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*Research article*

# Blow-up of solutions for coupled wave equations with damping terms and derivative nonlinearities

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**Abstract:** This work was concerned with the weakly coupled system of semi-linear wave equations with time dependent speeds of propagation, damping terms, and derivative nonlinear terms in generalized Einstein-de Sitter space-time on $\mathbb{R}^n$. Under certain assumptions about the indexes $k_1$, $k_2$, coefficients $\mu_1$, $\mu_2$, and nonlinearity exponents $p$, $q$, applying the iteration technique, finite time blow-up of local solutions to the small initial value problem of the coupled system was investigated. Blow-up region and upper bound lifespan estimate of solutions to the problem were established. Compared with blow-up results in the previous literature, the new ingredient relied on that the blow-up region of solutions obtained in this work varies due to the influence of coefficients $k_1$, $k_2$.

**Keywords:** coupled wave equations; damping terms; iteration method; blow-up; lifespan estimate
**Mathematics Subject Classification:** 35L15, 35L70

## 1. Introduction

Nonlinear is one of the important scientific phenomena which widely exists in different physical systems. The nonlinear wave equation is a typical hyperbolic equation which possesses important physical background and theoretical significance. It can be used to explain and simulate many physical phenomena, such as the propagation of sound waves, small vibrations of elastic rods, and so on. The stability and fracture behavior of nonlinear problems are extremely important research topics. In the nonlinear systems, different initial exponential components can lead to different characteristics of their mathematical solutions. Due to the influence of nonlinear factors, waves will become steep during propagation until they break. Due to the existence of friction phenomena in reality, factors such as damping are inevitable, which could have a certain impact on the energy of physical systems. The coefficient in the damping term depending on time variable could affect the energy decay rate of the solution to the wave equation. Nonlinear external forces could affect the fracture behavior and lifespan estimation of the solution. Recently, many mathematicians pay attention to the blow-up and lifespan estimate of the solution to the nonlinear wave equation together with its generalized forms, which are related to the famous Strauss conjecture (see detailed illustrations in [1–3]).

In the present work, we study the Cauchy problem for a coupled system of semi-linear wave equations with damping terms and derivative type nonlinearities in Einstein-de Sitter space-time, which is shown as follows:

$$\begin{cases} u_{tt} - t^{-2k_1}\Delta u + \dfrac{\mu_1}{t}u_t = |v_t|^p, & (x,t) \in \mathbb{R}^n \times [1,\infty), \\ v_{tt} - t^{-2k_2}\Delta v + \dfrac{\mu_2}{t}v_t = |u_t|^q, & (x,t) \in \mathbb{R}^n \times [1,\infty), \\ u(x,1) = \varepsilon f_1(x), \ \ u_t(x,1) = \varepsilon g_1(x), & x \in \mathbb{R}^n, \\ v(x,1) = \varepsilon f_2(x), \ \ v_t(x,1) = \varepsilon g_2(x), & x \in \mathbb{R}^n. \end{cases} \tag{1.1}$$

Here, $n \geq 1$, $p, q > 1$, $k_i \in [0,1)$, $\mu_i \geq 0$, $i = 1,2$. $f_i(x)$, $g_i(x)\,(i = 1,\,2)$ are nonnegative functions which satisfy $\operatorname{supp}(f_i,\, g_i) \subset B_R(0)$ for $i = 1,\,2$, where $B_R(0) = \{x\big|\,|x| \leq R\}$, $R \geq 2$. $\varepsilon$ is a small positive parameter describing the size of initial data.

To begin, we recall some known results for the classical wave equation

$$\begin{cases} u_{tt} - \Delta u = f(u,\, u_t), & x \in \mathbb{R}^n,\ t > 0, \\ u(x,0) = \varepsilon f(x), \ \ u_t(x,0) = \varepsilon g(x), & x \in \mathbb{R}^n. \end{cases} \tag{1.2}$$

It is well-known that problem (1.2) with power nonlinearity $f(u,\, u_t) = |u|^p$ possesses the Strauss exponent $p_S(n)$ (see [2]). In addition, $p_S(1) = \infty$. When $n \geq 2$, $p_S(n)$ is the positive solution to quadratic equation

$$-(n-1)p^2 + (n+1)p + 2 = 0.$$

More precisely, a solution to the Cauchy problem of the wave equation with small initial values blows up in finite time when $1 < p \leq p_S(n)$. If $p > p_S(n)$, there exists a unique global (in time) solution. Lifespan estimate of solution to the initial boundary value problem of the variable coefficient wave equation with $f(u,\, u_t) = |u|^p$ on exterior domain in two dimensions is obtained by employing the Kato lemma (see [4]). Both the blow-up result and existence of the global solution to problem (1.2) with $f(u,\, u_t) = |u_t|^p$ are discussed in [5, 6]. Problem (1.2) with $f(u,\, u_t) = |u_t|^p$ possesses the Glassey index $p_{\mathrm{G}}(n) = 1 + \frac{2}{n-1}$. When $1 < p \leq p_{\mathrm{G}}(n)$, a solution of the equation blows up in finite time. When $p > p_{\mathrm{G}}(n)$, the problem admits global solution. Zhou et al. [7] established blow-up of the solution to the initial boundary value problem of the variable coefficient wave equation with derivative nonlinearity by solving inequalities of ordinary differential equations. Nonexistence of the global solution to problem (1.2) with $f(u,\, u_t) = |u_t|^p + |u|^q$ is derived (see [8, 9]). The proof is based on the Kato lemma and test function method. Existence of global solution to the wave equations is considered in [10–12]. Kitamura et al. [13] obtained the lifespan estimate of classical solution of nonlinear wave equation with spatial weight in one space dimension. Lai et al. [14] derived lifespan estimate of solution for 2-dimensional semi-linear wave equation in asymptotically Euclidean exterior domain. Formation of singularities of solution to the semi-linear wave equation in general dimensions are investigated (see [15–19]). Ming et al. [20] considered blow-up of solution to the semilinear Moore-Gibson-Thompson equation, which is structurally similar to the wave equation. Concerning the related study of hyperbolic type equations and other models, we refer readers to the related references [21–24] for more details.

Taking $k_1 = k_2 = \mu_1 = \mu_2 = 0$ in problem (1.1), we obtain the following small initial value problem for the coupled system with classical wave equations:

$$\begin{cases} u_{tt} - \Delta u = |v_t|^p, & (x,t) \in \mathbb{R}^n \times [0,\infty), \\ v_{tt} - \Delta v = |u_t|^q, & (x,t) \in \mathbb{R}^n \times [0,\infty), \\ (u, u_t, v, v_t)(x,0) = (\varepsilon u_0, \varepsilon u_1, \varepsilon v_0, \varepsilon v_1)(x), & x \in \mathbb{R}^n. \end{cases} \tag{1.3}$$

We bear in mind that problem (1.2) is a special case of problem (1.3) when $p = q$ and $u = v$. Now, we make brief review of the existing literature on the dynamic properties of solutions to the coupled system of wave equations. There are several research results on the blow-up of the solution to the problem (1.3). Nonexistence of the global solution to the coupled system is shown in [25, 26]. Ikeda et al. [27] studied the behavior of solutions to the problem in different coupling cases with a variety of nonlinear terms. Series blow-up results and upper bound lifespan estimates of solutions to the problem are obtained. Based on the above results, it can be seen that there is a $(p, q)$ critical curve, which describes the blow-up and existence of global solutions, namely,

$$\Omega(n, p, q) := \max\left(\Lambda(n, p, q),\ \Lambda(n, q, p)\right) = 0,$$

where

$$\Lambda(n, p, q) = \frac{p+1}{pq-1} - \frac{n-1}{2}, \tag{1.4}$$

and $p, q > 1$. When $\Omega(n, p, q) < 0$, there are global solutions to the Cauchy problem (1.3). When $\Omega(n, p, q) \geq 0$, no matter how small the initial values are, the solutions always blow up in finite time. Upper bound estimate of the lifespan of solutions satisfies:

$$T(\varepsilon) \leq \begin{cases} C\varepsilon^{-\Omega(n,p,q)^{-1}}, & \Omega(n,p,q) > 0, \\ \exp\left(C\varepsilon^{-(pq-1)}\right), & \Omega(n,p,q) = 0,\ p \neq q, \\ \exp\left(C\varepsilon^{-(p-1)}\right), & \Omega(n,p,q) = 0,\ p = q. \end{cases} \tag{1.5}$$

Kubo et al. [28] proved the existence of the global solution to the problem in the three-dimensional and radial symmetry case.

