Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(10): 26836–26853.
DOI: 10.3934/math.20241306
Received: 18 July 2024
Revised: 05 September 2024
Accepted: 09 September 2024
Published: 14 September 2024

Research article

Representations of quadratic Heisenberg-Weyl algebras and polynomials in the fourth Painlevé transcendent

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Abstract: We provide new insights into the solvability property of a Hamiltonian involving the fourth Painlevé transcendent and its derivatives. This Hamiltonian is third-order shape invariant and can also be interpreted within the context of second supersymmetric quantum mechanics. In addition, this Hamiltonian admits third-order lowering and raising operators. We have considered the case when this Hamiltonian is irreducible, i.e., when no special solutions exist for given parameters α and β of the fourth Painlevé transcendent $P_{IV}(x, \alpha, \beta)$. This means that the Hamiltonian does not admit a potential in terms of rational functions (or the hypergeometric type of special functions) for those parameters. In such irreducible cases, the ladder operators are as well involving the fourth Painlevé transcendent and its derivative. An important case for which this occurs is when the second parameter (i.e., β) of the fourth Painlevé transcendent $P_{IV}(x, \alpha, \beta)$ is strictly positive, $\beta > 0$. This Hamiltonian was studied for all hierarchies of rational solutions that come in three families connected to the generalized Hermite and Okamoto polynomials. The explicit form of ladder, the associated wavefunctions involving exceptional orthogonal polynomials, and recurrence relations were also completed described. Much less is known for the irreducible case, in particular the excited states. Here, we developed a description of the induced representations based on various commutator identities for the highest and lowest weight type representations. We also provided for such representations a new formula concerning the explicit form of the related excited states from the point of view of the Schrödinger equation as two-variables polynomials that involve the fourth Painlevé transcendent and its derivative.

Keywords: exactly solvable; quantum Hamiltonians; polynomial Heisenberg-Weyl algebras; Painlevé transcendents; ladder operators; supersymmetric quantum mechanics
Mathematics Subject Classification: 33E17, 37J37, 81Q80, 81U15

1. Introduction

In the context of quantum Hamiltonians, various approaches to find the solution of their corresponding Schrödinger equation have been introduced. Some of those approaches rely on analytical and algebraic definitions of exact and quasi-exact solvability [1–3]. They build on different ideas such as the existence of an underlying hidden algebra, Bethe Ansatz equations, and invariant spaces of polynomials. Other approaches can be used to solve quantum mechanical systems which include Darboux-Crum and Krein-Adler types of transformations and their related intertwining relations and superalgebras [4], factorization relations [5], and ladder operators [6–8]. An important class of algebraic approaches concerns Liouville integrability and superintegrability where Abelian and non-Abelian algebraic structures within the Lie theory or one of its generalizations play the role of symmetry algebras [9]. In this paper, we will consider the case of ladder operators, and particular, of third-order ladder operators and their related Hamiltonians. In such setting, the Hamiltonian is not described by an algebraic form but written in terms of the fourth Painlevé transcendent $P_{IV}(x,\alpha,\beta)$ [10–13]. Some insight into zero modes was provided. This one-dimensional Hamiltonian which also appears in context of superintegrable systems as a building block, which makes it an important model [14]. Further details on the zero modes of its related superpartner were also provided.

The case where α and β take values for which the fourth Painlevé transcendent admits a rational solution was completely studied in recent papers. Those cases which relate to Okamoto polynomials [15, 16] and generalized Hermite polynomials [17] occur for the following choices of parameters α and β:

$$\alpha = 2m + n, \;\; \beta = -2(n - \frac{1}{3})^2, \quad \alpha - m - 2n, \;\; \beta = -2(m - \frac{1}{3})^2$$

and

$$\alpha = n - m, \;\; \beta = -2(m + n + \frac{1}{3})^2.$$

The connection of those cases with exceptional orthogonal polynomials was also studied. This paper will deal with irreducible cases i.e., when the fourth Painlevé transcendent does not admit reduction.

For a broader discussion of the problem of classifying systematically systems with ladder operators we refer the reader to [18, 19]. Deformed Heisenberg-Weyl algebras (span$\{c, c^\dagger, H, 1\}$) take the generic form

$$[H, c] = -ac, \tag{1.1a}$$

$$[H, c^\dagger] = ac^\dagger, \tag{1.1b}$$

$$[c, c^\dagger] = F(H), \tag{1.1c}$$

where a is a real constant, H is a Hamiltonian, c is a lowering operator and $c^\dagger$ is a raising operators. $F(H)$ is a function of H and can be, in applications, a rational or even exponential function. One important example is the case of q-deformed algebras. Usually, in applications in quantum physics, $F(H)$ is a polynomial and then the algebra is called a polynomial Heisenberg-Weyl algebra. Some aspects of infinite and finite dimensional representations were discussed in the literature [20, 21] and constraints for the existence of finite dimensional unitary representations were provided. Those constraints take the form of systems of algebraic equations. Examples of representations that decompose into combinations of finite and infinite representations were obtained [11, 12]. In applications, the corresponding generators c and $c^\dagger$ take the form of higher order differential operators. Despite the apparent simplicity of such algebraic structures, with only three generators only and three defining commutator relations, they relate to large classes of isospectral and almost isospectral deformations of one-dimensional Hamiltonians generalizing the harmonic and singular oscillators (including polynomial and nonpolynomial deformations of the potential) [12]. They include Painlevé transcendents (fourth [11, 14, 18] and fifth [12, 22]) and generalizations [19]. It was demonstrated how in such construction, already at the 6th degree, the associated Chazy equation that defines the higher transcendental functions in which the Hamiltonian can be written is outside known Painlevé transcendents reductions [19]. First-degree operators and second-degree operators lead to Lie algebra, i.e., when

$$F(H) = f_0$$

and

$$F(H) = f_0 + f_1 H.$$

The class of polynomial Heisenberg-Weyl algebras contains all generating spectrum algebras for quantum models related to exceptional Hermite and Laguerre polynomials [15], which then make the study of the related polynomial Heisenberg-Weyl algebra interesting from the perspective of the connection with orthogonal polynomials. Various other polynomial deformations of Lie algebras and their representations have attracted interest over the years, including deformation of super Lie algebras [23], Hopf structures [24], quadratic [25] and cubic [26] algebras.

