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Bi-univalent functions subordinated to a three leaf function induced by multiplicative calculus

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Abstract: Our aim was to develop a new class of bi starlike functions by utilizing the concept of subordination, driven by the idea of multiplicative calculus, specifically multiplicative derivatives. Several restrictions were imposed, which were indeed strict constraints, because we have tried to work within the current framework or the design of analytic functions. To make the study more versatile, we redefined our new class of function with Miller-Ross Poisson distribution (MRPD), in order to increase the study's adaptability. We derived the first coefficient estimates and Fekete-Szegő inequalities for functions in this new class. To demonstrate the characteristics, we have provided a few examples.

Keywords: analytic function; bi-univalent function; convolution; Miller-Ross function; multiplicative calculus; subordination; Poisson distribution
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1. Introduction and definition

Let $\mathcal{A}$ represent the class of functions of the form

$$\varphi(\xi) = \xi + a_2\xi^2 + a_3\xi^3 + \ldots = \xi + \sum_{n=2}^{\infty} a_n\xi^n, \tag{1.1}$$

which are analytic in the open unit disc $\mathbf{\Delta} = \{\xi : |\xi| < 1\}$, and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Further, let $\mathcal{S}$ denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathbf{\Delta}$. The two well-known subclasses of $\mathcal{S}$, namely the classes of starlike and convex functions will be denoted by $\mathcal{S}^*$ and C respectively. Refer to [1, 2] for formal definitions of various subclasses of $\mathcal{S}$.

Based on Koebe's one-quarter theorem [1], every $\varphi \in \mathcal{S}$ has the compositional inverse φ^{-1} satisfying

$$\varphi^{-1}(\varphi(\xi)) = \xi, \ (\xi \in \Delta) \text{ and } \varphi(\varphi^{-1}(w)) = w, \ \ (w \in \Delta_\rho),$$

where $\rho \geq \frac{1}{4}$ is the radius of the image $\varphi(\Delta)$. From [3, p. 57], it is known that $\varphi^{-1}(w)$ has the normalized Taylor-Maclaurin series

$$\varphi^{-1}(w) = w + \sum_{n=2}^{\infty} b_n w^n, \ \ (w \in \Delta_\rho), \tag{1.2}$$

where

$$b_n = \frac{(-1)^{n+1}}{n!}|A_{ij}|,$$

and $|A_{ij}|$ is the $(n-1)^{th}$ order determinant whose entries are denoted by

$$|A_{ij}| = \begin{cases} [(i-j+1)n + j - 1]a_{i-j+2}, & \text{if } \ i+1 \geq j, \\ 0, & \text{if } \ i+1 < j. \end{cases}$$

Then,

$$\chi(w) = \varphi^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - \left(5a_2^2 - 5a_2a_3 + a_4\right)w^4 + \cdots. \tag{1.3}$$

Bi-starlike functions of the order $\alpha (0 < \alpha \leq 1)$ denoted by $\mathcal{S}^*_\Sigma(\alpha)$ and bi-convex functions of the order α denoted by $\mathcal{CV}_\Sigma(\alpha)$ were presented by Brannan and Taha in [4]. The first two Taylor-Maclaurin coefficients, namely $|a_2|$ and $|a_3|$, were shown to have non-sharp estimates for each of the function classes $\mathcal{S}^*_\Sigma(\alpha)$ and $\mathcal{CV}_\Sigma(\alpha)$ [4, 5]. Unfortunately, there is still an unresolved problem for each of the Taylor-Maclaurin coefficients $|a_n|$ $(n \in \mathbb{N} \setminus \{1, 2\})$. After studying many interesting subclasses of Σ, a number of authors (see [6–14] and the references cited therein) came to the conclusion that the estimations of the first two Taylor-Maclaurin coefficients, $|a_2|$ and $|a_3|$, are not sharp.

Let $\mathcal{P}$ signify the category of functions that are analytic in $\mathbf{\Delta}$ with $p(0) = 1$ and $Re\{p(\xi)\} > 0$ for all ξ in $\mathbf{\Delta}$. Throughout this paper, due to Ma and Minda [15], we let ψ to be an analytic function belonging to the class $\mathcal{P}$, and which has a series expansion of the form

$$\psi(\xi) = 1 + M_1\xi + \xi^2 + M_3\xi^3 + \cdots, \qquad (M_1 > 0; \ \xi \in \mathbf{\Delta}). \tag{1.4}$$

Using the notion of subordination, several scholars have recently examined the subclass of starlike functions $\mathcal{S}^*(\psi)$ subjected to satisfying the following criteria,

$$\mathcal{S}^*(\psi) = \left\{ \varphi \in \mathcal{A} : \frac{\xi \varphi'(\xi)}{\varphi(\xi)} \prec \psi(\xi) \right\}. \tag{1.5}$$

Recently, the notion of subordination has been used to develop several analytic function classes based on the geometrical interpretation of their image domains, such as the right half plane, circular disc, oval and petal type domains, conic domain, generalized conic domains, and the leaf-like domain, by varying ψ in (1.5). Lately, Gandhi [16] defined the class of starlike functions connected with three leaf functions as:

$$\mathcal{S}^*_{3\mathcal{L}} = \left\{ \varphi \in \mathcal{A} : \frac{\xi \varphi'(\xi)}{\varphi(\xi)} \prec 1 + \frac{4}{5}\xi + \frac{1}{5}\xi^4, \xi \in \mathbf{\Delta} \right\},$$

and studied certain subclasses of analytic functions defined by subordination to the three-leaf function.

