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*Research article*

# Some properties of the generalized max Frank matrices

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**Abstract:** In this paper, we introduce a new generalization of the Frank matrix, which is a lower Hessenberg matrix called the generalized max $r$-Frank matrix. We obtain a recurrence relation provided by the characteristic polynomial, inverse, determinant, and norm properties of this matrix. We also present an example to illustrate the results obtained.

**Keywords:** Frank matrix; max matrix; max $r$-Frank matrix; determinant; inverse; norm
**Mathematics Subject Classification:** 15A09, 15A15, 15A60, 15B99

## 1. Introduction

Special matrices are a hotly studied subject in the research area of matrix theory. Especially, special matrices whose entries are well-known number sequences and polynomials have become a very interesting research subject in recent years, and some scholars have obtained some good results in this area. A lot of research examined the norms of the special matrices involving famous number sequences and polynomials. They found various properties of these matrices, such as lower bounds, upper bounds, and exact values for the spectral norms, eigenvalues, Euclidean norms, determinants, and permanents.

Min and max matrices are one of the most researched and known types of structured matrices, and they are widely used. Min and max matrices with minimum and maximum entries were first introduced by Pólya and Szegö [19] as

$$M_{min} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & \cdots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & n \end{bmatrix} \quad \text{and} \quad M_{max} = \begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & 2 & 3 & \cdots & n \\ 3 & 3 & 3 & \cdots & n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n & n & n & \cdots & n \end{bmatrix},$$

respectively. Min and max matrices arise in many different theoretical and practical fields, such as the one in which Bhat [2] defined meet matrices for the first time, and min matrices are used as an example in the same paper. Fonseca [4] uses the matrix inverses of some min and max matrices to investigate their eigenvalues. In [7], underestimating a min matrix's smallest eigenvalue yields boundaries for the values of trigonometric functions. Mattila and Haukkanen [12] studied some properties of the following types of min and max matrices.

$$M_{min} = \begin{bmatrix} a_1 & a_1 & a_1 & \cdots & a_1 \\ a_1 & a_2 & a_2 & \cdots & a_2 \\ a_1 & a_2 & a_3 & \cdots & a_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_n \end{bmatrix} \quad \text{and} \quad M_{max} = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ a_2 & a_2 & a_3 & \cdots & a_n \\ a_3 & a_3 & a_3 & \cdots & a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & a_n & a_n & \cdots & a_n \end{bmatrix}.$$

Kizilateş and Terzioglu [11] define $r$-min and $r$-max matrices and give the determinants, inverses, norms, and factorizations of these matrices. Frank defined [5] the matrix of order $n$

$$F_n = \begin{bmatrix} n & n-1 & 0 & \cdots & 0 \\ n-1 & n-1 & n-2 & \cdots & 0 \\ n-2 & n-2 & n-2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix},$$

which is called the Frank matrix. A Frank matrix is also a special max matrix. As test matrices for eigenprograms, the Frank matrix is frequently utilized. This is as a result of the well- and poorly-conditioned eigenvalues in the Frank matrix [3]. Hake [6] obtained the determinant, inverse, $LU$-decomposition, and characteristic polynomials of the Frank matrix. Varah [22] showed a generalization of the Frank matrix and computed its eigensystem. The generalized Frank matrix is defined [14] as

$$F_{a_n} = \begin{bmatrix} a_n & a_{n-1} & 0 & \cdots & 0 \\ a_{n-1} & a_{n-1} & a_{n-2} & \cdots & 0 \\ a_{n-2} & a_{n-2} & a_{n-2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ a_1 & a_1 & a_1 & \cdots & a_1 \end{bmatrix}_{n\times n}.$$

