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Research article

Investigating the characteristics of Clifford hypersurfaces and the unit sphere via a minimal immersion in S^{n+1}

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Abstract: In this article, we find the different sufficient conditions for a compact minimal hypersurface M of the unit sphere $S^{n+1}, n \in \mathbb{Z}^+$ to be the Clifford hypersurface $S^\ell(\sqrt{\frac{\ell}{n}}) \times S^m(\sqrt{\frac{m}{n}})$, where $\ell, m \in \mathbb{Z}^+$, $\ell + m = n$ or the sphere S^n. This classification is achieved by applying constraints to the tangent and normal components of the immersion.

Keywords: hypersurface; minimal hypersurface; Clifford hypersurface; conformal vector field; curvature tensor field; Ricci operator
Mathematics Subject Classification: 53C05, 53C20, 53C40

1. Introduction

In 1873, Clifford-Klein space forms made their way into mathematics history with a talk given by W. K. Clifford at the British Association for the Advancement of Sciences meeting in Bradford in September 1873 and a paper he published in June of the same year. Clifford's talk was titled on a surface of zero curvature and finite extension, and this is the only information that is available in the meeting proceedings. However, we have further information about it because to F. Klein, who attended Clifford's discussion and provided various versions of it [11]. In the context of elliptic geometry—which Clifford conceived in Klein's way as the geometry of the part of projective space limited by a purely imaginary quartic—Clifford described a closed surface which is locally flat, the today so-called Clifford surface (this name was introduced by Klein [11]). This surface is constructed by using Clifford parallels; Bianchi later provided a description by moving a circle along an elliptic straight line in such a way that it is always orthogonal to the straight line. So Clifford's surface is the analogue of a cylinder; but since it closed - it is often called a torus.

Let (M, g) be a compact minimal hypersurface of the unit sphere S^{n+1} with the immersion $\psi : M \to S^{n+1}$. Then we have the immersion $\overline{\psi} = \iota \circ \psi : M \to R^{n+2}$ in the Euclidean space R^{n+2}, where $\iota : S^{n+1} \to R^{n+2}$ is the inclusion map. The problem of finding sufficient conditions for the hypersurface M of the unit sphere S^{n+1} to be the Clifford hypersurface $S^{\ell}(\sqrt{\frac{\ell}{n}}) \times S^{m}(\sqrt{\frac{m}{n}})$, where $\ell, m \in \mathbb{Z}^+, \ell + m = n$ or the unit sphere S^n is one of an by the interesting questions in the differential geometry and specifically in the geometry of the hypersurfaces in a sphere. Many authors including the author of the article with others studied this problem in various ways (cf. [1, 4–8]). For notation and background information, the interested reader is referred to [2, 3].

We denote by $A = A_N$ and $A_{\overline{N}} = -I$ the shape operators of the immersion ψ and $\overline{\psi}$ corresponding to the unit normal vector field $N \in \mathfrak{X}(S^{n+1})$ and $\overline{N} \in \mathfrak{X}(R^{n+2})$, respectively, where $\mathfrak{X}(S^{n+1})$ and $\mathfrak{X}(R^{n+2})$ are the Lie algebras of smooth vector field on S^{n+1} and R^{n+2}, respectively.

Note that we can express the immersion $\overline{\psi}$ as

$$\overline{\psi} = v + f\overline{N} = u + \rho N + f\overline{N},$$

where v is the vector field tangential to S^{n+1}, u is the vector field tangential to M, $\rho =< \overline{\psi}, N >$, $f =< \overline{\psi}, \overline{N} >$ and $<,>$ is the Euclidean metric.

In [7], we obtained the Wang-type inequality [12] for compact minimal hypersurfaces in the unit sphere S^{2n+1} with Sasakian structure and used those inequalities to characterize minimal Clifford hypersurfaces in the unit sphere. Indeed, we obtained two different characterisations (see [7, Theorems 1 and 2]).

