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Research article

Finite-time stability analysis of singular neutral systems with time delay

Sheng Wang[1] **and Shaohua Long**[2,*]

[1] School of Science, Hunan Institute of Technology, Hengyang 421002, China
[2] School of Science, Chongqing University of Technology, Chongqing 400054, China

* **Correspondence:** Email: longshaohua0732@163.com, longshaohua0732@cqut.edu.cn.

Abstract: This paper studies the finite-time stability problem for a class of singular neutral systems by using the Lyapunov-Krasovskii function approach and regular neutral system theory. The considered systems involve not only the delayed version of the state, but also the delayed version of the derivative of the state. Some sufficient conditions are presented to ensure that the considered systems are regular, impulse-free, and finite-time stable. Three numerical examples are given to illustrate the effectiveness of the proposed methods.

Keywords: singular neutral system; finite-time stability; time delay; Lyapunov-Krasovskii function
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1. Introduction

Singular systems are also called differential-algebraic systems and can be employed to model many practical systems, such as economical systems [1], electrical circuits [2], and engineering systems [2]. It is worth pointing out that a singular system may have a unique solution, multiple solutions, or no solutions. In addition, impulse phenomenon may exist in singular systems. Hence, for a singular system, we often need to check if it is regular and impulse-free. During the past several decades, extensive attention has been given to this class of systems, and many results have been presented [3–7].

It is known that a system is called a time delay system if it contains the delayed version of the state. Time delay systems have been studied extensively [5–8]. Neutral systems are special time delay systems because a neutral system often contains not only the delayed version of the state, but also the delayed version of the derivative of the state. Due to this special characteristic, neutral systems are very important in modeling many practical systems, such as lossless transmission lines [9], chemical processes [10], population dynamics [11], and partial element equivalent circuits [12]. In the past several decades, much attention has been paid to neutral systems [13–16].

Many practical systems, like robot operating systems [17] and network communication systems [18], often work in a short time. For systems that work in a short time, the finite-time stability problem is very important. A system is said to be finite-time stable if, given a bound on the initial condition, the norm of its state vector does not exceed a certain threshold during a specified time interval [19]. The finite-time stability problem has been one of the research focuses in the past several decades, and many results have been proposed [20–24].

There are many papers that have considered singular neutral systems. The stability of singular neutral systems with constant delays was investigated in [25,26], while the stability of singular neutral systems with time-varying delay was considered in [27]. References [28] and [29] paid attention to the robust stabilization and the PD feedback H_∞ controller design for uncertain singular neutral systems, respectively. The stabilization of neutral singular Markovian jump systems was studied in [30–32]. References [33] and [34] studied the exponential stability and the asynchronous H_∞ controller design for neutral singular Markovian jump systems, respectively. However, the finite-time stability of singular neutral systems was not investigated in [25–34].

There are also many papers that investigated the finite-time stability problem of singular systems. The finite-time stability of singular systems with time delay was investigated in [35,36]. The authors of [37] studied the finite-time stability of singular nonlinear switched time delay systems by using a singular value decomposition approach. The finite-time stability of singular systems with time-varying delay was considered in [38,39]. References [40–42] paid attention to the finite-time stability of singular Markovian jump systems. However, none of the systems considered in [35–42] involve the delayed version of the derivative of its state.

It is worth pointing out that references [25–42] did not study the finite-time stability problem of singular neutral systems. Compared with the finite-time stability problem of singular systems, the finite-time stability problem of singular neutral systems is more complicated. One of the reasons is that we need to deal with the derivative of the state.

To the best of our knowledge, few results in the existing literature have dealt with the finite-time stability problem of singular neutral systems. This paper studies the finite-time stability problem for a class of singular neutral systems with time delay. By utilizing the Lyapunov-Krasovskii function approach, some sufficient conditions are proposed to ensure that the considered systems are regular, impulse-free, and finite-time stable. Three numerical examples are presented to illustrate the effectiveness of the proposed methods. The main contributions of this paper can be summarized as follows.

(i) References [25–42] did not study the finite-time stability problem of singular neutral systems. To the best of our knowledge, few results in the existing literature dealt with this problem for singular neutral systems. This paper investigates this problem for singular neutral systems.

(ii) An equivalent form of the considered system is well utilized in this paper (see Remark 3.3 for more information). A good sufficient condition ensuring that the considered system is finite-time stable is presented.

(iii) In this paper, the regular neutral system theory is also used to study the finite-time stability problem of the considered system. The sufficient condition obtained by using the regular neutral system theory is sometimes better. In addition, few papers in the existing literature have employed regular neutral system theory to study the asymptotic stability of singular neutral systems. This paper presents an improved result on the asymptotic stability of a class of singular neutral systems.

Notations: $\mathfrak{C}$ denotes the set of all complex numbers. The n-dimensional Euclidean space and the set of all $m \times n$ real matrices are denoted by $\mathcal{R}^n$ and $\mathcal{R}^{m\times n}$, respectively. I represents the identity matrix with appropriate dimensions, while I_n denotes the $n \times n$ identity matrix. In a symmetric matrix, the symbol "$*$" represents the symmetric element. $\Xi(a)$ denotes the largest integer smaller than scalar a. $\|\cdot\|$ means the spectral norm of a matrix. We use $\det(\cdot)$, $\lambda_{\min}(\cdot)$, and $\lambda_{\max}(\cdot)$ to represent the determinant, the smallest eigenvalue, and the largest eigenvalue of a matrix, respectively. For matrix A, A^T stands for the transpose of matrix A, $A > 0$ denotes that A is positive definite, and $A \geq 0$ means that A is semi-positive definite.

2. Preliminaries

In this paper, we study the following singular neutral system with time delay:

$$\begin{cases} E\dot{z}(t) - C\dot{z}(t-\eta) = Az(t) + Dz(t-\eta), \\ z(s) = \phi(s), \quad s \in [-\eta, 0]. \end{cases} \tag{2.1}$$

In system (2.1), E is a known real singular matrix, and satisfies $0 < \text{rank}(E) = \varpi < n$. C, A, and D are known real matrices. In addition, E, C, A, and D belong to $\mathcal{R}^{n\times n}$. $z(t) \in \mathcal{R}^n$ represents the state vector of the system. The vector $\phi(s)$ ($s \in [-\eta, 0]$) represents the initial condition of the system and we suppose that $\dot{\phi}(s)$ ($s \in [-\eta, 0]$) is continuous. $\eta > 0$ denotes the time delay and is a given scalar.

When $C = 0$, system (2.1) reduces to the following singular system:

$$\begin{cases} E\dot{z}(t) = Az(t) + Dz(t-\eta), \\ z(s) = \phi(s), \quad s \in [-\eta, 0]. \end{cases} \tag{2.2}$$

In the following, we will give some definitions which will be used later. Definition 2.1 is related to system (2.2) and will be employed to check if system (2.1) is regular and impulse-free.

Definition 2.1. *[1] (i) System (2.2) is said to be regular if there exists a scalar $s \in \mathfrak{C}$ satisfying* $\det(sE - A) \neq 0$.

(ii) System (2.2) is said to be impulse-free if the equation $\deg(\det(sE - A)) = \text{rank}(E)$ *holds, where* $\deg(\det(sE - A))$ *denotes the degree of* $\det(sE - A)$*, and* $\det(sE - A)$ *is a univariate polynomial in s.*

Definition 2.2. *For a given matrix $U > 0$ and given scalars $T > 0$, $m_1 > 0$, $m_2 > 0$, and $m_3 > 0$ ($m_3 \geq m_1$), system (2.1) is said to be finite-time stable with respect to (m_1, m_2, m_3, T, U) if the following holds:*

$$\{\sup_{-\eta\leq s\leq 0}\{z^T(s)Uz(s)\} \leq m_1,\ \sup_{-\eta\leq s\leq 0}\{\dot{z}^T(s)E^TUE\dot{z}(s)\} \leq m_2\} \Rightarrow z^T(t)Uz(t) \leq m_3,\ \forall t \in [0, T]. \tag{2.3}$$

The following lemmas will be employed to derive the main results of this paper.

Lemma 2.1. *[27] For a given matrix $Q > 0$ and given scalars η_1 and η_2 ($\eta_2 > \eta_1$), if the vector function χ: $[\eta_1, \eta_2] \to \mathcal{R}^n$ ensures that the integrations $\int_{\eta_1}^{\eta_2} \chi^T(s)Q\chi(s)ds$ and $\int_{\eta_1}^{\eta_2} \chi(s)ds$ are well defined, then*

$$-\int_{\eta_1}^{\eta_2} \chi^T(s)Q\chi(s)ds \leq -\frac{1}{\eta_2 - \eta_1}\int_{\eta_1}^{\eta_2} \chi^T(s)dsQ\int_{\eta_1}^{\eta_2} \chi(s)ds.$$

Lemma 2.2. *[27] Suppose that matrices R, L, and $Q > 0$ have appropriate dimensions. Then, the inequality $R^TL + L^TR \leq R^TQR + L^TQ^{-1}L$ holds.*

Remark 2.1. *It is worth pointing out that singular neutral systems have wide applications in practical systems. Please see Remark 1 of [27] for more information. For the systems mentioned in Remark 1 of [27], sometimes we may need to focus on the transient behaviors (for example, finite-time stability) of them over a finite-time interval. To the best of our knowledge, few papers in the existing literature studied the finite-time stability problem of singular neutral systems. Therefore, it is significant and necessary to study this problem for singular neutral systems.*

3. Main results

The state decomposition method is often used to study singular systems [43–45]. We will give an equivalent form of system (2.1) by first using the state decomposition method.

From $0 < \text{rank}(E) = \varpi < n$, we can find two invertible matrices J and Y satisfying $\overline{E} = JEY = \begin{bmatrix} I_\varpi & 0 \\ 0 & 0 \end{bmatrix}$. In addition, set

$$\overline{C} = JCY = \begin{bmatrix} \overline{C}_1 & \overline{C}_2 \\ \overline{C}_3 & \overline{C}_4 \end{bmatrix}, \overline{A} = JAY = \begin{bmatrix} \overline{A}_1 & \overline{A}_2 \\ \overline{A}_3 & \overline{A}_4 \end{bmatrix}, \overline{D} = JDY = \begin{bmatrix} \overline{D}_1 & \overline{D}_2 \\ \overline{D}_3 & \overline{D}_4 \end{bmatrix}. \tag{3.1}$$

Define $\omega(t) = Y^{-1}z(t) = \begin{bmatrix} \omega_1(t) \\ \omega_2(t) \end{bmatrix}$. According to (3.1), system (2.1) can be decomposed into

$$\begin{cases} \dot{\omega}_1(t) = \overline{A}_1\omega_1(t) + \overline{A}_2\omega_2(t) + \overline{D}_1\omega_1(t-\eta) + \overline{D}_2\omega_2(t-\eta) + \overline{C}_1\dot{\omega}_1(t-\eta) + \overline{C}_2\dot{\omega}_2(t-\eta), \\ 0 = \overline{A}_3\omega_1(t) + \overline{A}_4\omega_2(t) + \overline{D}_3\omega_1(t-\eta) + \overline{D}_4\omega_2(t-\eta) + \overline{C}_3\dot{\omega}_1(t-\eta) + \overline{C}_4\dot{\omega}_2(t-\eta), \\ \omega(s) = Y^{-1}\phi(s), \quad s \in [-\eta, 0]. \end{cases} \tag{3.2}$$

Remark 3.1. *From (3.2), it can be seen that system (2.1) is decomposed into two subsystems. One is described by the equation* $\dot{\omega}_1(t) = \overline{A}_1\omega_1(t)+\overline{A}_2\omega_2(t)+\overline{D}_1\omega_1(t-\eta)+\overline{D}_2\omega_2(t-\eta)+\overline{C}_1\dot{\omega}_1(t-\eta)+\overline{C}_2\dot{\omega}_2(t-\eta)$, *while the other is described by the equation* $0 = \overline{A}_3\omega_1(t)+\overline{A}_4\omega_2(t)+\overline{D}_3\omega_1(t-\eta)+\overline{D}_4\omega_2(t-\eta)+\overline{C}_3\dot{\omega}_1(t-\eta) + \overline{C}_4\dot{\omega}_2(t-\eta)$. *According to (3.2), it can be also seen that the vector* $\omega(t)$ *is decomposed into* $\omega_1(t)$ *and* $\omega_2(t)$. *We will employ (3.2) to study the finite-time stability of system (2.1), which means that the state decomposition method is utilized in this paper.*

The following theorem presents a sufficient condition such that system (2.1) is regular, impulse-free, and finite-time stable with respect to (m_1, m_2, m_3, T, U).