Such structures share similarities with the Heisenberg-Weyl algebra of the harmonic oscillator that it generalizes, which admits infinite dimensional representations. Polynomial Heisenberg-Weyl algebras allow for combinations of infinite and finite dimensional unitary representations. The representations are in general difficult to characterize and it may depend upon solving systems of algebraic equations in order to find zero modes of lowering and raising generators, as both lowering and raising operators may admit several zero modes. Those lowering and raising operators are realized via higher order differential operators.

Hamiltonians possessing ladder operators with polynomial Heisenberg-Weyl algebras are also relevant in regard to constructing multidimensional Hamiltonians possessing integrals of motion, both integrable and superintegrable [19]. Then progress on developing representations of the polynomial Heisenberg algebra can then have application in different contexts such as quantum integrable and superintegrable systems. We have recently demonstrated such algebraic structures give rise to complicated patterns of indecomposable representations taking the form of 2-chains when one considers certain Hamiltonians related to second order differential equations associated with exceptional Hermite polynomials and having a ladder with polynomial Heisenberg algebra. This case corresponds to a particular case of a rational solution for the model with the fourth Painlevé transcendent for specific values of the parameters α and β. This illustrates how the representation theory for those is rich.

This paper will deal with the case of induced representation from lowest states, i.e., zero modes of the lowering operator and how the raising operator allows us to generate additional states from the zero modes for a given Hamiltonian. Moreover, it will also deal with the case of highest weight for which the induced representations are achieved via a zero mode of the raising operator and then the action of the lowering operator provides the sequence of states from the point of view of the Hamiltonian. Here, the ladder operators, zero modes, and Hamiltonian will involve the fourth Painlevé transcendent and then the induced representation will facilitate the construction of the states. We will establish several commutator identities involving monomials which have not been explored in the literature. Using these representations, we will present explicit construction using differential operators.

In Section 2, we present the general form of a polynomial Heisenberg algebra. In Section 3, we present induced constructions and commutator identities. In Section 4, we apply the constructiosn of the Hamiltonian with third-order shape invariance in terms of the fourth Painlevé transcedent. In Sections 5 and 6, we present an explicit form of the induced construction with regard to the polynomials written in terms of the fourth Painlevé transcendent and its derivative for the lowest and highest weight types.

2. Polynomial Heisenberg algebra

In many systems, the Schrödinger equation includes an algebraic potential, i.e., in terms of polynomials or rational functions, and then explicit solutions for the Schrödinger equation

$$H\psi_n = E_n\psi_n \tag{2.1}$$

can be associated with the theory of hypergeometric functions, and possibly with generalizations in terms of confluent Heun or Heun equations via appropriate transformations. In some cases, when the potential depends on trigonometric or even elliptic functions, an algebraic form for the potential may still be used via some algebraization transformations. However, even with an algebraic form, this is still a difficult problem to obtain exact solutions for the Schrödinger equation. This can be facilitated by using various algebraic methods and in particular the connection with ladder operators and algebraic structures, such as the Heisenberg-Weyl type algebras, may allow us to get the complete spectrum from an initial state (zero modes) or even many zero modes, as higher order operators can annihilate multiple states. The associated representations of the deformed Heisenberg-Weyl algebra can be constructed via the associated special functions and related recurrence relations. In a later section, we will discuss how, for Hamiltonians without an algebraic form and connecting with higher transcendental functions, one cannot rely on classical approaches to ordinary differential equations but still on an algebraic approach.

Another aspect of Heisenberg-Weyl algebra is the existence of the Casimir invariant that can be exploited in regard to the descriptions of the representations [12,14,20,21]. A Casimir invariant of the polynomial Heisenberg-Weyl algebra is a polynomial with the generators $\{c, c^\dagger, H\}$ where

$$K = K(c, c^\dagger, H) = \sum_{i+j+k\leq n} \alpha_{ijk} H^i c^j (c^\dagger)^k \tag{2.2}$$

with α_{ijk} being constant such that

$$[K, H] = 0, \quad [K, c] = 0, \quad [K, c^\dagger] = 0. \tag{2.3}$$

For the quadratic case

$$F(H) = b_2 H^2 + b_1 H + b_0,$$

the Casimir takes the form

$$K = cc^\dagger - \frac{b_2}{3a}H^3 - \frac{(b_1 + ab_2)}{2a}H^2 - \frac{(6b_0 + 3ab_1 + a^2b_2)}{6a}H. \tag{2.4}$$

It can be demonstrated that in fact, the Casimir takes the following form in general

$$K = cc^\dagger - M(H), \tag{2.5}$$

where the polynomial $M(H + a)$ is constrained by

$$F(H) = M(H) - M(H - a). \tag{2.6}$$

This polynomial $M(H)$ can be constructed from the coefficients of the polynomial $F(H)$ of the commutator relations $[c, c^\dagger]$. As the Hamiltonian, the lowering and raising operators are differential operators this also implies that further relations in the realization can be obtained such as product relations (alternatively to only commutator relations):

$$c^\dagger c = M(H), \quad cc^\dagger = M(H + a). \tag{2.7}$$

This also implies, in the differential operator realization, that the Casimir invariant reduces to a polynomial of the Hamiltonian. Those additional relations in the realizations can be used to deduce the weight (or energy from the point of view of the corresponding Schrödinger equation) of the zero modes. A zero mode is an eigenstate of the Hamiltonian such that the action of raising or lowering (or both) is vanishing.

