left(\eta, \zeta_0 - \mathfrak{g}(\zeta) + {}^{\mathfrak{AB}}{}_0\mathfrak{I}_\eta^\mu \upsilon_\zeta(\eta), \upsilon_\zeta(\eta)\right),\ \eta \in J. \tag{4.4}$$

Proof. Corresponding to $\zeta \in E$, we put

$$ {}^{\mathfrak{AB}}{}_{0}\mathfrak{D}^{\mu}_{\eta}\zeta(\eta) = \upsilon_{\zeta}(\eta),\ \eta \in J. \tag{4.5}$$

Operating ${}^{\mathfrak{AB}}{}_{0}\mathfrak{I}^{\mu}_{\eta}$ on both sides of Eq (4.5), and using Lemma 2.1 and the nonlocal initial condition (1.3), we obtain (4.3).

Utilizing Eqs (4.3) and (4.5) in Eq (4.1), one can see that $\upsilon_{\zeta}(\eta)$ satisfies the functional equation (4.4).

On the other hand, suppose that $\zeta \in E$ is a solution of functional equation (4.3). Operating ${}^{\mathfrak{AB}}{}_{0}\mathfrak{D}^{\mu}_{\eta}$ on both sides of Eq (4.3) and using Lemma 2.1, we get

$$ {}^{\mathfrak{AB}}{}_{0}\mathfrak{D}^{\mu}_{\eta}\zeta(\eta) = {}^{\mathfrak{AB}}{}_{0}\mathfrak{D}^{\mu}_{\eta}\left[\zeta_0 - \mathfrak{g}(\zeta) + {}^{\mathfrak{AB}}{}_{0}\mathfrak{I}^{\mu}_{\eta}\upsilon_{\zeta}(\eta)\right] = {}^{\mathfrak{AB}}{}_{0}\mathfrak{D}^{\mu}_{\eta}\left[{}^{\mathfrak{AB}}{}_{0}\mathfrak{I}^{\mu}_{\eta}\upsilon_{\zeta}(\eta)\right] = \upsilon_{\zeta}(\eta),\ \eta \in J. \tag{4.6}$$

By utilizing Eqs (4.3) and (4.6), the functional represented by Eq (4.10) reduces to (4.1). Next, we verify that non-local initial condition stated in Eq (4.2). Since $\mathfrak{F}(0, \zeta(0), 0) = 0$, from Eq (4.3), we have

$$\begin{aligned} \zeta(0) &= \zeta_0 - \mathfrak{g}(\zeta) + \left[\frac{1-\mu}{\mathfrak{A}(\mu)}\upsilon_{\zeta}(\eta) + \frac{\mu}{\mathfrak{A}(\mu)\Gamma(\mu)}\int_0^{\eta}(\eta-\xi)^{\mu-1}\upsilon_{\zeta}(\xi)d\xi.\right]_{\eta=0} \\ &= \zeta_0 - \mathfrak{g}(\zeta) + \frac{1-\mu}{\mathfrak{A}(\mu)}\upsilon_{\zeta}(0) = \zeta_0 - \mathfrak{g}(\zeta) + \frac{1-\mu}{\mathfrak{A}(\mu)}\mathfrak{F}(0, \zeta(0), 0) \\ &= \zeta_0 - \mathfrak{g}(\zeta), \end{aligned}$$

which is the nonlocal condition (4.2).

The subsequent theorem establishes both the existence and uniqueness results for non-local initial value problem (4.1)-(4.2).

Theorem 4.1. *Let the function $\mathfrak{F} \in C^1(J \times E \times E, E)$ satisfies hypothesis (H1) and the function $\mathfrak{g}$ satisfies the condition*

(H2) $\|\mathfrak{g}(\zeta) - \mathfrak{g}(\zeta_1)\| \le \mathfrak{K}\|\zeta - \zeta_1\|_C,\ \ \mathfrak{K} \in R.$

Then, the non-local initial value problem (4.1)-(4.2) *has a solution provided*

$$\mathfrak{K}(1+\mathfrak{J}) < 1, \ \textit{and}\ \ \frac{\mathfrak{M}}{1-\mathfrak{N}} < \frac{\mathfrak{A}(\mu)}{1-\mu}, \tag{4.7}$$

where

$$\mathfrak{J} = \frac{\mathfrak{M}\mathfrak{A}(\mu)}{(1-\mathfrak{N})\mathfrak{A}(\mu) - \mathfrak{M}(1-\mu)}E_{\mu}\left(\frac{\mu\mathfrak{M}T^{\mu}}{(1-\mathfrak{N})\mathfrak{A}(\mu) - \mathfrak{M}(1-\mu)}\right). \tag{4.8}$$

Proof. Considering Lemma 4.1, we define the mapping $\tilde{\mathfrak{T}} : C \to C, C = C(J, E)$ as follows:

$$(\tilde{\mathfrak{T}}\zeta)(\eta) = \zeta_0 - \mathfrak{g}(\zeta) + {}^{\mathfrak{AB}}{}_{0}\mathfrak{I}^{\mu}_{\eta}\upsilon_{\zeta}(\eta),\ \eta \in J, \tag{4.9}$$

where υ_{ζ} is a solution of functional equation

$$\upsilon_{\zeta}(\eta) = \mathfrak{F}\left(\eta, \zeta_0 - \mathfrak{g}(\zeta) + {}^{\mathfrak{AB}}{}_{0}\mathfrak{I}^{\mu}_{\eta}\upsilon_{\zeta}(\eta), \upsilon_{\zeta}(\eta)\right),\ \eta \in J. \tag{4.10}$$