Theorem 3.1. *Suppose that* X *is a given matrix and satisfies* $E^TX = 0$ *and* $\text{rank}(X) = n - \varpi$. *Given scalars* $\mu \geq 0$, $\eta > 0$, $\widehat{h} > 0$, $\overline{h} \geq 0$, $\widehat{g}_k > 0\,(k = 1, 2, \cdots, 5)$, $\overline{g}_k \geq 0\,(k = 1, 2, \cdots, 5)$, $m_1 > 0$, $m_2 > 0$, $m_3 > 0\,(m_3 > m_1)$, $T > 0$, $\rho_1 > 0$, *and* $\rho_2 > 0$, *system (2.1) is regular, impulse-free, and finite-time stable with respect to* (m_1, m_2, m_3, T, U) *if* $\overline{C}_2 = 0$, $\overline{C}_4 = 0$, *and there exist matrices* $F > 0$, $G_k > 0\,(k = 1, 2, \cdots, 5)$, $Z_k\,(k = 1, 2, \cdots, 5)$, *and* W *such that*

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & A^TZ_3 + G_2 & (FE + XW + Z_1)^TCYJ + A^TZ_4 & \Gamma_{15} \\ * & \Gamma_{22} & D^TZ_3 - G_2 & Z_2^TCYJ + D^TZ_4 & \Gamma_{25} \\ * & * & -\frac{G_3}{\eta} & Z_3^TCYJ & -Z_3^T \\ * & * & * & \Gamma_{44} & \Gamma_{45} \\ * & * & * & * & \Gamma_{55} \end{bmatrix} < 0, \tag{3.3}$$

$$\overline{g}_k I < G_k < \widehat{g}_k I, \quad \forall k \in \{1, 2, \cdots, 5\}, \tag{3.4}$$

$$\overline{h} I < F < \widehat{h} I, \tag{3.5}$$

$$\|\overline{U}_1^{\frac{1}{2}} \overline{F}^{-\frac{1}{2}}\|^2 h_4 + \|\overline{U}_4^{\frac{1}{2}}\|^2 \overline{\Re}^2 \leq m_3 \ (when\ \overline{U}_2 = 0), \tag{3.6}$$

$$\min\left\{\delta_1 h_4 + \|(\overline{U}_4 + \rho_1 I)^{\frac{1}{2}}\|^2 \overline{\Re}^2, \delta_2 h_4 + \|(\overline{U}_4 + \rho_2 \overline{U}_2^T \overline{U}_2)^{\frac{1}{2}}\|^2 \overline{\Re}^2\right\} \leq m_3 \ (when\ \overline{U}_2 \neq 0), \tag{3.7}$$

where

$$\Gamma_{11} = (FE + XW + Z_1)^T A + A^T (FE + XW + Z_1) + G_1 + \eta G_3 - \mu E^T F E - E^T \frac{G_5}{\eta} E,$$

$$\Gamma_{12} = (FE + XW + Z_1)^T D + A^T Z_2 + E^T \frac{G_5}{\eta} E, \qquad \Gamma_{15} = A^T (G_4 + \eta G_5 + Z_5) - Z_1^T,$$

$$\Gamma_{22} = -G_1 + D^T Z_2 + Z_2^T D - E^T \frac{G_5}{\eta} E, \qquad \Gamma_{25} = D^T (G_4 + \eta G_5 + Z_5) - Z_2^T,$$

$$\Gamma_{44} = -G_4 + (CYJ)^T Z_4 + Z_4^T CYJ, \quad \Gamma_{45} = (CYJ)^T (G_4 + \eta G_5 + Z_5) - Z_4^T,$$

$$\Gamma_{55} = -G_4 - \eta G_5 - Z_5 - Z_5^T,$$

$$\sigma_k = \|G_k^{\frac{1}{2}} U^{-\frac{1}{2}}\|^2 \ (k = 1, 2, \cdots, 5),$$

$$h_1 = \|F^{\frac{1}{2}} E U^{-\frac{1}{2}}\|^2 m_1, \ h_2 = (\eta\sigma_1 + \eta^2 \sigma_2 + \frac{\eta^2}{2}\sigma_3) m_1, \ h_3 = (\eta\sigma_4 + \frac{\eta^2}{2}\sigma_5) m_2, \ h_4 = e^{\mu T} \sum_{k=1}^{3} h_k,$$

$$H_1 = \overline{A}_1 - \overline{A}_2 \overline{A}_4^{-1} \overline{A}_3, \ H_2 = \overline{D}_1 - \overline{A}_2 \overline{A}_4^{-1} \overline{D}_3, \ H_3 = \overline{D}_2 - \overline{A}_2 \overline{A}_4^{-1} \overline{D}_4, \ H_4 = \overline{C}_1 - \overline{A}_2 \overline{A}_4^{-1} \overline{C}_3,$$

$$L_1 = -\overline{A}_4^{-1} \overline{A}_3, \quad L_2 = -\overline{A}_4^{-1} \overline{D}_3, \quad L_3 = -\overline{A}_4^{-1} \overline{D}_4, \quad L_4 = -\overline{A}_4^{-1} \overline{C}_3,$$

$$\Delta_1 = \overline{\Delta} \begin{bmatrix} \overline{A}_1 \\ \overline{A}_3 \end{bmatrix}, \ \Delta_2 = \overline{\Delta} \begin{bmatrix} \overline{D}_1 \\ \overline{D}_3 \end{bmatrix}, \ \Delta_3 = \overline{\Delta} \begin{bmatrix} \overline{D}_2 \\ \overline{D}_4 \end{bmatrix}, \ \Delta_4 = \overline{\Delta} \begin{bmatrix} \overline{C}_1 \\ \overline{C}_3 \end{bmatrix}, \ \overline{\Delta} = \begin{bmatrix} I_{\varpi} & -\overline{A}_2 \\ 0 & -\overline{A}_4 \end{bmatrix}^{-1},$$

$$\kappa_1 = \|Y^{-1} U^{-\frac{1}{2}}\|^2 m_1,$$

$$\kappa_2 = \|J U^{-\frac{1}{2}}\|^2 m_2,$$

$$\aleph_1 = \|\Delta_1 \overline{F}_1^{-\frac{1}{2}}\| \sqrt{h_4} + \|[\Delta_2 \quad \Delta_3]\| \sqrt{\kappa_1} + \|\Delta_4\| \sqrt{\kappa_2},$$

$$\Upsilon_1 = \|H_1 \overline{F}_1^{-\frac{1}{2}}\| \sqrt{h_4} + \|[H_2 \quad H_3]\| \sqrt{\kappa_1} + \|H_4\| \sqrt{\kappa_2},$$

$$\Re_1 = \|L_1 \overline{F}_1^{-\frac{1}{2}}\| \sqrt{h_4} + \|[L_2 \quad L_3]\| \sqrt{\kappa_1} + \|L_4\| \sqrt{\kappa_2},$$

$$\aleph_p = (\|\Delta_1 \overline{F}_1^{-\frac{1}{2}}\| + \|\Delta_2 \overline{F}_1^{-\frac{1}{2}}\|) \sqrt{h_4} + \min\{\|\Delta_3\| \Re_{p-1} + \|\Delta_4\| \Upsilon_{p-1}, \|[\Delta_4 \quad \Delta_3]\| \aleph_{p-1}\}, \ 2 \leq p \leq \overline{n} + 1,$$

$$\Upsilon_p = (\|H_1 \overline{F}_1^{-\frac{1}{2}}\| + \|H_2 \overline{F}_1^{-\frac{1}{2}}\|) \sqrt{h_4} + \min\{\|H_3\| \Re_{p-1} + \|H_4\| \Upsilon_{p-1}, \|[H_4 \quad H_3]\| \aleph_{p-1}\}, \ 2 \leq p \leq \overline{n} + 1,$$

$$\Re_p = (\|L_1 \overline{F}_1^{-\frac{1}{2}}\| + \|L_2 \overline{F}_1^{-\frac{1}{2}}\|) \sqrt{h_4} + \min\left\{\|L_3\| \Re_{p-1} + \|L_4\| \Upsilon_{p-1}, \|[L_4 \quad L_3]\| \aleph_{p-1}\right\}, \ 2 \leq p \leq \overline{n} + 1,$$

$$\delta_1 = \|(\overline{U}_1 + \frac{1}{\rho_1} \overline{U}_2 \overline{U}_2^T)^{\frac{1}{2}} \overline{F}^{-\frac{1}{2}}\|^2, \qquad \delta_2 = \|(\overline{U}_1 + \frac{1}{\rho_2} I)^{\frac{1}{2}} \overline{F}^{-\frac{1}{2}}\|^2,$$

$$\overline{F} = \begin{bmatrix} \overline{F}_1 & \overline{F}_2 \\ * & \overline{F}_4 \end{bmatrix} = J^{-T} F J^{-1}, \quad \begin{bmatrix} \overline{U}_1 & \overline{U}_2 \\ * & \overline{U}_4 \end{bmatrix} = Y^T U Y, \quad \overline{n} = \Xi\left(\frac{T}{\eta}\right),$$

$$\overline{\Re} = \max\{\Re_1, \Re_2, \cdots, \Re_{\overline{n}+1}\}.$$

Proof. First, we prove that system (2.1) is regular and impulse-free.

From (3.3), we can have $\begin{bmatrix} \Gamma_{11} & \Gamma_{15} \\ * & \Gamma_{55} \end{bmatrix} < 0$. Then, according to $G_1 + \eta G_3 + A^T(G_4 + \eta G_5)A > 0$, it can be deduced that

$$\begin{aligned} \overline{\Gamma}_{11} &= \begin{bmatrix} I & A^T \end{bmatrix} \begin{bmatrix} \Gamma_{11} & \Gamma_{15} \\ * & \Gamma_{55} \end{bmatrix} \begin{bmatrix} I \\ A \end{bmatrix} - G_1 - \eta G_3 - A^T(G_4 + \eta G_5)A \\ &= (FE + XW)^T A + A^T(FE + XW) - \mu E^T F E - E^T \frac{G_5}{\eta} E < 0. \end{aligned} \tag{3.8}$$

Define $\overline{W} = WY = \begin{bmatrix} \overline{W}_1 & \overline{W}_2 \\ \overline{W}_3 & \overline{W}_4 \end{bmatrix}$, $\overline{X} = J^{-T}X = \begin{bmatrix} \overline{X}_1 & \overline{X}_2 \\ \overline{X}_3 & \overline{X}_4 \end{bmatrix}$, and $\overline{G}_5 = J^{-T}G_5J^{-1} = \begin{bmatrix} \overline{G}_{51} & \overline{G}_{52} \\ * & \overline{G}_{54} \end{bmatrix}$. According to $E^TX = 0$, we get $\overline{E}^T\overline{X} = 0$, which implies that $\overline{X}_1 = 0$ and $\overline{X}_2 = 0$. Then, from (3.8), we can have

$$\begin{aligned} Y^T\overline{\Gamma}_{11}Y &= Y^T((FE + XW)^T A + A^T(FE + XW) - \mu E^T F E - E^T \frac{G_5}{\eta} E)Y \\ &= (\overline{F}\,\overline{E} + \overline{X}\,\overline{W})^T \overline{A} + \overline{A}^T(\overline{F}\,\overline{E} + \overline{X}\,\overline{W}) - \mu \overline{E}^T \overline{F}\,\overline{E} - \overline{E}^T \frac{\overline{G}_5}{\eta} \overline{E} \\ &= \begin{bmatrix} \Lambda_1 & \Lambda_2 \\ * & \Lambda_4 \end{bmatrix} < 0, \end{aligned} \tag{3.9}$$

where $\Lambda_4 = \overline{A}_4^T(\overline{X}_3\overline{W}_2 + \overline{X}_4\overline{W}_4) + (\overline{X}_3\overline{W}_2 + \overline{X}_4\overline{W}_4)^T\overline{A}_4$.

From (3.9), it can be deduced that $\Lambda_4 < 0$. According to $\Lambda_4 < 0$, we can get $\det(\overline{A}_4) \neq 0$, which implies that $\det(sE - A) \not\equiv 0$ and $\deg(\det(sE - A)) = \text{rank}(E)$.