Mersin and Bahşi [16] investigated the bounds for the maximum eigenvalues of the special cases of the generalized Frank matrices, which are called Fibonacci Frank and Lucas Frank matrices. The $r$-Frank matrix is defined [21] as

$$F_{a_n}^r = \begin{bmatrix} a_n & a_{n-1} & 0 & \cdots & 0 \\ ra_{n-1} & a_{n-1} & a_{n-2} & \cdots & 0 \\ ra_{n-2} & ra_{n-2} & a_{n-2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ ra_1 & ra_1 & ra_1 & \cdots & a_1 \end{bmatrix}_{n\times n}.$$

Mersin [13] obtained some results about the eigenvalues of minimum matrices by using Sturm's theorem. Mersin and Bahşi [15] applied the Sturm theorem to the generalized Frank matrix, which is a special form of the Hessenberg matrix, and examined its eigenvalues by using the Sturm property. We see that a great deal of research has been done on the aforementioned matrices in the literature [1, 9, 10, 17, 18].

The Hadamard product of $A = \left(a_{ij}\right)_{n\times n}$ and $B = \left(b_{ij}\right)_{n\times n}$ is defined by $A \circ B = \left(a_{ij}b_{ij}\right)_{n\times n}$. The Hadamard inverse of the $A$ is denoted by $A^{\circ -1} = \left(a_{ij}^{-1}\right)$, where $a_{ij} \neq 0$ [20]. The Euclidean norm of the matrix $A$ is defined as [8]

$$\|A\|_E = \left(\sum_{i,j=1}^{n} \left|a_{ij}\right|^2\right)^{1/2},$$

the singular values of the matrix $A$ is

$$\sigma_i = \sqrt{\lambda_i(A^*A)},$$

where $\lambda_i$ is an eigenvalue of $A^*A$ and $A^*$ is conjugate transpose of matrix $A$. The square roots of the maximum eigenvalues of $A^*A$ are called the spectral norm of $A$ and are induced by $\|A\|_2$. The following inequality holds:

$$\frac{1}{\sqrt{n}}\|A\|_E \leq \|A\|_2 \leq \|A\|_E .$$

In the light of the above-mentioned studies, we define the generalized max $r$-Frank matrix, which is the general form of the Frank matrix, based on the relationship between the matrices mentioned above. We examine a max matrix and obtain some of its linear algebraic properties, such as determinants, inverses, and some norms for the generalized max $r$-Frank matrix and its reciprocal ones. Finally, we give an example to illustrate the results obtained.

## 2. Main results

In this section, since the Frank matrix is derived from the minimum matrix, we will define a variant of the Frank matrix derived from the maximum matrix. When constructing the Frank matrix, we consider that the elements of the minimum matrix are symmetric with respect to the anti-diagonal and transformed into a lower-Hessenberg matrix. As a result of these explanations, we can give the following definition.

The generalized max $r$-Frank matrix is defined

$$^{m_r}F_{a_n} = \begin{bmatrix} a_n & a_n & 0 & \cdots & 0 & 0 \\ ra_n & a_{n-1} & a_{n-1} & \cdots & 0 & 0 \\ ra_n & ra_{n-1} & a_{n-2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ ra_n & ra_{n-1} & ra_{n-2} & \cdots & a_2 & a_2 \\ ra_n & ra_{n-1} & ra_{n-2} & \cdots & ra_2 & a_1 \end{bmatrix}_{n\times n}. \tag{2.1}$$

where $a_i = \{a_1, a_2, a_3, ..., a_n\}$ is a finite sequence with any $a_i$ real numbers. If $r = 1$ and $a_i = \{1, 2, 3, ..., n\}$, then the generalized max $r$-Frank matrices are called max Frank matrices.

**Theorem 2.1.** *Let $P_{r,n}(\lambda)$ be the characteristic polynomial of the matrix $(^{m_r}F_{a_n})$ and $n \geq 3$. Then, $P_{r,n}(\lambda)$ satisfies:*

$$[P_{r,n}(\lambda)]_n = (a_1 - a_2 - \lambda)\,[P_{r,n}(\lambda)]_{n-1} - (ra_2 - a_2 + \lambda)\,a_2[P_{r,n}(\lambda)]_{n-2} \tag{2.2}$$

with initial $P_{r,1}(\lambda) = a_1 - \lambda$ and $P_{r,2}(\lambda) = \lambda^2 - \lambda\,(a_1 + a_2) - ra_2^2 + a_1a_2$.