In problem (1.1), let $k_1 = k_2 = -m,\ m > 0, \mu_1 = \mu_2 = 0$. We arrive at the following small initial value problem for coupled Tricomi equations:

$$\begin{cases} u_{tt} - t^{2m}\Delta u = |v_t|^p, & (x,t) \in \mathbb{R}^n \times [0,\infty), \\ v_{tt} - t^{2m}\Delta v = |u_t|^q, & (x,t) \in \mathbb{R}^n \times [0,\infty), \\ (u, u_t, v, v_t)(x,0) = (\varepsilon u_0, \varepsilon u_1, \varepsilon v_0, \varepsilon v_1)(x), & x \in \mathbb{R}^n. \end{cases} \tag{1.6}$$

When $p = q$, and $u = v$ in problem (1.6), we obtain the Cauchy problem for the single Tricomi equation, namely,

$$\begin{cases} u_{tt} - t^{2m}\Delta u = f(u,\ u_t), & x \in \mathbb{R}^n,\ t > 0, \\ u(x,0) = \varepsilon f(x), \quad u_t(x,0) = \varepsilon g(x), & x \in \mathbb{R}^n. \end{cases} \tag{1.7}$$

The Tricomi equation arises in gas dynamic problems, which is associated with the gas flows with nearly sonic speed describing the transition from subsonic flow to supersonic flow. Let us recall some related investigation of the Cauchy problem for the Tricomi equation. Problem (1.7) with $f(u,\ u_t) = |u|^p$ possesses the Strauss critical exponent $p_S(m,n)$, which is the biggest root of quadratic equation

$$-[n-1+(1-\frac{2}{m+2})]p^2+[n+1-3(1-\frac{2}{m+2})]p+2=0.$$

Applying the iteration argument, Lin et al. [29] derived blow-up dynamics and lifespan estimates of solutions to problem (1.7) with $f(u,\ u_t) = |u|^p$ in the subcritical and critical cases. Lifespan estimate of solution to problem (1.7) with $f(u,\ u_t) = |u_t|^p$ is deduced by applying the test function technique ($\Phi(x,t) = -t^{-2m}\partial_t(\eta_M^{2p'}(t)l(t))\phi(x)$) (see [30]). Chen et al. [31] investigated the blow-up result and upper bound lifespan estimate of solution to problem (1.7) with $f(u,\ u_t) = |u_t|^p + |u|^q$, where the iteration method is used. It is worth it to mention that $p_S(0,n)$ coincides with the Strauss critical exponent $p_S(n)$ of the classical wave equation. We refer the readers to [31–34] for the relevant illustrations.

Recently, the study of the Cauchy problem of the semi-linear Tricomi equation with damping term and mass term

$$\begin{cases} u_{tt} - t^{2m}\Delta u + g(u_t) + h(u) = f(u,\ u_t), & x\in\mathbb{R}^n,\ t>0, \\ u(x,0) = \varepsilon u_0(x), \quad u_t(x,0) = \varepsilon u_1(x), & x\in\mathbb{R}^n \end{cases} \tag{1.8}$$

attracts more attention, where $g(u_t)$ is the damping term and $h(u)$ is the mass term. For the case of $m = 0$ in problem (1.8), using the test function technique, Ikeda et al. [35] studied formation of singularity for the solution to the semi-linear wave equation with damping term and mass term in the subcritical and critical cases. Lai et al. [36] showed the blow-up result and lifespan estimate of solutions to problem (1.8) with space dependent damping term, potential term, and power nonlinearity by employing the test function technique. Ming et al. [37] discussed the Cauchy problem of the semi-linear wave equation with scattering time dependent damping term and divergence form nonlinearities. Upper bound lifespan estimate of solution in the subcritical and critical cases is deduced by making use of the rescaled test function approach and iteration method. Hamouda et al. [38, 39] verified the blow-up result of the solution to problem (1.8) with $-1 < m < 0$, $g(u_t) = \mu t^{-1}u_t$, $h(u) = 0$, and derivative nonlinearity in the case of $t > 1$. That is,

$$\begin{cases} u_{tt} - t^{-2k}\Delta u + \frac{\mu}{t}u_t = |u_t|^p, & x\in\mathbb{R}^n,\ t>1, \\ u(x,0) = \varepsilon u_0(x), \quad u_t(x,0) = \varepsilon u_1(x), & x\in\mathbb{R}^n, \end{cases} \tag{1.9}$$

where $k \in (0,1)$. Under certain assumptions, Hamouda et al. [39] applied the test function method to derive the functional differential inequality of solution in the generalized Einstein-de Sitter space-time. Blow-up index $p = p_E(n,k,\mu) = 1 + \frac{2}{(1-k)(n-1)+k+\mu}$ is presented. When $1 < p \le p_E(n,k,\mu)$, a solution of the equation blows up in finite time. In the special case $\mu = 2$, Hamouda et al. [38] derived the blow-up result for the semi-linear wave equation in generalized Einstein-de Sitter space-time with derivative type nonlinearity by taking advantage of the integral representation formula for the solution and Yagdjian integral transform technique. Palmieri [40] deduces formation of singularity for solution to problem (1.8) with $m > -1$, $g(u_t) = \mu t^{-1}u_t$, and $h(u) = \nu^2 t^{-2}u$ for $t > 1$ by applying the iteration method, where the nonlinear term is power type. Similar to the approach utilized in [39], Hamouda et al. [41] considered formation of singularity of the solution to the corresponding small initial boundary value problem (1.8) with $m > 0$, $g(u_t) = \mu t^{-1} u_t$, and $h(u) = 0$ on exterior domain. Blow-up dynamics of the small initial value problem of the Tricomi equation with the general damping term, and mass term are presented (see [42]). The related results of generalized variable coefficients wave equations are obtained in [43–46]. We also refer the interested readers to the references [47–51] for more details.

Now, we turn to problem (1.6). Under certain hypothetical conditions, there is a $(p, q)$ curve (which has not been proved to be critical or not) regarding the formation of singularity of solutions to system (1.6), that is,

$$\Omega(n, m, p, q) := \max\left(\Lambda(n, m, p, q),\ \Lambda(n, m, q, p)\right) = 0,$$

where $\Lambda(n, m, p, q) = \frac{p+1}{pq-1} - \frac{n(m+1)-2m-1}{2}$. When $\Omega(n, m, p, q) \geq 0$, the solutions $(u, v)$ of the problem blow up in finite time $T(\varepsilon)$, which satisfies:

$$T(\varepsilon) \leq \begin{cases} C\varepsilon^{-\Omega(n,m,p,q)}, & \Omega(n, m, p, q) > 0, \\ \exp\left(C\varepsilon^{-(pq-1)}\right), & \Omega(n, m, p, q) = 0,\ p \neq q, \\ \exp\left(C\varepsilon^{-(p-1)}\right), & \Omega(n, m, p, q) = 0,\ p = q. \end{cases}$$

For the mixed form of nonlinear terms in problem (1.6), relevant research of upper bound lifespan estimates of solutions can be referred to [52, 53]. The related study of the blow-up of solutions to problem (1.6) containing scale invariant damping terms and mass terms is shown in [54].

In problem (1.1), letting $k_1 = k_2 = 0$ and modifying the time coefficient of damping terms, we obtain the small initial value problem of coupled system with scale invariant damping terms as follows:

$$\begin{cases} u_{tt} - \Delta u + \dfrac{\mu_1}{1+t} u_t = |v_t|^p, & (x, t) \in \mathbb{R}^n \times [0, \infty), \\ v_{tt} - \Delta v + \dfrac{\mu_2}{1+t} v_t = |u_t|^q, & (x, t) \in \mathbb{R}^n \times [0, \infty), \\ (u, u_t, v, v_t)(x, 0) = (\varepsilon u_0, \varepsilon u_1, \varepsilon v_0, \varepsilon v_1)(x), & x \in \mathbb{R}^n. \end{cases} \tag{1.10}$$

Chen and Palmieri [55] first show the critical curve of system (1.10) in the effective case by applying the contraction mapping technique and test function method. By constructing appropriate functions, Hamouda et al. [56, 57] obtained the coupled integral inequalities. Blow-up of solutions to problem (1.10) is established by taking advantage of the iterative method. In addition, lifespan estimation $T(\varepsilon)$ of solutions satisfies:

$$T(\varepsilon) \leq \begin{cases} C\varepsilon^{-(\Omega(n,\mu_1,\mu_2,p,q))^{-1}}, & \Omega(n, \mu_1, \mu_2, p, q) > 0, \\ \exp\left(C\varepsilon^{-(pq-1)}\right), & \Omega(n, \mu_1, \mu_2, p, q) = 0,\ p \neq q, \\ \exp\left(C\varepsilon^{-\min\left(\frac{pq-1}{p+1}, \frac{pq-1}{q+1}\right)}\right), & \Omega(n, \mu_1, \mu_2, p, q) = 0,\ p = q, \end{cases}$$

where $\Omega(n, \mu_1, \mu_2, p, q) = \max\left(\Lambda(n + \mu_1, p, q),\ \Lambda(n + \mu_2, q, p)\right)$ (here, $\Lambda(n, p, q)$ is defined by (1.4)). Blow-up region of solutions to the problem is presented. In problem (1.10), by replacing the coefficient of damping terms $\frac{\mu_i}{1+t}$ $(i = 1, 2)$ with general nonnegative continuous and integrable functions $b_i(t)$ $(i = 1, 2)$, we obtain a general weakly coupled system of semi-linear damped wave equations. By applying the multiplier technique, Palmieri and Takamura [58] proved that there is a $(p, q)$ critical curve for solutions to the problem. Compared with that of problem (1.3), the $(p, q)$ critical curve does not change. Upper bound estimate for the lifespan of solution is the same as (1.5). The studies of the blow-up and existence of global solutions to coupled systems with other types of damping terms and nonlinear terms are presented in [27, 59–61].