1.1. Miller-Ross function (MRF)

Miller and Ross [17] proposed the special function as the basis of the solution of fractional order initial value problem, which is called the Miller-Ross function defined as

$$\mathbf{E}_{\nu,\mu}(\xi) = \xi^{\nu} e^{\mu\xi} \Theta^*(\nu, \mu\xi),$$

where Θ^* is the incomplete gamma function ([17], p. 314). Using the properties of the incomplete gamma functions, the Miller-Ross function (**MRF**) can easily be written as

$$\mathbf{E}_{\nu,\mu}(\xi) := \xi^{\nu} \sum_{n=0}^{\infty} \frac{(\mu\xi)^n}{\Gamma(n+\nu+1)}, \ \nu, \mu, \xi \in \mathbb{C}, \text{ with } \operatorname{Re}\nu > 0, \ \operatorname{Re}\mu > 0, \tag{1.6}$$

which can be stated as

$$\mathbf{E}_{\nu,\mu}(\xi) \equiv \xi^{\nu} \mathbf{E}_{1,1+\nu}(\mu\xi),$$

where $\mathbf{E}_{1,1+\nu}(\mu\xi)$ is the Mittag-Leffler function (**MLF**) of two parameters [18]. Some of the special values of the **MRF** can be given as follows:

$$\begin{aligned} \mathbf{E}_{\nu,\mu}(0) &= 0, \quad \operatorname{Re}(\nu) > 0 \\ \mathbf{E}_{0,\mu}(\xi) &= e^{\mu\xi}, \\ \mathbf{E}_{0,1}(\xi) &= e^{\xi}. \end{aligned}$$

Recently, Eker and Ece [19] showed that for $\mu > 0$ and if $\nu > 2\mu - 1$, then the normalized Miller-Ross function $\mathbf{E}_{\nu,\mu}$ is univalent and starlike in $\mathbf{\Delta}_{\frac{1}{2}} = \{\xi \in \mathbb{C} : |\xi| < \frac{1}{2}\}$. They also proved that if $\nu > (2 + \sqrt{2})\mu - 1$, then the normalized **MRF** is univalent and convex in $\mathbf{\Delta}_{\frac{1}{2}}$. For more details, we refer the reader to Miller and Ross [17].

In geometric function theory, the elementary distributions such as the Pascal, Poisson, logarithmic, binomial, and beta negative binomial have been partially studied from a theoretical point of view. For a detailed study, we refer the readers to [20–24]. The probability mass function of the Miller-Ross-type Poisson distribution (**MRPD**) is given by

$$\mathfrak{P}_{\nu,\mu}(m,k) := \frac{(m\mu)^k\, m^\nu}{\mathbf{E}_{\nu,\mu}(m)\Gamma(\nu+k+1)}, \quad k = 0, 1, 2, 3, \cdots \tag{1.7}$$

where $\nu > -1, \mu > 0$, $m > 0$, and $\mathbf{E}_{\nu,\mu}$ is the **MRF** given in (1.6). The Miller-Ross-type Poisson distribution is given by

$$\mathfrak{M}^m_{\nu,\mu}(\xi) = \xi + \sum_{n=2}^{\infty} \frac{(m\mu)^{n-1}\, m^\nu}{\mathbf{E}_{\nu,\mu}(m)\Gamma(\nu+n)}\xi^n. \tag{1.8}$$

The study of operators plays an important role in geometric function theory. Many differential and integral operators can be written in terms of convolution of certain analytic functions. It is observed that this formalism brings an ease in further mathematical exploration and also helps to understand the geometric and symmetric properties of such operators better. We consider the following operator:

$$\begin{aligned} \mathfrak{Y}^m_{\nu,\mu}\varphi(\xi) &= \varphi(\xi) * \mathfrak{M}^m_{\nu,\mu}(\xi) \\ &= \xi + \sum_{n=2}^{\infty} \Upsilon_n a_n \xi^n, \end{aligned} \tag{1.9}$$

where

$$\Upsilon_n = \frac{(m\mu)^{n-1}\, m^\nu}{\mathbf{E}_{\nu,\mu}(m)\Gamma(\nu+n)}, \tag{1.10}$$

and the symbol $*$ specifies the convolution (Hadamard product) of two series.

1.2. Multiplicative calculus

Bashirov, Kurpinar, and Özyapıin [25, pg. 37], (also see [26–28]) highlighted the importance of a calculus called *multiplicative calculus*, which is not versatile with respect to applications when compared with classical calculus, but is, nevertheless, very interesting and a useful mathematical tool for economics and finance. For a positive, real valued function $\varphi^* : \mathbb{R} \longrightarrow \mathbb{R}$, the multiplicative derivative is defined by

$$\varphi^*(x) = \lim_{h\to 0}\left(\frac{\varphi(x+h)}{\varphi(x)}\right)^{\frac{1}{h}} = e^{\frac{\varphi'(x)}{\varphi(x)}} = e^{[\ln\varphi(x)]'},$$

where $\varphi'(x)$ is the classical derivative. In a similar way, the n-th $*$-derivative of φ, which is denoted by $\varphi^{*(n)}$ for $n = 0, 1, \ldots$, with $\varphi^{*(0)} = \varphi$, can be defined by $\varphi^{*(n)} = e^{[\ln\varphi(x)]^{(n)}}$, provided the n-th derivative of φ at x exists.

The $*$-derivative of φ at a point ξ in a neighborhood of the complex plane where it is non-vanishing is given by

$$\varphi^*(\xi) = e^{\varphi'(\xi)/\varphi(\xi)} \quad \text{and} \quad \varphi^{*(n)}(\xi) = e^{\left[\varphi'(\xi)/\varphi(\xi)\right]^{(n)}},\ n = 1,\ 2, \ldots.$$

Motivated by the definition of a $*$-derivative, Karthikeyan and Murugusundaramoorthy in [29] (also see [30]) introduced and studied a class $\mathcal{R}(\psi)$ consisting of functions $\varphi \in \mathcal{A}$ satisfying the subordination condition

$$\frac{\xi e^{\frac{\xi^2\varphi'(\xi)}{\varphi(\xi)}}}{\varphi(\xi)} \prec \psi(\xi), \tag{1.11}$$