In this paper we will consider another approach, i.e., rather than building on Casimir, explicit realizations or factorization relations will be found using induced representation constructions. This is an approach that is also based on establishing identities and commutator identities of the monomials of the generators of the underlying quadratic algebra. Those formula will allow us in later sections, to provide further understanding of the explicit wavefunctions for the corresponding Hamiltonian and Schrödinger equation.

3. Induced representations and the algebraic definition of states

The notion of induced representations has been widely studied in different context and in particular in regard to Lie algebras. However, regarding polynomial algebra, much less is known. Constraints on the existence of zero modes of $c^\dagger$ and c can be achieved via $cc^\dagger$ and $c^\dagger c$, and their equivalence, in the differential operators realization, as polynomials of the Hamiltonian only. Here our approach differs and will concern lowest and highest weight constructions.

3.1. Highest and lowest weight constructions

We will define states via action on $\psi_0^{(i)}$ of the lowering operator c and action of the Hamiltonian, which plays a role analogous of a Cartan generator with

$$H\psi_0^{(i)} = E_0^{(i)}\psi_0^{(i)}, \quad c\psi_0^{(i)} = 0. \tag{3.1}$$

Here $\psi_0^{(i)}$ (with $i = 1, ..., l$) takes into account the possibility of a set of l zero modes, i.e., the set of states annihilated by c, which are the lowest weight states. Only one of them will be the ground state. Then we use the action of raising operators in the following way:

$$\psi_n^{(i)} = (c^\dagger)^n \psi_0^{(i)}. \tag{3.2}$$

The construction of the induced representation consists in establishing the action of the generators on $\psi_n^{(i)}$:

$$c\psi_n^{(i)} = \alpha_n \psi_{n-1}^{(i)}, \quad c^\dagger \psi_n^{(i)} = \psi_{n+1}^{(i)}, \quad H\psi_n^{(i)} = \beta_n \psi_n^{(i)}. \tag{3.3}$$

The following identities can be demonstrated:

$$[H, (c^\dagger)^k] = m(k)(c^\dagger)^k, \quad [c, (c^\dagger)^k] = (c^\dagger)^{k-1} R_k(H). \tag{3.4}$$

We will provide further details in the next subsection and determine explicitly the polynomial of H denoted $R_k(H)$. This polynomial depends on the index k. This allows us to demonstrate the following results:

$$c\psi_n^{(i)} = [c, (c^\dagger)^n]\psi_0^{(i)} = R_n(E_0^i)\psi_{n-1}^{(i)}, \tag{3.5}$$

$$H\psi_n^{(i)} = (c^\dagger)^n H\psi_0^{(i)} + [H, (c^\dagger)^n]\psi_0^{(i)} = (E_0^{(i)} + m(n))\psi_n^{(i)}. \tag{3.6}$$

Using induced representations from the highest weight of the form

$$H\phi_0^{(i)} = E_0^{(i)}\phi_0^{(i)}, \quad c^\dagger \phi_0^{(i)} = 0, \tag{3.7}$$

and taking

$$\phi_n^{(i)} = (c)^n \phi_0^{(i)}, \tag{3.8}$$

the related actions of the generators are given by

$$c\phi_n^{(i)} = \phi_{n+1}^{(i)}, \quad c^\dagger \phi_n^{(i)} = \tilde{\alpha}_n \phi_{n-1}^{(i)}, \quad H\phi_n^{(i)} = \tilde{\beta}_n \phi_n^{(i)}. \tag{3.9}$$

The results rely on establishing the following identities:

$$[H, c^k] = p(k)c^k, \quad [c^\dagger, c^k] = c^{k-1} S_k(H). \tag{3.10}$$

We will also establish, in the next subsection, details on the polynomial of H denoted as $S_k(H)$. This polynomial also depends explicitly on k. As a direct consequence, one can obtain

$$c^\dagger \phi_n^{(i)} = c^{n-1} S_n(H)\phi_0^{(i)}, \tag{3.11}$$

$$H\phi_n^{(i)} = (E_0^{(i)} + p(n))\phi_n^{(i)}. \tag{3.12}$$

One advantage to relying on induced representations in the context of polynomial Heisenberg algebra is that for ladder operators which take the form of differential operators of degree 3 and higher, they are associated with higher transcendental functions. This means, in such cases, the Hamiltonian and ladder operators involve special functions only defined via nonlinear differential equations. Among them the well-known Painlevé transcendents and higher order analog. However, this means that action of the ladder can no longer be straightforwardly calculated as for a harmonic oscillator and its rational deformation, and the zero mode is written in terms of those higher transcendental functions leading to a solution only determined via iterative action of the ladder operators and thus via the induced representations. However, as commutator identities are determined the representation can also be determined, explicitly. In Section 6, we will provide details on how those representations correspond in terms of explicit polynomials of those higher transcendental functions and their derivatives.

3.2. Commutator identities for quadratic Heisenberg-Weyl algebra

The purpose of this section is to consider the case of quadratic Heisenberg-Weyl algebra which connects with case of the fourth Painlevé transcendent and its related Hamiltonian and its ladder operators of third order. We consider the following general quadratic Heisenberg-Weyl algebra formed by $\{H, c, c^\dagger, 1\}$ and the relations (1.1a), (1.1b), and

$$[c, c^\dagger] = b_2 H^2 + b_1 H + b_0. \tag{3.13}$$

As consequence of the defining commutator relations, we obtain

$$[H, (c^\dagger)^k] = 2k(c^\dagger)^k, \tag{3.14}$$

$$[H, c^k] = -2k(c)^k. \tag{3.15}$$

Those formulas indicate similarities with weights in the context of Lie algebras and that the Hamiltonian H then plays an analogous role as a Cartan generator for the well-known Lie algebra $sl(2)$. Other commutator identities can also be demonstrated in different ways and in particular without relying on explicit differential operators realization or factorization relations. It can be shown that the commutator of c with a monomial of $c^\dagger$ can be rewritten in the following way