In this paper, our main aim is to obtain the classification by imposing conditions over the tangent and normal components of the immersion. Precisely, we will prove that if $\rho^2(1-\beta)+\varphi^2(\alpha-1\alpha) \geq 0$ and $Z(\varphi) = \{x \in M : \varphi(x) = 0\}$ is a discrete set where α and β are two constants satisfies $(n-1)\alpha \leq Ric \leq \beta, \beta \leq 1$, then M isometric to the Clifford hypersurface $S^{\ell}(\sqrt{\frac{\ell}{n}}) \times S^{m}(\sqrt{\frac{m}{n}})$, where $\ell, m \in \mathbb{Z}^+, \ell + m = n$ (cf. Theorem 3.1). Also, in this paper, we will show that if M has constant scalar curvature S with u is a nonzero vector field and $\nabla\rho = \lambda \nabla f$, $\lambda \in R^*$, then M isometric to the Clifford hypersurface $S^{\ell}(\sqrt{\frac{\ell}{n}}) \times S^{m}(\sqrt{\frac{m}{n}})$, where $\ell, m \in \mathbb{Z}^+, \ell + m = n$ (cf. Theorem 5.2). Also, we will study the cases:

(i) when v is a nonzero vector field normal to M or tangent to M,
(ii) if u is a nonzero conformal vector field,
(iii) the case if v is a nonzero vector field with ρ is a constant or f is a constant.

2. Preliminaries

Let (M, g) be a compact minimal hypersurface of the unit sphere S^{n+1}, $n \in \mathbb{Z}^+$ with the immersion $\psi : M \to S^{n+1}$ and let $\overline{\psi} = \iota \circ \psi : M \to R^{n+2}$. We shall denoted by g the induced metric on the hypersurface M as well as the induced metric on S^{n+1}. Also, we denote by ∇, $\overline{N}$ and D the Riemannian connections on M, S^{n+1} and R^{n+2}, respectively. Let $N \in \mathfrak{X}(S^{n+1})$ and $\overline{N} \in \mathfrak{X}(R^{n+2})$ be the unit normal vector fields on S^{n+1} and R^{n+2}, respectively and let A_N and $A_{\overline{N}} = -I$ be the shape operators of the immersions ψ and $\overline{\psi}$, respectively.

The curvature tensor field of the hypersurface M is given by the Gauss formula:

$$R(X, Y)Z = g(Y, Z)X - g(X, Z)Y + g(AY, Z)AX - g(AX, Z)AY, \tag{2.1}$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

The global tensor field for orthonormal frame of vector field $\{e_1, \ldots, e_n\}$ on M^n is defined as

$$Ric(X, Y) = \sum_{i=1}^{n}\{g(R(e_i, X)Y, e_i)\}, \tag{2.2}$$

for all $X, Y \in \chi(M)$, the above tensor is called the Ricci tensor.

From (2.1) and (2.2) we can derive the expression for Ricci tensor as follows:

$$Ric(X, Y) = (n-1)g(X, Y) - g(AX, AY), \tag{2.3}$$

If we fix a distinct vector e_u from $\{e_1, \ldots, e_n\}$ on M, suppose which is u. Then, the Ricci curvature Ric is defined by

$$Ric = \sum_{p=1, p\neq u}^{n}\left\{g(R(e_i, e_u)e_u, e_i)\right\} = (n-1)||u||^2 - ||Au||^2, \tag{2.4}$$

and the scalar curvature S of M is given by

$$S = n(n-1) - ||A||^2, \tag{2.5}$$

where $||A||$ is the length of the shape operator A.

The Ricci operator Q is the symmetric tensor field defined by

$$QX = \sum_{i=1}^{n} R(X, e_i)e_i, \tag{2.6}$$

where $\{e_1, ..., e_n\}$ is a local orthonormal frame and it is well known that the Ricci operator Q satisfies $g(QX, Y) = Ric(X, Y)$ for all $X, Y \in \mathfrak{X}(M)$. Also, it is known that

$$\sum(\nabla Q)(e_i, e_i) = \frac{1}{2}(\nabla S), \tag{2.7}$$

where the covariant derivative $(\nabla Q)(X, Y) = \nabla_X QY - Q(\nabla_X Y)$ and ∇S is the gradient of the scalar curvature S.

The Codazzi equation of the hypersurface is

$$(\nabla A)(X, Y) = (\nabla A)(Y, X), \tag{2.8}$$

for all $X, Y \in \mathfrak{X}(M)$, where the covariant derivative $(\nabla A)(X, Y) = \nabla_X AY - A(\nabla_X Y)$. A smooth vector field ζ is called conformal vector field if its flow consists of conformal transformations or equivalently,

$$Ł_\zeta g = 2\tau g,$$

where $Ł_\zeta g$ is the Lie derivative of g with respect to ζ.