By $\overline{C}_2 = 0$ and $\overline{C}_4 = 0$, it can be seen that $C = CYJE$. Set $\varphi(t) = E\dot{z}(t)$. Then, system (2.1) can be rewritten as

$$\begin{cases} \varphi(t) = E\dot{z}(t), \\ \varphi(t) - CYJ\varphi(t - \eta) = Az(t) + Dz(t - \eta). \end{cases} \tag{3.10}$$

Set $\psi(t) = \begin{bmatrix} z(t) \\ \varphi(t) \end{bmatrix}$, $\widehat{E} = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}$, $\widehat{A} = \begin{bmatrix} 0 & I \\ A & -I \end{bmatrix}$, and $\widehat{D} = \begin{bmatrix} 0 & 0 \\ D & CYJ \end{bmatrix}$. Then, according to (3.10), we can get

$$\widehat{E}\,\dot{\psi}(t) = \widehat{A}\psi(t) + \widehat{D}\psi(t - \eta). \tag{3.11}$$

According to $\widehat{E} = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}$ and $\widehat{A} = \begin{bmatrix} 0 & I \\ A & -I \end{bmatrix}$, we can deduce

$$\begin{aligned} &\det(s\widehat{E} - \widehat{A}) \\ &= \det\left(\begin{bmatrix} sE & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & I \\ A & -I \end{bmatrix}\right) = \det\left(\begin{bmatrix} sE & -I \\ -A & I \end{bmatrix}\right) = \det\left(\begin{bmatrix} sE - A & 0 \\ -A & I \end{bmatrix}\right) = \det(sE - A). \end{aligned} \tag{3.12}$$

By (3.12) and $\det(sE - A) \not\equiv 0$, we can have $\det(s\widehat{E} - \widehat{A}) \not\equiv 0$. Note (3.12), $\deg(\det(sE - A)) = \text{rank}(E)$, and $\text{rank}(E) = \text{rank}(\widehat{E})$. It can then be deduced that $\deg(\det(s\widehat{E} - \widehat{A})) = \text{rank}(\widehat{E})$. According to Definition 2.1, it can be concluded that system (3.11) is regular and impulse-free. Because system (3.11) is equivalent to system (2.1), we can obtain that system (2.1) is regular and impulse-free. □

Next, we prove that system (2.1) is finite-time stable with respect to (m_1, m_2, m_3, T, U). Choose the following Lyapunov-Krasovskii function for system (2.1):

$$V(t) = V_1(t) + V_2(t) + V_3(t), \tag{3.13}$$

where

$$\begin{aligned}
V_1(t) &= z^T(t)E^T FEz(t),\\
V_2(t) &= \int_{t-\eta}^{t} z^T(s)G_1 z(s)ds + \int_{t-\eta}^{t} z^T(s)ds\, G_2 \int_{t-\eta}^{t} z(s)ds + \int_{-\eta}^{0}\int_{t+\beta}^{t} z^T(s)G_3 z(s)dsd\beta,\\
V_3(t) &= \int_{t-\eta}^{t} \dot{z}^T(s)E^T G_4 E\dot{z}(s)ds + \int_{-\eta}^{0}\int_{t+\beta}^{t} \dot{z}^T(s)E^T G_5 E\dot{z}(s)dsd\beta.
\end{aligned}$$

By (3.13) and Lemma 2.1, we can have

$$\begin{aligned}
V_1(0) &= z^T(0)E^T FEz(0)\\
&= z^T(0)U^{\frac{1}{2}}U^{-\frac{1}{2}}E^T FEU^{-\frac{1}{2}}U^{\frac{1}{2}}z(0)\\
&\le \lambda_{\max}\left(U^{-\frac{1}{2}}E^T FEU^{-\frac{1}{2}}\right) \sup_{-\eta\le s\le 0}\{z^T(s)Uz(s)\}\\
&\le h_1,
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
V_2(0) &= \int_{-\eta}^{0} z^T(s)G_1 z(s)ds + \int_{-\eta}^{0} z^T(s)ds\, G_2 \int_{-\eta}^{0} z(s)ds + \int_{-\eta}^{0}\int_{\beta}^{0} z^T(s)G_3 z(s)dsd\beta\\
&\le \int_{-\eta}^{0} z^T(s)G_1 z(s)ds + \eta\int_{-\eta}^{0} z^T(s)G_2 z(s)ds + \int_{-\eta}^{0}\int_{\beta}^{0} z^T(s)G_3 z(s)dsd\beta\\
&\le\eta \sup_{-\eta\le s\le 0}\{z^T(s)G_1 z(s)\} + \eta^2 \sup_{-\eta\le s\le 0}\{z^T(s)G_2 z(s)\} + \frac{\eta^2}{2}\sup_{-\eta\le s\le 0}\{z^T(s)G_3 z(s)\}\\
&=\eta \sup_{-\eta\le s\le 0}\{z^T(s)U^{\frac{1}{2}}U^{-\frac{1}{2}}G_1U^{-\frac{1}{2}}U^{\frac{1}{2}}z(s)\} + \eta^2 \sup_{-\eta\le s\le 0}\{z^T(s)U^{\frac{1}{2}}U^{-\frac{1}{2}}G_2U^{-\frac{1}{2}}U^{\frac{1}{2}}z(s)\}\\
&\quad + \frac{\eta^2}{2}\sup_{-\eta\le s\le 0}\{z^T(s)U^{\frac{1}{2}}U^{-\frac{1}{2}}G_3U^{-\frac{1}{2}}U^{\frac{1}{2}}z(s)\}\\
&\le\eta\|G_1^{\frac{1}{2}}U^{-\frac{1}{2}}\|^2 m_1 + \eta^2\|G_2^{\frac{1}{2}}U^{-\frac{1}{2}}\|^2 m_1 + \frac{\eta^2}{2}\|G_3^{\frac{1}{2}}U^{-\frac{1}{2}}\|^2 m_1 = h_2,
\end{aligned} \tag{3.15}$$

$$\begin{aligned}
V_3(0) &= \int_{-\eta}^{0} \dot{z}^T(s)E^T G_4 E\dot{z}(s)ds + \int_{-\eta}^{0}\int_{\beta}^{0} \dot{z}^T(s)E^T G_5 E\dot{z}(s)dsd\beta\\
&\le \eta \sup_{-\eta\le s\le 0}\{\dot{z}^T(s)E^T G_4 E\dot{z}(s)\} + \frac{\eta^2}{2}\sup_{-\eta\le s\le 0}\{\dot{z}^T(s)E^T G_5 E\dot{z}(s)\}
\end{aligned}$$

$$\leq \eta \sup_{-\eta\leq s\leq 0}\{\dot{z}^T(s)E^TU^{\frac{1}{2}}U^{-\frac{1}{2}}G_4U^{-\frac{1}{2}}U^{\frac{1}{2}}E\dot{z}(s)\} + \frac{\eta^2}{2}\sup_{-\eta\leq s\leq 0}\{\dot{z}^T(s)E^TU^{\frac{1}{2}}U^{-\frac{1}{2}}G_5U^{-\frac{1}{2}}U^{\frac{1}{2}}E\dot{z}(s)\}$$

$$\leq \eta\|G_4^{\frac{1}{2}}U^{-\frac{1}{2}}\|^2m_2 + \frac{\eta^2}{2}\|G_5^{\frac{1}{2}}U^{-\frac{1}{2}}\|^2m_2 = h_3. \tag{3.16}$$

From (3.13) and Lemma 2.1, we can also deduce

$$\begin{aligned}
\dot{V}_1(t) =&2\dot{z}^T(t)E^TFEz(t)\\
=&2[Az(t) + Dz(t-\eta) + C\dot{z}(t-\eta)]^TFEz(t)\\
=&2[Az(t) + Dz(t-\eta) + CYJE\dot{z}(t-\eta)]^TFEz(t),
\end{aligned}\tag{3.17}$$

$$\begin{aligned}
\dot{V}_2(t) =&z^T(t)G_1z(t) - z^T(t-\eta)G_1z(t-\eta) + 2[z^T(t) - z^T(t-\eta)]G_2\int_{t-\eta}^t z(s)ds\\
&+ \eta z^T(t)G_3z(t) - \int_{t-\eta}^t z^T(s)G_3z(s)ds\\
\leq& z^T(t)G_1z(t) - z^T(t-\eta)G_1z(t-\eta) + 2[z^T(t) - z^T(t-\eta)]G_2\int_{t-\eta}^t z(s)ds\\
&+ \eta z^T(t)G_3z(t) - \frac{1}{\eta}\int_{t-\eta}^t z^T(s)dsG_3\int_{t-\eta}^t z(s)ds,
\end{aligned}\tag{3.18}$$

$$\begin{aligned}
\dot{V}_3(t) =&\dot{z}^T(t)E^TG_4E\dot{z}(t) - \dot{z}^T(t-\eta)E^TG_4E\dot{z}(t-\eta) + \eta\dot{z}^T(t)E^TG_5E\dot{z}(t) - \int_{t-\eta}^t \dot{z}^T(s)E^TG_5E\dot{z}(s)ds\\
\leq&\dot{z}^T(t)E^T(G_4 + \eta G_5)E\dot{z}(t) - \dot{z}^T(t-\eta)E^TG_4E\dot{z}(t-\eta) - \frac{1}{\eta}\int_{t-\eta}^t \dot{z}^T(s)dsE^TG_5E\int_{t-\eta}^t \dot{z}(s)ds\\
=&\dot{z}^T(t)E^T(G_4 + \eta G_5)E\dot{z}(t) - \dot{z}^T(t-\eta)E^TG_4E\dot{z}(t-\eta)\\
&- \frac{1}{\eta}[z(t) - z(t-\eta)]^TE^TG_5E[z(t) - z(t-\eta)].
\end{aligned}\tag{3.19}$$

By $E^TX = 0$, it can be seen that $E^TXWz(t) \equiv 0$, which implies that

$$2[Az(t) + Dz(t-\eta) + CYJE\dot{z}(t-\eta)]^TXWz(t) \equiv 0. \tag{3.20}$$

From $Az(t) + Dz(t-\eta) + C\dot{z}(t-\eta) - E\dot{z}(t) \equiv 0$, it can be obtained that

$$2[Az(t) + Dz(t-\eta) + CYJE\dot{z}(t-\eta) - E\dot{z}(t)]^T[Z_1 \quad Z_2 \quad Z_3 \quad Z_4 \quad Z_5]\xi^T(t) \equiv 0, \tag{3.21}$$

where $\xi(t) = \left[z^T(t) \quad z^T(t-\eta) \quad \int_{t-\eta}^t z^T(s)ds \quad \dot{z}^T(t-\eta)E^T \quad \dot{z}^T(t)E^T\right]$.

By (3.13) and (3.17)–(3.21), we can obtain

$$\dot{V}(t) - \mu V(t) \leq \dot{V}(t) - \mu z^T(t)E^TFEz(t) \leq \xi^T(t)\Gamma\xi(t) \leq 0. \tag{3.22}$$

According to (3.22) and $e^{-\mu t}(\dot{V}(t) - \mu V(t)) = \frac{d(e^{-\mu t}V(t))}{dt}$, we can have

$$\int_0^t e^{-\mu s}(\dot{V}(s) - \mu V(s))ds = e^{-\mu t}V(t) - V(0) \leq 0. \tag{3.23}$$

For any $t \in (0,\ T]$, from (3.13)–(3.16) and (3.23), we can get $V(t) \leq e^{\mu t}V(0) \leq e^{\mu T}\sum_{k=1}^{3} h_k = h_4$. Define $\omega(t) = Y^{-1}z(t) = \begin{bmatrix} \omega_1(t) \\ \omega_2(t) \end{bmatrix}$. According to $V(t) \leq h_4$ and $Y^T E^T FEY = \begin{bmatrix} \overline{F}_1 & 0 \\ 0 & 0 \end{bmatrix}$, it can be seen that

$$\begin{aligned} \omega_1^T(t)\overline{F}_1\omega_1(t) &= \omega^T(t)Y^T E^T FEY\omega(t) \\ &= z^T(t)Y^{-T}Y^T E^T FEYY^{-1}z(t) \\ &= z^T(t)E^T FEz(t) \\ &\leq V(t) \leq h_4, \quad \forall t \in (0,\ T]. \end{aligned} \tag{3.24}$$

The following two inequalities hold:

$$\begin{aligned} \omega^T(t)\omega(t) &= z^T(t)Y^{-T}Y^{-1}z(t) \\ &= z^T(t)U^{\frac{1}{2}}U^{-\frac{1}{2}}Y^{-T}Y^{-1}U^{-\frac{1}{2}}U^{\frac{1}{2}}z(t) \\ &\leq \lambda_{\max}(U^{-\frac{1}{2}}Y^{-T}Y^{-1}U^{-\frac{1}{2}}) \sup_{-\eta \leq s \leq 0}\{z^T(s)Uz(s)\} \\ &\leq \|Y^{-1}U^{-\frac{1}{2}}\|^2 m_1 = \kappa_1, \quad \forall t \in [-\eta, 0], \end{aligned} \tag{3.25}$$

$$\begin{aligned} \dot{\omega}_1^T(t)\dot{\omega}_1(t) &= \dot{z}^T(t)Y^{-T}\overline{E}^T\overline{E}Y^{-1}\dot{z}(t) \\ &= \dot{z}^T(t)Y^{-T}Y^T E^T J^T JEYY^{-1}\dot{z}(t) \\ &= \dot{z}^T(t)E^T J^T JE\dot{z}(t) \\ &\leq \|JU^{-\frac{1}{2}}\|^2 m_2 = \kappa_2, \quad \forall t \in [-\eta, 0]. \end{aligned} \tag{3.26}$$

By $\overline{C}_2 = 0$ and $\overline{C}_4 = 0$, system (2.1) is equivalent to the following:

$$\begin{cases} \dot{\omega}_1(t) = \overline{A}_1\omega_1(t) + \overline{A}_2\omega_2(t) + \overline{D}_1\omega_1(t-\eta) + \overline{D}_2\omega_2(t-\eta) + \overline{C}_1\dot{\omega}_1(t-\eta), \\ 0 = \overline{A}_3\omega_1(t) + \overline{A}_4\omega_2(t) + \overline{D}_3\omega_1(t-\eta) + \overline{D}_4\omega_2(t-\eta) + \overline{C}_3\dot{\omega}_1(t-\eta). \end{cases} \tag{3.27}$$