Since the wave speeds change with time in problem (1.1), we transform the damping terms to obtain scale invariant dampings, namely, $u_1(x, \tau_1) = u(x, t)$, $v_1(x, \tau_2) = v(x, t)$, where

$$\tau_i = \phi_{k_i}(t) := \frac{t^{1-k_i}}{1-k_i}, \quad i = 1, 2. \tag{1.11}$$

Then, $u_1(x, \tau_1)$ and $v_1(x, \tau_2)$ satisfy the following equations, respectively:

$$\begin{aligned}
&(u_1)_{\tau_1\tau_1} - \Delta u_1 + \frac{\mu_1 - k_1}{(1-k_1)\tau_1}(u_1)_{\tau_1} \\
&= (1-k_1)^{\frac{2k_1}{1-k_1}}(1-k_2)^{\frac{k_2 p}{k_2-1}}\tau_1^{\frac{2k_1}{1-k_1}}\tau_2^{\frac{k_2 p}{k_2-1}}\left|(v_1)_{\tau_2}\right|^p, \\
&(v_1)_{\tau_2\tau_2} - \Delta v_1 + \frac{\mu_2 - k_2}{(1-k_2)\tau_2}(v_1)_{\tau_2} \\
&= (1-k_2)^{\frac{2k_2}{1-k_2}}(1-k_1)^{\frac{k_1 q}{k_1-1}}\tau_2^{\frac{2k_2}{1-k_2}}\tau_1^{\frac{k_1 q}{k_1-1}}\left|(u_1)_{\tau_1}\right|^q.
\end{aligned}$$

In fact, we make the transformation of time variable $t$ in (1.11), so that the ranges of compact support of $u(x, t)$ and $v(x, t)$ are

$$\operatorname{supp}(u, v) \subset \left\{(x, t) \in \mathbb{R}^n \times [1, \infty) \middle| \, |x| \leq \phi_{k_i}(t) + R\right\}, \quad i = 1,\ 2.$$

Inspired by the works in [38, 39, 41, 42, 54, 56, 62], our target in this paper is to investigate blow-up and lifespan estimates of solutions to the weakly coupled system of the semi-linear wave equation with scale invariant dampings and derivative nonlinearities in generalized Einstein-de Sitter space-time. We observe that nonexistence of the global solution to the Cauchy problem of the semi-linear wave equation with derivative nonlinearity in the generalized Einstein-de Sitter space-time is derived in [38]. The proof is based on the integral representation formula for the solution to the corresponding linear problem in the one-dimensional case, which is determined through the Yagdjian integral transform approach. It is worth noticing that Hassen et al. [42] verified the nonexistence of global solution to the generalized Tricomi equation with scale invariant damping, mass term, and derivative nonlinearity in the subcritical and critical cases. Upper bound lifespan estimates of solution to the problem are obtained by making use of the method of solving inequalities of ordinary differential equations. Employing the similar method in [42], Hamouda et al. [41] established the blow-up dynamic and lifespan estimate of solutions to the initial boundary problem for the generalized Tricomi equation with scale invariant damping and derivative nonlinearity on exterior domain. For problem (1.9), Hamouda et al. [39] exploited the test function method to establish functional differential inequality of the solution in the generalized Einstein-de Sitter space-time. Blow-up region and upper bound lifespan estimates of solutions to problem (1.9) are established. Hamouda et al. [56] obtained the blow-up of solutions to the Cauchy problem of the semi-linear wave equation with scale invariant damping terms (namely, problem (1.10)) by deriving the coupled integral inequalities and using the iterative approach. Improvement on the blow-up results of solutions to the weakly coupled system of wave equations with scale invariant damping terms, mass terms, and derivative nonlinearities are considered (see [62]). Taking advantage of the test function technique, Hassen et al. [54] investigated the blow-up phenomenons of solutions to the weakly coupled system of the generalized Tricomi equation with scale invariant dampings, mass terms, and derivative type nonlinear terms. From our observation, there is no related results for blow-up and lifespan estimate of solutions to problem (1.1) with scale invariant damping terms and derivative nonlinearities in the generalized Einstein-de Sitter space-time. The main difficulties in this work are to establish appropriate iterative frameworks in the coupled system case and lower bounds for the constructed functionals, which are used to derive the finite time blow-up and upper bound lifespan estimate of solutions. We extend the problem studied in [39] to problem (1.1) by utilizing the test function method, which is different from the constructed functions in [54, 62] (see Theorem 1.1). The results in Theorem 1.1 coincide with the results in [39] when $p = q$ and $k_1 = k_2$. As a consequence, the results in [39] are a special case of the results in Theorem 1.1. Compared with blow-up results in the work [56], when $k_i = 0,\ i = 1, 2$ in problem (1.1) (that is, $\sigma_i = \mu_i$), blow-up region $\Omega(n, \sigma_1, \sigma_2, p, q)$ of problem (1.1) is same as that in [56]. The blow-up region of solutions obtained in this paper varies due to the influence of the coefficients $k_1,\ k_2$. To the best of our knowledge, the results in Theorem 1.1 are new.

For the problem (1.1), first of all, we present the definition of its energy solutions.

**Definition 1.1.** Let $(f_i, g_i) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, $i = 1, 2$. The solutions $(u, v)$ are called energy solutions of problem (1.1) on $[1, T)$, if

$$\begin{cases} (u, v) \in C([1, T), H^1(\mathbb{R}^n)) \cap C^1([1, T), L^2(\mathbb{R}^n)), \\ u_t \in L^q_{loc}((1, T) \times \mathbb{R}^n),\ v_t \in L^p_{loc}((1, T) \times \mathbb{R}^n), \end{cases}$$

and the following two equalities are satisfied:

$$\begin{aligned} &\int_{\mathbb{R}^n} u_t(x, t)\Phi(x, t)dx - \varepsilon \int_{\mathbb{R}^n} g_1(x)\Phi(x, 1)dx \\ &- \int_1^t \int_{\mathbb{R}^n} u_t(x, s)\Phi_t(x, s)dxds + \int_1^t s^{-2k_1} \int_{\mathbb{R}^n} \nabla u(x, s)\nabla\Phi(x, s)dxds \\ &+ \int_1^t \int_{\mathbb{R}^n} \frac{\mu_1}{s} u_t(x, s)\Phi(x, s)dxds \\ &= \int_1^t \int_{\mathbb{R}^n} |v_t(x, s)|^p\, \Phi(x, s)dxds, \end{aligned}$$

$$\begin{aligned} &\int_{\mathbb{R}^n} v_t(x, t)\tilde{\Phi}(x, t)dx - \varepsilon \int_{\mathbb{R}^n} g_2(x)\tilde{\Phi}(x, 1)dx \\ &- \int_1^t \int_{\mathbb{R}^n} v_t(x, s)\tilde{\Phi}_t(x, s)dxds + \int_1^t s^{-2k_2} \int_{\mathbb{R}^n} \nabla v(x, s)\nabla\tilde{\Phi}(x, s)dxds \\ &+ \int_1^t \int_{\mathbb{R}^n} \frac{\mu_2}{s} v_t(x, s)\tilde{\Phi}(x, s)dxds \\ &= \int_1^t \int_{\mathbb{R}^n} |u_t(x, s)|^q\, \tilde{\Phi}(x, s)dxds, \end{aligned}$$

where $\Phi, \tilde{\Phi} \in C_0^\infty(\mathbb{R}^n \times [1, T))$.

The main result of this paper is stated below.