$$\begin{aligned} [c, (c^\dagger)^n] &= (c^\dagger)^{n-1}((a_0 + a_1 n)H^2 + (a_2 + a_3 n + a_4 n^2)H + (a_5 + a_6 n + a_7 n^2 + a_8 n^3) \\ &= (c^\dagger)^{n-1} R_n(H), \end{aligned} \tag{3.16}$$

where the coefficients a_0 to a_8 are given by

$$\begin{aligned} &a_0 = 0, \quad a_1 = b_2, \quad a_2 = 0, \quad a_3 = -ab_2 + b_1, \quad a_4 = ab_2, \quad a_5 = 0, \\ &a_6 = \frac{1}{6}(a^2 b_2 + 6b_0 - 3ab_1), \quad a_7 = \frac{1}{2}(-a^2 b_2 + ab_1), \quad a_8 = \frac{a^2}{3} b_2. \end{aligned} \tag{3.17}$$

We can also establish a similar formula for the commutator of $c^\dagger$ with a monomial of c:

$$\begin{aligned} [c^\dagger, c^n] &= (c)^{n-1}((a_0 + a_1 n)H^2 + (a_2 + a_3 n + a_4 n^2)H + (a_5 + a_6 n + a_7 n^2 + a_8 n^3) \\ &= (c)^{n-1} S_n(H), \end{aligned} \tag{3.18}$$

where the coefficients a_0 to a_8 take the form

$$\begin{aligned} &a_0 = 0, \quad a_1 = -b_2, \quad a_2 = 0, \quad a_3 = -ab_2 - b_1, \quad a_4 = ab_2, \quad a_5 = 0, \\ &a_6 = \frac{1}{6}(-a^2 b_2 - 6b_0 - 3ab_1), \quad a_7 = \frac{1}{2}(a^2 b_2 + ab_1), \quad a_8 = -\frac{a^2}{3} b_2. \end{aligned} \tag{3.19}$$

In the case of the Lie algebra $sl(2)$, the commutator identities of the triplet formed by the Cartan, the raising generator and the lowering generator play a role in construction of induced representations. It is then expected that the polynomial identities will play a similar role for quadratic algebras.

4. Representations Painlevé IV: irreducible case $\beta > 0$

In this section, we recall some results related to third-order shape invariance and construction based on second-order supersymmetric quantum mechanics [11, 12, 14, 22]. The one-dimensional Hamiltonian with a third-order ladder operator has the form (with $\lambda = 1$, see [11, 14])

$$H = -\frac{d^2}{dx^2} - 2f' + 4f^2 + 4xf + x^2 - 1. \tag{4.1}$$

The function

$$f = P_{IV}(x, \alpha, \beta)$$

satisfies a second order nonlinear differential equation which can be written in terms of the fourth Painlevé equation. Here we will rely only on the function f and its defining equation given by

$$f'' = \frac{f'^2}{2f} + 6f^3 + 8xf^2 + 2(x^2 - (1+\alpha))f + \frac{\beta}{2f}. \tag{4.2}$$

The ladder operators c and $c^\dagger$ of third order take the form

$$c = M^+Q^-, \quad c^\dagger = Q^+M^-, \tag{4.3}$$

where the operators $M^\pm$ and $Q^\pm$ are given by

$$M^+ = \partial_x^2 + h(x)\partial_x + g(x) = (\partial_x + W_1)(\partial_x + W_2), \tag{4.4}$$

$$M^- = \partial_x^2 - \partial_x h(x) + g(x) = (-\partial_x + W_2)(-\partial_x + W_1), \tag{4.5}$$

$$Q^+ = (\partial + W_3), \quad Q^- = (-\partial + W_3), \tag{4.6}$$

and the functions W_i (for $i = 1, 2, 3$) take the form

$$W_1 = -f + \frac{f' - \sqrt{-\beta}}{2f}, \quad W_2 = -f - \frac{f' - \sqrt{-\beta}}{2f}, \quad W_3 = -2f - x. \tag{4.7}$$

The fact that the parameter takes positive or negative values is an important feature. In the case $\beta < 0$, the operators M^+ and M^- well factorize into first-order operators which allows, in the context of supersymmetric quantum mechanics, them to be interpreted as physical intermediate Hamiltonians. In the case of $\beta > 0$ the second-order operators M^+ and M^- would be referred as irreducible. Also, the choice of sign for the parameter d has other consequences. Only in the case of $\beta < 0$, the fourth Painlevé transcendent admits families of rational solutions known as Okamoto and generalized Hermite polynomials. Those special cases were studied in a complete way in recent papers [16, 17]. The case of $\beta > 0$ is of importance as it corresponds to a physical model that does not reduce to algebraic form which is an interesting feature among the realm of exactly solvable quantum systems.

In order to establish the explicit form for the states of the induced representation, we will need to rely on further derivatives of this nonlinear equation:

$$f^{(n)} = (\frac{f'^2}{2f} + 6f^3 + 8xf^2 + 2(x^2 - (1+\alpha))f + \frac{\beta}{2f})^{(n-2)}. \tag{4.8}$$

Here $()^{(n)}$ is the nth derivative. We can then establish the following quadratic Heisenberg-Weyl algebra [12, 14, 22]:

$$[H, c] = -2c, \tag{4.9a}$$
$$[H, c^\dagger] = 2c^\dagger, \tag{4.9b}$$
$$[c^\dagger, c] = -2(3H^2 - (4\alpha + 2)H + \alpha^2 + \beta), \tag{4.9c}$$

which correspond to the following choice of structure:

$$b_2 = -6, \quad b_1 = 2(4\alpha + 2), \quad b_2 = -2(\alpha^2 + \beta), \tag{4.10}$$

where α and d are parameters related to the fourth Painlevé transcendent.