For a smooth function k, we denote by ∇k the gradient of k and we define the Hessian operator $A_k : \mathfrak{X}(M) \to \mathfrak{X}(M)$ by $A_k X = \nabla_X \nabla k$. Also, we denote by $\triangle$ the Laplace operator acting on $C^\infty(M)$ the set of all smooth functions on M. It is well known that the sufficient and necessary condition for a connected and complete n-dimensional Riemannian manifold (M, g) to be isometric to the sphere $S^n(c)$, is there is a non-constant smooth function $k \in C^\infty(M)$ satisfying $A_k = -ckI$, which is called Obata's equation.

Now, we will introduce some lemmas that we will use to prove the results of this paper:

Lemma 2.1. *(Bochner's Formula) [9] Let (M, g) be a compact Riemannian manifold and $h \in C^\infty(M)$. Then,*

$$\int_M \{Ric(\nabla h, \nabla h) + \|A_h\|^2 - (\triangle h)^2\} = 0.$$

Lemma 2.2. *[9] Let (M, g) be a Riemannian manifold and $h \in C^\infty(M)$. Then*

$$\sum_{i=1}^{n} (\nabla A_h)(e_i, e_i) = Q(\nabla h) + \nabla(\triangle h),$$

where $\{e_1, ..., e_n\}$ is a local orthonormal frame and $(\nabla A_h)(X, Y) = \nabla_X A_h(Y) - A_h(\nabla_X Y), X, Y \in \mathfrak{X}(M)$.

Lemma 2.3. *Let (M, g) be a compact minimal hypersurface of the unit sphere S^{n+1}, $n \in \mathbb{Z}^+$ with the immersion $\psi : M \to S^{n+1}$ and let $\overline{\psi} = \iota \circ \psi : M \to R^{n+2}$. Then*

(i) $\nabla_X u = (1 - f)X + \rho AX$ for any $X \in \mathfrak{X}(M)$, $\nabla \rho = -Au$ and $\nabla f = u$.
(ii) $\triangle \rho = -\rho \|A\|^2$ and $\triangle f = n(1 - f)$.
(iii) $\int \{\rho trA^3 + (\frac{n}{2} - 1)(1 - f)\} = 0$ and $\int \rho^2 \|A\|^2 = \int \|Au\|^2$.
(iv) Let $\varphi = 1 - f$. Then $\nabla \varphi = -u, \triangle \varphi = -n\varphi$ and $\int \|u\|^2 = n \int \varphi^2$, where v is the vector field tangential to S^{n+1}, u is the vector field tangential to M, $\rho =< \overline{\psi}, N >$, $f =< \overline{\psi}, \overline{N} >$ and $<, >$ is the Euclidean metric on R^{n+2}.

Proof. (i) Note that as $\overline{\psi} = u + \rho N + f\overline{N}$, for any $X \in \mathfrak{X}(M)$:

$$\begin{aligned} X &= D_X u + X(\rho)N + \rho D_X N + X(f)\overline{N} + f D_X \overline{N} \\ &= \nabla_X u + g(AX, u)N - g(X, u)\overline{N} + X(\rho)N - \rho AX + X(f)\overline{N} + fX, \end{aligned}$$

by equating tangential and normal component, we get

$$\nabla_X u = (1 - f)X + \rho AX,$$

$$\nabla \rho = -Au,$$

and

$$\nabla f = u.$$

(ii) As M is a minimal hypersurface of S^{n+1},

$$\triangle \rho = -\sum g(\nabla_{e_i} Au, e_i) = -\sum [g((1 - f)e_i + \rho Ae_i, Ae_i) + g(u, \nabla_{e_i} Ae_i)] = -\rho \|A\|^2,$$

and

$$\triangle f = \sum g(\nabla_{e_i} u, e_i) = -\sum [g((1 - f)e_i + \rho Ae_i, e_i) = n(1 - f).$$