**Theorem 1.1.** *Let $\Omega(n,\sigma_1,\sigma_2,p,q) = \max(\Lambda(n+\sigma_1,p,q),\Lambda(n+\sigma_2,q,p)) \geq 0$, where $\Lambda(n,p,q)$ is defined by* (1.4)*, $\sigma_i = k_i + \mu_i$, $i = 1,2$. Suppose that $f_i \in H^1(\mathbb{R}^n)$, $g_i \in L^2(\mathbb{R}^n)$ are non-negative functions which are not identically zero. If $(u,v)$ are the energy solutions of problem* (1.1) *which satisfy* $\operatorname{supp}(u,v) \subset \left\{(x,t) \in \mathbb{R}^n \times [1,\infty) \middle| |x| \leq \phi_{k_i}(t) + R\right\}$*, then there exists a positive constant $\varepsilon_0(n,k_1,k_2,\mu_1,\mu_2,f_1,f_2,g_1,g_2,T_3)$ ($T_3$ is a positive constant), such that for all $\varepsilon \in (0,\varepsilon_0]$, the upper bound estimate of lifespan $T(\varepsilon)$ of $(u,v)$ satisfies:*

$$T(\varepsilon) \leq \begin{cases} C\varepsilon^{\frac{pq-1}{A_1-B_1}}, & \Omega(n,\sigma_1,\sigma_2,p,q) = \Lambda(n+\sigma_1,p,q) \geq 0,\ \sigma_1 \geq \sigma_2, \\ C\varepsilon^{\frac{pq-1}{A_2-B_2}}, & \Omega(n,\sigma_1,\sigma_2,p,q) = \Lambda(n+\sigma_2,q,p) \geq 0,\ \sigma_1 \leq \sigma_2, \\ \exp\left(C\varepsilon^{-\min\left(\frac{pq-1}{p+1},\frac{pq-1}{q+1}\right)}\right), & \Lambda(n+\sigma_1,p,q) = \Lambda(n+\sigma_2,q,p) = 0, \end{cases} \tag{1.12}$$

where the constant $C$ is independent of $\varepsilon$, $A_1 = \frac{n-1}{2}(pq-1) + \frac{\sigma_2}{2}pq + \frac{\sigma_1}{2}p$, $B_1 = \frac{\sigma_2}{2}p + \frac{\sigma_1}{2} + p + 1$, $A_2 = \frac{n-1}{2}(pq-1) + \frac{\sigma_1}{2}pq + \frac{\sigma_2}{2}q$, $B_2 = \frac{\sigma_1}{2}q + \frac{\sigma_2}{2} + q + 1$.

**Remark 1.1.** It is worth it to mention that the upper bound lifespan estimate (1.12) of solutions to problem (1.1) in Theorem 1.1 coincides with the results in [39] when $p = q$ and $k_1 = k_2$. In addition, when $k_i = 0$, $i = 1, 2$ in problem (1.1) (that is, $\sigma_i = \mu_i$), blow-up region $\Omega(n,\sigma_1,\sigma_2,p,q)$ of problem (1.1) is the same as that in [56]. Therefore, the results in [39, 56] could be viewed as special cases of the results in Theorem 1.1. Moreover, the results in Theorem 1.1 complement the case that propagation speeds of generalized coupled wave equations with dampings and derivative type nonlinearities contain power functions with negative exponents.

## 2. Related lemmas

Suppose that $\rho_i(t)$ $(i = 1,2)$ satisfies the following differential equation:

$$\frac{d^2}{dt^2}\rho_i(t) - t^{-2k_i}\rho_i(t) - \frac{d}{dt}\left(\frac{\mu_i}{t}\rho_i(t)\right) = 0, \quad t \geq 1,\ i = 1,2.$$

According to [34, 39], $\rho_i(t)$ can be expressed as

$$\rho_i(t) = t^{\frac{1+\mu_i}{2}} K_{\frac{\mu_i-1}{2(1-k_i)}}\left(\phi_{k_i}(t)\right), \quad t \geq 1,\ i = 1,2, \tag{2.1}$$

where $\phi_{k_i}(t)$ are defined by (1.11),

$$K_\nu(t) = \int_0^\infty \exp(-t\cosh\xi)\cosh(\nu\xi)d\xi, \quad \nu \in \mathbb{R}.$$

It holds that

$$\lim_{t\to\infty} \frac{t^{k_i}\rho_i'(t)}{\rho_i(t)} = -1, \quad i = 1,2, \tag{2.2}$$

$$\rho_1(t)e^{\phi_{k_1}(t)} \geq C_0^{-1}t^{\frac{k_1+\mu_1}{2}}, \quad \rho_2^{-p}(t)e^{-p\phi_{k_1}(t)} \geq C_0^{-p}t^{\frac{-p(k_1+\mu_1)}{2}}, \tag{2.3}$$

where $C_0$ is a positive constant. Similarly, we acquire

$$\rho_2(t)e^{\phi_{k_2}(t)} \geq \widetilde{C}_0^{-1}t^{\frac{k_2+\mu_2}{2}}, \quad \rho_1^{-q}(t)e^{-q\phi_{k_2}(t)} \geq \widetilde{C}_0^{-q}t^{\frac{-q(k_2+\mu_2)}{2}},$$

where $\widetilde{C_0}$ is a positive constant.

We define the function $\varphi_i(x)\,(i=1,2)$ as follows:

$$\varphi_i(x)=\begin{cases} e^x+e^{-x}, & n=1,\\ \displaystyle\int_{\mathbb{S}^{n-1}} e^{x\cdot\omega}d\omega, & n\geq 2.\end{cases}$$

Therefore, we obtain $\Delta\varphi_i(x)=\varphi_i(x)$. Let $\Psi_i(x,t)=\varphi_i(x)\rho_i(t)\,(i=1,2)$. It holds that $\Delta\Psi_i(t)=\Psi_i(x,t)$ and

$$\frac{d^2}{dt^2}\Psi_i(x,t)-t^{-2k_i}\Delta\Psi_i(x,t)-\frac{d}{dt}\left(\frac{\mu_i}{t}\Psi_i(x,t)\right)=0.$$

**Lemma 2.1.** *[17] Let $r>1$. It holds that*

$$\begin{aligned}&\int_{\{|x|\leq\phi_{k_i}(t)+R\}}\Psi_i^r(x,t)dx\\ &\leq C_1\rho_i^r(t)e^{r\phi_{k_i}(t)}\left(1+\phi_{k_i}(t)\right)^{\frac{(2-r)(n-1)}{2}},\quad t\geq 1,\ i=1,2,\end{aligned}\tag{2.4}$$

where $C_1$ is a positive number $C_1=C(n,\mu_1,\mu_2,k_1,k_2,p,q,R,r)$.

Now, we define the following functions related to the solutions $u(x,t)$ and $v(x,t)$ of problem (1.1):

$$G_1(t)=\int_{\mathbb{R}^n}u(x,t)\Psi_1(x,t)dx,\quad \tilde{G}_1(t)=\int_{\mathbb{R}^n}u_t(x,t)\Psi_1(x,t)dx,$$

$$G_2(t)=\int_{\mathbb{R}^n}v(x,t)\Psi_2(x,t)dx,\quad \tilde{G}_2(t)=\int_{\mathbb{R}^n}v_t(x,t)\Psi_2(x,t)dx.$$

In the lemma below, we present lower bound estimates for $G_i(t)$ and $\tilde{G}_i(t)$.

**Lemma 2.2.** *[39] Suppose that $(u,v)$ are solutions to problem (1.1). The initial value functions $f_i,g_i\,(i=1,2)$ satisfy the assumptions of Theorem 1.1. Then, there exists a constant $T_0=T_0(k_1,k_2,\mu_1,\mu_2)>2$ such that*

$$G_i(t)\geq C_{G_i}\varepsilon t^{k_i},\quad t\geq T_0,\ i=1,2,$$

where $C_{G_i}=C_{G_i}(f_1,g_1,f_2,g_2,n,k_1,k_2,\mu_1,\mu_2,R)$.

**Lemma 2.3.** *[39] Suppose that $(u,v)$ are solutions to problem (1.1). The initial value functions $f_i,g_i\,(i=1,2)$ satisfy the assumptions of Theorem 1.1. Then, there exists $T_1=T_1(k_1,k_2,\mu_1,\mu_2)>T_0$ such that*

$$\tilde{G}_i(t)\geq C_{\tilde{G}_i}\varepsilon,\quad t\geq T_1,\ i=1,2,$$

where $C_{\tilde{G}_i}=C_{\tilde{G}_i}(f_1,g_1,f_2,g_2,n,k_1,k_2,\mu_1,\mu_2,R)$. It holds that

$$\tilde{G}_1(t)+\left[\frac{\mu_1}{t}-\frac{\rho_1'(t)}{\rho_1(t)}\right]G_1(t)=\int_1^t\int_{\mathbb{R}^n}|v_t(x,s)|^p\,\Psi_1(x,s)dxds+\varepsilon C(f_1,g_1),\tag{2.5}$$

$$\tilde{G}_1'(t) + \left[\frac{\mu_1}{t} - \frac{\rho_1'(t)}{\rho_1(t)}\right]\tilde{G}_1(t) = t^{-2k_1}G_1(t) + \int_{\mathbb{R}^n} |v_t(x,t)|^p\,\Psi_1(x,t)dx, \tag{2.6}$$

where $\rho_1(t)$ is defined by (2.1).

**Remark 2.1.** For $\tilde{G}_2(t)$ and $\tilde{G}_2'(t)$, we acquire similar forms of (2.5) and (2.6) with suitable modification, which are omitted here for simplification.