Another framework in regard to a higher order ladder was studied in [19]. This has led to connection to Painlevé via the Chazy equations and their reductions to Painlevé transcendents.

4.1. Lowest weight induced representations

Considering the case where it is irreducible, i.e., when the fourth Painlevé transcendent $P_{IV}(x, \alpha, \beta)$ does not admit any special solution in terms of rational or hypergeometric functions in the case of $\beta > 0$. In one important case $\beta > 0$, the operators M^- and M^+ do not have Hermitian intermediate Hamiltonians and the second-order Darboux transformation can no longer be interpreted as two first-order Darboux transformations. The factorized form can nevertheless be used to solve the equation related to the zero mode of the lowering operator. We will generate an infinite dimensional representations of lowest and highest weight type. The irreducible case considered occurs when β is positive. Then one zero mode satisfies

$$c\psi_0^{(1)} = 0$$

and

$$\psi_n = (c^\dagger)^n \psi_0^{(1)}, \tag{4.11}$$

and the action of the lowering and raising operator is given by

$$c^\dagger \psi_n^{(1)} = \psi_{n+1}^{(1)}, \tag{4.12}$$
$$c\psi_n^{(1)} = \left[f(E_0^{(1)}\right]\psi_{n-1}^{(1)}. \tag{4.13}$$

Following from general construction, the commutator identities take the form

$$[c, (c^\dagger)^n] = (c^\dagger)^{n-1}(f_n(H)), \tag{4.14}$$

$$f_n(H) = (-2\beta n - 2n(2 - 2n + \alpha)^2) - 2n(-8 + 6n - 4\alpha)H - 6H^2. \tag{4.15}$$

This leads to the explicit formula when acting on the zero mode

$$[c, (c^\dagger)^n]\psi_0^{(1)} = (c^\dagger)^{n-1}(f_n(E_0^{(1)}))\psi_0^{(0)}, \tag{4.16}$$

and in this case as

$$E_0^{(1)} = 0,$$

the function $f_n(E_0^{(1)})$ can be written as

$$f_n(E_0^{(1)}) = -2n(\beta + (2 - 2n + \alpha)^2). \tag{4.17}$$

The state formed by the lowest weight representation with interpretation has eigenstates of the corresponding Schrodinger equation as

$$H\psi_n = (E_0^{(1)} + 2n)\psi_n. \tag{4.18}$$

4.2. Highest weight induced representations

Then one zero mode satisfies

$$c^\dagger \phi_0^{(1)} = 0$$

and

$$\phi_n = (c)^n \phi_0^{(1)}, \tag{4.19}$$

and the explicit action is

$$c\phi_n^{(1)} = \phi_{n+1}^{(1)}, \tag{4.20}$$

$$c^\dagger \phi_n^{(1)} = \left[f(E_0^{(1)}\right]\psi_{n-1}^{(1)}. \tag{4.21}$$

We then obtain the commutator identities:

$$[c^\dagger, (c)^n] = (c)^{n-1}(f_n(H)), \tag{4.22}$$

$$f_n(H) = 2n(\beta + 4(n-1)n + 4(n-1)\alpha + \alpha^2) + 2n(4 - 6n - 4\alpha)H + 6nH^2. \tag{4.23}$$

When acting on the zero mode of the raising operator, we obtain

$$[c^\dagger, (c)^n]\phi_0^{(1)} = (c)^{n-1}(f_n(E_0^{(0)}))\phi_0^{(1)}. \tag{4.24}$$

As the energy of this zero mode is

$$E_0^{(1)} = \alpha - \sqrt{-\beta},$$

we need to consider irreducible cases among $\beta < 0$. For this case,

$$f_n(E_0^{(1)}) = 4n(-\beta + 2n^2 + n(-2 + 3\sqrt{-\beta} - \alpha) - \sqrt{-\beta}(2 + \alpha)). \tag{4.25}$$

This provides the construction of the induced representations in the highest weight case. The states formed by the highest weight representation have also an interpretation as eigenstates of the corresponding Schrödinger equation due to

$$H\phi_n = (E_0^{(1)} - 2n)\phi_n. \tag{4.26}$$

5. States for the lowest weight representation as polynomials of fourth Painlevé transcendents

In previous sections, an algebraic description of the chain of states via induced representations was presented. One can also obtain explicitly the action of the different generators via various commutator identities. However, viewed as explicit expressions in terms of the function f (i.e., the fourth Painlevé transcendent), it is a highly nontrivial problem. This open problem has not been looked at in the literature (only for different families of rational solutions) and also hypergeometric type was studied (but not the irreducible case). We consider the case where the second order operators do not factorize into two first-order Darboux supercharges, i.e., the irreducible case $\beta > 0$ where the only one physical zero mode is obtained:

$$\psi_0^{(1)} = e^{\int W_3(x')dx'}. \tag{5.1}$$

We then obtain, from the formula related to induced representations,

$$\psi_1^{(1)} = c^\dagger \psi_0^{(1)}, \tag{5.2}$$

and the following explicit expression

$$\psi_1^{(1)} = \frac{1}{2f^3} e^{\int W_3(x')dx'} \Big(20xf^5 + 8f^6 + 4f^4(-3 + 4x^2 - 3f') + f'(\beta + f'^2) + f^3(-8x + 4x^3 - 8xf' - 2f'') + f(\beta x + xf'^2 - 2f'f'') + f^2(2\beta + 2f'^2 - 2xf'' + f''')\Big). \tag{5.3}$$

Then, using consequences of the defining second-order nonlinear equation for the fourth Painlevé transcendent

$$f^{(3)} = \left(\frac{f'^2}{2f} + 6f^3 + 8xf^2 + 2(x^2 - (1+\alpha))f + \frac{\beta}{2f}\right)', \tag{5.4}$$

where $()'$ denotes $()_x$ and the equation of the fourth Painevé transcendent is

$$f^{(2)} = \frac{f'^2}{2f} + 6f^3 + 8xf^2 + 2(x^2 - (1+\alpha))f + \frac{\beta}{2f}, \tag{5.5}$$

we get an expression only in terms of f and f'. This allows us to rewrite $\psi_1^{(1)}$ as

$$\psi_1^{(1)} = \frac{e^{\int W_3(x')dx'}}{2f} \left(\beta - 4f(x+f)(-\alpha + f(x+f) + f'^2\right). \tag{5.6}$$