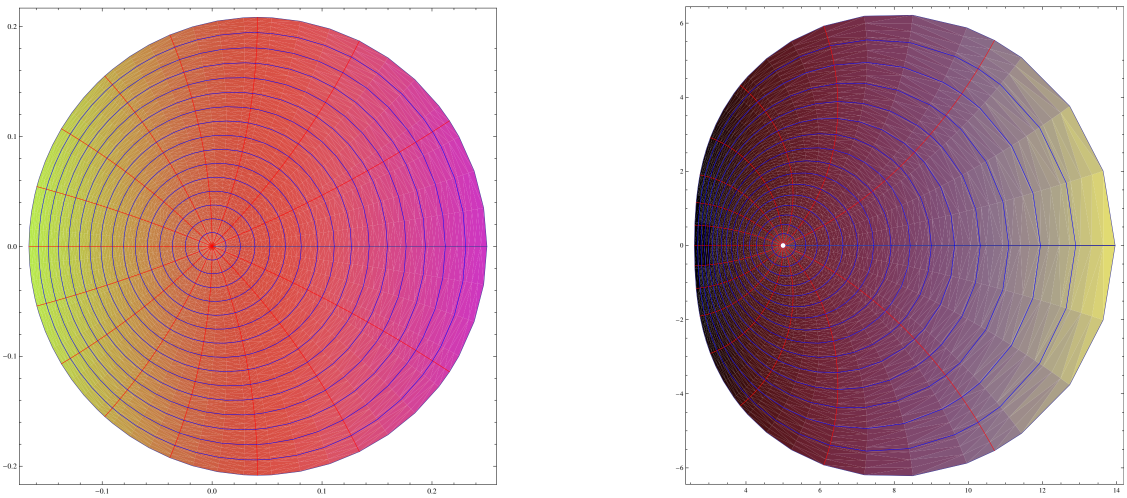
where ψ is defined as in (1.4). Similarly, we let $\mathcal{BR}(\psi)$ to denote the class of functions satisfying the conditions

$$\frac{\xi e^{\frac{\xi^2\varphi'(\xi)}{\varphi(\xi)}}}{\varphi(\xi)} \prec \psi(\xi), \quad \text{and} \quad \frac{we^{\frac{w^2\chi'(w)}{\chi(w)}}}{\chi(w)} \prec \psi(w). \tag{1.12}$$

Example 1.1. In this example, we will illustrate that a function $\varphi \in \mathcal{S}$ satisfying the condition (1.11) does not imply that its inverse function would satisfy the condition (1.11). Let $\varphi(\xi) = \frac{\xi}{5-\xi}$. The function $\varphi(\xi) = \frac{\xi}{(5-\xi)}$ is convex univalent and maps $\mathbf{\Delta}$ onto a circular-shaped region in the w-plane, see Figure 1(a). Whereas, the inverse function of φ is given by $\chi(w) = \frac{5w}{w+1}$. $\chi(w)$ is convex univalent in $\mathbf{\Delta}$ and maps the unit disc onto the left-hand side of $\frac{2}{5}$, see Figure 2(a). For $\chi = \varphi^{-1}$, let

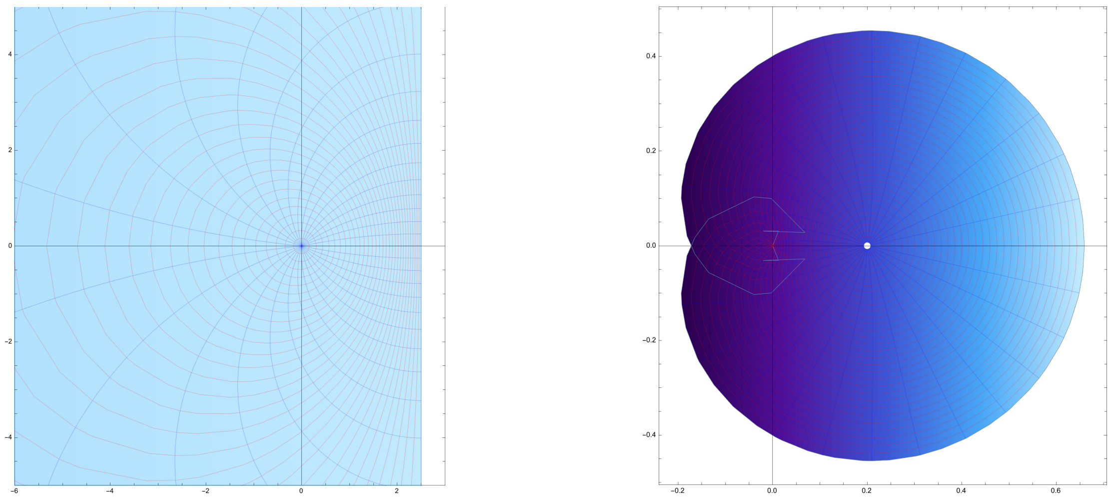
$$\Omega(\xi) = \frac{\xi e^{\frac{\xi^2\varphi'(\xi)}{\varphi(\xi)}}}{\varphi(\xi)} = (5-\xi)e^{\frac{5\xi}{5-\xi}} \quad \text{and} \quad \Upsilon(w) = \frac{we^{\frac{w^2\chi'(w)}{\chi(w)}}}{\chi(w)} = \frac{(1+w)}{5}e^{\frac{w}{1+w}}.$$

We can see that the function $\Omega(\xi)$ maps the unit disc onto a cardioid region in the right-half plane, see Figure 1(b). In addition, the function $\Upsilon(w) = \frac{(1+w)}{5}e^{\frac{w}{1+w}}$ maps the unit disc onto a cardioid region but the image does not lie in the right-half plane, see Figure 2(b). Further, Figures 1 and 2 illustrate that the function $\varphi(\xi) = \frac{\xi}{5-\xi}$ is in class $\mathcal{R}(\psi)$ but does not belong to $\mathcal{BR}(\psi)$.