(iii)

$$\begin{aligned} divA \nabla \rho &= -\sum [g((1 - f)e_i + \rho Ae_i, A^2 e_i) + g(u, \nabla_{e_i} A^2 e_i)] \\ &= -(1 - f)\|A\|^2 - \rho trA^3 + \frac{1}{2} \nabla S \end{aligned}$$

$$= (1-f)(S-n(n-1)) - \rho trA^3 + \frac{1}{2}divSu - \frac{n}{2}S(1-f)$$
$$= (1-\frac{n}{2})(1-f)S - n(n-1)(1-f) - \rho trA^3 + \frac{1}{2}divSu.$$

So, if M is a compact, we get

$$\int\{\rho trA^3 + (\frac{n}{2}-1)(1-f)S\} = 0.$$

Also, note that

$$\frac{1}{2}\triangle\rho^2 = -\rho^2\|A\|^2 + \|Au\|^2.$$

So, since M is a compact, we get

$$\int\rho^2\|A\|^2 = \int\|Au\|^2.$$

(iv) Let $\varphi = 1-f$. Then,

$$\nabla\varphi = -\nabla f = -u.$$

Also,

$$\triangle\varphi = -\triangle f = -n\varphi$$

Also, note that

$$\frac{1}{2}\triangle\varphi^2 = -n\varphi^2 + \|u\|^2.$$

Since M is a compact, we get

$$\int\|u\|^2 = n\int\varphi^2.$$

□

Note that as $\triangle f = n(1-f)$, f is a constant if and only if $f = 1$. In Section 3, we study the case when $Z(\varphi) = \{x \in M : \varphi(x) = 0\}$ is a discrete set and $\rho^2(1-\beta) + \varphi^2(\alpha-1) \geq 0$, where α and β are two constants satisfying $(n-1)\alpha \leq Ric \leq (n-1)\beta$, $\beta < 1$. In Section 4, we study the cases v is a nonzero vector field with f or ρ is a constant, the cases v is a nonzero vector field tangent or normal to the minimal hypersurface M and the case if u is a nonzero conformal vector field. In Section 5, we study the case under the restriction $Au = \lambda u$, $\lambda \in R$.

3. Minimal hypersurface with $Z(\varphi)$ is a discrete set

Note that on using Lemma 2.3(iii), we get

$$0 = \int\{\rho^2\|A\|^2 - \|Au\|^2\}.$$

Combining Lemma 2.3(iv) with Eq (2.2), we conclude

$$0 = \int\{\rho^2(n(n-1)-S) + Ric(u,u) - n(n-1)\varphi^2\}.$$

Let α and β be two constants satisfying (they exist owing to compactness of M) $(n-1)\alpha \leq Ric \leq (n-1)\beta$, $\beta < 1$. Then, using above equation, we get

$$0 \geq \int \{n(n-1)\rho^2 - n(n-1)\beta\rho^2 + (n-1)\alpha\|u\|^2 - n(n-1)\varphi^2\}$$
$$= n(n-1)\int \{\rho^2(1-\beta) + \varphi^2(\alpha-1)\}.$$

Assume that $\rho^2(1-\beta) + \varphi^2(\alpha-1) \geq 0$, which in view of the above inequality implies

$$\rho^2 = (\frac{1-\alpha}{1-\beta})\varphi^2.$$

Assume that $Z(\varphi)$ is a discrete set, then on using $(\frac{\rho}{\varphi})^2 = \frac{1-\alpha}{1-\beta}$ on $M - Z(\varphi)$. As ρ and φ are continuous functions and $Z(\varphi)$ is a discrete set we get $(\frac{\rho}{\varphi})^2 = \frac{1-\alpha}{1-\beta}$ on M. So $\rho = \kappa\varphi$, $\kappa = \sqrt{\frac{1-\alpha}{1-\beta}}$ is a constant. Thus, by Lemma 2.3(i), we have

$$\|A\|^2\rho = n\kappa\varphi,$$

and hence

$$\kappa\varphi(\|A\|^2 - n) = 0,$$

thus either $\kappa = 0$ or $\varphi(\|A\|^2 - n) = 0$. If $\kappa = 0$, then $\alpha = 1$ and therefore M isometric to the unit sphere S^n and it will imply $\beta = 1$, which is a contradiction with our assumption $\beta \neq 1$. So $\varphi(\|A\|^2 - n) = 0$, but as $\varphi \neq 0$ on $M - Z(\varphi)$ and $Z(\varphi)$ is a discrete set we get $\|A\|^2 = n$ on all M, by continuity of the function $\|A\|^2$, and therefore M isometric to the Clifford hypersurface $S^\ell(\sqrt{\frac{\ell}{n}}) \times S^m(\sqrt{\frac{m}{n}})$, where $\ell, m \in \mathbb{Z}^+, \ell + m = n$. Thus, we have proved the following theorem:

Theorem 3.1. *Let M be a compact connected minimal hypersurface of S^{n+1} and α and β be two constants such that $(n-1)\alpha \leq Ric \leq (n-1)\beta$, $\beta < 1$. If $\rho^2(1-\beta) + \varphi^2(\alpha-1) \geq 0$ and $Z(\varphi) = \{x \in M : \varphi(x) = 0\}$ is discrete, then M isometric to the Clifford hypersurface $S^\ell(\sqrt{\frac{\ell}{n}}) \times S^m(\sqrt{\frac{m}{n}})$, where $\ell,\ m \in \mathbb{Z}^+,\ \ell + m = n$.*

4. Minimal hypersurface with v is a nonzero vector field

In this section, we study the cases v is a nonzero vector field that is either tangent or normal to the minimal hypersurface M. Also, we will study the cases v is a nonzero vector field with f or ρ is a constant, and the case if u is a nonzero conformal vector field.

Theorem 4.1. *Let M be a complete minimal simply connected hypersurface of S^{n+1}.*

(i) If v is a nonzero vector field tangent to M, then M isometric to the unit sphere S^n.
(ii) If v is a nonzero vector field normal to M, then M isometric to the unit sphere S^n.

Proof. (i) As $\rho = 0$, using Lemma 2.3(i) and (iv), we get

$$\nabla_X(\nabla\varphi) = \varphi X,$$

for any $X \in \mathfrak{X}(M)$. So $A_\varphi X = -\varphi X$ for any $X in \mathfrak{X}(M)$. If φ is a constant then $u = 0$ and $v = 0$, which is a contradiction. So φ is nonconstant function satisfies the Obata's equation and therefore M isometric to the unit sphere S^n.
(ii) Note that as $u = 0$ and $divu = n(1 - f)$ (Lemma 3.2(ii)), we get $f = 1$, so by Lemma 3.1(i), we have $\rho AX = 0$ for all $X \in \mathfrak{X}(M)$, but $\rho \neq 0$ since v is nonzero vector field. So $AX = 0$ for all $X \in \mathfrak{X}(M)$ and therefore M isometric to the unit sphere S^n. □

Theorem 4.2. *Let M be a complete minimal simply connected hypersurface of S^{n+1}.*

(i) If v is a nonzero vector field and ρ is a constant, then M isometric to the unit sphere S^n.
(ii) If v is a nonzero vector field and f is a constant, then M isometric to the unit sphere S^n.

Proof. (i) If $\rho \neq 0$, then by using Lemma 2.3(ii), we get $\rho\|A\|^2 = 0$ and therefore M isometric to the unit sphere S^n. If $\rho = 0$, then by using Lemma 2.3(i) and (iv) we get $\nabla_X u = -\varphi X$ for any $X \in \mathfrak{X}(M)$. So $A_\varphi X = -\varphi X$ for any $X \in \mathfrak{X}(M)$. If φ is a constant then $u = 0$ and $v = 0$, which is a contradiction. So φ is nonconstant function satisfies the Obata's equation and therefore M isometric to the unit sphere S^n.
(ii) If f is a constant, then $u = \nabla f = 0$, so $\nabla \rho = -Au = 0$, so ρ is a constant and hence $\rho\|A\|^2 = 0$, so either $\rho = 0$ or M isometric to the unit sphere S^n. Assume, $\rho = 0$ then $u = \nabla f = 0$ and thus $v = 0$, which is a contradiction. So M isometric to the unit sphere S^n. □

Theorem 4.3. *Let M be a complete minimal simply connected hypersurface of S^{n+1}. If u is a nonzero conformal vector field, then M isometric to the unit sphere S^n.*

Proof. Assume u is a conformal vector field with potential map σ then for any $X, Y \in \mathfrak{X}(M)$:

$$2\sigma g(X,Y) = g(\nabla_X u, Y) + g(\nabla_Y u, X) = 2(1-f)g(X,Y) + 2\rho g(AX, Y).$$

So $\rho AX = (\sigma + f - 1)X$ for any $X \in \mathfrak{X}(M)$.