According to $\det(\overline{A}_4) \neq 0$, it is easy to see that system (3.27) can be rewritten as follows:

$$\begin{aligned} \begin{bmatrix} \dot{\omega}_1(t) \\ \omega_2(t) \end{bmatrix} &= \overline{\Delta}\begin{bmatrix} \overline{A}_1 \\ \overline{A}_3 \end{bmatrix}\omega_1(t) + \overline{\Delta}\begin{bmatrix} \overline{D}_1 \\ \overline{D}_3 \end{bmatrix}\omega_1(t-\eta) + \overline{\Delta}\begin{bmatrix} \overline{D}_2 \\ \overline{D}_4 \end{bmatrix}\omega_2(t-\eta) + \overline{\Delta}\begin{bmatrix} \overline{C}_1 \\ \overline{C}_3 \end{bmatrix}\dot{\omega}_1(t-\eta) \\ &= \Delta_1\omega_1(t) + \Delta_2\omega_1(t-\eta) + \Delta_3\omega_2(t-\eta) + \Delta_4\dot{\omega}_1(t-\eta). \end{aligned} \tag{3.28}$$

From (3.27) and $\det(\overline{A}_4) \neq 0$, it can also be deduced that

$$\begin{cases} \begin{aligned} \dot{\omega}_1(t) &= (\overline{A}_1 - \overline{A}_2\overline{A}_4^{-1}\overline{A}_3)\omega_1(t) + (\overline{D}_1 - \overline{A}_2\overline{A}_4^{-1}\overline{D}_3)\omega_1(t-\eta) + (\overline{D}_2 - \overline{A}_2\overline{A}_4^{-1}\overline{D}_4)\omega_2(t-\eta) \\ &\quad +(\overline{C}_1 - \overline{A}_2\overline{A}_4^{-1}\overline{C}_3)\dot{\omega}_1(t-\eta) \\ &= H_1\omega_1(t) + H_2\omega_1(t-\eta) + H_3\omega_2(t-\eta) + H_4\dot{\omega}_1(t-\eta), \\ \omega_2(t) &= -\overline{A}_4^{-1}\overline{A}_3\omega_1(t) - \overline{A}_4^{-1}\overline{D}_3\omega_1(t-\eta) - \overline{A}_4^{-1}\overline{D}_4\omega_2(t-\eta) - \overline{A}_4^{-1}\overline{C}_3\dot{\omega}_1(t-\eta) \\ &= L_1\omega_1(t) + L_2\omega_1(t-\eta) + L_3\omega_2(t-\eta) + L_4\dot{\omega}_1(t-\eta). \end{aligned} \end{cases} \tag{3.29}$$

Set $\theta(t) = \begin{bmatrix} \dot{\omega}_1(t) \\ \omega_2(t) \end{bmatrix}$. For any $t \in (0, \eta]$, it can be deduced from (3.24)–(3.26), (3.28), and (3.29) that

$$\begin{cases} \|\theta(t)\| & \le \|\Delta_1 \overline{F}_1^{-\frac{1}{2}}\| \, \|\overline{F}_1^{\frac{1}{2}} \omega_1(t)\| + \|[\Delta_2 \quad \Delta_3]\| \, \|\omega(t-\eta)\| + \|\Delta_4\| \, \|\dot{\omega}_1(t-\eta)\| \\ & \le \|\Delta_1 \overline{F}_1^{-\frac{1}{2}}\| \sqrt{h_4} + \|[\Delta_2 \quad \Delta_3]\| \sqrt{\kappa_1} + \|\Delta_4\| \sqrt{\kappa_2} = \aleph_1, \\ \|\dot{\omega}_1(t)\| & \le \|H_1 \overline{F}_1^{-\frac{1}{2}}\| \, \|\overline{F}_1^{\frac{1}{2}} \omega_1(t)\| + \|[H_2 \quad H_3]\| \, \|\omega(t-\eta)\| + \|H_4\| \, \|\dot{\omega}_1(t-\eta)\| \\ & \le \|H_1 \overline{F}_1^{-\frac{1}{2}}\| \sqrt{h_4} + \|[H_2 \quad H_3]\| \sqrt{\kappa_1} + \|H_4\| \sqrt{\kappa_2} = \Upsilon_1, \\ \|\omega_2(t)\| & \le \|L_1 \overline{F}_1^{-\frac{1}{2}}\| \, \|\overline{F}_1^{\frac{1}{2}} \omega_1(t)\| + \|[L_2 \quad L_3]\| \, \|\omega(t-\eta)\| + \|L_4\| \, \|\dot{\omega}_1(t-\eta)\| \\ & \le \|L_1 \overline{F}_1^{-\frac{1}{2}}\| \sqrt{h_4} + \|[L_2 \quad L_3]\| \sqrt{\kappa_1} + \|L_4\| \sqrt{\kappa_2} = \Re_1. \end{cases} \tag{3.30}$$

For any $t \in (\eta, 2\eta]$, we can get

$$\begin{cases} \|\theta(t)\| & \le \|\Delta_1 \overline{F}_1^{-\frac{1}{2}}\| \, \|\overline{F}_1^{\frac{1}{2}} \omega_1(t)\| + \|\Delta_2 \overline{F}_1^{-\frac{1}{2}}\| \, \|\overline{F}_1^{\frac{1}{2}} \omega_1(t-\eta)\| + \|\Delta_3 \omega_2(t-\eta) + \Delta_4 \dot{\omega}_1(t-\eta)\| \\ & \le (\|\Delta_1 \overline{F}_1^{-\frac{1}{2}}\| + \|\Delta_2 \overline{F}_1^{-\frac{1}{2}}\|) \sqrt{h_4} + \min\{\|\Delta_3\| \Re_1 + \|\Delta_4\| \Upsilon_1, \|[\Delta_4 \quad \Delta_3]\| \aleph_1\} = \aleph_2, \\ \|\dot{\omega}_1(t)\| & \le \|H_1 \overline{F}_1^{-\frac{1}{2}}\| \, \|\overline{F}_1^{\frac{1}{2}} \omega_1(t)\| + \|H_2 \overline{F}_1^{-\frac{1}{2}}\| \, \|\overline{F}_1^{\frac{1}{2}} \omega_1(t-\eta)\| + \|H_3 \omega_2(t-\eta) + H_4 \dot{\omega}_1(t-\eta)\| \\ & \le (\|H_1 \overline{F}_1^{-\frac{1}{2}}\| + \|H_2 \overline{F}_1^{-\frac{1}{2}}\|) \sqrt{h_4} + \min\{\|H_3\| \Re_1 + \|H_4\| \Upsilon_1, \|[H_4 \quad H_3]\| \aleph_1\} = \Upsilon_2, \\ \|\omega_2(t)\| & \le \|L_1 \overline{F}_1^{-\frac{1}{2}}\| \, \|\overline{F}_1^{\frac{1}{2}} \omega_1(t)\| + \|L_2 \overline{F}_1^{-\frac{1}{2}}\| \, \|\overline{F}_1^{\frac{1}{2}} \omega_1(t-\eta)\| + \|L_3 \omega_2(t-\eta) + L_4 \dot{\omega}_1(t-\eta)\| \\ & \le (\|L_1 \overline{F}_1^{-\frac{1}{2}}\| + \|L_2 \overline{F}_1^{-\frac{1}{2}}\|) \sqrt{h_4} + \min\{\|L_3\| \Re_1 + \|L_4\| \Upsilon_1, \|[L_4 \quad L_3]\| \aleph_1\} = \Re_2. \end{cases} \tag{3.31}$$

For any $t \in (\overline{n}\eta, (\overline{n}+1)\eta]$, it can be deduced that

$$\begin{cases} \|\theta(t)\| & \le \|\Delta_1 \overline{F}_1^{-\frac{1}{2}}\| \, \|\overline{F}_1^{\frac{1}{2}} \omega_1(t)\| + \|\Delta_2 \overline{F}_1^{-\frac{1}{2}}\| \, \|\overline{F}_1^{\frac{1}{2}} \omega_1(t-\eta)\| + \|\Delta_3 \omega_2(t-\eta) + \Delta_4 \dot{\omega}_1(t-\eta)\| \\ & \le (\|\Delta_1 \overline{F}_1^{-\frac{1}{2}}\| + \|\Delta_2 \overline{F}_1^{-\frac{1}{2}}\|) \sqrt{h_4} + \min\{\|\Delta_3\| \Re_{\overline{n}} + \|\Delta_4\| \Upsilon_{\overline{n}}, \|[\Delta_4 \quad \Delta_3]\| \aleph_{\overline{n}}\} = \aleph_{\overline{n}+1}, \\ \|\dot{\omega}_1(t)\| & \le \|H_1 \overline{F}_1^{-\frac{1}{2}}\| \, \|\overline{F}_1^{\frac{1}{2}} \omega_1(t)\| + \|H_2 \overline{F}_1^{-\frac{1}{2}}\| \, \|\overline{F}_1^{\frac{1}{2}} \omega_1(t-\eta)\| + \|H_3 \omega_2(t-\eta) + H_4 \dot{\omega}_1(t-\eta)\| \\ & \le (\|H_1 \overline{F}_1^{-\frac{1}{2}}\| + \|H_2 \overline{F}_1^{-\frac{1}{2}}\|) \sqrt{h_4} + \min\{\|H_3\| \Re_{\overline{n}} + \|H_4\| \Upsilon_{\overline{n}}, \|[H_4 \quad H_3]\| \aleph_{\overline{n}}\} = \Upsilon_{\overline{n}+1}, \\ \|\omega_2(t)\| & \le \|L_1 \overline{F}_1^{-\frac{1}{2}}\| \, \|\overline{F}_1^{\frac{1}{2}} \omega_1(t)\| + \|L_2 \overline{F}_1^{-\frac{1}{2}}\| \, \|\overline{F}_1^{\frac{1}{2}} \omega_1(t-\eta)\| + \|L_3 \omega_2(t-\eta) + L_4 \dot{\omega}_1(t-\eta)\| \\ & \le (\|L_1 \overline{F}_1^{-\frac{1}{2}}\| + \|L_2 \overline{F}_1^{-\frac{1}{2}}\|) \sqrt{h_4} + \min\{\|L_3\| \Re_{\overline{n}} + \|L_4\| \Upsilon_{\overline{n}}, \|[L_4 \quad L_3]\| \aleph_{\overline{n}}\} = \Re_{\overline{n}+1}. \end{cases} \tag{3.32}$$

For any $t \in (0,\ T]$, according to (3.30)–(3.32), it can be seen that

$$\|\omega_2(t)\| \le \max\{\Re_1, \Re_2, \cdots, \Re_{\overline{n}+1}\} = \overline{\Re}. \tag{3.33}$$

When $\overline{U}_2 = 0$, it can be obtained from (3.24) and (3.33) that

$$\begin{aligned} z^T(t) U z(t) &= \omega^T(t) Y^T U Y \omega(t) \\ &= \omega^T(t) \overline{U} \omega(t) \\ &= \omega_1^T(t) \overline{U}_1 \omega_1(t) + \omega_2^T(t) \overline{U}_4 \omega_2(t) \\ &\le \|\overline{U}_1^{\frac{1}{2}} \overline{F}^{-\frac{1}{2}}\|^2 h_4 + \|\overline{U}_4^{\frac{1}{2}}\|^2 \overline{\Re}^2 \le m_3, \quad \forall t \in (0,\ T]. \end{aligned} \tag{3.34}$$

If $\overline{U}_2 \neq 0$, according to (3.24), (3.33), and Lemma 2.2, it can be seen that

$$\begin{aligned} z^T(t)Uz(t) &= \omega^T(t)\overline{U}\omega(t) \\ &= \omega_1^T(t)\overline{U}_1\omega_1(t) + 2\omega_1^T(t)\overline{U}_2\omega_2(t) + \omega_2^T(t)\overline{U}_4\omega_2(t) \\ &\leq \omega_1^T(t)(\overline{U}_1 + \frac{1}{\rho_1}\overline{U}_2\overline{U}_2^T)\omega_1(t) + \omega_2^T(t)(\overline{U}_4 + \rho_1 I)\omega_2(t) \\ &\leq \|(\overline{U}_1 + \frac{1}{\rho_1}\overline{U}_2\overline{U}_2^T)^{\frac{1}{2}}\overline{F}^{-\frac{1}{2}}\|^2 h_4 + \|(\overline{U}_4 + \rho_1 I)^{\frac{1}{2}}\|^2\overline{\mathfrak{R}}^2, \ \ \forall t \in (0,\ T]. \end{aligned} \tag{3.35}$$

When $\overline{U}_2 \neq 0$, from (3.24), (3.33), and $z^T(t)Uz(t) = \omega_1^T(t)\overline{U}_1\omega_1(t) + 2\omega_1^T(t)\overline{U}_2\omega_2(t) + \omega_2^T(t)\overline{U}_4\omega_2(t)$, we can also have

$$\begin{aligned} z^T(t)Uz(t) &\leq \omega_1^T(t)(\overline{U}_1 + \frac{1}{\rho_2}I)\omega_1(t) + \omega_2^T(t)(\overline{U}_4 + \rho_2\overline{U}_2^T\overline{U}_2)\omega_2(t) \\ &\leq \|(\overline{U}_1 + \frac{1}{\rho_2}I)^{\frac{1}{2}}\overline{F}^{-\frac{1}{2}}\|^2 h_4 + \|(\overline{U}_4 + \rho_2\overline{U}_2^T\overline{U}_2)^{\frac{1}{2}}\|^2\overline{\mathfrak{R}}^2, \ \ \forall t \in (0,\ T]. \end{aligned} \tag{3.36}$$

According to $m_3 \geq m_1$, (3.6), (3.7), and (3.34)–(3.36), it can be seen that $z^T(t)Uz(t) \leq m_3$ holds for any $t \in [0,\ T]$, which implies that this theorem holds. The proof is completed.