We demonstrate that the mapping $\tilde{\mathfrak{T}}$ fulfills all the conditions outlined in Lemma 2.4. For the continuity of $\tilde{\mathfrak{T}}$, consider any sequence $\{\zeta_m\}$ in C converging to $\zeta \in C$. Then, for any $\eta \in J$, using hypothesis (H2), we obtain

$$\begin{aligned} \|(\tilde{\mathfrak{T}}\zeta_m)\eta - (\tilde{\mathfrak{T}}\zeta)\eta\| &\leq \|\mathfrak{g}(\zeta_m) - \mathfrak{g}(\zeta)\| + \| {}^{\mathfrak{AB}}{}_0\mathfrak{I}^{\mu}_{\eta}(\upsilon_{\zeta_m}(\eta) - \upsilon_{\zeta}(\eta))\| \\ &\leq \mathfrak{K}\|\zeta_m - \zeta\|_C + {}^{\mathfrak{AB}}{}_0\mathfrak{I}^{\mu}_{\eta}\|\upsilon_{\zeta_m}(\eta) - \upsilon_{\zeta}(\eta))\|, \end{aligned} \tag{4.11}$$

where υ_{ζ_m} and υ_{ζ} are the solutions of functional equations

$$\upsilon_{\zeta_m}(\eta) = \mathfrak{F}\left(\eta, \zeta_0 - \mathfrak{g}(\zeta_m) + {}^{\mathfrak{AB}}{}_0\mathfrak{I}^{\mu}_{\eta}\upsilon_{\zeta_m}(\eta), \upsilon_{\zeta_m}(\eta)\right),\ \eta \in J,$$

and

$$\upsilon_{\zeta}(\eta) = \mathfrak{F}\left(\eta, \zeta_0 - \mathfrak{g}(\zeta) + {}^{\mathfrak{AB}}{}_0\mathfrak{I}^{\mu}_{\eta}\upsilon_{\zeta}(\eta), \upsilon_{\zeta}(\eta)\right),\ \eta \in J,$$

respectively. Using the hypothesis (H1), for any $\eta \in J$, we obtain

$$\begin{aligned} \|\upsilon_{\zeta_m}(\eta) - \upsilon_{\zeta}(\eta)\| &\leq \mathfrak{M}\|\mathfrak{g}(\zeta_m) - \mathfrak{g}(\zeta)\| + \mathfrak{M}\ {}^{\mathfrak{AB}}{}_0\mathfrak{I}^{\mu}_{\eta}\|\upsilon_{\zeta_m}(\eta) - \upsilon_{\zeta}(\eta)\| + \mathfrak{N}\|\upsilon_{\zeta_m}(\eta) - \upsilon_{\zeta}(\eta)\| \\ &\leq \mathfrak{M}\mathfrak{K}\|\zeta_m - \zeta\|_C + \mathfrak{M}\ {}^{\mathfrak{AB}}{}_0\mathfrak{I}^{\mu}_{\eta}\|\upsilon_{\zeta_m}(\eta) - \upsilon_{\zeta}(\eta)\| + \mathfrak{N}\|\upsilon_{\zeta_m}(\eta) - \upsilon_{\zeta}(\eta)\|. \end{aligned}$$

This gives

$$\|\upsilon_{\zeta_m}(\eta) - \upsilon_{\zeta}(\eta)\| \leq \frac{1}{1-\mathfrak{N}}\left(\mathfrak{M}\mathfrak{K}\|\zeta_m - \zeta\|_C + \mathfrak{M}\ {}^{\mathfrak{AB}}{}_0\mathfrak{I}^{\mu}_{\eta}\|\upsilon_{\zeta_m}(\eta) - \upsilon_{\zeta}(\eta)\|\right).$$

Application of Lemma 2.2 with $c(\eta) = \dfrac{\mathfrak{M}\mathfrak{K}\|\zeta_m - \zeta\|_C}{1-\mathfrak{N}}$ and $d(\eta) = \dfrac{\mathfrak{M}}{1-\mathfrak{N}}$ gives

$$\|\upsilon_{\zeta_m}(\eta) - \upsilon_{\zeta}(\eta)\| \leq \frac{\mathfrak{M}\mathfrak{K}B(\mu)}{(1-\mathfrak{N})B(\mu) - \mathfrak{M}(1-\mu)}\mathbb{E}_{\mu}\left(\frac{\mu\mathfrak{M}\eta^{\mu}}{(1-\mathfrak{N})B(\mu) - \mathfrak{M}(1-\mu)}\right)\|\zeta_m - \zeta\|_C,\ \eta \in J.$$

This gives

$$\|\upsilon_{\zeta_m} - \upsilon_{\zeta}\|_C \leq \frac{\mathfrak{M}\mathfrak{K}B(\mu)}{(1-\mathfrak{N})B(\mu) - \mathfrak{M}(1-\mu)}\mathbb{E}_{\mu}\left(\frac{\mu\mathfrak{M}T^{\mu}}{(1-\mathfrak{N})B(\mu) - \mathfrak{M}(1-\mu)}\right)\|\zeta_m - \zeta\|_C.$$

Since $\|\zeta_m - \zeta\|_C \to 0$, from above inequality we get $\|\upsilon_{\zeta_m} - \upsilon_{\zeta}\|_C \to 0$. Taking into account the previous estimates, using the inequality (4.11), we obtain $\|\tilde{\mathfrak{T}}\zeta_m - \tilde{\mathfrak{T}}\zeta\|_C \to 0$. Therefore, $\tilde{\mathfrak{T}}$ is continuous.

Define

$$\tilde{R} = \frac{(\|\zeta_0\| + \|\mathfrak{g}(0)\|)(1+\mathfrak{I}) + \mathfrak{I}\mathfrak{M}_{\mathfrak{F}}}{1 - \mathfrak{K}(1+\mathfrak{I})},\ \text{where } \mathfrak{M}_{\mathfrak{F}} = \sup_{\eta\in J}\|\mathfrak{F}(\eta,0,0)\|.$$

Condition (3.8) leads us to conclude $\tilde{R} > 0$. With this particular choice of $\tilde{R} > 0$, we define the set $B_{\tilde{R}} = \left\{\zeta \in C : \|\zeta\|_C \leq \tilde{R}\right\}$.