(a) $\varphi(\xi) = \frac{\xi}{5-\xi}$ **(b)** $\Omega(\xi) = (5-\xi)e^{\frac{5\xi}{5-\xi}}$

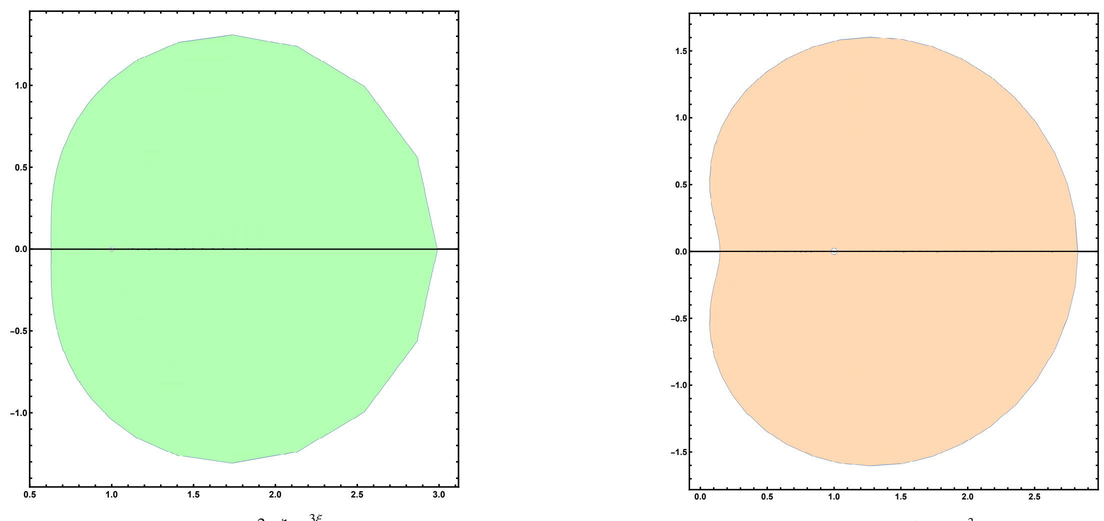
Figure 1. The images of $|\xi| < 1$ under $\varphi(\xi) = \frac{\xi}{5-\xi}$ and $\Omega(\xi)$, respectively.



(a) $\chi(w) = \frac{5w}{1+w}$ **(b)** $\Upsilon(w) = \frac{(1+w)}{5} e^{\frac{w}{1+w}}$

Figure 2. The images of the unit disc under $\chi(w) = \frac{5w}{1+w}$ and $\Upsilon(w) = \frac{(1+w)}{5} e^{\frac{w}{1+w}}$, respectively.

Example 1.2. In this example, we will show that the class $\mathcal{BR}(\psi)$ is non-empty. Let $\varphi(\xi) = \frac{3\xi}{3-\xi}$. The function $\varphi(\xi) = \frac{3\xi}{(3-\xi)}$ satisfies the normalization $\varphi(0) = \varphi'(\xi) - 1 = 0$. Whereas, the inverse function of φ is given by $\chi(w) = \frac{3w}{w+3}$. Figure 3 illustrates that the function $\varphi(\xi) = \frac{3\xi}{3-\xi}$ is in class $\mathcal{BR}(\psi)$.



(a) $L(\xi) = \frac{3-\xi}{3} e^{\frac{3\xi}{3-\xi}}$ **(b)** $M(w) = \frac{3+w}{3} e^{\frac{3w}{3+w}}$

Figure 3. The images of the unit disc under $L(\xi) = \frac{3-\xi}{3} e^{\frac{3\xi}{3-\xi}}$ and $M(w) = \frac{3+w}{3} e^{\frac{3w}{3+w}}$, respectively.

Motivated by the study on bi-univalent functions, see [7–14,31–34] and the references cited therein, and by the definition of $\mathcal{BR}(\psi)$, in Sections 2 and 3 of this article, we will obtain the initial coefficients of $|a_2|$, $|a_3|$, and the bounds on Fekete-Szegö results, of the function class defined in Definitions 2.1 and 3.1 related to the three leaf function $\Xi(\xi) = 1 + \xi + \frac{1}{5}\xi^4, \xi \in \mathbf{\Delta}$.

2. Bi-starlike functions $\mathfrak{Y}^m_{\nu,\mu}(\Xi)$

Definition 2.1. For $\nu > -1, \mu > 0,\ m > 0$ and $\mathfrak{Y}^m_{\nu,\mu}\varphi(\xi)$ defined as in (1.9), we denote the family $\mathfrak{Y}^m_{\nu,\mu}(\Xi)$ to be the class of functions $\varphi \in \mathcal{A}$ which satisfy the following conditions

$$\frac{\xi\, e^{\frac{\xi^2(\mathfrak{Y}^m_{\nu,\mu}\varphi(\xi))'}{\mathfrak{Y}^m_{\nu,\mu}\varphi(\xi)}}}{\left[\mathfrak{Y}^m_{\nu,\mu}\varphi(\xi)\right]} \prec \Xi(\xi) = 1 + \xi + \frac{1}{5}\xi^4, \xi \in \mathbf{\Delta},$$

and

$$\frac{w\, e^{\frac{w^2(\mathfrak{Y}^m_{\nu,\mu}\chi(w))'}{\mathfrak{Y}^m_{\nu,\mu}\chi(w)}}}{\left[\mathfrak{Y}^m_{\nu,\mu}\chi(w)\right]} \prec \Xi(w) = 1 + w + \frac{1}{5}w^4, w \in \mathbf{\Delta},$$

where $\chi = \varphi^{-1}$ is defined as in (1.3).

Remark 2.1. Notice that the function $\Xi(\xi)$ defined in Definition 2.1 is different from the function used in $\mathcal{S}^*_{3\mathcal{L}}$. The deviation was necessary so that we could obtain the coefficient inequalities for class $\mathfrak{Y}^m_{\nu,\mu}(\Xi)$. The function $\Xi(\xi) = 1 + \xi + \frac{1}{5}\xi^4$ is in the class $\mathcal{P}$ and maps the unit disc onto the three leaf region in the right-half plane, see Figure 4.

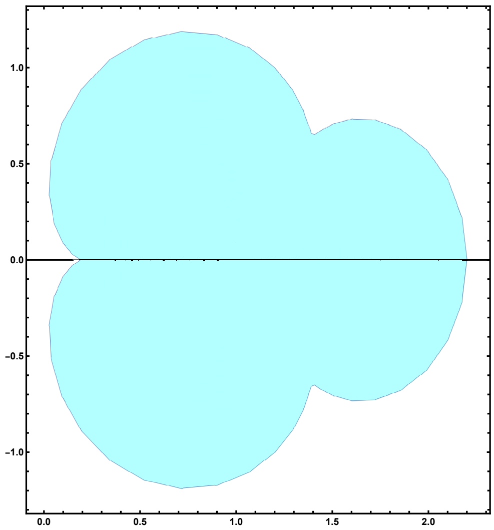


Figure 4. The image of the unit disc under $\Xi(\xi) = 1 + \xi + \frac{1}{5}\xi^4$.