Remark 3.2. *Set* $\widehat{\aleph}_1 = \aleph_1$ *and* $\widehat{\aleph}_p = (\|\Delta_1\overline{F}_1^{-\frac{1}{2}}\| + \|\Delta_2\overline{F}_1^{-\frac{1}{2}}\|)\sqrt{h_4} + \|[\Delta_4 \quad \Delta_3]\|\widehat{\aleph}_{p-1}$*, where* $2 \leq p \leq \overline{n}+1$*. In addition, set* $\overline{\aleph} = \max\{\widehat{\aleph}_1, \widehat{\aleph}_2, \cdots, \widehat{\aleph}_{\overline{n}+1}\}$*.*

By using (3.28) along with (3.29), we obtain $\|\omega_2(t)\| \leq \overline{\mathfrak{R}}$*. If we only use (3.28) and do not use (3.29), we will obtain* $\|\omega_2(t)\| \leq \|\theta(t)\| \leq \overline{\aleph}$*. It can be proved that* $\overline{\aleph} \leq \overline{\mathfrak{R}}$*. In addition, if we only use (3.29), the obtained upper bound of* $\|\omega_2(t)\|$ *is also greater than or equal to that obtained by using (3.28) along with (3.29). Hence, the upper bound of* $\|\omega_2(t)\|$ *obtained by using (3.28) along with (3.29) is better.*

In Theorem 3.1, (3.27) is an equivalent form of system (2.1) and we get (3.28) and (3.29) from (3.27). Because the upper bound of $\|\omega_2(t)\|$ *obtained by using (3.28) along with (3.29) is better, (3.27) is well utilized in Theorem 3.1.*

Remark 3.3. *In Theorem 3.1, condition (3.3) is a linear matrix inequality and helps us to obtain an upper bound of* $\omega_1^T(t)\omega_1(t)$*. In addition, condition (3.3) also helps us to obtain* $\det(\overline{A}_4) \neq 0$*. The invertible matrix* $\overline{A}_4$ *and the condition that* $\overline{C}_2 = 0$ *and* $\overline{C}_4 = 0$ *guarantee system (2.1) to be regular and impulse-free.*

In the numerical simulations, conditions (3.4) and (3.5) may help us to obtain a smaller upper bound of $z^T(t)Uz(t)$ *by adjusting the values of parameters* $\overline{g}_k$ $(k \in \{1, 2, \cdots, 5\})$*,* $\widehat{g}_k$ $(k \in \{1, 2, \cdots, 5\})$*,* $\overline{h}$*, and* $\widehat{h}$*.*

Condition (3.6) shows that the upper bound of $z^T(t)Uz(t)$ *obtained by Theorem 3.1 is* $\|\overline{U}_1^{\frac{1}{2}}\overline{F}^{-\frac{1}{2}}\|^2 h_4 + \|\overline{U}_4^{\frac{1}{2}}\|^2\overline{\mathfrak{R}}^2$ *when* $\overline{U}_2 = 0$*. Then, if* $\|\overline{U}_1^{\frac{1}{2}}\overline{F}^{-\frac{1}{2}}\|^2 h_4 + \|\overline{U}_4^{\frac{1}{2}}\|^2\overline{\mathfrak{R}}^2$ *is not greater than the given positive scalar* m_3*, we can conclude that system (2.1) is finite-time stable with respect to* (m_1, m_2, m_3, T, U) *when* $\overline{U}_2 = 0$*. Similarly, condition (3.7) illustrates that the upper bound of* $z^T(t)Uz(t)$ *obtained by Theorem 3.1 is* $\min\left\{\delta_1 h_4 + \|(\overline{U}_4 + \rho_1 I)^{\frac{1}{2}}\|^2\overline{\mathfrak{R}}^2, \delta_2 h_4 + \|(\overline{U}_4 + \rho_2\overline{U}_2^T\overline{U}_2)^{\frac{1}{2}}\|^2\overline{\mathfrak{R}}^2\right\}$ *when* $\overline{U}_2 \neq 0$*. Then, if* $\min\left\{\delta_1 h_4 + \|(\overline{U}_4 + \rho_1 I)^{\frac{1}{2}}\|^2\overline{\mathfrak{R}}^2, \delta_2 h_4 + \|(\overline{U}_4 + \rho_2\overline{U}_2^T\overline{U}_2)^{\frac{1}{2}}\|^2\overline{\mathfrak{R}}^2\right\}$ *is not greater than the given positive* *scalar* m_3*, we can conclude that system (2.1) is finite-time stable with respect to* (m_1, m_2, m_3, T, U) *when* $\overline{U}_2 \neq 0$.

The following theorem gives a different sufficient condition such that system (2.1) is regular, impulse-free, and finite-time stable.

Theorem 3.2. *Suppose that* $\sup_{-\eta \leq s \leq 0} \{z^T(s)Uz(s)\} \leq m_1$*, where* $U > 0$ *and* $m_1 > 0$*. In addition, suppose that* X *is a given matrix and satisfies* $E^T X = 0$ *and* $rank(X) = n - \varpi$*. Given scalars* $\mu \geq 0$, $\eta > 0$, $\widehat{h} > 0$, $\overline{h} \geq 0$, $\widehat{g}_k > 0\,(k = 1, 2, 3)$, $\overline{g}_k \geq 0\,(k = 1, 2, 3)$, $m_3 > 0\,(m_3 \geq m_1)$, $T > 0$, $\rho_1 > 0$, *and* $\rho_2 > 0$*, system (2.1) is regular, impulse-free, and satisfies* $z^T(t)Uz(t) \leq m_3\,(\forall t \in [0, T])$ *if* $\overline{C}_3 = 0$, $\overline{C}_4 = 0$*, and there exist matrices* $F > 0$, $G_k > 0\,(k = 1, 2, 3)$, $Z_k\,(k = 1, 2, 3, 4)$*, and* W *such that*

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & A^T Z_3 + G_2 & A^T Z_4 - Z_1^T \\ * & \Gamma_{22} & D^T Z_3 - G_2 & D^T Z_4 - Z_2^T \\ * & * & -\frac{G_3}{\eta} & -Z_3^T \\ * & * & * & -Z_4 - Z_4^T \end{bmatrix} < 0, \tag{3.37}$$

$$\overline{g}_k I < G_k < \widehat{g}_k I, \quad \forall k \in \{1, 2, 3\}, \tag{3.38}$$

$$\overline{h} I < F < \widehat{h} I, \tag{3.39}$$

$$\|\overline{U}_1^{\frac{1}{2}} \overline{F}^{-\frac{1}{2}}\|^2 \overline{\Upsilon}^2 + \|\overline{U}_4^{\frac{1}{2}}\|^2 \overline{\mathfrak{R}}^2 \leq m_3 \ (\textit{when } \overline{U}_2 = 0), \tag{3.40}$$

$$\min\left\{\delta_1 \overline{\Upsilon}^2 + \|(\overline{U}_4 + \rho_1 I)^{\frac{1}{2}}\|^2 \overline{\mathfrak{R}}^2, \delta_2 \overline{\Upsilon}^2 + \|(\overline{U}_4 + \rho_2 \overline{U}_2^T \overline{U}_2)^{\frac{1}{2}}\|^2 \overline{\mathfrak{R}}^2\right\} \leq m_3 \ (\textit{when } \overline{U}_2 \neq 0), \tag{3.41}$$

where

$$\begin{aligned}
&\Gamma_{11} = (FE + XW + Z_1)^T A + A^T (FE + XW + Z_1) + G_1 + \eta G_3 - \mu E^T F E, \\
&\Gamma_{12} = (FE + XW + Z_1)^T D - A^T F C + A^T Z_2, \\
&\Gamma_{22} = -G_1 - D^T F C - C^T F D + D^T Z_2 + Z_2^T D, \\
&\sigma_k = \|G_k^{\frac{1}{2}} U^{-\frac{1}{2}}\|^2 \, (k = 1, 2, 3), \\
&h_1 = (\|F^{\frac{1}{2}} E U^{-\frac{1}{2}}\| + \|F^{\frac{1}{2}} C U^{-\frac{1}{2}}\|)^2 m_1, \quad h_2 = (\eta\sigma_1 + \eta^2\sigma_2 + \frac{\eta^2}{2}\sigma_3) m_1, \quad h_3 = e^{\mu T}(h_1 + h_2), \\
&L_1 = -\overline{A}_4^{-1} \overline{A}_3, \quad L_2 = -\overline{A}_4^{-1} \overline{D}_3, \quad L_3 = -\overline{A}_4^{-1} \overline{D}_4, \\
&\kappa_1 = \|Y^{-1} U^{-\frac{1}{2}}\|^2 m_1, \\
&\Upsilon_1 = \sqrt{h_3} + \|\overline{F}_1^{\frac{1}{2}} [\overline{C}_1 \quad \overline{C}_2]\| \sqrt{\kappa_1}, \\
&\mathfrak{R}_1 = \|L_1 \overline{F}_1^{-\frac{1}{2}}\| \Upsilon_1 + \|[L_2 \quad L_3]\| \sqrt{\kappa_1}, \\
&\Upsilon_p = \sqrt{h_3} + \|\overline{F}_1^{\frac{1}{2}} \overline{C}_1 \overline{F}_1^{-\frac{1}{2}}\| \Upsilon_{p-1} + \|\overline{F}_1^{\frac{1}{2}} \overline{C}_2\| \mathfrak{R}_{p-1}, \ 2 \leq p \leq \overline{n} + 1, \\
&\mathfrak{R}_p = \|L_1 \overline{F}_1^{-\frac{1}{2}}\| \Upsilon_p + \|L_2 \overline{F}_1^{-\frac{1}{2}}\| \Upsilon_{p-1} + \|L_3\| \mathfrak{R}_{p-1}, \ 2 \leq p \leq \overline{n} + 1, \\
&\delta_1 = \|(\overline{U}_1 + \frac{1}{\rho_1} \overline{U}_2 \overline{U}_2^T)^{\frac{1}{2}} \overline{F}^{-\frac{1}{2}}\|^2, \qquad \delta_2 = \|(\overline{U}_1 + \frac{1}{\rho_2} I)^{\frac{1}{2}} \overline{F}^{-\frac{1}{2}}\|^2, \\
&\overline{F} = J^{-T} F J^{-1} = \begin{bmatrix} \overline{F}_1 & \overline{F}_2 \\ * & \overline{F}_4 \end{bmatrix}, \quad \begin{bmatrix} \overline{U}_1 & \overline{U}_2 \\ * & \overline{U}_4 \end{bmatrix} = Y^T U Y, \quad \overline{n} = \Xi\left(\frac{T}{\eta}\right),
\end{aligned}$$

$$\overline{\Re} = \max\{\Re_1, \Re_2, \cdots, \Re_{\overline{n}+1}\}, \qquad \overline{\Upsilon} = \max\{\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_{\overline{n}+1}\}.$$

Proof. By using a method similar to that used in the proof of Theorem 3.1, we can have that $\overline{A}_4$ is invertible, which implies that $\det(sE - A) \not\equiv 0$ and $\deg(\det(sE - A)) = \mathrm{rank}(E)$.