*Proof.* The characteristic polynomial of $({}^{r}F_n)$ is

$$P_{r,n}(\lambda) = det(({}^{m_r}F_{a_n}) - \lambda I) = \begin{vmatrix} a_n - \lambda & a_n & 0 & \cdots & 0 & 0 \\ ra_n & a_{n-1} - \lambda & a_{n-1} & \cdots & 0 & 0 \\ ra_n & ra_{n-1} & a_{n-2} - \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ ra_n & ra_{n-1} & ra_{n-2} & \cdots & a_2 - \lambda & a_2 \\ ra_n & ra_{n-1} & ra_{n-2} & \cdots & ra_2 & a_1 - \lambda \end{vmatrix}.$$

We apply some elementary column operators to $P_{r,n}(\lambda)$, and we have that

$$P_{r,n}(\lambda) = \\ \begin{vmatrix} a_n - \lambda & a_n & 0 & \cdots & 0 & 0 \\ ra_n & a_{n-1} - \lambda & a_{n-1} & \cdots & 0 & 0 \\ 0 & ra_{n-1} - a_{n-1} + \lambda & a_{n-2} - a_{n-1} - \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & \cdots & a_2 - a_3 - \lambda & a_2 \\ 0 & 0 & 0 & \cdots & ra_2 - a_2 + \lambda & a_1 - a_2 - \lambda \end{vmatrix}.$$

By expanding this determinant by the last row and after by the last column, we obtain

$$[P_{r,n}(\lambda)]_n = (a_1 - a_2 - \lambda)\,[P_{r,n}(\lambda)]_{n-1} - (ra_2 - a_2 + \lambda)\,a_2[P_{r,n}(\lambda)]_{n-2}$$

with initial $P_{r,1}(\lambda) = a_1 - \lambda$ and $P_{r,2}(\lambda) = \lambda^2 - \lambda\,(a_1 + a_2) - ra_2^2 + a_1a_2$. □

**Theorem 2.2.** *Assume that* $({}^{m_r}F_{a_n})$ *is a matrix in* (2.1)*, and*

$$= \begin{bmatrix} a_n & M \\ rN & {}^{m_r}F_{a_{n-1}} \end{bmatrix}$$

with $N = [a_n \quad a_n \quad a_n \quad ... \quad a_n]^T_{(n-1)\times 1}$, and $M = [a_n \quad 0 \quad 0 \quad ... \quad 0]_{1\times(n-1)}$. If $({}^{m_r}F_{a_n})$ is non-singular matrix, then the inverse of $({}^{m_r}F_{a_n})$ is

$$({}^{m_r}F^{-1}_{a_n}) = \begin{bmatrix} u & -urM({}^{m_r}F^{-1}_{a_{n-1}}) \\ -u({}^{m_r}F^{-1}_{a_{n-1}})N & ({}^{m_r}F^{-1}_{a_{n-1}}) + ur({}^{m_r}F^{-1}_{a_{n-1}})NM({}^{m_r}F^{-1}_{a_{n-1}}) \end{bmatrix}, \tag{2.3}$$

where $u = \frac{a_{n-1}}{a_na_{n-1} - ra_n^2}$.