Lemma 2.1. *[2] Let $\mathcal{P}$ be the family of all functions h that are analytic in $\mathbf{\Delta}$ with $\Re(h(\xi)) > 0$ and is given by*

$$h(\xi) = 1 + p_1\xi + p_2\xi^2 + \cdots, \qquad (\xi \in \Delta).$$

Then,

$$|p_k| \leq 2, \forall\, k.$$

Theorem 2.2. *Let $\varphi \in \mathfrak{Y}^m_{\nu,\mu}(\Xi)$ and let χ be the inverse of φ defined by* (1.3) *as*

$$\chi(w)=\varphi^{-1}(w)=w-a_2w^2+(2a_2^2-a_3)w^2-\left(5a_2^2-5a_2a_3+a_4\right)w^4+\cdots,\quad (|w|<r;\ r\geq 1/4).$$

Then,

$$|a_2| \;\leq\; \min\left\{\frac{2}{|\Upsilon_2|},\,\frac{\sqrt{2}}{\Upsilon_2},\,\sqrt{\frac{5}{2\left|3\Upsilon_2^2-\Upsilon_3\right|}}\right\}, \tag{2.1}$$

and

$$|a_3| \;\leq\; \min\left\{\frac{2}{|\Upsilon_2^2|}+\frac{2}{|\Upsilon_3|},\,\frac{5}{2\left|3\Upsilon_2^2-\Upsilon_3\right|}+\frac{2}{|\Upsilon_3|}\right\}, \tag{2.2}$$

where

$$\Upsilon_2=\frac{(m\mu)\,m^\nu}{\mathbf{E}_{\nu,\mu}(m)\Gamma(\nu+2)}\ \textit{and}\ \Upsilon_3=\frac{(m\mu)^2\,m^\nu}{\mathbf{E}_{\nu,\mu}(m)\Gamma(\nu+3)}. \tag{2.3}$$

Proof. Define the functions $p(\xi)$ and $q(w)$ by

$$p(\xi):=\frac{1+u(\xi)}{1-u(\xi)}=1+\vartheta_1\xi+\vartheta_2\xi^2+\cdots,$$

and

$$q(w):=\frac{1+v(w)}{1-v(w)}=1+\upsilon_1w+\upsilon_2w^2+\cdots,$$

where $u(\xi)$ and $v(w)$ are analytic in Δ, with $u(0)=0$, $v(0)=0$, and $|u(\xi)|<1$, $|v(w)|<1$ for all $\xi,w\in\Delta$. Then, $p(\xi)$ and $q(w)$ are analytic in Δ with $p(0)=1=q(0)$. Equivalently,

$$u(\xi):=\frac{p(\xi)-1}{p(\xi)+1}=\frac{1}{2}\left[\vartheta_1\xi+\left(\vartheta_2-\frac{\vartheta_1^2}{2}\right)\xi^2+\cdots\right],$$

and

$$v(w):=\frac{q(w)-1}{q(w)+1}=\frac{1}{2}\left[\upsilon_1w+\left(\upsilon_2-\frac{\upsilon_1^2}{2}\right)w^2+\cdots\right].$$

Since $u,v:\Delta\to\Delta$, the functions p,q have a positive real part and

$$|\vartheta_i|\leq 2\quad \textit{and}\quad |\upsilon_i|\leq 2\quad \textit{for each}\quad i\geq 2. \tag{2.4}$$

Now,

$$\begin{aligned}\Xi(u(\xi)) &= 1+u(\xi)+\frac{1}{5}\,(u(\xi))^4\\ &= 1+\frac{1}{2}\vartheta_1\xi+\frac{1}{2}\left(\vartheta_2-\frac{1}{2}\vartheta_1^2\right)\xi^2+\cdots,\end{aligned} \tag{2.5}$$

and

$$\Xi(v(w))=1+\frac{1}{2}\upsilon_1w+\frac{1}{2}\left(\upsilon_2-\frac{1}{2}\upsilon_1^2\right)w^2+\cdots. \tag{2.6}$$

Since $\varphi \in \mathfrak{Y}^m_{\nu,\mu}(\Xi)$, we have

$$\frac{\xi\, e^{\frac{\xi^2[\mathfrak{Y}^m_{\nu,\mu}\varphi(\xi)]'}{\mathfrak{Y}^m_{\nu,\mu}\varphi(\xi)}}}{\left[\mathfrak{Y}^m_{\nu,\mu}\varphi(\xi)\right]} = \Xi[u(\xi)], \tag{2.7}$$

and

$$\frac{w\, e^{\frac{w^2[\mathfrak{Y}^m_{\nu,\mu}\chi(w)]'}{\mathfrak{Y}^m_{\nu,\mu}\chi(w)}}}{\left[\mathfrak{Y}^m_{\nu,\mu}\chi(w)\right]} = \Xi[v(w)], \tag{2.8}$$

where ξ and w belongs to $\mathbf{\Delta}$. The left-hand side of (2.7) is given by

$$\frac{\xi e^{\frac{\xi^2[\mathfrak{Y}^m_{\nu,\mu}\varphi(\xi)]'}{\mathfrak{Y}^m_{\nu,\mu}\varphi(\xi)}}}{\left[\mathfrak{Y}^m_{\nu,\mu}\varphi(\xi)\right]} = 1 + [1 - a_2\Upsilon_2]\,\xi + \left[\frac{1}{2} + \Upsilon_2^2 a_2^2 - \Upsilon_3 a_3\right]\xi^2$$
$$+\left[\frac{1}{6} + \frac{a_2\Upsilon_2}{2} - a_2^2\Upsilon_2^2 - a_2^3\Upsilon_2^3 + a_3\Upsilon_3 + 2a_2a_3\Upsilon_2\Upsilon_3 - a_4\Upsilon_4\right]\xi^3 + \cdots . \tag{2.9}$$