## 3. Proof of Theorem 1.1

Let $\alpha$ be a positive constant. Multiplying both sides of (2.5) by $\alpha\frac{\rho_1'(t)}{\rho_1(t)}$ and using (2.6), we arrive at

$$\begin{aligned}
&\tilde{G}_1'(t) + \left(\frac{\mu_1}{t} - (1-\alpha)\frac{\rho_1'(t)}{\rho_1(t)}\right)\tilde{G}_1(t)\\
&= -\varepsilon\alpha\frac{\rho_1'(t)}{\rho_1(t)}C(f_1,g_1) + \left(t^{-2k_1} + \alpha\frac{\rho_1'(t)}{\rho_1(t)}\left(\frac{\mu_1}{t} - \frac{\rho_1'(t)}{\rho_1(t)}\right)\right)G_1(t)\\
&\quad + \int_{\mathbb{R}^n} |v_t(x,t)|^p\,\Psi_1(x,t)dx\\
&\quad - \alpha\frac{\rho_1'(t)}{\rho_1(t)}\int_1^t\int_{\mathbb{R}^n} |v_t(x,s)|^p\,\Psi_1(x,s)dxds, \quad t \geq 1.
\end{aligned}$$

Noticing the condition (2.2), we choose $T_2 \geq T_1$ such that

$$\begin{aligned}
&\tilde{G}_1'(t) + \left(\frac{\mu_1}{t} - (1-\alpha)\frac{\rho_1'(t)}{\rho_1(t)}\right)\tilde{G}_1(t)\\
&\geq \frac{\varepsilon\alpha t^{-k_1}}{2}C(f_1,g_1) + (1-4\alpha)t^{-2k_1}G_1(t) + \int_{\mathbb{R}^n} |v_t(x,t)|^p\,\Psi_1(x,t)dx\\
&\quad + \frac{\alpha t^{-k_1}}{2}\int_1^t\int_{\mathbb{R}^n} |v_t(x,s)|^p\,\Psi_1(x,s)dxds, \quad t \geq T_2.
\end{aligned} \tag{3.1}$$

We set $\alpha \in \left(\frac{1}{7}, \frac{1}{4}\right)$. Using Lemma 2.2, formula (3.1) can be rewritten as

$$\begin{aligned}
&\tilde{G}_1'(t) + \left(\frac{\mu_1}{t} - (1-\alpha)\frac{\rho_1'(t)}{\rho_1(t)}\right)\tilde{G}_1(t)\\
&\geq \frac{\varepsilon\alpha t^{-k_1}}{2}C(f_1,g_1) + \int_{\mathbb{R}^n} |v_t(x,t)|^p\,\Psi_1(x,t)dx\\
&\quad + \frac{\alpha t^{-k_1}}{2}\int_1^t\int_{\mathbb{R}^n} |v_t(x,s)|^p\,\Psi_1(x,s)dxds, \quad t \geq T_2.
\end{aligned} \tag{3.2}$$

In the following, we set

$$L_1(t) = \frac{1}{16}\int_{T_3}^t\int_{\mathbb{R}^n} |v_t(x,s)|^p\,\Psi_1(x,s)dxds + \varepsilon C_2, \tag{3.3}$$

$$L_2(t) = \frac{1}{16}\int_{T_3}^t\int_{\mathbb{R}^n} |u_t(x,s)|^q\,\Psi_2(x,s)dxds + \varepsilon C_2,$$

where $C_2 = \min\left(\frac{\alpha C(f_1,g_1)}{4(1+\alpha)}, \frac{\alpha C(f_2,g_2)}{4(1+\alpha)}, C_{\tilde{G}_1}, C_{\tilde{G}_2}\right)$.

Let $F_i(t) = \tilde{G}_1(t) - L_1(t)\,(i = 1, 2)$. Making use of (3.2), we know

$$\begin{aligned}
&F_1'(t) + \left(\frac{\mu_1}{t} - (1-\alpha)\frac{\rho_1'(t)}{\rho_1(t)}\right)F_1(t)\\
&\geq \frac{15}{16}\int_{\mathbb{R}^n} |v_t(x,t)|^p\,\Psi_1(x,t)dx\\
&\quad + \left(\frac{\alpha}{2} - \frac{1}{16}\left(\frac{\mu_1}{t^{1-k_1}} - (1+\alpha)\frac{t^{k_1}\rho_1'(t)}{\rho_1(t)}\right)\right)\\
&\quad + \left(\frac{\alpha}{2}C(f_1,g_1) - C_2\left(\frac{\mu_1}{t^{1-k_1}} - (1+\alpha)\frac{t^{k_1}\rho_1'(t)}{\rho_1(t)}\right)\right)\varepsilon t^{-k_1}, \quad t \geq T_3,
\end{aligned}$$

where $T_3 \geq T_2$. When $t \geq T_3$, we arrive at the following inequalities:

$$\frac{\alpha}{2} - \frac{1}{16}t^{k_1}\left(\frac{\mu_1}{t} - (1+\alpha)\frac{\rho_1'(t)}{\rho_1(t)}\right) \geq 0,$$

$$\frac{\alpha}{2}C(f_1,g_1) - C_2 t^{k_1}\left(\frac{\mu_1}{t} - (1+\alpha)\frac{\rho_1'(t)}{\rho_1(t)}\right) \geq 0.$$

Thus, we obtain

$$F_1'(t) + \left(\frac{\mu_1}{t} - (1-\alpha)\frac{\rho_1'(t)}{\rho_1(t)}\right)F_1(t) \geq 0, \quad t \geq T_3. \tag{3.4}$$

Multiplying both sides of (3.4) by $\frac{t^{\mu_1}}{\rho_1^{1+\alpha}(t)}$ and integrating over $(T_3, t)$, we acquire

$$F_1(t) \geq F_1(T_3)\frac{T_3^{\mu_1}}{\rho_1^{1+\alpha}(T_3)}\frac{\rho_1^{1+\alpha}(t)}{t^{\mu_1}}, \quad t \geq T_3.$$

Combining the definition of $C_2$ and applying

$$F_1(T_3) = \tilde{G}_1(T_3) - L_1(T_3) \geq \tilde{G}_1(T_3) - C_{\tilde{G}_1}\varepsilon \geq 0,$$

we conclude $F_1(t) \geq 0$. It follows that

$$\tilde{G}_1(t) \geq L_1(t), \quad t \geq T_3.$$

Similarly, we have

$$\tilde{G}_2(t) \geq L_2(t), \quad t \geq T_3. \tag{3.5}$$

Making use of the Hölder inequality and (2.4) yields

$$\begin{aligned}
&\int_{\mathbb{R}^n} |v_t(x,t)|^p\,\Psi_1(x,t)dx\\
&\geq \left(\int_{\{|x|\leq\phi_{k_1}(t)+R\}} \Psi_2^{\frac{p}{p-1}}(x,t)\Psi_1^{-\frac{1}{p-1}}(x,t)dx\right)^{-(p-1)}\left(\int_{\mathbb{R}^n} v_t(x,t)\Psi_2(x,t)dx\right)^p\\
&\geq C_1^{1-p}\rho_1(t)\rho_2^{-p}(t)e^{-(p-1)\phi_{k_1}(t)}\left(1+\phi_{k_1}(t)\right)^{-\frac{(p-1)(n-1)}{2}}\tilde{G}_2^p(t).
\end{aligned} \tag{3.6}$$

Substituting (2.4) into (3.6) leads to

$$\int_{\mathbb{R}^n} |v_t(x,t)|^p \, \Psi_1(x,t)dx$$
$$\geq C_0^{-p-1} C_1^{1-p} t^{-\frac{(k_1+\mu_1)(p-1)}{2}} (1+\phi_{k_1}(t))^{-\frac{(p-1)(n-1)}{2}} \tilde{G}_2^p(t).$$

Combining (3.3) and (3.5), we have

$$L_1'(t) \geq C_3 t^{-\frac{(p-1)(k_1+\mu_1+n-1)}{2}} L_2^p(t), \quad t \geq T_3, \tag{3.7}$$

where $C_3 = C_3(p,q)$. Similarly, we obtain

$$L_2'(t) \geq C_3 t^{-\frac{(q-1)(k_2+\mu_2+n-1)}{2}} L_1^q(t), \quad t \geq T_3. \tag{3.8}$$

Integrating (3.7) and (3.8) over $(T_3, t)$ gives rise to

$$L_1(t) \geq C_4 \int_{T_3}^t (s+T_3)^{-\frac{(p-1)(k_1+\mu_1+n-1)}{2}} L_2^p(s)ds + \varepsilon C_3, \quad t \geq T_3, \tag{3.9}$$

$$L_2(t) \geq C_4 \int_{T_3}^t (s+T_3)^{-\frac{(q-1)(k_2+\mu_2+n-1)}{2}} L_1^q(s)ds + \varepsilon C_3, \quad t \geq T_3. \tag{3.10}$$