Here the monomials present are $\{f^4, f^3, f^2, f, f'^2, 1\}$. For the next member of this sequence of states, we consider ψ_2. Starting recursively with the formula

$$f^{(6)} = \left(\frac{f'^2}{2f} + 6f^3 + 8xf^2 + 2(x^2 - (1+\alpha))f + \frac{\beta}{2f}\right)'''', \tag{5.7}$$

and corresponding equations for $f^{(5)}$, $f^{(4)}$, $f^{(3)}$, and $f^{(2)}$, we get for $\psi_2^{(1)}$ an expression only in terms of

f and f':

$$\psi_2^{(1)} = \frac{e^{\int W_3(x')dx'}}{f^2}\Big(16f^8 + 64xf^7 + (96x^2 - 32\alpha)f^6 + (32x + 64x^3 - 96x\alpha)f^5 \\ - 8f^4f'^2(-32 - 8\beta + 64x^2 + 16x^4 - 96x^2\alpha + 16\alpha^2)f^4 - 16xf^3f'^2 \\ + (-32x - 16\beta x + 32x^3 - 32x\alpha - 32x^3\alpha + 32x\alpha^2)f^3 + (-8x^2 + 8\alpha)f^2f' - 16f^2f' \\ + (-8\beta - 8\beta x^2 + 16\alpha + 8\beta\alpha - 32x^2\alpha - 8\alpha^2 + 16x^2\alpha^2)f^2 + (-8x + 8x\alpha)ff'^2 \\ + (-8\beta x + 8\beta x\alpha)f + f'^4 + 2\beta f'^2 + \beta^2\Big). \tag{5.8}$$

Considering the induced representation

$$\psi_n^{(1)} = c^n\psi_0^{(1)} \tag{5.9}$$

via explicit calculations from $\psi_2^{(1)}, \psi_3^{(1)}, ..., \psi_8^{(1)}$, we can obtain the expansion in terms of f and f' only. However, if the expression can be determined explicitly, the formula become quite large. It was shown via software and symbolic calculations that, for $\psi_n^{(1)}$ (up to $n = 8$), we get polynomials in terms of f and f' with a coefficient depending on x of the form

$$\psi_n^{(1)} = \frac{e^{\int W_3(x')dx'}}{f^n}\left(\sum_{j=0}^{n}\sum_{i=0}^{4n-4j}\alpha_{1,ij}(x)Q_{1,ij}(f,f') + \sum_{j=0}^{n-1}\sum_{i=2}^{4n-6-4j}\alpha_{2,ij}(x)Q_{2,ij}(f,f')\right),$$

where

$$Q_{1,ij}(f,f') = f^i f'^{2j}, \quad j = 0,...,n; i = 0,...,4n-4j,$$
$$Q_{2,ij}(f,f') = f^i f'^{2j+1}, \quad i = 0,...,n-1; j = 2,...,4n-6-4j,$$

where $\alpha_{l,ij}(x)$ are polynomials in x for $l = 1,2$ of degree and at most $2n$. The problem of establishing the formula in general is quite complicated due to the growth of the number of terms and complexity. Some details on the structure of $Q_{1,ij}(f,f')$ and $Q_{2,ij}(f,f')$ are given in the Figure 1. The points in Figure 1 represent the distribution of the monomials in terms of f and f'.

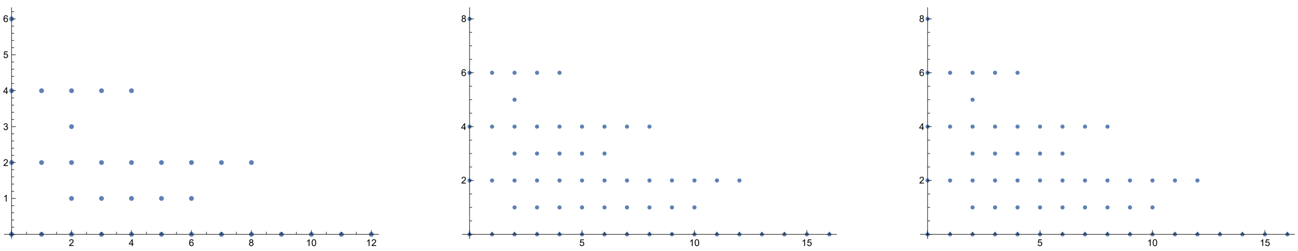

Figure 1. Plot (in the x and y-axis) of the exponents i and $2j$ (or $2j+1$) present for the expansion in terms of f and f' for $Q_{1,ij}(f,f')$ and $Q_{2,ij}(f,f')$. The graph corresponds to ψ_2–ψ_4.