Equating the coefficients of ξ, ξ^2, w, and w^2 in (2.7) and (2.8), we have

$$[1 - a_2\Upsilon_2] = \frac{1}{2}\vartheta_1, \tag{2.10}$$

$$\left[\frac{1}{2} + \Upsilon_2^2 a_2^2 - \Upsilon_3 a_3\right] = \frac{1}{2}\left(\vartheta_2 - \frac{\vartheta_1^2}{2}\right), \tag{2.11}$$

$$[1 + a_2\Upsilon_2] = \frac{1}{2}\upsilon_1, \tag{2.12}$$

and

$$\left[\frac{1}{2} + \Upsilon_2^2 a_2^2 - \Upsilon_3(2a_2^2 - a_3)\right] = \frac{1}{2}\left(\upsilon_2 - \frac{\upsilon_1^2}{2}\right). \tag{2.13}$$

From (2.10) and (2.12), we have

$$\vartheta_1 + \upsilon_1 = 4. \tag{2.14}$$

From (2.10),

$$|\Upsilon_2 a_2| \le \frac{1}{2}|\vartheta_1| + 1 = 2.$$

Thus,

$$|a_2| \le \frac{2}{|\Upsilon_2|}. \tag{2.15}$$

Squaring and adding Eqs (2.10) and (2.12), we get

$$\begin{aligned} 1 + \Upsilon_2^2 a_2^2 &= \frac{1}{8}\left(\vartheta_1^2 + \upsilon_1^2\right) \qquad (2.16)\\ |a_2^2| &\le \frac{1}{\Upsilon_2^2}\left(\frac{1}{8}\left(|\vartheta_1^2| + |\upsilon_1^2|\right) + 1\right). \end{aligned}$$

Using the known inequalities $|\vartheta_n| \leq 2$ and $|\upsilon_n| \leq 2$ for all $n \geq 2$, we get

$$\begin{aligned} |a_2|^2 &\leq \frac{2}{\Upsilon_2^2}, \\ |a_2| &\leq \frac{\sqrt{2}}{\Upsilon_2}. \end{aligned} \tag{2.17}$$

Adding Eqs (2.11) and (2.13), and then by using (2.16) in the resulting equation, we have

$$\begin{aligned} 1 + 2\left[\Upsilon_2^2 - \Upsilon_3\right] a_2^2 &= \frac{1}{2}(\vartheta_2 + \upsilon_2) - \frac{1}{4}\left(\vartheta_1^2 + \upsilon_1^2\right), \\ 2\left[\Upsilon_2^2 - \Upsilon_3\right] a_2^2 + 2\Upsilon_2^2 a_2^2 &= \frac{1}{2}(\vartheta_2 + \upsilon_2) - 3, \\ a_2^2 &= \frac{1}{2\left[3\Upsilon_2^2 - \Upsilon_3\right]}\left(\frac{1}{2}(\vartheta_2 + \upsilon_2) - 3\right). \end{aligned} \tag{2.18}$$

Applying the triangle inequality to (2.18) and using (2.4),

$$\begin{aligned} |a_2|^2 &\leq \frac{5}{2\left|3\Upsilon_2^2 - \Upsilon_3\right|}, \\ |a_2| &\leq \sqrt{\frac{5}{2\left|3\Upsilon_2^2 - \Upsilon_3\right|}}. \end{aligned} \tag{2.19}$$

To obtain (2.2), subtracting (2.13) with (2.11), we get

$$\begin{aligned} a_3 &= a_2^2 - \frac{(\vartheta_2 - \upsilon_2)}{4\Upsilon_3} + \frac{1}{8\Upsilon_3}\left(\vartheta_1^2 - \upsilon_1^2\right) \\ |a_3| &\leq |a_2|^2 + \frac{(|\vartheta_2| + |\upsilon_2|)}{4\Upsilon_3} + \frac{1}{8\Upsilon_3}\left(|\vartheta_1^2| + |\upsilon_1^2|\right) \\ &\leq |a_2|^2 + \frac{2}{\Upsilon_3}. \end{aligned} \tag{2.20}$$

Now, using (2.17) and (2.20) in the above equality, we can obtain the result (2.2). □

3. Sakacugchi-type bi-univalent functions $\mathfrak{V}_s^m(\nu, \mu; \Xi)$

By proposing the family $\mathcal{S}_s^* = \left\{\varphi \in \mathcal{S} : \Re e \frac{2\xi\varphi'(\xi)}{\varphi(\xi) - \varphi(-\xi)} > 0, \ \ \xi \in \mathbf{\Delta}\right\}$ of starlike functions concerning symmetric points in 1959, Sakaguchi [35] generalized the family $\mathcal{S}^*$ of starlike functions. In this section, we will define a class of analytic functions with respect to symmetric points analogous to the class $\mathfrak{Y}_{\nu,\mu}^m(\Xi)$. But such an analogous class requires some deviations in the analytic characterizations that were used in the Definition 2.1.

Definition 3.1. For $\nu > -1, \mu > 0, \ m > 0$ and $\mathfrak{Y}_{\nu,\mu}^m \varphi(\xi)$ defined as in (1.9), we denote the class of analytic functions $\varphi \in \mathcal{A}$ to be in $\mathfrak{V}_s^m(\nu, \mu; \Xi)$ if following conditions are satisfied

$$\frac{2\xi e^{\frac{\xi[\mathfrak{Y}_{\nu,\mu}^m \varphi(\xi)]'}{\mathfrak{Y}_{\nu,\mu}^m \varphi(\xi)}}}{e\left[\mathfrak{Y}_{\nu,\mu}^m \varphi(\xi) - \mathfrak{Y}_{\nu,\mu}^m \varphi(-\xi)\right]} \prec \Xi(\xi) = 1 + \xi + \frac{1}{5}\xi^4, \xi \in \mathbf{\Delta},$$

and

$$\frac{2we^{\frac{w[\mathfrak{Y}^m_{\nu,\mu}\chi(w)]'}{\mathfrak{Y}^m_{\nu,\mu}\chi(w)}}}{e\left[\mathfrak{Y}^m_{\nu,\mu}\chi(w)-\mathfrak{Y}^m_{\nu,\mu}\chi(-w)\right]} \prec \Xi(w)=1+w+\frac{1}{5}w^4, w\in\mathbf{\Delta},$$

where $\chi=\varphi^{-1}$ and $e=\exp(1)$.