If $\rho = 0$, then M isometric to the unit sphere S^n (by Theorem 4.2(i)).

If $\rho \neq 0$, then $A = FI$, $F = \frac{\kappa+f-1}{\rho}$, that is M is a totally umplical hypersurface of S^{n+1} but M is minimal hypersurface of S^{n+1} so M isometric to the unit sphere S^n. □

5. Minimal hypersurface with $Au = \lambda u, \lambda \in R$

Theorem 5.1. *Let M be a complete minimal simply connected hypersurface of S^{n+1}. If $Au = \lambda u$, $\lambda \in R$ and $\triangle\rho \neq 0$, then*

$$\|A\|^2 = \lambda^2 \frac{(n-1)\|u\|^2 + n\varphi^2}{n(n-1)\varphi^2 - (n-1-\lambda^2)\|u\|^2}.$$

Proof. We know that

$$\|A_\rho\|^2 = \sum g(A_\rho e_i, A_\rho e_i) = \lambda^2 \sum g((1-f)e_i + \rho A e_i, (1-f)e_i + \rho A e_i) = \lambda^2[n\varphi^2 + \rho^2\|A\|^2].$$

Also,

$$\|A_f\|^2 = \sum g(A_f e_i, A_f e_i) = \sum g((1-f)e_i + \rho A e_i, (1-f)e_i + \rho A e_i) = n\varphi^2 + \rho^2\|A\|^2.$$

By using Lemma 2.3(iii), we get

$$\int \rho^2\|A\|^2 = \lambda^2 \int \|A\|^2.$$

Using the Bochner's Formula (Lemma 2.1) for the smooth function ρ:

$$\begin{aligned} 0 &= \int \{Ric(\nabla\rho, \nabla\rho) + \|A_\rho\|^2 - (\triangle_\rho)^2\} \\ &= \int \{\lambda^2(n-1-\lambda^2)\|u\|^2 + \lambda^2 n\varphi^2 + \lambda^4\|u\|^2 - \rho^2\|A\|^4\} \\ &= \int \{\lambda^2(n-1)\|u\|^2 + \lambda^2 n\varphi^2 - \rho^2\|A\|^4\}. \end{aligned}$$

This implies

$$\rho^2\|A\|^4 = \lambda^2[(n-1)\|u\|^2 + n\varphi^2].$$

Now, using the Bochner's Formula (Lemma 2.1) for the smooth function f:

$$0 = \int \{Ric(\nabla f, \nabla f) + \|A_f\|^2 - (\triangle_f)^2\} = \int \{(n-1-\lambda^2)\|u\|^2 + \rho^2\|A\|^2 - n(n-1)\varphi^2\}.$$

Now as $\triangle\rho \neq 0$, $\rho \triangle \rho \neq 0$ and hence $\rho^2\|A\|^2 \neq 0$ and therefore

$$\|A\|^2 = \lambda^2 \frac{(n-1)\|u\|^2 + n\varphi^2}{n(n-1)\varphi^2 - (n-1-\lambda^2)\|u\|^2}.$$

□

Theorem 5.2. *Let M be a complete minimal simply connected hypersurface of S^{n+1} with constant scalar curvature S. If u is a nonzero vector field, $Au = \lambda u$, $\lambda \in R^*$, then M isometric to the Clifford hypersurface $S^\ell(\sqrt{\frac{\ell}{n}}) \times S^m(\sqrt{\frac{m}{n}})$, where $\ell, m \in \mathbb{Z}^+$, $\ell + m = n$.*

Proof. We notice that

$$\sum(\nabla A_\rho)(e_i, e_i) = -\sum \nabla_{e_i} \nabla_{e_i} Au = -\lambda \sum[-e_i(f)e_i + e_i(\rho)Ae_i] = \lambda(1+\lambda^2)u.$$