Next, we demonstrate that $\tilde{\mathfrak{T}}(B_{\tilde{R}}) \subseteq B_{\tilde{R}}$. Let any $\eta \in J$ and $\zeta \in B_r$. Then applying hypothesis (H2), we obtain

$$\begin{aligned} \|(\tilde{\mathfrak{T}}\zeta)\eta\| &\leq \|\zeta_0\| + \|\mathfrak{g}(\zeta) - \mathfrak{g}(0)\| + \|\mathfrak{g}(0)\| + {}^{\mathfrak{AB}}{}_0\mathfrak{I}^{\mu}_{\eta}\|\upsilon_{\zeta}(\eta)\| \\ &\leq \|\zeta_0\| + \mathfrak{K}\|\zeta\|_C + \|\mathfrak{g}(0)\| + {}^{\mathfrak{AB}}{}_0\mathfrak{I}^{\mu}_{\eta}\|\upsilon_{\zeta}(\eta)\| \\ &\leq \|\zeta_0\| + \mathfrak{K}R + \|\mathfrak{g}(0)\| + {}^{\mathfrak{AB}}{}_0\mathfrak{I}^{\mu}_{\eta}\|\upsilon_{\zeta}\|. \end{aligned} \tag{4.12}$$

For any $\eta \in J$, we discover that

$$\begin{aligned}\|\upsilon_\zeta(\eta)\| &\le \|\mathfrak{F}\left(\eta, \zeta_0 - \mathfrak{g}(\zeta) + {}^{\mathfrak{AB}}{}_0\mathfrak{I}^{\mu}_{\eta}\upsilon_\zeta(\eta), \upsilon_\zeta(\eta)\right) - \mathfrak{F}(\eta, 0, 0)\| + \|\mathfrak{F}(\eta, 0, 0)\| \\ &\le \mathfrak{M}\|\zeta_0 - \mathfrak{g}(\zeta) + {}^{\mathfrak{AB}}{}_0\mathfrak{I}^{\mu}_{\eta}\upsilon_\zeta(\eta)\| + \mathfrak{N}\|\upsilon_\zeta(\eta)\| + \mathfrak{M}_{\mathfrak{F}} \\ &\le \mathfrak{M}\left(\|\zeta_0\| + \mathfrak{K}R + \|\mathfrak{g}(0)\| + {}^{\mathfrak{AB}}{}_0\mathfrak{I}^{\mu}_{\eta}\|\upsilon_\zeta(\eta)\|\right) + \mathfrak{N}\|\upsilon_\zeta(\eta)\| + \mathfrak{M}_{\mathfrak{F}}.\end{aligned}$$

Therefore,

$$\|\upsilon_\zeta(\eta)\| \le \frac{1}{1-\mathfrak{N}}\left\{\mathfrak{M}_{\mathfrak{F}} + \mathfrak{M}\left(\|\zeta_0\| + \mathfrak{K}R + \|\mathfrak{g}(0)\|\right) + \mathfrak{M}\,{}^{\mathfrak{AB}}{}_0\mathfrak{I}^{\mu}_{\eta}\|\upsilon_\zeta(\eta)\|\right\}.$$

As an application of Lemma 2.2 with $c(\eta) = \frac{\mathfrak{M}_{\mathfrak{F}} + \mathfrak{M}(\|\zeta_0\| + \mathfrak{K}R + \|\mathfrak{g}(0)\|)}{1-\mathfrak{N}}$ and $d(\eta) = \frac{\mathfrak{M}}{1-\mathfrak{N}}$, we get

$$\|\upsilon_\zeta(\eta)\| \le \frac{\left[\mathfrak{M}_{\mathfrak{F}} + \mathfrak{M}\left(\|\zeta_0\| + \mathfrak{K}R + \|\mathfrak{g}(0)\|\right)\right]\mathfrak{A}(\mu)}{\mathfrak{A}(\mu)(1-\mathfrak{N}) - (1-\mu)\mathfrak{M}}\,\mathbb{E}_\mu\left(\frac{\mu\mathfrak{M}\eta^\mu}{B(\mu)(1-\mathfrak{N}) - \mathfrak{M}(1-\mu)}\right),\ \eta \in J.$$

This gives

$$\|\upsilon_\zeta(\eta)\| \le \frac{\left[\mathfrak{M}_{\mathfrak{F}} + \mathfrak{M}\left(\|\zeta_0\| + \mathfrak{K}R + \|\mathfrak{g}(0)\|\right)\right]\mathfrak{A}(\mu)}{\mathfrak{A}(\mu)(1-\mathfrak{N}) - (1-\mu)\mathfrak{M}}\,\mathbb{E}_\mu\left(\frac{\mu\mathfrak{M}T^\mu}{B(\mu)(1-\mathfrak{N}) - \mathfrak{M}(1-\mu)}\right),\ \eta \in J.$$

Utilizing the value of $\mathfrak{J}$ defined in (4.8), from above inequality, we obtain

$$\|\upsilon_\zeta(\eta)\| \le \frac{\left(\|\zeta_0\| + \mathfrak{K}R + \|\mathfrak{g}(0)\| + \mathfrak{M}_{\mathfrak{F}}\right)\mathfrak{J}}{\frac{1-\mu}{\mathfrak{A}(\mu)} + \frac{\mu}{\mathfrak{A}(\mu)\Gamma(\mu)}\frac{T^\mu}{\mu}} := \delta.$$

Thus from inequality (4.12), we have

$$\begin{aligned}\|(\tilde{\mathfrak{T}}\zeta)\eta\| &\le \|\zeta_0\| + \mathfrak{K}R + \|\mathfrak{g}(0)\| + {}^{\mathfrak{AB}}{}_0\mathfrak{I}^{\mu}_{\eta}\delta \\ &= \|\zeta_0\| + \mathfrak{K}R + \|\mathfrak{g}(0)\| + \delta\left\{\frac{1-\mu}{\mathfrak{A}(\mu)} + \frac{\mu}{\mathfrak{A}(\mu)\Gamma(\mu)}\int_0^\eta (\eta-\xi)^{\mu-1}d\xi\right\} \\ &\le \mathfrak{K}R + \|\zeta_0\| + \|\mathfrak{g}(0)\| + \delta\left\{\frac{1-\mu}{\mathfrak{A}(\mu)} + \frac{\mu}{\mathfrak{A}(\mu)\Gamma(\mu)}\frac{T^\mu}{\mu}\right\} \\ &\le \mathfrak{K}R + \|\zeta_0\| + \|\mathfrak{g}(0)\| + \left(\|\zeta_0\| + \mathfrak{K}R + \|\mathfrak{g}(0)\| + \mathfrak{M}_{\mathfrak{F}}\right)\mathfrak{J} \\ &= \mathfrak{K}R(1+\mathfrak{J}) + \left\{(\|\zeta_0\| + \|\mathfrak{g}(0)\|)(1+\mathfrak{J}) + \mathfrak{J}\mathfrak{M}_{\mathfrak{F}}\right\} \\ &= \mathfrak{K}R(1+\mathfrak{J}) + R\left\{1 - \mathfrak{K}(1+\mathfrak{J})\right\} = \tilde{R}.\end{aligned}$$

Thus, $\|\tilde{\mathfrak{T}}(\zeta)\|_C \le \tilde{R}$ for any $\zeta \in B_{\tilde{R}}$ and the proof of $\tilde{\mathfrak{T}}(B_{\tilde{R}}) \subseteq B_{\tilde{R}}$ is completed. For any $\zeta \in B_{\tilde{R}}$, we have $\|\tilde{\mathfrak{T}}\zeta\|_C \le \tilde{R}$ and hence $\tilde{\mathfrak{T}}(B_{\tilde{R}})$ is bounded.