Remark 3.1. Note that apart from replacing the denominator, $e^{\frac{\xi^2[\mathfrak{Y}^m_{\nu,\mu}\varphi(\xi)]'}{\mathfrak{Y}^m_{\nu,\mu}\varphi(\xi)}}$ in the class $\mathfrak{Y}^m_{\nu,\mu}(\Xi)$ has been replaced with $e^{\frac{\xi[\mathfrak{Y}^m_{\nu,\mu}\varphi(\xi)]'}{\mathfrak{Y}^m_{\nu,\mu}\varphi(\xi)}}$. This adaptation was required to avoid the redundancy.

Theorem 3.1. *Let $\varphi\in\mathfrak{V}^m_s(\nu,\mu;\Xi)$ and let χ be the inverse of φ given by* (1.3)*, which is*

$$\chi(w)=\varphi^{-1}(w)=w-a_2w^2+(2a_2^2-a_3)w^3-\left(5a_2^2-5a_2a_3+a_4\right)w^4+\cdots.$$

Then,

$$|a_2|\le\min\left\{\frac{1}{|\Upsilon_2|},\ \sqrt{\frac{2}{\left|\Upsilon_3+\Upsilon_2^2\right|}}\right\}, \tag{3.1}$$

and

$$|a_3|\le\min\left\{\frac{1}{|\Upsilon_3|}+\frac{1}{|\Upsilon_2|^2},\ \frac{2}{\left|2\Upsilon_3+\Upsilon_2^2\right|}+\frac{1}{|\Upsilon_3|}\right\}, \tag{3.2}$$

where $\Upsilon_n=\frac{(m\mu)^{n-1}\ m^{\nu}}{\mathbf{E}_{\nu,\mu}(m)\Gamma(\nu+n)}$.

Proof. $\varphi\in\mathfrak{V}^m_s(\nu,\mu;\Xi)$, and then we have

$$\frac{2\xi e^{\frac{\xi[\mathfrak{Y}^m_{\nu,\mu}\varphi(\xi)]'}{\mathfrak{Y}^m_{\nu,\mu}\varphi(\xi)}}}{e\left[\mathfrak{Y}^m_{\nu,\mu}\varphi(\xi)-\mathfrak{Y}^m_{\nu,\mu}\varphi(-\xi)\right]}=\psi\left[\frac{p(\xi)-1}{p(\xi)+1}\right], \tag{3.3}$$

and

$$\frac{2we^{\frac{w[\mathfrak{Y}^m_{\nu,\mu}\chi(w)]'}{\mathfrak{Y}^m_{\nu,\mu}\chi(w)}}}{e\left[\mathfrak{Y}^m_{\nu,\mu}\chi(w)-\mathfrak{Y}^m_{\nu,\mu}\chi(-w)\right]}=\psi\left[\frac{q(w)-1}{q(w)+1}\right], \tag{3.4}$$

where z and w belong to Δ. Through expansion and simplification, the left-hand side of (3.3) will yield

$$\frac{2\xi e^{\frac{\xi[\mathfrak{Y}^m_{\nu,\mu}\varphi(\xi)]'}{\mathfrak{Y}^m_{\nu,\mu}\varphi(\xi)}}}{e\left[\mathfrak{Y}^m_{\nu,\mu}\varphi(\xi)-\mathfrak{Y}^m_{\nu,\mu}\varphi(-\xi)\right]}=1+a_2\Upsilon_2\xi+\frac{1}{2}\left[2\Upsilon_3a_3-\Upsilon_2^2a_2^2\right]\xi^2$$
$$+\frac{1}{6}\left[a_2^3\Upsilon_2^3-12a_2a_3\Upsilon_2\Upsilon_3+18a_4\Upsilon_4\right]\xi^3+\cdots. \tag{3.5}$$

Equating the coefficients of ξ, ξ^2, w, and w^2 in (3.5) and (2.5), we have

$$a_2\Upsilon_2=\frac{1}{2}\vartheta_1, \tag{3.6}$$

$$\frac{1}{2}\left[2\Upsilon_3 a_3 - \Upsilon_2^2 a_2^2\right] = \frac{1}{2}\left(\vartheta_2 - \frac{\vartheta_1^2}{2}\right), \tag{3.7}$$

$$-a_2\Upsilon_2 = \frac{1}{2}\nu_1, \tag{3.8}$$

and

$$\frac{1}{2}\left[2\Upsilon_3(2a_2^2 - a_3) - \Upsilon_2^2 a_2^2\right] = \frac{1}{2}\left(\nu_2 - \frac{\nu_1^2}{2}\right). \tag{3.9}$$

From (3.6) and (3.8), we have

$$\vartheta_1 = -\nu_1. \tag{3.10}$$

Squaring and adding (3.6) and (3.8),

$$2a_2^2\Upsilon_2^2 = \frac{1}{4}\left(\vartheta_1^2 + \nu_1^2\right). \tag{3.11}$$

Thus,

$$a_2^2 = \frac{1}{8\Upsilon_2^2}\left(\vartheta_1^2 + \nu_1^2\right). \tag{3.12}$$

Adding Eqs (3.7) and (3.9), we have

$$a_2^2\left[2\Upsilon_3 - \Upsilon_2^2\right] = \frac{1}{2}(\vartheta_2 + \nu_2) - \frac{1}{4}\left(\vartheta_1^2 + \nu_1^2\right), \tag{3.13}$$

and then by using (3.11), we have

$$a_2^2 = \frac{(\vartheta_2 + \nu_2)}{2\left[2\Upsilon_3 + \Upsilon_2^2\right]}. \tag{3.14}$$