*Proof.* We can prove the theorem by using the induction method on $n$. We obtain $n = 2$.

$$({}^{m_r}F_{a_2}^{-1}) = \frac{1}{det({}^{m_r}F_{a_2})}\begin{bmatrix} a_1 & -ra_2 \\ -a_2 & a_2 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{a_1}{a_2a_1-ra_2^2} & -\frac{ra_2}{a_2a_1-ra_2^2} \\ -\frac{a_2}{a_2a_1-ra_2^2} & \frac{a_2}{a_2a_1-ra_2^2} \end{bmatrix}.$$

On the other hand, for $n = 2$ in our assertion of equality (2.3), we have that

$$\begin{bmatrix} u & -urM({}^{m_r}F_{a_1}^{-1}) \\ -u({}^{m_r}F_{a_1}^{-1})N & ({}^{m_r}F_{a_1}^{-1}) + ur({}^{m_r}F_{a_1}^{-1})NM({}^{m_r}F_{a_1}^{-1}) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{a_1}{a_2a_1-ra_2^2} & -\frac{ra_1}{a_2a_1-ra_2^2}a_2\frac{1}{a_1} \\ -\frac{a_1}{a_2a_1-ra_2^2}\frac{1}{a_1}a_2 & \frac{1}{a_1} + \frac{a_1}{a_2a_1-ra_2^2}\frac{1}{a_1}ra_2a_2\frac{1}{a_1} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{a_1}{a_2a_1-ra_2^2} & -\frac{ra_2}{a_2a_1-ra_2^2} \\ -\frac{a_2}{a_2a_1-ra_2^2} & \frac{a_2}{a_2a_1-ra_2^2} \end{bmatrix}.$$

It is seen that equality (2.3) is achieved in these two cases. Thus, our assertion is true for $n = 2$. Assume that our claim is true for $(n-1)$, then by the identity $({}^{m_r}F_{a_{n-1}}^{-1})({}^{m_r}F_{a_{n-1}}) = I_{n-1}$, we have

$$({}^{m_r}F_{a_{n-1}}^{-1})\begin{bmatrix} a_{n-1} \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{(n-1)\times 1} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{(n-1)\times 1}.$$

To this end, we show that the result is true for $n$. By multiplying $({}^{m_r}F_{a_n}^{-1})$ and $({}^{m_r}F_{a_n})$, we obtain

$$\begin{bmatrix} u & -urM({}^{m_r}F_{a_{n-1}}^{-1}) \\ -u({}^{m_r}F_{a_{n-1}}^{-1})N & ({}^{m_r}F_{a_{n-1}}^{-1}) + ur({}^{m_r}F_{a_{n-1}}^{-1})NM({}^{m_r}F_{a_{n-1}}^{-1}) \end{bmatrix}\begin{bmatrix} a_n & M \\ rN & {}^{m_r}F_{a_{n-1}} \end{bmatrix}$$
$$= \left[\begin{matrix} ua_n + (-urM({}^{m_r}F_{a_{n-1}}^{-1})rN) \\ -u({}^{m_r}F_{a_{n-1}}^{-1})Na_n + (({}^{m_r}F_{a_{n-1}}^{-1}) + ur({}^{m_r}F_{a_{n-1}}^{-1})NM({}^{m_r}F_{a_{n-1}}^{-1}))rN \end{matrix}\right.$$
$$\left.\begin{matrix} uM + (-urM({}^{m_r}F_{a_{n-1}}^{-1})({}^{m_r}F_{a_{n-1}})) \\ -u({}^{m_r}F_{a_{n-1}}^{-1})NM + (({}^{m_r}F_{a_{n-1}}^{-1}) + ur({}^{m_r}F_{a_{n-1}}^{-1})NM({}^{m_r}F_{a_{n-1}}^{-1}))^{m_r}F_{a_{n-1}} \end{matrix}\right]$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & I_{n-1} \end{bmatrix}.$$

Thus, the proof is complete. □

**Theorem 2.3.** *The determinant of the matrix* $({}^{m_r}F_{a_n})$

$$det([{}^{m_r}F_{a_n}]_n) = (a_1 - a_2)det([{}^{m_r}F_{a_n}]_{n-1}) - (ra_2 - a_2)a_2det([{}^{m_r}F_{a_n}]_{n-2}) \tag{2.4}$$

is valid.