Set $\varphi(t) = z(t - \eta)$. Then, system (2.1) can be rewritten as

$$\begin{cases} \varphi(t) = z(t - \eta), \\ E\dot{z}(t) - C\dot{\varphi}(t) = Az(t) + Dz(t - \eta). \end{cases} \tag{3.42}$$

Set $\psi(t) = \begin{bmatrix} z(t) \\ \varphi(t) \end{bmatrix}$, $\widehat{E} = \begin{bmatrix} E & -C \\ 0 & 0 \end{bmatrix}$, $\widehat{A} = \begin{bmatrix} A & 0 \\ 0 & -I \end{bmatrix}$, and $\widehat{D} = \begin{bmatrix} D & 0 \\ I & 0 \end{bmatrix}$. Then, from (3.42), we can get

$$\widehat{E}\,\dot{\psi}(t) = \widehat{A}\psi(t) + \widehat{D}\psi(t - \eta). \tag{3.43}$$

According to $\widehat{E} = \begin{bmatrix} E & -C \\ 0 & 0 \end{bmatrix}$ and $\widehat{A} = \begin{bmatrix} A & 0 \\ 0 & -I \end{bmatrix}$, it can be deduce that

$$\det(s\widehat{E} - \widehat{A}) = \det\left(\begin{bmatrix} sE & -sC \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} A & 0 \\ 0 & -I \end{bmatrix}\right) = \det\left(\begin{bmatrix} sE - A & -sC \\ 0 & I \end{bmatrix}\right) = \det(sE - A). \tag{3.44}$$

From $\overline{C}_3 = 0$ and $\overline{C}_4 = 0$, it can be deduced that $\mathrm{rank}(JEY) = \mathrm{rank}(J[E \quad -C]Y)$, which implies that rank $(E) = \mathrm{rank}(\widehat{E})$. According to (3.44) and $\det(sE - A) \not\equiv 0$, it can be concluded that $\det(s\widehat{E} - \widehat{A}) \not\equiv 0$. In addition, by (3.44), rank $(E) = \mathrm{rank}(\widehat{E})$, and $\deg(\det(sE - A)) = \mathrm{rank}(E)$, we can get $\deg(\det(s\widehat{E} - \widehat{A})) = \mathrm{rank}(\widehat{E})$. According to Definition 2.1, it can be concluded that system (3.43) is regular and impulse-free, which implies that system (2.1) is regular and impulse-free.

Next, we prove that system (2.1) satisfies $z^T(t)Uz(t) \leq m_3\,(\forall t \in [0, T])$. Choose the following Lyapunov-Krasovskii function for system (2.1):

$$V(t) = V_1(t) + V_2(t), \tag{3.45}$$

where

$$\begin{aligned} V_1(t) &= [Ez(t) - Cz(t - \eta)]^T F[Ez(t) - Cz(t - \eta)], \\ V_2(t) &= \int_{t-\eta}^{t} z^T(s)G_1 z(s)ds + \int_{t-\eta}^{t} z^T(s)ds\, G_2 \int_{t-\eta}^{t} z(s)ds + \int_{-\eta}^{0}\int_{t+\beta}^{t} z^T(s)G_3 z(s)dsd\beta. \end{aligned}$$

Set $\gamma = \frac{\|F^{\frac{1}{2}}CU^{-\frac{1}{2}}\|}{\|F^{\frac{1}{2}}EU^{-\frac{1}{2}}\|}$. By (3.45) and Lemma 2.2, we can have

$$\begin{aligned} V_1(0) &= [Ez(0) - Cz(-\eta)]^T F[Ez(0) - Cz(-\eta)] \\ &\leq (1 + \gamma)z^T(0)E^T FEz(0) + (1 + \frac{1}{\gamma})z^T(-\eta)C^T FCz(-\eta) \\ &= (1 + \gamma)z^T(0)U^{\frac{1}{2}}U^{-\frac{1}{2}}E^T FEU^{-\frac{1}{2}}U^{\frac{1}{2}}z(0) + (1 + \frac{1}{\gamma})z^T(-\eta)U^{\frac{1}{2}}U^{-\frac{1}{2}}C^T FCU^{-\frac{1}{2}}U^{\frac{1}{2}}z(-\eta) \end{aligned}$$

$$\leq (1+\gamma)\|F^{\frac{1}{2}}EU^{-\frac{1}{2}}\|^2 \sup_{-\eta\leq s\leq 0}\{z^T(s)Uz(s)\} + (1+\frac{1}{\gamma})\|F^{\frac{1}{2}}CU^{-\frac{1}{2}}\|^2 \sup_{-\eta\leq s\leq 0}\{z^T(s)Uz(s)\}$$
$$\leq h_1, \tag{3.46}$$

$$\begin{aligned}
V_2(0) &= \int_{-\eta}^{0} z^T(s)G_1z(s)ds + \int_{-\eta}^{0} z^T(s)ds\, G_2 \int_{-\eta}^{0} z(s)ds + \int_{-\eta}^{0}\int_{\beta}^{0} z^T(s)G_3z(s)dsd\beta \\
&\leq \int_{-\eta}^{0} z^T(s)G_1z(s)ds + \eta\int_{-\eta}^{0} z^T(s)G_2z(s)ds + \int_{-\eta}^{0}\int_{\beta}^{0} z^T(s)G_3z(s)dsd\beta \\
&\leq \eta\sigma_1 \sup_{-\eta\leq s\leq 0}\{z^T(s)Uz(s)\} + \eta^2\sigma_2 \sup_{-\eta\leq s\leq 0}\{z^T(s)Uz(s)\} + \frac{\eta^2}{2}\sigma_3 \sup_{-\eta\leq s\leq 0}\{z^T(s)Uz(s)\} \\
&\leq \eta\sigma_1 m_1 + \eta^2\sigma_2 m_1 + \frac{\eta^2}{2}\sigma_3 m_1 = h_2.
\end{aligned} \tag{3.47}$$

The following hold:

$$\begin{aligned}
\dot{V}_1(t) &= 2[E\dot{z}(t) - C\dot{z}(t-\eta)]^T F[Ez(t) - Cz(t-\eta)] \\
&= 2[Az(t) + Dz(t-\eta)]^T F[Ez(t) - Cz(t-\eta)],
\end{aligned} \tag{3.48}$$

$$\begin{aligned}
\dot{V}_2(t) &= z^T(t)G_1z(t) - z^T(t-\eta)G_1z(t-\eta) + 2[z^T(t) - z^T(t-\eta)]G_2\int_{t-\eta}^{t} z(s)ds \\
&\quad + \eta z^T(t)G_3z(t) - \int_{t-\eta}^{t} z^T(s)G_3z(s)ds \\
&\leq z^T(t)G_1z(t) - z^T(t-\eta)G_1z(t-\eta) + 2[z^T(t) - z^T(t-\eta)]G_2\int_{t-\eta}^{t} z(s)ds \\
&\quad + \eta z^T(t)G_3z(t) - \frac{1}{\eta}\int_{t-\eta}^{t} z^T(s)dsG_3\int_{t-\eta}^{t} z(s)ds.
\end{aligned} \tag{3.49}$$

By $E^TX = 0$, it can be seen that $C^TX = 0$ and $\dot{z}^T(t)E^TXWz(t) \equiv 0$, which implies that

$$2[Az(t) + Dz(t-\eta)]^T XWz(t) \equiv 0. \tag{3.50}$$

From $Az(t) + Dz(t-\eta) + C\dot{z}(t-\eta) - E\dot{z}(t) \equiv 0$, it can be obtained that

$$2[Az(t) + Dz(t-\eta) + C\dot{z}(t-\eta) - E\dot{z}(t)]^T[Z_1 \quad Z_2 \quad Z_3 \quad Z_4]\xi^T(t) \equiv 0, \tag{3.51}$$

where $\xi(t) = [z^T(t) \quad z^T(t-\eta) \quad \int_{t-\eta}^{t} z^T(s)ds \quad (E\dot{z}(t) - C\dot{z}(t-\eta))^T]$.

For any $t \in (0,\ T]$, from (3.45)–(3.47), we can get $V(t) \leq e^{\mu t}V(0) \leq h_3$. Define $\omega(t) = Y^{-1}z(t) = \begin{bmatrix} \omega_1(t) \\ \omega_2(t) \end{bmatrix}$. According to $V(t) \leq h_3$, it can be seen that

$$\begin{aligned}
&[\omega_1(t) - \overline{C}_1\omega_1(t-\eta) - \overline{C}_2\omega_2(t-\eta)]^T\overline{F}_1[\omega_1(t) - \overline{C}_1\omega_1(t-\eta) - \overline{C}_2\omega_2(t-\eta)] \\
&= [Ez(t) - Cz(t-\eta)]^T J^T J^{-T} F J^{-1} J[Ez(t) - Cz(t-\eta)] \\
&= [Ez(t) - Cz(t-\eta)]^T F[Ez(t) - Cz(t-\eta)]
\end{aligned}$$

$$\leq V(t)$$
$$\leq h_3, \quad \forall t \in (0,\ T]. \tag{3.52}$$

By $\overline{C}_3 = 0$ and $\overline{C}_4 = 0$, system (2.1) is equivalent to the following:

$$\begin{cases} \dot{\omega}_1(t) - \overline{C}_1\dot{\omega}_1(t-\eta) - \overline{C}_2\dot{\omega}_2(t-\eta) = \overline{A}_1\omega_1(t) + \overline{A}_2\omega_2(t) + \overline{D}_1\omega_1(t-\eta) + \overline{D}_2\omega_2(t-\eta), \\ 0 = \overline{A}_3\omega_1(t) + \overline{A}_4\omega_2(t) + \overline{D}_3\omega_1(t-\eta) + \overline{D}_4\omega_2(t-\eta). \end{cases} \tag{3.53}$$

From (3.53), it can be deduced that

$$\begin{aligned} \omega_2(t) &= -\overline{A}_4^{-1}\overline{A}_3\omega_1(t) - \overline{A}_4^{-1}\overline{D}_3\omega_1(t-\eta) - \overline{A}_4^{-1}\overline{D}_4\omega_2(t-\eta) \\ &= L_1\omega_1(t) + L_2\omega_1(t-\eta) + L_3\omega_2(t-\eta). \end{aligned} \tag{3.54}$$

According to (3.25) involved in the proof of Theorem 3.1, it can be seen that $\omega^T(t)\omega(t) \leq \kappa_1$ ($\forall t \in [-\eta, 0]$). Then, for any $t \in (0, \eta]$, it can be deduced from (3.52) and (3.54) that

$$\begin{cases} \|\overline{F}_1^{\frac{1}{2}}\omega_1(t)\| \leq \sqrt{h_3} + \|\overline{F}_1^{\frac{1}{2}}[\overline{C}_1 \quad \overline{C}_2]\|\,\|\omega(t-\eta)\| \leq \sqrt{h_3} + \|\overline{F}_1^{\frac{1}{2}}[\overline{C}_1 \quad \overline{C}_2]\|\sqrt{\kappa_1} = \Upsilon_1, \\ \|\omega_2(t)\| \leq \|L_1\overline{F}_1^{-\frac{1}{2}}\|\,\|\overline{F}_1^{\frac{1}{2}}\omega_1(t)\| + \|[L_2 \quad L_3]\|\,\|\omega(t-\eta)\| \leq \|L_1\overline{F}_1^{-\frac{1}{2}}\|\Upsilon_1 + \|[L_2 \quad L_3]\|\sqrt{\kappa_1} = \Re_1. \end{cases} \tag{3.55}$$

For any $t \in (\eta, 2\eta]$, we can get

$$\begin{cases} \|\overline{F}_1^{\frac{1}{2}}\omega_1(t)\| \leq \sqrt{h_3} + \|\overline{F}_1^{\frac{1}{2}}\overline{C}_1\overline{F}_1^{-\frac{1}{2}}\|\Upsilon_1 + \|\overline{F}_1^{\frac{1}{2}}\overline{C}_2\|\Re_1 = \Upsilon_2, \\ \|\omega_2(t)\| \leq \|L_1\overline{F}_1^{-\frac{1}{2}}\|\Upsilon_2 + \|L_2\overline{F}_1^{-\frac{1}{2}}\|\Upsilon_1 + \|L_3\|\Re_1 = \Re_2. \end{cases} \tag{3.56}$$

For any $t \in (\overline{n}\eta, (\overline{n}+1)\eta]$, it can be deduced that

$$\begin{cases} \|\overline{F}_1^{\frac{1}{2}}\omega_1(t)\| \leq \sqrt{h_3} + \|\overline{F}_1^{\frac{1}{2}}\overline{C}_1\overline{F}_1^{-\frac{1}{2}}\|\Upsilon_{\overline{n}} + \|\overline{F}_1^{\frac{1}{2}}\overline{C}_2\|\Re_{\overline{n}} = \Upsilon_{\overline{n}+1}, \\ \|\omega_2(t)\| \leq \|L_1\overline{F}_1^{-\frac{1}{2}}\|\Upsilon_{\overline{n}+1} + \|L_2\overline{F}_1^{-\frac{1}{2}}\|\Upsilon_{\overline{n}} + \|L_3\|\Re_{\overline{n}} = \Re_{\overline{n}+1}. \end{cases} \tag{3.57}$$

According to (3.34)–(3.36) involved in the proof of Theorem 3.1 and $m_3 \geq m_1$, it can be seen that $z^T(t)Uz(t) \leq m_3$ holds for any $t \in [0,\ T]$, which implies that this theorem holds. The proof is completed. □

The following theorem also presents a sufficient condition such that system (2.1) is regular, impulse-free, and finite-time stable.