In the light of the known inequalities given in (2.4), (3.14) reduces to the result (3.1). To obtain (3.2), subtracting (3.9) with (3.7), we get

$$a_3 = a_2^2 + \frac{(\vartheta_2 - \nu_2)}{4\Upsilon_3}. \tag{3.15}$$

Using (3.12) in (3.15), we obtain

$$a_3 = \frac{\left(\vartheta_1^2 + \nu_1^2\right)}{8\Upsilon_2^2} + \frac{(\nu_2 - \vartheta_2)}{4\Upsilon_3}. \tag{3.16}$$

Similarly, applying (3.14) in (3.15), we obtain

$$a_3 = \frac{(\vartheta_2 + \nu_2)}{2\left[2\Upsilon_3 + \Upsilon_2^2\right]} + \frac{(\vartheta_2 - \nu_2)}{4\Upsilon_3}. \tag{3.17}$$

In view of (3.16) and (3.17), we can obtain the result (3.2) using (2.4).

□

3.1. Fekete-Szegö problem

Utilizing a_2^2 and a_3 values, and motivated by Zaprawa's recent work [33] as given in the below lemma, we prove the Fekete-Szegö problem for $\varphi \in \mathfrak{Y}_{\nu,\mu}^m(\Xi)$ in the following theorem.

Lemma 3.2. *[33] Let $l_1, l_2 \in \mathbb{R}$ and $p_1, p_2 \in \mathbb{C}$. If $|p_1|, |p_2| < \zeta$, then,*

$$|(l_1 + l_2)p_1 + (l_1 - l_2)p_2| \leq \begin{cases} 2|l_1|\zeta & , \quad |l_1| \geq |l_2| \\ 2|l_2|\zeta & , \quad |l_1| \leq |l_2| \end{cases}$$

Theorem 3.3. *For $\hbar \in \mathbb{R}$, and let $\varphi \in \mathfrak{Y}_{\nu,\mu}^m(\Xi)$ be of the form* (1.1)*. Then,*

$$\left|a_3 - \hbar a_2^2\right| \leq \begin{cases} \frac{1}{|\Upsilon_3|}, & 0 \leq |\phi(\hbar, \wp)| \leq \frac{1}{4\Upsilon_3}, \\ \\ 4\left|\phi(\hbar, \xi)\right|, & |\phi(\hbar, \wp)| \geq \frac{1}{4\Upsilon_3}. \end{cases}$$

Proof. It follows from (3.14) and (3.16) that

$$\begin{aligned} a_3 - \hbar a_2^2 &= \frac{(\vartheta_2 - \nu_2)}{4\Upsilon_3} + (1 - \hbar)\, a_2^2 \\ &= \frac{(\vartheta_2 - \nu_2)}{4\Upsilon_3} + (1 - \hbar)\, \frac{(\vartheta_2 + \nu_2)}{2\left[2\Upsilon_3 + \Upsilon_2^2\right]} \\ &= \left(\frac{(1 - \hbar)}{2\left[2\Upsilon_3 + \Upsilon_2^2\right]} + \frac{1}{4\Upsilon_3}\right)\vartheta_2 + \left(\frac{(1 - \hbar)}{2\left[2\Upsilon_3 + \Upsilon_2^2\right]} - \frac{1}{4\Upsilon_3}\right)\nu_2, \end{aligned}$$

where

$$\phi(\hbar, \wp) = \frac{(1 - \hbar)}{2\left[2\Upsilon_3 + \Upsilon_2^2\right]}.$$

According to Lemma 3.2, we get

$$\left|a_3 - \hbar a_2^2\right| \leq \begin{cases} \frac{1}{|\Upsilon_3|}, & 0 \leq |\phi(\hbar, \wp)| \leq \frac{1}{4\Upsilon_3}, \\ \\ 4\left|\phi(\hbar, \xi)\right|, & |\phi(\hbar, \wp)| \geq \frac{1}{4\Upsilon_3}. \end{cases}$$

□

Fixing $\hbar = 1$ in Theorem 3.3, we get the following result:

Corollary 3.4. *If $\varphi \in \mathfrak{Y}_{\nu,\mu}^m(\Xi)$ is of the form* (1.1)*, then,*

$$\left|a_3 - a_2^2\right| \leq \frac{1}{|\Upsilon_3|}.$$

4. Conclusions

We endeavor to create a fresh category of bi-starlike function classes $\mathfrak{Y}^m_{\nu,\mu}(\Xi)$ and $\mathfrak{B}^m_s(\nu,\mu;\Xi)$ subordinating to a three leaf domain, guided by the principles of multiplicative calculus, particularly multiplicative derivatives. In an effort to enhance the flexibility of our research, we redefine our innovative function class using the Miller-Ross Poisson distribution. We have obtained the initial coefficient estimates and Fekete-Szegő inequalities for functions belonging to this novel class. For different choices of the function parameters involved in the Definitions 2.1 and 3.1, the function classes $\mathfrak{Y}^m_{\nu,\mu}(\Xi)$ and $\mathfrak{B}^m_s(\nu,\mu;\Xi)$ reduces to classes having good geometrical implications but do not reduce to well-known classes like starlike, convex, and spiral-like. So, our main results have a lot of applications, but here we restricted ourselves to pointing out only a few of them. Moreover, one can extend the study in the future for new subclasses of bi-univalent functions influenced by multiplicative calculus, subordinating with different choices of the function Ξ like Gregory coefficients [9], Van der Pol numbers (VPN) [36], and a Vertical strip domain [37].

Author contributions

G. Murugusundaramoorthy, K. Vijaya and K. R. Karthikeyan: conceptualization, formal analysis, validation, methodology, data interpretation and visualization, resouces, writing original draft, manuscript review and editing, supervision; Sheza M. El-Deeb: formal analysis, investigation, validation, data interpretation and visualization, manuscript review and editing, supervision, project administration; Jong-Suk Ro: methodology, formal analysis, validation, writing original draft, data interpretation, supervision, project administration, resource management for the project and funding acquisition to pay APC. The authors have read and agreed to the published version of the manuscript.

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Conflict of interest

All authors declare that they have no conflict of interest.

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