To prove the equicontinuity of $\tilde{\mathfrak{T}}(B_{\tilde{R}})$, take any $\eta_1, \eta_2 \in J$ with $\eta_1 < \eta_2$. Then, we have

$$\begin{aligned}&\|(\tilde{\mathfrak{T}}\zeta)(\eta_1) - (\tilde{\mathfrak{T}}\zeta)(\eta_2)\| \\ &= \left\|\left(\zeta_0 - \mathfrak{g}(\zeta) + {}^{\mathfrak{AB}}{}_0\mathfrak{I}^{\mu}_{\eta}\upsilon_\zeta(\eta)\right)_{\eta=\eta_1} - \left(\zeta_0 - \mathfrak{g}(\zeta) + {}^{\mathfrak{AB}}{}_0\mathfrak{I}^{\mu}_{\eta}\upsilon_\zeta(\eta)\right)_{\eta=\eta_2}\right\| \\ &\le \frac{(1-\mu)}{\mathfrak{A}(\mu)}\|\upsilon_\zeta(\eta_1) - \upsilon_\zeta(\eta_2)\| + \frac{\mu}{\mathfrak{A}(\mu)\Gamma(\mu)}\int_0^{\eta_1}\left\{(\eta_1-\xi)^{\mu-1} - (\eta_2-\xi)^{\mu-1}\right\}\|\upsilon_\zeta(\xi)\|d\xi\end{aligned}$$

$$
\begin{aligned}
&+\frac{\mu}{\mathfrak{A}(\mu)\Gamma(\mu)}\int_{\eta_1}^{\eta_2}(\eta_2-\xi)^{\mu-1}\|\upsilon_\zeta(\xi)\|d\xi\\
&\le\frac{(1-\mu)}{\mathfrak{A}(\mu)}\|\upsilon_\zeta(\eta_1)-\upsilon_\zeta(\eta_2)\|+\frac{\delta\mu}{\mathfrak{A}(\mu)\Gamma(\mu)}\int_0^{\eta_1}\left\{(\eta_1-\xi)^{\mu-1}-(\eta_2-\xi)^{\mu-1}\right\}d\xi\\
&+\frac{\delta\mu}{\mathfrak{A}(\mu)\Gamma(\mu)}\int_{\eta_1}^{\eta_2}(\eta_2-\xi)^{\mu-1}d\xi\\
&\le\frac{(1-\mu)}{\mathfrak{A}(\mu)}\|\upsilon_\zeta(\eta_1)-\upsilon_\zeta(\eta_2)\|+\frac{\delta\mu}{\mathfrak{A}(\mu)\Gamma(\mu+1)}\left\{-(\eta_2-\eta_1)^{\mu}-\eta_1^{\mu}+\eta_2^{\mu}\right\}+\frac{\delta\mu}{\mathfrak{A}(\mu)\Gamma(\mu+1)}(\eta_2-\eta_1)^{\mu}.
\end{aligned}
$$

Since $\zeta\in B_R$, from above inequality it follows that $\|(\tilde{\mathfrak{T}}\zeta)\eta_2-(\tilde{\mathfrak{T}}\zeta)\eta_1\|\to 0$ whenever $\|\eta_2-\eta_1\|\to 0$. This proves that $\tilde{\mathfrak{T}}(B_R)$ is equicontinuous.

Finally, we show that function $\tilde{\mathfrak{T}}: B_R\to B_R$ satisfies the following requirement:

$$\nu_c(\tilde{\mathfrak{T}}(\mathfrak{V}))\le k\nu_c(\mathfrak{V}),$$

for any closed subset $\mathfrak{V}$ of B_R and $0<k<1$. Utilizing the properties of μ as outlined in Lemma 2.3, we deduce

$$
\begin{aligned}
\nu(\tilde{\mathfrak{T}}(\zeta)(\eta):\zeta\in\mathfrak{V}) &= \nu\left(\zeta_0-\mathfrak{g}(\zeta)+{}^{\mathfrak{AB}}{}_0\mathfrak{I}_\eta^\mu\upsilon_\zeta(\eta):\zeta\in\mathfrak{V}\right)\\
&= \nu(\zeta_0-\mathfrak{g}(\zeta))+\nu({}^{\mathfrak{AB}}{}_0\mathfrak{I}_\eta^\mu\upsilon_\zeta(\eta)))\\
&= \nu({}^{\mathfrak{AB}}{}_0\mathfrak{I}_\eta^\mu\upsilon_\zeta(\eta)))\le{}^{\mathfrak{AB}}{}_0\mathfrak{I}_\eta^\mu\nu(\upsilon_\zeta(\eta)).
\end{aligned}
$$

Thus,

$$\nu(\tilde{\mathfrak{T}}(\zeta)(\eta):\zeta\in\mathfrak{V})\le{}^{\mathfrak{AB}}{}_0\mathfrak{I}_\eta^\mu\nu(\upsilon_\zeta(\eta)). \tag{4.13}$$

Using functional relation (4.10), we find

$$
\begin{aligned}
\nu(\upsilon_\zeta(\eta)) &= \nu\left(\mathfrak{F}\left(\eta,\zeta_0-\mathfrak{g}(\zeta)+{}^{\mathfrak{AB}}{}_0\mathfrak{I}_\eta^\mu\upsilon_\zeta(\eta),\upsilon_\zeta(\eta)\right)\right)\\
&\le\mathfrak{M}\nu(\zeta_0-\mathfrak{g}(\zeta)+{}^{\mathfrak{AB}}{}_0\mathfrak{I}_\eta^\mu\upsilon_\zeta(\eta))+\mathfrak{N}\nu(\upsilon_\zeta(\eta))\\
&\le\mathfrak{M}\,{}^{\mathfrak{AB}}{}_0\mathfrak{I}_\eta^\mu\nu(\upsilon_\zeta(\eta))+\mathfrak{N}\nu(\upsilon_\zeta(\eta)).
\end{aligned}
$$