Also for any $X \in \chi(M)$, we have

$$g(Q\nabla\rho, X) = -\lambda Ric(u, X) = -\lambda(n-1-\lambda^2)g(u, X).$$

Thus

$$Q\nabla\rho = -\lambda(n-1-\lambda^2)u,$$

and

$$\nabla(\triangle\rho) = -\nabla(\rho\|A\|^2) = -[\rho\nabla\|A\|^2 - \lambda\|A\|^2 u].$$

Using Lemma 2.2, we get

$$\lambda(1+\lambda^2)u = -\lambda(n-1-\lambda^2)u - \rho\nabla\|A\|^2 + \lambda\|A\|^2 u.$$

But $||A||^2$ is a constant, since the scalar curvature S is a constant (see Eq (2.2)). Thus

$$\lambda(1+\lambda^2)u = \lambda(1+\lambda^2)u - \lambda nu + \lambda||A||^2 u,$$

and so

$$\lambda(n - ||A||^2)u = 0,$$

since $\lambda \neq 0$ and u is a nonzero vector field, $||A||^2 = n$ and therefore M isometric to the Clifford hypersurface $S^\ell(\sqrt{\frac{\ell}{n}}) \times S^m(\sqrt{\frac{m}{n}})$, where $\ell, m \in \mathbb{Z}^+, \ell + m = n$. □

Remark 5.1. *Note that the structure of [10] can be viewed as an example of the current article's structure in specific cases. In other words, the structure used for the article [10] can be recovered specifically if we select* $l = 1, m = 2$ *and* $n = 3$. *Because of this, the structure used in this article is the generalized case of [10].*

Conflict of interest

The authors declare no conflict of interest.

References

1. I. Al-Dayel, S. Deshmukh, O. Belova, A remarkable property of concircular vector fields on a Riemannian manifold, *Mathematics*, **8** (2020), 469. http://dx.doi.org/10.3390/math8040469
2. S. Chern, M. do Carmo, S. Kobayashi, Minimal submanifolds of a sphere with second fundamental form of constant length, In: *Functional analysis and related fields*, Berlin: Springer, 1970, 59–75. http://dx.doi.org/10.1007/978-3-642-48272-4_2
3. J. Simons, Minimal varieties in Riemannian manifolds, *Ann. Math.*, **88** (1968), 62–105. http://dx.doi.org/10.2307/1970556
4. S. Deshmukh, Characterizing spheres and Euclidean spaces by conformal vector fields, *Ann. Mat.*, **196** (2017), 2135–2145. http://dx.doi.org/10.1007/s10231-017-0657-0
5. S. Deshmukh, A note on hypersurfaces in a sphere, *Monatsh. Math.*, **174** (2014), 413–426. http://dx.doi.org/10.1007/s00605-013-0549-3
6. S. Deshmukh, A note on compact hypersurfaces in a Euclidean space, *CR Math.*, **350** (2012), 971–974. http://dx.doi.org/10.1016/j.crma.2012.10.027
7. S. Deshmukh, I. Al-Dayel, A not on minimal hypersurface of an odd dimensional sphere, *Mathematics*, **8** (2020), 294. http://dx.doi.org/10.3390/math8020294
8. S. Deshmukh, I. Al-Dayel, Characterizing spheres by an immersion in Euclidean spaces, *Arab Journal of Mathematical Sciences*, **23** (2017), 85–93. http://dx.doi.org/10.1016/j.ajmsc.2016.09.002
9. S. Deshmukh, I. Al-Dayel, Curvature bounds for the spectrum of a compact Riemannian manifold of constant scalar curvature, *J. Geom. Anal.*, **15** (2005), 589–606. http://dx.doi.org/10.1007/BF02922246

10. H. Li, A characterization of Clifford minimal hypersurfaces in S^4, *Proc. Amer. Math. Soc.*, **123** (1995), 3183–3187. http://dx.doi.org/10.1090/S0002-9939-1995-1277113-6

11. F. Klein, Zur nicht-euklidischen Geometrie, *Math. Ann.*, **37** (1890), 544–572. http://dx.doi.org/10.1007/BF01724772

12. Q. Wang, Rigidity of Clifford minimal hypersurfaces, *Monatsh. Math.*, **140** (2003), 163–167. http://dx.doi.org/10.1007/s00605-002-0039-5

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