In the following, we define

$$\sigma_i = k_i + \mu_i, \quad (i = 1,\ 2). \tag{3.11}$$

### *3.1. Case $\Omega(n, \sigma_1, \sigma_2, p, q) > 0$*

We note the constant $T_3 > 2$. Using the definition of $\sigma_i$ $(i = 1,\ 2)$, (3.9) and (3.10) can be rewritten as follows:

$$L_1(t) \geq C_4 \int_{T_3}^t (s+T_3)^{-\frac{n-1}{2}(p-1)+\frac{\sigma_1}{2}-\frac{\sigma_1}{2}p} L_2^p(s)ds + C_3\varepsilon, \quad t \geq T_3, \tag{3.12}$$

$$L_2(t) \geq C_4 \int_{T_3}^t (s+T_3)^{-\frac{n-1}{2}(q-1)+\frac{\sigma_2}{2}-\frac{\sigma_2}{2}q} L_1^q(s)ds + C_3\varepsilon, \quad t \geq T_3. \tag{3.13}$$

Suppose the condition $\Omega(n, \sigma_1, \sigma_2, p, q) = \Lambda(n+\sigma_1, p, q) > 0$. In the following, we assume

$$L_1(t) \geq \alpha_j (t+T_3)^{-\beta_j} (t-T_3)^{\gamma_j}, \quad t \geq T_3, \tag{3.14}$$

where $\left\{\alpha_j\right\}_{j\in\mathbb{N}}$, $\left\{\beta_j\right\}_{j\in\mathbb{N}}$, $\left\{\gamma_j\right\}_{j\in\mathbb{N}}$ are nonnegative sequences of real numbers.

When $j = 0$, we set $\alpha_0 = C_2\varepsilon, \beta_0 = 0, \gamma_0 = 0$. We intend to prove that (3.14) is true for all $j \geq 0$. Substituting (3.14) into (3.13), we get

$$\begin{aligned}
L_2(t) &\geq C_4 \int_{T_3}^t (s+T_3)^{-\frac{n-1}{2}(q-1)+\frac{\sigma_2}{2}-\frac{\sigma_2}{2}q} L_1^q(s)ds \\
&\geq C_4 \alpha_j^q \int_{T_3}^t (s+T_3)^{-\frac{n-1}{2}(q-1)+\frac{\sigma_2}{2}-\frac{\sigma_2}{2}q-q\beta_j} (s-T_3)^{q\gamma_j} ds \\
&\geq C_4 \alpha_j^q (t+T_3)^{-\frac{n-1}{2}(q-1)-\frac{\sigma_2}{2}q-q\beta_j} \int_{T_3}^t (s-T_3)^{q\gamma_j+\frac{\sigma_2}{2}} ds \\
&= C_4 \alpha_j^q (q\gamma_j + \frac{\sigma_2}{2} + 1)^{-1} (t+T_3)^{-\frac{n-1}{2}(q-1)-\frac{\sigma_2}{2}q-q\beta_j} (t-T_3)^{q\gamma_j+\frac{\sigma_2}{2}+1},
\end{aligned}$$

where $t \geq T_3$. Substituting the above lower bound estimate for $L_2(t)$ into (3.12) yields

$$\begin{aligned}
L_1(t) &\geq C_4 \int_{T_3}^{t} (s+T_3)^{-\frac{n-1}{2}(p-1)+\frac{\sigma_1}{2}-\frac{\sigma_1}{2}p} L_2^p(s)ds \\
&\geq \frac{C_4^{p+1}\alpha_j^{pq}}{(q\gamma_j+\frac{\sigma_2}{2}+1)^p} \int_{T_3}^{t} (s+T_3)^{-\frac{n-1}{2}(pq-1)-\frac{\sigma_2}{2}pq-\frac{\sigma_1}{2}p-pq\beta_j} \\
&\quad \times (s-T_3)^{pq\gamma_j+\frac{\sigma_2}{2}p+\frac{\sigma_1}{2}+p}\, ds \\
&\geq \frac{C_4^{p+1}\alpha_j^{pq}}{(q\gamma_j+\frac{\sigma_2}{2}+1)^p} (t+T_3)^{-\frac{n-1}{2}(pq-1)-\frac{\sigma_2}{2}pq-\frac{\sigma_1}{2}p-pq\beta_j} \\
&\quad \times \int_{T_3}^{t} (s-T_3)^{pq\gamma_j+\frac{\sigma_2}{2}p+\frac{\sigma_1}{2}+p}\, ds \\
&= \frac{C_4^{p+1}\alpha_j^{pq}}{(q\gamma_j+\frac{\sigma_2}{2}+1)^p(pq\gamma_j+\frac{\sigma_2}{2}p+\frac{\sigma_1}{2}+p+1)} \\
&\quad \times (t+T_3)^{-\frac{n-1}{2}(pq-1)-\frac{\sigma_2}{2}pq-\frac{\sigma_1}{2}p-pq\beta_j} \\
&\quad \times (t-T_3)^{pq\gamma_j+\frac{\sigma_2}{2}p+\frac{\sigma_1}{2}+p+1},\ t \geq T_3.
\end{aligned}$$

Making use of (3.14), we have

$$\alpha_{j+1} = C_4^{p+1}(\frac{\sigma_2}{2}+1+q\gamma_j)^{-p}(\frac{\sigma_2}{2}p+\frac{\sigma_1}{2}+p+1+pq\gamma_j)\alpha_j^{pq}, \tag{3.15}$$

$$\beta_{j+1} = \frac{n-1}{2}(pq-1)+\frac{\sigma_2}{2}pq+\frac{\sigma_1}{2}p+pq\beta_j,$$

$$\gamma_{j+1} = \frac{\sigma_2}{2}p+\frac{\sigma_1}{2}+p+1+pq\gamma_j. \tag{3.16}$$

Let $A_1 = \frac{n-1}{2}(pq-1)+\frac{\sigma_2}{2}pq+\frac{\sigma_1}{2}p$, $\beta_0 = 0$. Employing (3.16) leads to

$$\begin{aligned}
\beta_j &= A_1 + pq\beta_{j-1} \\
&= A_1 \sum_{k=0}^{j-1}(pq)^k + (pq)^j\beta_0.
\end{aligned} \tag{3.17}$$

In a similar way, we have

$$\gamma_j = B_1\frac{(pq)^j-1}{pq}, \tag{3.18}$$

where $B_1 = \frac{\sigma_2}{2}p+\frac{\sigma_1}{2}+p+1$.

According to $\alpha_j$, $\gamma_j \leq \frac{B_1}{pq-1}(pq)^j$, and (3.15), we acquire

$$\begin{aligned}
\alpha_j &= C_4^{p+1}(\frac{\sigma_2}{2}+1+q\gamma_{j-1})^{-p}(\frac{\sigma_2}{2}p+\frac{\sigma_1}{2}+p+1+pq\gamma_{j-1})^{-1}\alpha_{j-1}^{pq} \\
&\geq C_4^{p+1}(\frac{\sigma_2}{2}p+\frac{\sigma_1}{2}+p+1+pq\gamma_{j-1})^{-(p+1)}\alpha_{j-1}^{pq} \\
&\geq \underbrace{C_4^{p+1}\left(\frac{B_1}{pq-1}\right)^{-(p+1)}}_{C_5}(pq)^{-(p+1)j}\alpha_{j-1}^{pq} \\
&= C_5(pq)^{-(p+1)j}\alpha_{j-1}^{pq}.
\end{aligned} \tag{3.19}$$

Taking the logarithm of both sides of (3.19), we derive

$$\begin{aligned}\log\alpha_j &\geq pq\log\alpha_{j-1} - j\log(pq)^{p+1} + \log C_5\\ &\geq (pq)^2\log\alpha_{j-2} - (j+(j-1)pq)\log(pq)^{p+1} + (1+pq)\log C_5\\ &\geq (pq)^j\log\alpha_0 - \left(\sum_{k=0}^{j-1}(j-k)(pq)^k\right)\log(pq)^{p+1} + \left(\sum_{k=0}^{j-1}(pq)^k\right)\log C_5\\ &= (pq)^j\left(\log\alpha_0 - \frac{pq}{(pq-1)^2}\log(pq)^{p+1} + \frac{\log C_5}{pq-1}\right)\\ &\quad+ \underbrace{(j+1)\frac{\log(pq)^{p+1}}{pq-1} + \frac{\log(pq)^{p+1}}{(pq-1)^2} - \frac{\log C_5}{pq-1}}_{\geq 0}.\end{aligned}$$