Theorem 3.3. *Suppose that* $\sup_{-\eta\leq s\leq 0}\{z^T(s)Uz(s)\} \leq m_1$ *and* $\sup_{-\eta\leq s\leq 0}\{\dot{z}^T(s)U\dot{z}(s)\} \leq m_2$*, where* $U > 0$, $m_1 > 0$ *and* $m_2 > 0$*. In addition, suppose that* X *is a given matrix and satisfies* $E^TX = 0$ *and* $rank(X) = n - \varpi$*. Given scalars* $\mu \geq 0$, $\eta > 0$, $\widehat{h} > 0$, $\overline{h} \geq 0$, $\widehat{g}_k > 0\,(k = 1,2,\cdots,5)$, $\overline{g}_k \geq 0\,(k = 1,2,\cdots,5)$, $m_3 > 0\,(m_3 > m_1)$*, and* $T > 0$*, system (1) is regular, impulse-free, and satisfies* $z^T(t)Uz(t) \leq m_3$ ($\forall t \in [0,T]$) *if* $\overline{C}_3 = 0$, $\overline{C}_4 = 0$, $\det(\overline{A}_4) \neq 0$*, and there exist matrices* $F > 0$, $G_k > 0\,(k = 1,2,\cdots,5)$, $Z_k\,(k = 1,2,\cdots,5)$*, and* W *such that*

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & L_1^TZ_3 + G_2 & (F+Z_1)^TL_3 + L_1^TZ_4 & \Gamma_{15} \\ * & \Gamma_{22} & L_2^TZ_3 - G_2 & Z_2^TL_3 + L_2^TZ_4 & \Gamma_{25} \\ * & * & -\frac{G_3}{\eta} & Z_3^TL_3 & -Z_3^T \\ * & * & * & \Gamma_{44} & \Gamma_{45} \\ * & * & * & * & \Gamma_{55} \end{bmatrix} < 0, \tag{3.58}$$

$$\overline{g}_k I < G_k < \widehat{g}_k I, \quad \forall k \in \{1, 2, \cdots, 5\}, \tag{3.59}$$

$$\overline{h} I < F < \widehat{h} I, \tag{3.60}$$

$$\|U^{\frac{1}{2}} F^{-\frac{1}{2}}\|^2 h_4 \le m_3, \tag{3.61}$$

where

$$\Gamma_{11} = L_1^T(F + Z_1) + (F + Z_1)^T L_1 + (XW)^T A + A^T(XW) + G_1 + \eta G_3 - \mu F - \frac{G_5}{\eta},$$

$$\Gamma_{12} = (F + Z_1)^T L_2 + (XW)^T D + L_1^T Z_2 + \frac{G_5}{\eta}, \qquad \Gamma_{15} = L_1^T(G_4 + \eta G_5 + Z_5) - Z_1^T,$$

$$\Gamma_{22} = -G_1 + L_2^T Z_2 + Z_2^T L_2 - \frac{G_5}{\eta}, \qquad \Gamma_{25} = L_2^T(G_4 + \eta G_5 + Z_5) - Z_2^T,$$

$$\Gamma_{44} = -G_4 + L_3^T Z_4 + Z_4^T L_3, \quad \Gamma_{45} = L_3^T(G_4 + \eta G_5 + Z_5) - Z_4^T,$$

$$\Gamma_{55} = -G_4 - \eta G_5 - Z_5 - Z_5^T, \quad \sigma_k = \|G_k^{\frac{1}{2}} U^{-\frac{1}{2}}\|^2 \ (k = 1, 2, \cdots, 5),$$

$$h_1 = \|F^{\frac{1}{2}} U^{-\frac{1}{2}}\|^2 m_1, \quad h_2 = (\eta\sigma_1 + \eta^2\sigma_2 + \frac{\eta^2}{2}\sigma_3)m_1, \quad h_3 = (\eta\sigma_4 + \frac{\eta^2}{2}\sigma_5)m_2, \quad h_4 = e^{\mu T}\sum_{k=1}^{3} h_k,$$

$$L_1 = (JE + M_2JA)^{-1}M_1JA, \quad L_2 = (JE + M_2JA)^{-1}M_1JD, \quad L_3 = (JE + M_2JA)^{-1}(JC - M_2JD),$$

$$M_1 = \begin{bmatrix} I_{\varpi} & 0 \\ 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 0 \\ 0 & I_{n-\varpi} \end{bmatrix}.$$

Proof. By using a method similar to that used in the proof of Theorem 3.2, we can have that system (2.1) is regular and impulse-free.

Next, we prove that system (2.1) satisfies $z^T(t)Uz(t) \le m_3\ (\forall t \in [0, T])$. Choose the following Lyapunov-Krasovskii function for system (2.1):

$$V(t) = V_1(t) + V_2(t) + V_3(t), \tag{3.62}$$

where

$$V_1(t) = z(t)^T F z(t),$$

$$V_2(t) = \int_{t-\eta}^{t} z^T(s)G_1 z(s)ds + \int_{t-\eta}^{t} z^T(s)ds\, G_2 \int_{t-\eta}^{t} z(s)ds + \int_{-\eta}^{0}\int_{t+\beta}^{t} z^T(s)G_3 z(s)dsd\beta,$$

$$V_3(t) = \int_{t-\eta}^{t} \dot{z}^T(s)G_4\dot{z}(s)ds + \int_{-\eta}^{0}\int_{t+\beta}^{t} \dot{z}^T(s)G_5\dot{z}(s)dsd\beta.$$

Similar to (3.15) and (3.16), we can get $V_2(0) \le h_2$ and $V_3(0) \le h_3$. In addition, by (3.62), we can have

$$V_1(0) = z^T(0)Fz(0) = z^T(0)U^{\frac{1}{2}}U^{-\frac{1}{2}}FU^{-\frac{1}{2}}U^{\frac{1}{2}}z(0) \le \lambda_{\max}\left(U^{-\frac{1}{2}}FU^{-\frac{1}{2}}\right)\sup_{-\eta\le s\le 0}\{z^T(s)Uz(s)\} \le h_1. \tag{3.63}$$

From (3.53), it can be obtained that

$$\begin{cases} \dot{\omega}_1(t) - \overline{C}_1\dot{\omega}_1(t-\eta) - \overline{C}_2\dot{\omega}_2(t-\eta) = \overline{A}_1\omega_1(t) + \overline{A}_2\omega_2(t) + \overline{D}_1\omega_1(t-\eta) + \overline{D}_2\omega_2(t-\eta), \\ 0 = \overline{A}_3\dot{\omega}_1(t) + \overline{A}_4\dot{\omega}_2(t) + \overline{D}_3\dot{\omega}_1(t-\eta) + \overline{D}_4\dot{\omega}_2(t-\eta). \end{cases} \tag{3.64}$$

According to (3.64), it can be seen that $(JE + M_2JA)\dot{z}(t) - (JC - M_2JD)\dot{z}(t-\eta) = M_1JAz(t) + M_1JDz(t-\eta)$. By $\det(\overline{A}_4) \neq 0$, it can be deduced that $JE + M_2JA$ is invertible. Then, we can have

$$\dot{z}(t) = L_1z(t) + L_2z(t-\eta) + L_3\dot{z}(t-\eta). \tag{3.65}$$

From (3.62) and (3.65), we can deduce

$$\dot{V}_1(t) = 2\dot{z}^T(t)Fz(t) = 2[L_1z(t) + L_2z(t-\eta) + L_3\dot{z}(t-\eta)]^T Fz(t), \tag{3.66}$$

$$\begin{aligned}\dot{V}_2(t) \leq\ & z^T(t)G_1z(t) - z^T(t-\eta)G_1z(t-\eta) + 2[z^T(t) - z^T(t-\eta)]G_2\int_{t-\eta}^{t} z(s)ds \\ & + \eta z^T(t)G_3z(t) - \frac{1}{\eta}\int_{t-\eta}^{t} z^T(s)dsG_3\int_{t-\eta}^{t} z(s)ds,\end{aligned} \tag{3.67}$$

$$\dot{V}_3(t) \leq \dot{z}^T(t)(G_4 + \eta G_5)\dot{z}(t) - \dot{z}^T(t-\eta)G_4\dot{z}(t-\eta) - \frac{1}{\eta}[z(t) - z(t-\eta)]^T G_5[z(t) - z(t-\eta)]. \tag{3.68}$$

By $E^TX = 0$, it can be seen that $C^TX = 0$ and $\dot{z}^T(t)E^TXWz(t) \equiv 0$, which implies that

$$2[Az(t) + Dz(t-\eta)]^T XWz(t) \equiv 0. \tag{3.69}$$

From $L_1z(t) + L_2z(t-\eta) + L_3\dot{z}(t-\eta) - \dot{z}(t) \equiv 0$, it can be seen that

$$2[L_1z(t) + L_2z(t-\eta) + L_3\dot{z}(t-\eta) - \dot{z}(t)]^T[Z_1 \quad Z_2 \quad Z_3 \quad Z_4 \quad Z_5]\xi^T(t) \equiv 0, \tag{3.70}$$

where $\xi(t) = \left[z^T(t) \quad z^T(t-\eta) \quad \int_{t-\eta}^{t} z^T(s)ds \quad \dot{z}^T(t-\eta) \quad \dot{z}^T(t)\right]$.

Similar to the proof of Theorem 3.1, it can be obtained that $z^T(t)Fz(t) \leq V(t) \leq e^{\mu T}\sum_{k=1}^{3} h_k = h_4$ $(\forall t \in (0,\ T])$. Then, we can get

$$z^T(t)Uz(t) = z^T(t)F^{\frac{1}{2}}F^{-\frac{1}{2}}UF^{-\frac{1}{2}}F^{\frac{1}{2}}z(t) \leq \|U^{\frac{1}{2}}F^{-\frac{1}{2}}\|^2h_4 \leq m_3, \ \ \forall t \in (0,\ T]. \tag{3.71}$$

According to $m_3 \geq m_1$ and (3.71), it can be seen that $z^T(t)Uz(t) \leq m_3$ holds for any $t \in [0,\ T]$, which implies that this theorem holds. The proof is completed. □

Remark 3.4. *It can be seen that (3.65) is a regular neutral system. Therefore, Theorem 3.3 employs regular neutral system theory to study the finite-time stability of singular neutral system (2.1). In addition, Remark 3.3 can also help us to understand the conditions of Theorems 3.2 and 3.3 easily. Therefore, we omit the explanations for the conditions of Theorems 3.2 and 3.3 in this paper.*

Remark 3.5. *In Theorem 3.3, we suppose that* $\sup_{-\eta\leq s\leq 0}\{z^T(s)Uz(s)\} \leq m_1$ *and* $\sup_{-\eta\leq s\leq 0}\{\dot{z}^T(s)U\dot{z}(s)\} \leq m_2$. *If m_2 is unknown, Theorem 3.3 can be not employed to analyze the finite-time stability problem of system (2.1). In this case, we can use Theorem 3.2 to analyze the finite-time stability problem of system (2.1) when $\overline{C}_3 = 0$ and $\overline{C}_4 = 0$. If m_2 is known, we can use not only Theorem 3.2, but also Theorem 3.3 to analyze the finite-time stability problem of system (2.1) when $\overline{C}_3 = 0$ and $\overline{C}_4 = 0$, and Theorem 3.3 is often better than Theorem 3.2.*

Based on Theorem 3.3, we can have the following result on the asymptotic stability of system (2.1). For the definition of the asymptotic stability of system (2.1), please see Definition 2 of [27].

Corollary 3.1. *Suppose that X is a given matrix and satisfies $E^TX = 0$ and rank$(X) = n - \varpi$. Given scalars $\eta > 0$, system (2.1) is regular, impulse-free, and stable if $\overline{C}_3 = 0$, $\overline{C}_4 = 0$, $\det(\overline{A}_4) \neq 0$, and there exist matrices $F > 0$, $G_k > 0\,(k = 1, 2, \cdots, 5)$, $Z_k\,(k = 1, 2, \cdots, 5)$, and W such that*

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & L_1^T Z_3 + G_2 & (F + Z_1)^T L_3 + L_1^T Z_4 & \Gamma_{15} \\ * & \Gamma_{22} & L_2^T Z_3 - G_2 & Z_2^T L_3 + L_2^T Z_4 & \Gamma_{25} \\ * & * & -\frac{G_3}{\eta} & Z_3^T L_3 & -Z_3^T \\ * & * & * & \Gamma_{44} & \Gamma_{45} \\ * & * & * & * & \Gamma_{55} \end{bmatrix} < 0, \tag{3.72}$$

where

$$\Gamma_{11} = L_1^T(F + Z_1) + (F + Z_1)^T L_1 + (XW)^T A + A^T(XW) + G_1 + \eta G_3 - \frac{G_5}{\eta},$$
$$\Gamma_{12} = (F + Z_1)^T L_2 + (XW)^T D + L_1^T Z_2 + \frac{G_5}{\eta}, \qquad \Gamma_{15} = L_1^T(G_4 + \eta G_5 + Z_5) - Z_1^T,$$
$$\Gamma_{22} = -G_1 + L_2^T Z_2 + Z_2^T L_2 - \frac{G_5}{\eta}, \qquad \Gamma_{25} = L_2^T(G_4 + \eta G_5 + Z_5) - Z_2^T,$$
$$\Gamma_{44} = -G_4 + L_3^T Z_4 + Z_4^T L_3, \quad \Gamma_{45} = L_3^T(G_4 + \eta G_5 + Z_5) - Z_4^T,$$
$$\Gamma_{55} = -G_4 - \eta G_5 - Z_5 - Z_5^T,$$
$$L_1 = (JE + M_2JA)^{-1}M_1JA, \quad L_2 = (JE + M_2JA)^{-1}M_1JD, \quad L_3 = (JE + M_2JA)^{-1}(JC - M_2JD),$$
$$M_1 = \begin{bmatrix} I_\varpi & 0 \\ 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 0 \\ 0 & I_{n-\varpi} \end{bmatrix}.$$

Remark 3.6. *Not only Corollary 3.1 of this paper but also Corollary 1 of [27] can be applied to analyzing the asymptotic stability of system (2.1). It is worth pointing out that Corollary 1 of [27] is obtained by using singular system theory, while Corollary 3.1 of this paper is obtained by using regular neutral system theory. In addition, we find that Corollary 1 of this paper is often better than Corollary 1 of [27]. In the following, we will give an example to show the advantage of Corollary 3.1 of this paper. Please see Example 4.3 of this paper.*

4. Numerical examples

The following three examples are used to show the effectiveness of the results proposed in this paper.