This gives

$$\nu(\upsilon_\zeta(\eta))\le\frac{\mathfrak{M}}{1-\mathfrak{N}}\,{}^{\mathfrak{AB}}{}_0\mathfrak{I}_\eta^\mu\nu(\upsilon_\zeta(\eta))).$$

An application of Lemma 2.2 with $c(\eta)=0$ and $d(\eta)=\dfrac{\mathfrak{M}}{1-\mathfrak{N}}$ gives

$$\nu(\upsilon_\zeta(\eta))=0,\ \eta\in J. \tag{4.14}$$

Using the Eq (4.14), the inequality (4.13) becomes

$$\nu(\tilde{\mathfrak{T}}(\zeta)(\eta):\zeta\in\mathfrak{V})=0,\eta\in J,$$

and hence we obtain

$$\nu_c(\tilde{\mathfrak{T}}(\mathfrak{V}))=0. \tag{4.15}$$

For any $\zeta \in \mathfrak{V}$, we have $\nu(\zeta(\eta)) \geq 0,\ \eta \in J$, hence for any $k \in (0,1)$, we get $k\nu(\zeta(\eta)) \geq 0,\ \eta \in J$, which gives

$$k\nu_c(\mathfrak{V}) \geq 0. \tag{4.16}$$

Combining (4.15) and (4.16), we obtain

$$\nu_c(\tilde{\mathfrak{T}}(\mathfrak{V})) \leq k\nu_c(\mathfrak{V}).$$

This proves that ν is a set contraction. Accordingly, Lemma 2.4 states that $\tilde{\mathfrak{T}}$ has a fixed point, which is the desired solution of the problem (1.1) subject to non-local initial condition (1.3).

Theorem 4.2. *Assume that the hypotheses (H1) and (H2) hold. Then, the nonlinear implicit AB-FDEs* (4.1)-(4.2) *has a solution if* $\mathfrak{K} < 1$.

Proof. Consider the set $B_{\tilde{R}}$ defined in Theorem 4.1. Note that $B_{\tilde{R}}$ is bounded, closed and convex subset of a Banach space C such that $0 \in B_{\tilde{R}}$. Further, we consider the same operator $\tilde{\mathfrak{T}} : B_{\tilde{R}} \to B_{\tilde{R}}$ defined in the proof of Theorem 4.1. We prove that $\tilde{\mathfrak{T}}$ satisfies the requirement of Monach fixed point theorem.

The boundedness and continuity of $\tilde{\mathfrak{T}}$ have previously been demonstrated in the proof of Theorem 4.1. Thus, all we have to do is demonstrate that the implication

$$\mathfrak{V} = c\bar{o}nv\tilde{\mathfrak{T}}(\mathfrak{V}) \text{ or } \mathfrak{V} = \tilde{\mathfrak{T}}(\mathfrak{V}) \cup \{0\} \implies \nu_c(\mathfrak{V}) = 0$$

holds for every subset $\mathfrak{V}$ of $B_{\tilde{R}}$. Let $\mathfrak{V}$ equicontiniuos subset of $B_{\tilde{R}}$ such that $\mathfrak{V} \subset conv(\tilde{\mathfrak{T}}(\mathfrak{V}) \cup \{0\})$. Define

$$x : J \to [0,\infty) \text{ by } x(\eta) = \nu(\mathfrak{V}(\eta)).$$

It is obvious from Lemma 2.6 that x is continuous. Lemma 2.5, Remark 3.2, and the properties of the measure μ allow us to get, for any $\eta \in J$,

$$\begin{aligned} x(\eta) = \nu(\mathfrak{V}(\eta)) &= \nu\left(\tilde{\mathfrak{T}}(\mathfrak{V})(\eta) \cup \{0\}\right) \leq \nu\left(\tilde{\mathfrak{T}}(\mathfrak{V})(\eta)\right) \\ &= \nu\left(\zeta_0 - \mathfrak{g}(\mathfrak{V})(\eta) + {}^{\mathfrak{AB}}{}_0\mathfrak{I}^{\mu}_{\eta}\mathfrak{V}(\eta)\right) \\ &= \nu(\mathfrak{g}(\mathfrak{V})(\eta)) + \nu({}^{\mathfrak{AB}}{}_0\mathfrak{I}^{\mu}_{\eta}\mathfrak{V}(\eta)). \end{aligned}$$

As mentioned in Remark 3.2, based on hypothesis (H2), we can write

$$\nu(\mathfrak{g}(\mathfrak{V})(\eta)) \leq \mathfrak{K}\nu(\mathfrak{V}(\eta)).$$

Therefore,

$$x(\eta) \leq \mathfrak{K}\nu(\mathfrak{V}(\eta)) + {}^{\mathfrak{AB}}{}_0\mathfrak{I}^{\mu}_{\eta}\nu(\mathfrak{V}(\eta)) = \mathfrak{K}x(\eta) + {}^{\mathfrak{AB}}{}_0\mathfrak{I}^{\mu}_{\eta}x(\eta).$$

This gives

$$x(\eta) \leq \frac{1}{1-\mathfrak{K}}\, {}^{\mathfrak{AB}}{}_0\mathfrak{I}^{\mu}_{\eta}x(\eta),\ \eta \in J.$$

Applying Lemma 2.2 with $c(\eta) = 0$ and $d(\eta) = \dfrac{1}{1-\mathfrak{K}}$, we obtain $x(\eta) = 0$ for all $\eta \in J$. Theorem 2.5 is then used to determine that $\tilde{\mathfrak{T}}$ has a fixed point $x \in B_{\tilde{R}}$, which forms a solution of (1.1) subject to nonlocal initial condition (1.3).