Let $j \geq \tilde{j} = \max\left\{0, \frac{\log C_4}{\log(pq)^{p+1}} - \frac{pq}{pq-1}\right\}$. It follows that

$$\log\alpha_j \geq (pq)^j\log(C_6\varepsilon), \tag{3.20}$$

where $C_6 = C_3\,(pq)^{-\frac{pq(p+1)}{(pq-1)^2}}\,C_5^{\frac{1}{pq-1}}$. Finally, combining the formulas (3.14), (3.17), (3.18), and (3.20), we arrive at

$$\begin{aligned}L_1(t) &\geq \alpha_j(t+T_3)^{-A_1\frac{(pq)^j-1}{pq-1}}(t-T_3)^{B_1\frac{(pq)^j-1}{pq-1}}\\ &\geq \exp\Big((pq)^j\log(C_6\varepsilon)\Big)(t+T_3)^{-A_1\frac{(pq)^j-1}{pq-1}}(t-T_3)^{B_1\frac{(pq)^j-1}{pq-1}}\\ &= \exp\left((pq)^j\left(\log(C_6\varepsilon) - \frac{A_1}{pq-1}\log(t+T_3) + \frac{B_1}{pq-1}\log(t-T_3)\right)\right)\\ &\quad\times(t+T_3)^{\frac{A_1}{pq-1}}(t-T_3)^{-\frac{B_1}{pq-1}}, \quad j\geq\tilde{j},\ t\geq T_3.\end{aligned} \tag{3.21}$$

Let $t \geq 3T_3$. Hence, $2(t-T_3) \geq t+T_3$. According to (3.21), we have

$$\begin{aligned}L_1(t) &\geq \exp\left((pq)^j\log\left(2^{-\frac{B_1}{pq-1}}C_6\varepsilon(t+T_3)^{\frac{B_1-A_1}{pq-1}}\right)\right)\\ &\quad\times(t+T_3)^{\frac{A_1}{pq-1}}(t-T_3)^{-\frac{B_1}{pq-1}}, \quad j\geq\tilde{j},\ t\geq T_3.\end{aligned} \tag{3.22}$$

Combining the definition of $\Lambda(n+\sigma_1, p, q)$ and its nonnegativity, we know

$$\begin{aligned}\frac{B_1-A_1}{pq-1} &\geq -\frac{\sigma_2 p(q-1)+\sigma_1(p-1)}{2(pq-1)} + \frac{p+1}{pq-1} - \frac{n-1}{2}\\ &\geq \frac{\sigma_2 p(1-q)+\sigma_1(1-p)}{2(pq-1)} + \frac{\sigma_1}{2}\\ &= \frac{p(q-1)(\sigma_1-\sigma_2)}{2(pq-1)}.\end{aligned} \tag{3.23}$$

Suppose $\sigma_1 \geq \sigma_2$. Then, we have $\frac{B_1-A_1}{pq-1} \geq 0$. Because of $t+T_3 \geq 4T_3$, we choose $\varepsilon_0 = \varepsilon_0(n, k_1, k_2, \mu_1, \mu_2, f_1, f_2, g_1, g_2, T_3) > 0$ which is small enough such that $t+T_3 \geq 4T_3$. Thus, we have

$$\left(2^{-\frac{B_1}{pq-1}}C_6\varepsilon_0\right)^{-\left(\frac{B_1-A_1}{pq-1}\right)^{-1}} \geq 4T_3.$$

Hence, for all $\varepsilon \in (0, \varepsilon_0]$ and $t > \left(2^{-\frac{B_1}{pq-1}} C_6 \varepsilon\right)^{-\left(\frac{B_1-A_1}{pq-1}\right)^{-1}} \geq 4T_3$, we obtain $2^{-\frac{B_1}{pq-1}} C_6 \varepsilon t^{\frac{B_1-A_1}{pq-1}} > 1$. Therefore, sending $j \to \infty$ in (3.22), we arrive at the blow-up result of $L_1(t)$ in finite time. Blow-up of $L_2(t)$ and $L_1(t)$ occurs simultaneously. Let $2^{-\frac{B_1}{pq-1}} C_6 \varepsilon t^{\frac{B_1-A_1}{pq-1}} \leq 1$. Upper bound estimate for the lifespan of $L_1(t)$ and $L_2(t)$ is

$$T(\varepsilon) \leq C\varepsilon^{\frac{pq-1}{A_1-B_1}}, \tag{3.24}$$

where $A_1 = \frac{n-1}{2}(pq-1) + \frac{\sigma_2}{2}pq + \frac{\sigma_1}{2}p$, $B_1 = \frac{\sigma_2}{2}p + \frac{\sigma_1}{2} + p + 1$, $\sigma_1$, and $\sigma_2$ are defined by (3.11), and $A_1 - B_1 < 0$ if $\sigma_1 \geq \sigma_2$.

On the other hand, let $\Omega(n, \sigma_1, \sigma_2, p, q) = \Lambda(n + \sigma_2, q, p) > 0$, and using the same method, we see that $L_1(t)$ and $L_2(t)$ blow up in finite time. The upper bound estimate of their lifespan satisfies:

$$T(\varepsilon) \leq C\varepsilon^{\frac{pq-1}{A_2-B_2}}, \tag{3.25}$$

where $A_2 = \frac{n-1}{2}(pq-1) + \frac{\sigma_1}{2}pq + \frac{\sigma_2}{2}q$, $B_2 = \frac{\sigma_1}{2}q + \frac{\sigma_2}{2} + q + 1$, $\sigma_1$, and $\sigma_2$ are defined by (3.11), and $A_2 - B_2 < 0$ if $\sigma_2 \geq \sigma_1$.

### 3.2. *Case* $\Omega(n, \sigma_1, \sigma_2, p, q) = 0$, $\Lambda(n + \sigma_1, p, q) \neq \Lambda(n + \sigma_2, q, p)$

Without loss of generality, we may assume $\Omega(n, \sigma_1, \sigma_2, p, q) = \Lambda(n+\sigma_1, p, q) = 0 > \Lambda(n+\sigma_2, q, p)$. Then, we deduce from (3.23) that

$$\frac{B_1 - A_1}{pq-1} = \frac{\sigma_2 p(1-q) + \sigma_1(1-p)}{2(pq-1)} + \frac{\sigma_1}{2} = \frac{p(q-1)(\sigma_1 - \sigma_2)}{2(pq-1)}.$$

Because of $pq - p > 0$, we get $\frac{B_1-A_1}{pq-1} \geq 0$ if $\sigma_1 \geq \sigma_2$. Similar to the derivations in Subsection 3.1, we obtain the same estimate as in (3.24), namely,

$$T(\varepsilon) \leq C\varepsilon^{\frac{pq-1}{A_1-B_1}}.$$

On the other hand, letting $\Omega(n, \sigma_1, \sigma_2, p, q) = \Lambda(n + \sigma_2, q, p) = 0 > \Lambda(n + \sigma_1, p, q)$ and using the similar method, we see that the upper bound estimate for the lifespan of $L_1(t)$ and $L_2(t)$ is same as (3.25) if $\sigma_2 \geq \sigma_1$, that is,

$$T(\varepsilon) \leq C\varepsilon^{\frac{pq-1}{A_2-B_2}}.$$

### 3.3. *Case* $\Lambda(n + \sigma_1, p, q) = \Lambda(n + \sigma_2, q, p) = 0$

Using the condition $\Lambda(n + \sigma_1, p, q) = \Lambda(n + \sigma_2, q, p) = 0$, we rewrite (3.12) and (3.13) as

$$\begin{aligned} L_1(t) &\geq C_4 \int_{T_3}^t (s + T_3)^{-\frac{p^2-1}{pq-1}} L_2^p(s) ds + C_3\varepsilon \\ &\geq C_4 \int_{T_3}^t (s + T_3)^{-(p+1)} L_2^p(s) ds + C_3\varepsilon, \quad t \geq T_3, \end{aligned} \tag{3.26}$$

$$\begin{aligned} L_2(t) &\geq C_4 \int_{T_3}^t (s + T_3)^{-\frac{q^2-1}{pq-1}} L_1^q(s) ds + C_3\varepsilon \\ &\geq C_4 \int_{T_3}^t (s + T_3)^{-(q+1)} L_1^q(s) ds + C_3\varepsilon, \quad t \geq T_3. \end{aligned} \tag{3.27}$$

In the following, we estimate the lower bound of integrals in (3.26) and (3.27).