*Proof.* By using row-column operations to get $det({}^{m_r}F_{a_n})$, we obtain

$$det([{}^{m_r}F_{a_n}]_n) = (a_1 - a_2)det([{}^{m_r}F_{a_n}]_{n-1}) - (ra_2 - a_2)a_2det([{}^{m_r}F_{a_n}]_{n-2}).$$

□

**Theorem 2.4.** *The Euclidean norm of the matrix* $({}^{m_r}F_{a_n})$ *is*

$$||A||_E = \left(2\sum_{i=2}^{n}(a_i)^2 + r\sum_{i=1}^{n-1}(n-i)(a_{n+1-i})^2 + a_1\right)^{\frac{1}{2}}. \tag{2.5}$$

*Proof.* If we apply the definition of Euclidean norm to the matrix $({}^{m_r}F_{a_n})$, we obtain

$$||A||_E^2 = 2\sum_{i=2}^{n}(a_i)^2 + r\sum_{i=1}^{n-1}(n-i)(a_{n+1-i})^2 + a_1.$$

□

**Theorem 2.5.** *The upper bounds for the spectral norm of the matrix* $({}^{m_r}F_{a_n})$ *is*

$$||A||_2 \le a_n\sqrt{n(n-1)|r|^2 + n}, \quad |r| \ge 1,$$
$$||A||_2 \le a_n\sqrt{n(n-2)|r|^2 + n}, \quad |r| < 1.$$

*Proof.* The matrix $A$ is of the form

$$A = \begin{bmatrix} a_n & a_n & 0 & \cdots & 0 & 0 \\ ra_n & a_{n-1} & a_{n-1} & \cdots & 0 & 0 \\ ra_n & ra_{n-1} & a_{n-2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ ra_n & ra_{n-1} & ra_{n-2} & \cdots & a_2 & a_2 \\ ra_n & ra_{n-1} & ra_{n-2} & \cdots & ra_2 & a_1 \end{bmatrix}.$$

Then, we have,

$$||A||_E^2 = 2\sum_{i=2}^{n}(a_i)^2 + r\sum_{i=1}^{n-1}(n-i)(a_{n+1-i})^2.$$

Let the matrices $B$ and $C$ as

$$B = \begin{bmatrix} a_n & a_n & 0 & \cdots & 0 & 0 \\ a_n & a_{n-1} & a_{n-1} & \cdots & 0 & 0 \\ a_n & a_{n-1} & a_{n-2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & a_2 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & a_1 \end{bmatrix}_{n\times n}, C = \begin{bmatrix} 1 & 1 & 0 & \ldots & 0 \\ r & 1 & 1 & \ldots & 0 \\ r & r & & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ r & r & r & \ldots & 1 \\ r & r & r & \ldots & 1 \end{bmatrix}_{n\times n},$$

such that $A = B \circ C$. Hence, when $|r| \geq 1$, we obtain

$$c_1(B) = \max_j \sqrt{\sum_i |a_{ij}|^2} = \sqrt{na_n^2}.$$

$$r_1(C) = \max_i \sqrt{\sum_j |a_{ij}|^2} = \sqrt{(n-1)|r|^2 + 1}.$$

We have

$$\|A\|_2 \leq a_n \sqrt{n(n-1)|r|^2 + n}.$$

When $|r| < 1$, we also obtain

$$\|A\|_2 \leq a_n \sqrt{n(n-2)|r|^2 + n}.$$

□

## 3. A numerical example

In this section, we give a numerical example to verify our results. In the example to be given, the matrix (2.1), whose entries are Leonardo numbers, will be discussed for n = 5. The Leonardo sequence is defined by the following recurrence relation:

$$Le_{n+2} = Le_{n+1} + Le_n + 1, (n > 0),$$

where $Le_1 = Le_2 = 1$.