5. Examples

Example 5.1. Consider the following non-linear implicit AB-FDE:

$$ {}^{\mathfrak{AB}}{}_0\mathfrak{D}_\eta^{\frac{1}{2}}\zeta(\eta) = \frac{\zeta(\eta)}{5} + \left(\frac{9}{2}\eta^2\mathbb{E}_{\frac{1}{2},3}(-\sqrt{\eta}) - \frac{\eta^2+1}{5}\right) + \frac{1}{10}\left|{}^{\mathfrak{AB}}{}_0\mathfrak{D}_\eta^{\frac{1}{2}}\zeta(\eta)\right|, \eta \in J = [0,1], \tag{5.1} $$

with boundary condition

$$ \zeta(0) + 2\zeta(1) = 4. \tag{5.2} $$

Choose the normalizing function $\mathfrak{A}(\mu) = \mu - \mu^2 + 1,\ \mu \in [0,1]$. Then $B(0) = B(1) = 1$. Define, $\mathfrak{F} : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that

$$ \mathfrak{F}(\eta, \zeta, y) = \frac{\zeta}{5} + \left(\frac{9}{2}\eta^2\mathbb{E}_{\frac{1}{2},3}(-\sqrt{\eta}) - \frac{\eta^2+1}{5}\right) + \frac{1}{10}|y|. $$

Let any $(\eta_i, \zeta_i, y_i) \in J \times \mathbb{R} \times \mathbb{R}, (i = 1, 2)$. Then,

$$ \begin{aligned} &|\mathfrak{F}(\eta_1, \zeta_1, y_1) - \mathfrak{F}\eta_2, \zeta_2, y_2)| \\ &\leq \left|\left(\frac{\zeta_1}{5} + \left(\frac{9}{2}\eta_1^2\mathbb{E}_{\frac{1}{2},3}(-\sqrt{\eta_1}) - \frac{\eta_1^2+1}{5}\right) + \frac{1}{10}|y_1|\right) - \left(\frac{\zeta_2}{5} + \left(\frac{9}{2}\eta_2^2\mathbb{E}_{\frac{1}{2},3}(-\sqrt{\eta_2}) - \frac{\eta_2^2+1}{5}\right) + \frac{1}{10}|y_2|\right)\right| \\ &\leq \frac{1}{5}|\zeta_1 - \zeta_2| + \frac{1}{10}\left|\left(45\eta_1^2\mathbb{E}_{\frac{1}{2},3}(-\sqrt{\eta_1}) - 5\eta_1^2\right) - \left(45\eta_2^2\mathbb{E}_{\frac{1}{2},3}(-\sqrt{\eta_2}) - 5\eta_2^2\right)\right| + \frac{1}{10}||y_1| - |y_2|| \\ &\leq \frac{1}{5}|\zeta_1 - \zeta_2| + \frac{1}{10}\left|45\eta_1^2\mathbb{E}_{\frac{1}{2},3}(-\sqrt{\eta_1}) - 45\eta_2^2\mathbb{E}_{\frac{1}{2},3}(-\sqrt{\eta_2})\right| + \frac{1}{2}|\eta_2^2 - \eta_1^2| + \frac{1}{10}|y_1 - y_2| \\ &\leq \frac{1}{5}|\zeta_1 - \zeta_2| + \frac{1}{10}\left|45\eta_1^2\mathbb{E}_{\frac{1}{2},3}(-\sqrt{\eta_1}) - 45\eta_2^2\mathbb{E}_{\frac{1}{2},3}(-\sqrt{\eta_2})\right| + |\eta_2 - \eta_1| + \frac{1}{10}|y_1 - y_2|. \end{aligned} $$

Note that

$$ \begin{aligned} \frac{d}{d\eta}\left(45\eta^2\mathbb{E}_{\frac{1}{2},3}(-\sqrt{\eta})\right) &= 45\left\{2\eta\mathbb{E}_{\frac{1}{2},3}(-\sqrt{\eta}) + \eta^2\frac{d}{dt}\sum_{k=0}^{\infty}\frac{(-1)^k\eta^{\frac{k}{2}}}{\Gamma(\frac{k}{2}+3)}\right\} \\ &= 90t\mathbb{E}_{\frac{1}{2},3}(-\sqrt{\eta}) + 45\eta^2\sum_{k=1}^{\infty}\frac{(-1)^k\frac{k}{2}\eta^{\frac{k}{2}-1}}{\Gamma(\frac{k}{2}+3)} := p(\eta), \end{aligned} $$

which exists for all $\eta \in [0,1]$. Thus, $45\eta^2\mathbb{E}_{\frac{1}{2},2}(-\sqrt{\eta})$ is continuous and differentiable on [0, 1], by mean value theorem there exists ξ lying between η_1, η_2 such that

$$ \left|45\eta_1^2\mathbb{E}_{\frac{1}{2},2}(-\sqrt{\eta_1}) - 45\eta_2^2\mathbb{E}_{\frac{1}{2},2}(-\sqrt{\eta_2})\right| \leq p(\xi)\,|\eta_1 - \eta_2|. $$

Let $\mathfrak{L} = 1 + \dfrac{p}{10}$, where $p = \max_{\eta\in J} p(\eta)$. Then,

$$ |\mathfrak{F}(\eta_1, \zeta_1, y_1) - \mathfrak{F}(\eta_2, \zeta_2, y_2)| \leq \frac{1}{5}|\zeta_1 - \zeta_2| + \mathfrak{L}|\eta_2 - \eta_1| + \frac{1}{10}|y_1 - y_2|. $$

This proves that the function $\mathfrak{F}$ satisfies hypothesis (H1) with Lipschitz constants

$$\mathfrak{L} = 1 + \frac{p}{10},\ \mathfrak{M} = \frac{1}{5} \text{ and } \mathfrak{N} = \frac{1}{10}.$$

Then,

$$\mathfrak{N} + \mathfrak{M}\left(1 + \frac{|\beta|}{|\alpha+\beta|}\right)\left(\frac{1-\mu}{\mathfrak{A}(\mu)} + \frac{T^\mu}{\mathfrak{A}(\mu)\Gamma(\mu)}\right) = \frac{1}{10} + \frac{1}{5}\cdot\frac{2}{3}\left(\frac{\frac{1}{2}}{\frac{5}{4}} + \frac{1}{\Gamma(\frac{1}{2})}\right) = 0.3273 < 1.$$

Since the function $\mathfrak{F}$ meets every requirement of Theorem 3.1, the non-linear implicit AB-FDE (5.1) subject to boundary condition (5.2) has a solution according to Theorem 3.1. One can verify that

$$\zeta(\eta) = \eta^2 + 1,\ \eta \in [0, 1]$$

is the solution of the problem (5.1)-(5.2). Note that with this solution we find $\zeta(0) = 1$, which yields

$$\mathfrak{F}(0, \zeta(0), 0) = \frac{\zeta(0)}{5} - \frac{1}{5} = \frac{1}{5} - \frac{1}{5} = 0.$$