Suppose

$$L_1(t) \geq E_j(t+T_3)^{-p\alpha_j}\left(\log\frac{t}{T_3}\right)^{\beta_j}, \quad t \geq T_3, \tag{3.28}$$

where $\left\{E_j\right\}_{j\in\mathbb{N}}$, $\left\{\alpha_j\right\}_{j\in\mathbb{N}}$, $\left\{\beta_j\right\}_{j\in\mathbb{N}}$ are nonnegative real numbers sequences. When $j=0$, we derive $E_0 = C_3\varepsilon$, $\alpha_0=\beta_0=0$. Noticing $t+T_3 \leq 2t$, for all $t \geq T_3$, and substituting (3.28) into (3.27), we conclude

$$\begin{aligned}
L_2(t) &\geq C_4(t+T_3)^{-q}\int_{T_3}^{t}(s+T_3)^{-1}L_1^q(s)ds \\
&\geq C_4E_j^q(t+T_3)^{-q}\int_{T_3}^{t}(s+T_3)^{-1-pq\alpha_j}\left(\log\frac{s}{T_3}\right)^{q\beta_j}ds \\
&\geq 2^{-1}C_4E_j^q(t+T_3)^{-q-pq\alpha_j}\int_{T_3}^{t}\frac{1}{s}\left(\log\frac{s}{T_3}\right)^{q\beta_j}ds \\
&\geq 2^{-1}C_4T_3^{-1}E_j^q(q\beta_j+1)^{-1}(t+T_3)^{-q-pq\alpha_j}\left(\log\frac{t}{T_3}\right)^{q\beta_j+1},
\end{aligned} \tag{3.29}$$

where $t \geq T_3$. From (3.29) and (3.26), we deduce

$$\begin{aligned}
L_1(t) &\geq C_4(t+T_3)^{-p}\int_{T_3}^{t}(s+T_3)^{-1}L_2^p(s)ds \\
&\geq 2^{-p}C_4^{1+p}T_3^{-p}(q\beta_j+1)^{-p}E_j^{pq}(t+T_3)^{-p} \\
&\quad\times\int_{t}^{T_3}(s+T_3)^{-1-pq-p^2q\alpha_j}\left(\log\frac{s}{T_3}\right)^{pq\beta_j+p}ds \\
&\geq 2^{-1-p}C_4^{1+p}T_3^{-p}(q\beta_j+1)^{-p}E_j^{pq}(t+T_3)^{-p-pq-p^2q\alpha_j} \\
&\quad\times\int_{t}^{T_3}\frac{1}{s}\left(\log\frac{s}{T_3}\right)^{pq\beta_j+p}ds \\
&\geq 2^{-1-p}C_4^{1+p}T_3^{-p}(q\beta_j+1)^{-p}(pq\beta_j+p+1)^{-1}E_j^{pq} \\
&\quad\times(t+T_3)^{-p-pq-p^2q\alpha_j}\left(\log\frac{t}{T_3}\right)^{pq\beta_j+p+1}, \quad t \geq T_3.
\end{aligned}$$

Therefore, the following derivation holds for $j+1$, namely,

$$\alpha_{j+1} = -p-pq-p^2q\alpha_j,$$

$$\beta_{j+1} = p+1+pq\beta_j,$$

$$E_{j+1} = 2^{-1-p}C_4^{1+p}(q\beta_j+1)^{-p}(pq\beta_j+p+1)^{-1}E_j^{pq}.$$

As a consequence, we have

$$\begin{aligned}\alpha_j &= -p - pq - p^2 q\alpha_{j-1} \\ &= -p - pq + p^3 q + p^3 q^2 + p^4 q^2 \alpha_{j-2} \\ &= -(p+pq)\sum_{k=0}^{j-1}(-p^2 q)^k \\ &= \frac{p+pq}{p^2 q+1}\left((-p^2 q)^{j-1} - 1\right),\end{aligned} \tag{3.30}$$

$$\begin{aligned}\beta_j &= p + 1 + pq\beta_{j-1} \\ &= (p+1)(1+pq) + (pq)^2\beta_{j-2} \\ &= (p+1)\sum_{k=0}^{j-1}(pq)^k + (pq)^j\beta_0 \\ &= \frac{p+1}{pq-1}\left((pq)^j - 1\right),\end{aligned} \tag{3.31}$$

$$\begin{aligned}E_j &= 2^{-1-p}C_4^{1+p}(q\beta_{j-1}+1)^{-p}(pq\beta_{j-1}+p+1)^{-1}E_j^{pq} \\ &\geq 2^{-1-p}C_4^{1+p}(pq\beta_{j-1}+p+1)^{-1-p}E_{j-1}^{pq} \\ &= 2^{-1-p}C_4^{1+p}\beta_j^{-1-p}E_{j-1}^{pq} \\ &\geq \tilde{E}(pq)^{-(p+1)j}E_{j-1}^{pq},\end{aligned} \tag{3.32}$$

where $\tilde{E} = 2^{-1-p}C_4^{1+p}\left(\frac{p+1}{pq-1}\right)^{-1-p}$. Taking $j \geq \tilde{j} = \max\left\{0, \frac{\log\tilde{E}}{\log(pq)^{p+1}} - \frac{pq}{pq-1}\right\}$, we obtain

$$E_j \geq \exp\left((pq)^j \log(\tilde{\tilde{E}}\varepsilon)\right),$$

where $\tilde{\tilde{E}} = C_3\,(pq)^{-\frac{pq(p+1)}{(pq-1)^2}}\,\tilde{E}^{\frac{1}{pq-1}}$.

Combining (3.28) with (3.30)–(3.32), we have

$$\begin{aligned}L_1(t) &\geq E_j(t+T_3)^{-p\alpha_j}\left(\log\frac{t}{T_3}\right)^{\beta_j} \\ &\geq \exp\left((pq)^j\log(\tilde{\tilde{E}}\varepsilon)\left(\log\frac{t}{T_3}\right)^{\frac{p+1}{pq-1}\left((pq)^j-1\right)}(t+T_3)^{-\frac{p(p+pq)}{p^2q+1}\left((-p^2q)^{j-1}-1\right)}\right).\end{aligned}$$

Let $j = 2k-1,\ k \in \mathbb{N}^*$. For all $t \geq T_3,\ j \geq \tilde{j}$, we acquire

$$\begin{aligned}L_1(t) &\geq \exp\left((pq)^j\log(\tilde{\tilde{E}}\varepsilon)\right)\left(\log\frac{t}{T_3}\right)^{\frac{p+1}{pq-1}\left((pq)^j-1\right)} \\ &= \exp\left((pq)^j\log\left(\tilde{\tilde{E}}\varepsilon\left(\log\frac{t}{T_3}\right)^{\frac{p+1}{pq-1}}\right)\right)\left(\log\frac{t}{T_3}\right)^{-\frac{p-1}{pq-1}}.\end{aligned} \tag{3.33}$$

Choose $\varepsilon_0 = \varepsilon_0(n, k_1, k_2, \mu_1, \mu_2, f_1, f_2, g_1, g_2, T_3)$ small enough such that

$$\exp\left(\left(\tilde{\tilde{E}}\varepsilon_0\right)^{-\frac{pq-1}{p+1}}\right) \geq 1.$$

Thus, we have

$$\tilde{\tilde{E}}\varepsilon\left(\log\frac{t}{T_3}\right)^{\frac{p+1}{pq-1}} > 1$$

for all $\varepsilon \in (0, \varepsilon_0]$ and $t > T_3 \exp\left(\left(\tilde{\tilde{E}}\varepsilon_0\right)^{-\frac{pq-1}{p+1}}\right)$.

Therefore, from (3.33), we know that $L_1(t)$ and $L_2(t)$ blow up in finite time when $j \to \infty$. On the other hand, when $t \leq T_3 \exp\left(\left(\tilde{\tilde{E}}\varepsilon_0\right)^{-\frac{pq-1}{p+1}}\right)$, we derive that the upper bound estimate of the lifespan satisfies:

$$T(\varepsilon) \leq \exp\left(C\varepsilon^{-\frac{pq-1}{p+1}}\right),$$

where $C$ is a positive constant.

Similar to (3.28), we make the following assumption for $L_2(t)$, that is,

$$L_2(t) \geq E_j(t+T_3)^{-q\alpha_j}\left(\log\frac{t}{T_3}\right)^{\beta_j}, \quad t \geq T_3.$$

Similarly, we acquire that $L_1(t)$ and $L_2(t)$ blow up in finite time. The upper bound estimate of their lifespan satisfies

$$T(\varepsilon) \leq \exp\left(C\varepsilon^{-\frac{pq-1}{q+1}}\right),$$

where $C$ is a positive constant. The proof of Theorem 1.1 is finished. ■

## 4. Conclusions

This work is dedicated to investigating the weakly coupled system of semi-linear wave equations with time dependent speed of propagation, damping terms, and derivative nonlinear terms in generalized Einstein-de Sitter space-time on $\mathbb{R}^n$. Taking advantage of the iteration method, we deduce formation of singularity of solutions to the coupled system (1.1). Blow-up region and upper bound lifespan estimate of solutions to the problem are established. The blow-up region of solutions in this work varies due to the influence of coefficients $k_1$, $k_2$. It is worth it to mention that upper bound lifespan estimate (1.12) of solutions to problem (1.1) in Theorem 1.1 coincides with the results in [39] when $p = q$ and $k_1 = k_2$. When $k_i = 0$, $i = 1, 2$ in problem (1.1) (namely, $\sigma_i = \mu_i$), blow-up region $\Omega(n, \sigma_1, \sigma_2, p, q)$ of problem (1.1) is the same as that in [56]. In addition, the results in Theorem 1.1 complement the case that propagation speeds of generalized coupled wave equations with dampings and derivative type nonlinearities contain power functions with negative exponents.

## Author contributions

All authors contributed equally to this work. All authors have read and approved the final version of the manuscript for publication.

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## Conflict of interest

The authors declare that they have no conflict of interest.

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