6. States for the highest weight representation as polynomials of fourth Painlevé transcendents

In previous sections, an algebraic description of the chain of state via induced representation was presented with the lowest weight. One can also obtain explicitly the action of the different generators via various commutator identities for the case of the highest weight. However, again viewed as explicit expression in terms of the function f related to the fourth Painlevé transcendent, it is highly nontrivial. Due to the structure of the energy of the ground state, we need to consider $\beta < 0$:

$$\phi_0^{(1)} = e^{\int W_1(x')dx'} \tag{6.1}$$

and then we obtain from

$$\phi_1^{(1)} = c\phi_0^{(1)}, \tag{6.2}$$

explicitly

$$\begin{aligned}\phi_1^{(1)} =& -\frac{e^{\int W_1(x')dx'}}{2f^3}\Big(8xf^5 + 8f^6 - 4f^4(2+3f') + f'(\beta + f'^2) + f^3(4\sqrt{-\beta}x \\ &- 8f' - 2f'') - 2ff'f'' + f^2(-2\sqrt{-\beta} + 2\beta + 2f' + 2f'^2 + f''')\Big).\end{aligned} \tag{6.3}$$

Then using

$$f^{(3)} = \left(\frac{f'^2}{2f} + 6f^3 + 8xf^2 + 2(x^2 - (1+\alpha))f + \frac{\beta}{2f}\right)', \tag{6.4}$$

$$f^{(2)} = \frac{f'^2}{2f} + 6f^3 + 8xf^2 + 2(x^2 - (1+\alpha))f + \frac{\beta}{2f}, \tag{6.5}$$

we get the polynomial only in f and f' and then $\phi_1^{(1)}$ takes the form

$$\phi_1^{(1)} = -\frac{e^{\int W_1(x')dx'}}{2f}\left(-2\sqrt{-\beta} + \beta + 4f((1+\sqrt{-\beta})x - f(-1 + x^2 - \alpha + 2xf + f^2)) + f'(2+f')\right). \tag{6.6}$$

Considering $\phi_2^{(1)}$, we can obtain the following formula, again only in terms of f and f',

$$\begin{aligned}\phi_2^{(1)} = \frac{e^{\int W_1(x')dx'}}{f^2}\Big(& -4xf(x)^3(-2(\alpha(\sqrt{-\beta}+2) + 2\sqrt{-\beta} + 3) + 2(\sqrt{-\beta}+2)x^2 + d + f'(x)(f'(x)+2)) \\ &+ 2f(x)^2(\beta(\alpha - 3x^2 - 1) + 2(-2\alpha(\sqrt{-\beta}+1) + (5\sqrt{-\beta}+4)x^2 + \sqrt{-\beta}) \\ &- (-\alpha + x^2 - 1)f'(x)(f'(x)+2)) - 2f(x)^4(-2(\alpha(\alpha+2) + \sqrt{-\beta} + 2) \\ &+ 4x^2(\alpha + 2\sqrt{-\beta} + 5) + \beta + f'(x)(f'(x)+2) - 2x^4) + 2xf(x)((\sqrt{-\beta}+2)f'(x)(f'(x)+2) \\ &- 4\sqrt{-\beta} + d(\sqrt{-\beta}+4)) + \frac{1}{4}(\beta + f'(x)^2)(\beta - 4\sqrt{-\beta} + f'(x)(f'(x)+4)) \\ &+ 8xf(x)^5(-2\alpha - \sqrt{-\beta} + 2x^2 - 4) + 8f(x)^6(-\alpha + 3x^2 - 1) + 4f(x)^8 + 16xf(x)^7\Big).\end{aligned} \tag{6.7}$$

Considering the induced representation

$$\phi_n^{(1)} = c^n\phi_0^{(1)}, \tag{6.8}$$

it was shown via symbolic calculations that for $\psi_n^{(1)}$ (up to $n = 8$), we get polynomials in f and f' with a coefficient depending on x of the form

$$\phi_n^{(1)} = \frac{e^{\int W_1(x')dx'}}{f^n}\left(\sum_j^n \sum_{i=0}^{4n-4j} \alpha_{1,ij}(x)Q_{1,ij}(f,f') + \sum_j^{n-1} \sum_{i=0}^{4n-4-4j} \alpha_{2,ij}(x)Q_{2,ij}(f,f')\right),$$

where

$$Q_{1,ij}(f, f') = f^i f'^{2j}, \quad j = 0, ..., n; i = 0, ..., 4n - 4j,$$

$$Q_{2,ij}(f, f') = f^i f'^{2j+1}, \quad j = 0, ..., n - 1; j = 0, ..., 4n - 4 - 4j,$$

with $\alpha_{l,ij}(x)$ being polynomials in x for $l = 1, 2$ and of degree at most $2n$. The structure of those polynomials is also quite complicated. Some details on the structure of $Q_{1,ij}(f, f')$ and $Q_{2,ij}(f, f')$ is given in Figure 2. The points in Figure 2 represent the distribution of the monomials in terms of f and f'. In that way, the induced representation can be used to obtain quite nontrivial solutions from the point of view of the Schrödinger equation of the Hamiltonian and define polynomials in f and f' recursively via the action of the lowering or raising operators.

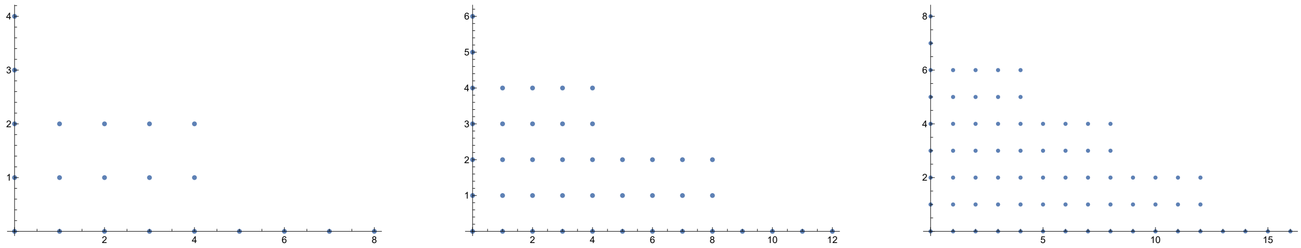

Figure 2. Plot (in the x- and y-axis) of the exponents i and $2j$ (or $2j + 1$) present for the expansion in terms of $f\ f'$ for $Q_{1,ij}(f, f')$ and $Q_{2,ij}(f, f')$. The graph corresponds to ψ_2–ψ_4.