Example 5.2. Consider the non-linear implicit AB-FDE (5.1) subject to nonlocal initial condition

$$\zeta(0) + \mathfrak{g}(\zeta) = 1.7, \tag{5.3}$$

where $g : C(J, \mathbb{R}) \to \mathbb{R}$ is the function defined by

$$\mathfrak{g}(\zeta) = 0.4\,\zeta(0.5) + 0.1\,\zeta(1).$$

For any $\zeta, \zeta_1 \in C(J, \mathbb{R})$, we have

$$|\mathfrak{g}(\zeta) - \mathfrak{g}(\zeta_1)| \le 0.5\,|\zeta - \zeta_1|.$$

This proves $\mathfrak{g}$ satisfies the hypothesis with $\mathfrak{K} = 0.5$. Next, consider

$$\begin{aligned}
\mathfrak{I} &= \frac{\mathfrak{M}\mathfrak{A}(\mu)}{(1-\mathfrak{N})\mathfrak{A}(\mu) - \mathfrak{M}(1-\mu)} E_\mu\left(\frac{\mu\mathfrak{M}T^\mu}{(1-\mathfrak{N})\mathfrak{A}(\mu) - \mathfrak{M}(1-\mu)}\right)\\
&= \frac{0.2\times 1.25}{0.9\times 1.25 - 0.2\times 0.5} E_{\frac{1}{2}}\left(\frac{0.5\times 0.2}{0.9\times 1.25 - 0.2\times 0.5}\right)\\
&= \frac{0.5}{1.025} E_{\frac{1}{2}}\left(\frac{0.5\times 0.2}{1.025}\right) = 0.4878 E_{\frac{1}{2}}(0.0975).
\end{aligned} \tag{5.4}$$

Using expansion of one parameter Mittag-Leffler function, we have

$$\begin{aligned}
E_{\frac{1}{2}}(0.0975) &= \sum_{k=0}^{\infty} \frac{0.0975^k}{\Gamma(\frac{k}{2}+1)}\\
&= \left(1 + \frac{0.0975^2}{1!} + \frac{0.0975^4}{2!} + \cdots\right) + \left(\frac{0.0975}{\Gamma(\frac{3}{2})} + \frac{0.0975^3}{\Gamma(\frac{5}{2})} + \frac{0.0975^5}{\Gamma(\frac{7}{2})} + \ldots\right)
\end{aligned}$$

$$
\begin{aligned}
&= e^{0.0975^2} + \frac{0.0975}{\frac{1}{2}\sqrt{\pi}}\left(1 + \frac{2}{3}0.0975^2 + \frac{2^2}{3\times 5}0.0975^4 + \frac{2^3}{3\times 5\times 7}0.0975^6 + \cdots\right)\\
&\le e^{0.0975^2} + \frac{2\times 0.0975}{\sqrt{\pi}}\left(1 - \frac{2\times 0.0975^2}{3}\right)^{-1} = 1.0095 + 0.11\times 0.9936^{-1}\\
&= 1.1202. \qquad (5.5)
\end{aligned}
$$

Utilizing estimation (5.5) in Eq (5.4), we obtain

$$\mathfrak{I} < 0.4878 \times 1.1202 = 0.5464.$$

Therefore,

$$\mathfrak{K}(1 + \mathfrak{I}) < 0.5 \times 1.5464 = 0.7732 < 1.$$

Meeting all the conditions outlined in Theorem 4.1, it follows that the non-linear implicit AB-FDE given by Eq (5.1), accompanied by the non-local initial condition (5.3), have a solution. Further, one can verify that

$$\zeta(\eta) = \eta^2 + 1,\ \eta \in [0, 1]$$

is the solution of non-linear implicit AB-FDEs (5.1) with non-local initial condition (5.3).

6. Conclusions

We successfully proved significant existence results for nonlinear implicit fractional differential equations involving the nonsingular version of the Caputo fractional derivative defined by Atangana and Baleanu. The investigation of implicit fractional differential equations encompassed both boundary and nonlocal conditions, shedding light on the complexities of such equations. The foundational tools employed in this analysis were Kuratowski's measure of non-compactness and associated fixed point theorems, namely Darbo's fixed point theorem and Mönch's fixed point theorem. To illustrate our findings, we provided concrete examples and solutions to the considered problems with both boundary and non-local initial conditions.

Authors contributions

K. D. Kucche, S. T. Sutar and K. S. Nisar: Conceptualization, Formal analysis, Investigations, Methodology, Validation, Visualization, Writing–original drat, Writing–review & editing. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

The authors declare no conflicts of interest.

References

1. K. Diethelm, *The analysis of fractional differential equations*, Berlin, Heidelberg: Springer, 2010. https://doi.org/10.1007/978-3-642-14574-2
2. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, 2006.
3. V. Lakshimikantham, A. S. Vatsala, Basic theory of fractional differential equations, *Nonlinear Anal.*, **69** (2008), 2677–2682. https://doi.org/10.1016/j.na.2007.08.042
4. V. Lakshmikamntham, S. Leela, J. V. Devi, *Theory of fractional dynamic systems*, Cambridge Scientific Publisher, 2009.
5. I. Podlubny, *Fractional differential equations*, Academic Press, 1999.
6. D. Baleanu, A. Jajarmi, S. S. Sajjadi, D. Mozyrska, A new fractional model and optimal control of a tumor-immune surveillance with non-singular derivative operator, *Chaos*, **29** (2019), 083127. https://doi.org/10.1063/1.5096159
7. A. Jajarmi, D. Baleanu, S. S. Sajjadi, J. H. Asad, A new feature of the fractional Euler-Lagrange equations for a coupled oscillator using a nonsingular operator approach, *Front. Phys.*, **7** (2019), 196. https://doi.org/10.3389/fphy.2019.00196
8. A. Jajarmi, S. Arshad, D. Baleanu, A new fractional modelling and control strategy for the outbreak of dengue fever, *Phys. A*, **535** (2019), 122524. https://doi.org/10.1016/j.physa.2019.122524
9. A. Jajarmi, B. Ghanbari, D. Baleanu, A new and efficient numerical method for the fractional modeling and optimal control of diabetes and tuberculosis co-existence, *Chaos*, **29** (2019), 093111. https://doi.org/10.1063/1.5112177
10. M. Caputo, M. Fabrizio, A new definition of fractional derivative without singular kernel, *Progr. Fract. Differ. Appl.*, **1** (2015), 73–85.
11. A. Atangana, D. Baleanu, New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model, *Thermal Sci.*, **20** (2016), 763–769. http://dx.doi.org/10.2298/TSCI160111018A
12. A. Atangana, S. I. Araz, Step forward on nonlinear differential equations with the Atangana-Baleanu derivative: inequalities, existence, uniqueness and method, *Chaos Solitons Fract.*, **173** (2023), 113700. https://doi.org/10.1016/j.chaos.2023.113700
13. C. Dineshkumar, R. Udhayakumar, V. Vijayakumar, K. S. Nisar, A. Shukla, A note concerning to approximate controllability of Atangana-Baleanu fractional neutral stochastic systems with infinite delay, *Chaos Solitons Fract.*, **157** (2022), 111916. https://doi.org/10.1016/j.chaos.2022.111916
14. K. D. Kucche, S. T. Sutar, Analysis of nonlinear fractional differential equations involving Atangana-Baleanu-Caputo derivative, *Chaos Solitons Fract.*, **143** (2021), 110556. https://doi.org/10.1016/j.chaos.2020.110556
15. S. T. Sutar, K. D. Kucche, On nonlinear hybrid fractional differential equations with Atangana-Baleanu-Caputo derivative, *Chaos Solitons Fract.*, **143** (2021), 110557. https://doi.org/10.1016/j.chaos.2020.110557