Example 4.1. *Suppose that system (2.1) has the following parameters:*

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; A = \begin{bmatrix} -0.35 & 0.37 \\ 0.75 & \beta \end{bmatrix}, \; D = \begin{bmatrix} -0.15 & 0.18 \\ 0.21 & -0.20 \end{bmatrix}, \; C = \begin{bmatrix} 0.22 & 0 \\ -0.16 & 0 \end{bmatrix}, \; U = \begin{bmatrix} 8 & 4 \\ 4 & 5 \end{bmatrix}.$$

Set $m_1 = 2.78$, $m_2 = 0.20$, $J = I$, $Y = I$, and $X = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ in this example. For convenience, set $\chi = \{1, 2, \cdots, 5\}$.

First, we calculate the minimum allowable m_3 for some different values of parameter β. Set $\eta =$ 0.51, $T = 15$, $\overline{g}_k = 0\,(k \in \chi)$, $\widehat{g}_k = 1.2\,(k \in \chi)$, $\widehat{h} = 10.2$, and $\mu = 0.0001$. In addition, set $\overline{h} = 2.44$, 6.43, 8.69, 9.90, *and* 10.19 *when* $\beta = -0.9, -1.0, -1.1, -1.2,$ *and* -1.3, *respectively. Table 1 lists the values of the minimum allowable* m_3 *obtained by using Theorem 3.1.*

Table 1. Minimum allowable m_3 for different values of β.

	$\beta = -0.9$	$\beta = -1.0$	$\beta = -1.1$	$\beta = -1.2$	$\beta = -1.3$
Minimum allowable m_3	29.6009	18.8339	15.6536	13.9280	12.8905

Second, we calculate the minimum allowable m_3 *for some different values of parameter* η. *Set* $\beta = -1.2$, $T = 15$, $\overline{g}_k = 0\,(k \in \chi)$, $\widehat{g}_k = 1.2\,(k \in \chi)$, $\widehat{h} = 10.2$, *and* $\mu = 0.0001$. *In addition, set* $\overline{h} = 10.19, 10.12, 9.70, 9.36,$ *and* 9.08 *when* $\eta = 0.21, 0.41, 0.61, 0.81,$ *and* 1.01, *respectively. Table 2 lists the values of the minimum allowable* m_3 *obtained by using Theorem 3.1.*

Table 2. Minimum allowable m_3 for different values of η.

	$\eta = 0.21$	$\eta = 0.41$	$\eta = 0.61$	$\eta = 0.81$	$\eta = 1.01$
Minimum allowable m_3	12.9122	13.5138	14.3896	15.4533	16.6553

Finally, we employ a figure to further show the effectiveness of Theorem 3.1. Set $T = 15$, $\eta = 0.51$, $\beta = -1.2$, *and* $\phi(s) = [0.76, -0.62]^T$ $(s \in [-0.51, 0])$. *In addition, set* $\vartheta(t) = z^T(t)Uz(t)$ $(t \in [0, T])$. *Figure 1 depicts the trajectory of* $\vartheta(t)$ $(t \in [0, T])$. *From Figure 1, it can be seen that* $\vartheta(t) \le 13.9280$ *for any* $t \in [0, T]$, *which shows the effectiveness of Theorem 3.1.*

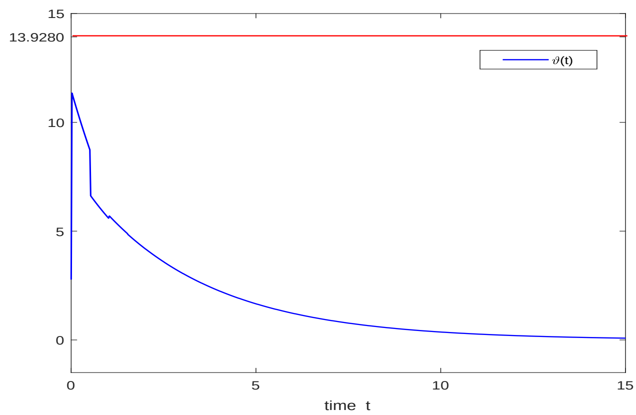

Figure 1. Trajectory of $\vartheta(t) = z^T(t)Uz(t)$ when $\beta = -1.2$ and $\eta = 0.51$.

Example 4.2. *Suppose that system (2.1) has the following parameters:*

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A = \begin{bmatrix} -0.26 & 0.28 \\ 0.89 & 1.27 \end{bmatrix}, D = \begin{bmatrix} -0.17 & 0.18 \\ -0.25 & 0.22 \end{bmatrix}, C = \begin{bmatrix} 0.29 & 0.21 \\ 0 & 0 \end{bmatrix}, U = \begin{bmatrix} 5 & -2 \\ -2 & 6 \end{bmatrix}.$$

Set $m_1 = 3.02$, $J = I$, $Y = I$, *and* $X = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ *in this example. For convenience, set* $\chi = \{1, 2, 3\}$.

First, we suppose that the upper bounds of $\sup_{-\eta\le s\le 0}\{\dot{z}^T(s)U\dot{z}(s)\}$ *are unknown. Our purpose is to calculate the minimum allowable* m_3 *for some different values of parameter* η*. We can employ Theorem 3.2 to analyze the finite-time stability problem of the system considered in this example. When the upper bounds of* $\sup_{-\eta\le s\le 0}\{\dot{z}^T(s)U\dot{z}(s)\}$ *are unknown, Theorem 3.3 is invalid. Set* $T = 15$, $\overline{g}_k = 0\,(k \in \chi)$, $\widehat{g}_k = 1.2\,(k \in \chi)$, $\overline{h} = 14.24$, $\widehat{h} = 20.2$, *and* $\mu = 0.0001$. *Table 3 lists the values of the minimum allowable* m_3 *obtained by using Theorem 3.2.*

Table 3. Minimum allowable m_3 for different values of η.

	η=0.5	η=1.0	η=1.5	η=2.0	η=2.5
Minimum allowable m_3	99.5581	103.0167	106.7578	109.3732	112.9402

Next, we suppose that $\sup_{-\eta\le s\le 0}\{\dot{z}^T(s)U\dot{z}(s)\} \le 0.12$. *Our purpose is to calculate the minimum allowable* m_3 *for some different values of parameter* η*. Set* $T = 15$, $\overline{g}_k = 0\,(k \in \chi)$, $\widehat{g}_k = 1.2\,(k \in \chi)$, $\widehat{h} = 20.2$, *and* $\mu = 0.0001$. *In addition, set* $\overline{h} = 7.57, 6.07, 4.96, 4.24,$ *and* 3.70 *when* $\eta = 0.5, 1.0, 1.5, 2.0,$ *and* 2.5, *respectively. Table 4 lists the values of the minimum allowable* m_3 *obtained by using Theorem 3.3.*

Table 4. Minimum allowable m_3 for different values of η.

	η=0.5	η=1.0	η=1.5	η=2.0	η=2.5
Minimum allowable m_3	7.2112	7.9938	10.2215	11.8405	15.4750

Finally, we employ a figure to further show the effectiveness of Theorems 3.2 and 3.3. Set $T = 15$, $\eta = 0.50$, *and* $\phi(s) = [0.33, -0.54]^T$ $(s \in [-0.50, 0])$. *In addition, set* $\vartheta(t) = z^T(t)Uz(t)$ $(t \in [0, T])$. *Figure 2 depicts the trajectory of* $\vartheta(t)$ $(t \in [0, T])$. *From Figure 2, it can be seen that* $\vartheta(t) \le 7.2112$ *for any* $t \in [0, T]$, *which demonstrates the effectiveness of Theorems 3.2 and 3.3.*

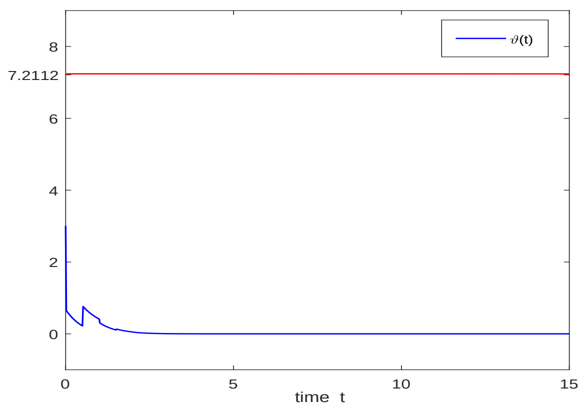

Figure 2. Trajectory of $\vartheta(t) = z^T(t)Uz(t)$ when $\eta = 0.50$.

Remark 4.1. *In Example 4.1, it can be seen that* $\overline{C} = JCY = C = \begin{bmatrix} 0.22 & 0 \\ -0.16 & 0 \end{bmatrix}$. *Because* $\overline{C}_3 \neq 0$, *Theorems 3.2 and 3.3 can be not applied to analyzing the finite-time stability of the system considered in Example 4.1. In addition, Theorem 3.1 can be not applied to analyzing the finite-time stability of the system considered in Example 4.2.*

For the system considered in Example 4.2, it can be seen from Tables 3 and 4 that Theorem 3.3 is better than Theorem 3.2 when $\sup_{-\eta \leq s \leq 0} \{\dot{z}^T(s)U\dot{z}(s)\} \leq 0.12$.

Remark 4.2. *The methods presented in this paper can be employed to analyze the finite-time stability problem of the systems considered in Examples 4.1 and 4.2. However, references [25–42] can be not applied to analyzing the finite-time stability problem of the systems considered in Examples 4.1 and 4.2 because [25–42] did not study the finite-time stability of singular neutral systems.*

Example 4.3. *Suppose that system (2.1) has the following parameters:*

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A = \begin{bmatrix} -0.25 & 0.30 \\ 0.86 & \beta \end{bmatrix}, D = \begin{bmatrix} -0.27 & 0.28 \\ -0.35 & 0.32 \end{bmatrix}, C = \begin{bmatrix} 0.89 & 0.81 \\ 0 & 0 \end{bmatrix}.$$

Set $J = I$, $Y = I$, *and* $X = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ *in this example.*

We pay attention to the asymptotic stability of the system considered in this example. We calculate the maximum allowable time delay η *for some different values of parameter* β. *Corollary 3.1 of this paper and Corollary 1 of [27] can be employed to analyze the stability problem of the system considered in this example. It can be seen from Table 5 that Corollary 3.1 of this paper is much better than Corollary 1 of [27].*

Table 5. Maximum allowable η for different values of β.

	β=0.90	β=1.10	β=1.30	β=1.50	β=1.70
Corollary 3.1 of this paper	0.9973	2.5280	4.2609	4.9613	5.3065
Corollary 1 of [27]	-	-	-	-	-

Symbol "-" means that the method is invalid when $\eta \geq 0.0001$

5. Conclusions

By using the Lyapunov-Krasovskii function approach and regular neutral system theory, this paper has investigated the finite-time stability for singular neutral systems with time delay. Some sufficient conditions have been proposed to guarantee the considered systems to be regular, impulse-free, and finite-time stable. Three numerical examples have been provided to show the effectiveness of the obtained results. It is noted that Theorem 3.1 is obtained under the condition that $\overline{C}_2 = 0$ and $\overline{C}_4 = 0$. Theorem 3.2, Theorem 3.3, and Corollary 3.1 are obtained under the condition that $\overline{C}_3 = 0$ and $\overline{C}_4 = 0$. One of our future works is to study the finite-time stability for singular neutral systems that do not satisfy the conditions just mentioned. In addition, studying the finite-time stability for other systems, such as positive singular neutral systems, fractional-order singular neutral systems, and T-S fuzzy singular neutral systems, is also our future work.

Author contributions

Sheng Wang: Formal analysis, writing-original draft, writing-review & editing; Shaohua Long: Conceptualization, methodology, formal analysis, writing-original draft, writing-review & editing, supervision, simulation. All authors have read and approved the final version of the manuscript for publication.

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Conflicts of interest

The authors have no relevant financial or non-financial interests to disclose.

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