In view of these results, the Hamiltonian in terms of the fourth Painlevé transcendent is exactly solvable, but the solution is not expressed in the usual way. The solvability property can be defined as the solution being expressed in terms of orthogonal polynomials and more generally as the solution of the hypergeometric equation (exact solvability) and Heun equations (quasi exact solvability). Here the solvability is provided via quadratic algebra and the possibility to define infinite dimensional representations.

7. N-dimensional superintegrable Hamiltonian with Painlevé transcendent and related states

From the point of view of quantum integrable and superintegrable systems, Hamiltonians involving higher transcendental functions are an important class. Some examples on two-dimensional spaces were obtained [14, 19]. A systems with N degrees of freedom is integrable if it admits N mutually commuting and well-defined independent integrals of motion (including the Hamiltonian). Superintegrable systems admit an additional set of k ($k = 1, ..., N - 1$) integrals which commute only with the Hamiltonian but not necessarily with other integrals. Those integrals are also independent and well-defined quantum mechanical operators. A system is maximally superintegrable if it allows a total of $2N - 1$ integrals (only N of them can be mutually commuting). A large body of literature exists on superintegrability and we refer the reader to [9] for more details of their classification and properties.

Based on the one-dimensional Hamiltonians with ladder operators, one can introduce N-dimensional versions. The approach consists in taking sums of copies of those one-dimensional Hamiltonians as building blocks in a similar manner as for the isotropic harmonic oscillator or the

Smorodinsky-Winternitz potential, i.e., considering

$$H = \sum_{i}^{N} H_i = \sum_{i}^{N} \left(-\frac{d^2}{dx_i^2} - 2f_i' + 4f_i^2 + 4x_i f_i + x_i^2 - 1 \right). \tag{7.1}$$

Here, all the parameters λ_i are set as $\lambda_i = 1$. The λ_i can be exploited to provide anisotropic versions. Here

$$f_i = P_{IV}(x_i, \alpha_i, \beta_i),$$

which means that it is not required for the construction to choose the same parameters α_i and β_i in different components; they can be taken independently. Integrals of motion related to the separation of variables in Cartesian coordinates can be generated directly:

$$H_i = -\frac{d^2}{dx_i^2} - 2f_i' + 4f_i^2 + 4x_i f_i + x_i^2 - 1, \quad i = 1, ..., N. \tag{7.2}$$

This means that the Hamiltonian is at least integrable as it allows the separation of variables in Cartesian coordinates. For each components H_i, which depends on the variable x_i only, the corresponding ladder is given by c_i and $c_i^\dagger$. The ladder operators can be used to obtain integrals of motion of the form

$$I_{ij} = c_i c_j^\dagger, \quad I_{ij}^\dagger = c_i^\dagger c_j, \quad i, j = 1, ..., N. \tag{7.3}$$

This provides maximal superintegrability (i.e., $2N - 1$ integrals) and the existence of an underlying symmetry algebra. This Hamiltonian generalizes the isotropic harmonic oscillator and the Smorodinsky-Winternitz systems in N dimensions. Here the constructions from the previous section can be used to obtain the states in an algebraic way (in the irreducible case) and to provide explicit expressions in terms of the fourth Painlevé transcendent and the derivative of the fourth Painlevé transcendent. The states relative to a given component using lowest weight representations are denoted by

$$\psi_{i;n_i}^1 = (c_i^\dagger)^{n_i} \psi_{i,0}^{(1)}. \tag{7.4}$$

The formula in the case of the lowest weight representation for all variables x_i is

$$\phi_{n_1,...,n_N} = \prod_{i}^{N} (c_i^\dagger)^{n_i} \phi_0^{[N]}, \tag{7.5}$$

where

$$\psi_0^{[N]} = \prod_{i=1}^{N} \psi_{i;0}^{(1)},$$

where

$$c_i \psi_{i;0} = 0,$$

and then by construction,

$$c_i \psi_0^{[N]} = 0, \quad \forall i = 1, ..., N.$$

The study of different integrable and superintegrable deformations is beyond the scope of this paper. However, this illustrates the wide applicability of the results from previous sections. So far only one example that would correspond to a particular choice of α and β in the context of three-dimensional Euclidean space was solved algebraically [27]. It was pointed out that the symmetry algebra is a generalization of the $su(3)$ algebra and the finite dimensional representations take the form of multiplets that can be decomposed into $su(3)$-like multiplets. However, as the parameters of the fourth Painlevé transcendent were such that rational solutions exist this allowed to make calculations using explicit realizations for the wavefunctions to build the representations. When the parameters are generic, as pointed out in earlier sections, induced representations need to be used and the explicit wavefunctions take a complicated form.

8. Conclusions

Polynomial Heisenberg-Weyl algebras and the corresponding constraints for the existence of finite dimensional representations have been studied. Those are important in applications to quantum mechanical systems as they correspond often to the degenerate spectrum and their related states decomposed into multiplets. Here we provided insight into another setting consisting of induced constructions of infinite dimensional representations and their related coefficients for the action of the generators. In order to establish explicit formula, we have used identities based on commutators of monomials of the generators and applied on the Hamiltonian related to the fourth Painlevé transcendent. This Hamiltonian possesses third-order ladder operators and also connects with certain deformations of the harmonic oscillator. The representations of highest and lowest weight types become in this setting two-variables polynomials in terms of the fourth Painlevé transcendent and its derivative. To our knowledge, those types of polynomials have not been studied in the literature. They may have broader applications in a similar way as other types of polynomials, for example, them Lamé polynomials which appear in different contexts of mathematical physics.

The algebraic methods developed in this paper may also have wider applications given that polynomial Heisenberg-Weyl algebras and other polynomial algebras appear not only in context of quantum mechanics, but other contexts of mathematical physics such as recoupling, Racah polynomials, and more generally, in connection with orthogonal polynomials [9, 28].

Acknowledgments

Ian Marquette was supported by the Australian Research Council Future Fellowship FT180100099.

Conflict of interest

The author declares no conflict of interest in this paper.

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