16. S. T. Sutar, K. D. Kucche, Existence and data dependence results for fractional differential equations involving Atangana-Baleanu derivative, *Rend. Circ. Mat. Palermo Seri. 2*, **71** (2022), 647–663. https://doi.org/10.1007/s12215-021-00622-w

17. A. Jajarmi, S. Arshad, D. Baleuno, A new fractional modelling and control strategy for the outbreak of dengue fever, *Phys. A*, **535** (2019), 122524. https://doi.org/10.1016/j.physa.2019.122524

18. L. Byszewski, Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem, *J. Math. Anal. Appl.*, **162** (1991), 494–505. https://doi.org/10.1016/0022-247X(91)90164-U

19. L. Byszewski, Existence and uniqueness of a classical solution to a functional-differential abstract nonlocal Cauchy problem, *Int. J. Stoch. Anal.*, **12** (1999), 91–97. https://doi.org/10.1155/S1048953399000088

20. L. Byszewski, H. Akca, Existence of solutions of a semilinear functional-differential evolution nonlocal problem, *Nonlinear Anal.*, **34** (1998), 65–72. https://doi.org/10.1016/S0362-546X(97)00693-7

21. A. Bednarz, L. Byszewski, Continuous dependence of mild solutions on initial nonlocal data, of the nonlocal semilinear functional-differential evolution Cauchy problems of the first and second order, *Czasopismo Techniczne*, **5** (2018), 141–148. https://doi.org/10.4467/2353737XCT.18.080.8562

22. K. Balachandran, M. Chandrasekaran, Existence of solutions of nonlinear integrodifferential equations with nonlocal condition, *Int. J. Stoch. Anal.*, **10** (1997), 279–288. https://doi.org/10.1155/S104895339700035X

23. K. Balachandran, Existence and uniqueness of mild and strong solutions of nonlinear integro-differential equations with nonlocal condition, *Differ. Equ. Dyn. Syst.*, **6** (1998), 159–165.

24. K. Deimling, *Nonlinear functional analysis*, Berlin, Heidelberg: Springer, 1985. https://doi.org/10.1007/978-3-662-00547-7

25. G. Darbo, Punti uniti in trasformazioni a codominio non compatto, *Rend. Semin. Mat. Univ. Padova*, **24** (1955), 84–92.

26. H. Monch, Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces, *Nonlinear Anal.*, **4** (1980), 985–999. https://doi.org/10.1016/0362-546X(80)90010-3

27. J. Appell, Implicit functions, nonlinear integral equations, and the measure of noncompactness of the superposition operator, *J. Math. Anal. Appl.*, **83** (1981), 251–263. https://doi.org/10.1016/0022-247X(81)90261-4

28. M. Sarwar, S. Hussain, K. Abodayeh, S. Moonsuwan, T. Sitthiwirattham, Controllability of semilinear noninstantaneous impulsive neutral stochastic differential equations via Atangana-Baleanu Caputo fractional derivative, *Alex. Eng. J.*, **94** (2024), 149–158. https://doi.org/10.1016/j.aej.2024.03.022

29. E. Thilakraj, K. Kaliraj, C. Ravichandran, M. Manjula, New investigation on controllability of Sobolev-type Volterra-Fredholm functional integro-differential equation with non-local condition, *Results Control Optim.*, **15** (2024), 100418. https://doi.org/10.1016/j.rico.2024.100418

30. K. D. Kucche, J. J. Nieto, V. Venktesh, Theory of nonlinear implicit fractional differential equations, *Differ. Equ. Dyn. Syst.*, **28** (2020), 1–17. https://doi.org/10.1007/s12591-016-0297-7

31. K. D. Kucche, S. T. Sutar, Stability via successive approximations for nonlinear implicit fractional differential equations, *Moroccan J. Pure Appl. Anal.*, **3** (2017), 36–54. https://doi.org/10.1515/mjpaa-2017-0004

32. S. T. Sutar, K. D. Kucche, Global existence and uniqueness for implicit differential equation of arbitrary order, *Fract. Differ. Calc.*, **5** (2015), 199–208. https://doi.org//10.7153/fdc-05-17

33. S. T. Sutar, K. D. Kucche, Implicit fractional differential equations in Banach spaces via Picard and weakly Picard operator theory, *Acta Math. Univ. Comenianae*, **4** (2021), 403–420.

34. T. Abdeljawad, D. Baleanu, Integration by parts and its applications of a new nonlocal fractional derivative with Mittag-Leffler nonsingular kernel, *J. Nonlinear Sci. Appl.*, **10** (2017), 1098–1107. http://dx.doi.org/10.22436/jnsa.010.03.20

35. C. Ravichandran, K. Logeswari, F. Jarad, New results on existence in the framework of Atangana-Baleanu derivative for fractional integro-differential equations, *Chaos Solitons Fract.*, **125** (2019), 194–200. https://doi.org/10.1016/j.chaos.2019.05.014

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