**Example 3.1.** Let

$$({}^{m_2}F_{Le_5}) = \begin{bmatrix} 9 & 9 & 0 & 0 & 0 \\ 2.9 & 5 & 5 & 0 & 0 \\ 2.9 & 2.5 & 3 & 3 & 0 \\ 2.9 & 2.5 & 2.3 & 1 & 1 \\ 2.9 & 2.5 & 2.3 & 2.1 & 1 \end{bmatrix}$$

be a matrix as in (2.1) for $r = 2$ and $n = 5$. For the characteristic polynomials of $({}^{m_2}F_{Le_5})_{i\leq 5}$, (2.2) yields for $(i \geq 2)$

$$[P_{r,i}(\lambda)]_i = (a_1 - a_2 - \lambda)\,[P_{r,i}(\lambda)]_{i-1} - (ra_2 - a_2 + \lambda)\,a_2[P_{r,i}(\lambda)]_{i-2}.$$

Thus, $P_{2,1}(\lambda) = 9 - \lambda$, $P_{2,2}(\lambda) = \lambda^2 - 14\lambda - 117$, $P_{2,3}(\lambda) = -\lambda^3 + 17\lambda^2 + 125\lambda + 9$, $P_{2,4}(\lambda) = \lambda^4 - 18\lambda^3 - 126\lambda^2 + 218\lambda + 1035$, $P_{2,5}(\lambda) = -\lambda^5 + 19\lambda^4 + 110\lambda^3 - 360\lambda^2 - 1169\lambda - 9$.

**Example 3.2.** The inverse of $({}^{m_2}F_{Le_5})$ can be calculated as

$$({}^{m_2}F_{Le_5}^{-1}) = \begin{bmatrix} u & -2uM({}^{m_2}F_{Le_4}^{-1}) \\ -u({}^{m_2}F_{Le_4}^{-1})N & ({}^{m_2}F_{Le_4}^{-1}) + 2u({}^{m_2}F_{Le_4}^{-1})NM({}^{m_2}F_{Le_4}^{-1}) \end{bmatrix},$$

where $u = \frac{Le_4}{Le_4 Le_5 - 2Le_5^2}$, $N = [Le_5 \quad Le_5 \quad Le_5 \quad Le_5]^T_{(4)\times 1}$ and $M = [Le_5 \quad 0 \quad 0 \quad 0]_{1\times(4)}$. Thus, after the necessary calculations, the inverse of $({}^{m_2}F^{-1}_{Le_5})$ is obtained as follows:

$$({}^{m_2}F^{-1}_{Le_5}) = \begin{bmatrix} -\frac{35}{9} & -3 & 5 & 15 & -15 \\ 4 & 3 & -5 & -15 & 15 \\ 10 & 8 & -13 & -39 & 39 \\ 0 & 0 & 0 & -1 & 1 \\ -30 & -24 & 38 & 116 & -115 \end{bmatrix}.$$

**Example 3.3.** The determinant of $({}^{m_2}F_{Le_5})$ can be calculated as

$$det({}^{m_2}F_{Le_5}) = \begin{vmatrix} 9 & 9 & 0 & 0 & 0 \\ 2.9 & 5 & 5 & 0 & 0 \\ 2.9 & 2.5 & 3 & 3 & 0 \\ 2.9 & 2.5 & 2.3 & 1 & 1 \\ 2.9 & 2.5 & 2.3 & 2.1 & 1 \end{vmatrix}.$$

$$det([{}^{m_2}F_{Le_5}]_5) = (Le_1 - Le_2)det([{}^{m_2}F_{Le_5}]_4) - (2Le_2 - Le_2)Le_2 det([{}^{m_2}F_{Le_5}]_3)$$

$$= (1-1)\begin{vmatrix} 9 & 9 & 0 & 0 \\ 2.9 & 5 & 5 & 0 \\ 2.9 & 2.5 & 3 & 3 \\ 2.9 & 2.5 & 2.3 & 1 \end{vmatrix} - (2.1-1)\begin{vmatrix} 9 & 9 & 0 \\ 2.9 & 5 & 5 \\ 2.9 & 2.5 & 3 \end{vmatrix} = -9.$$

**Example 3.4.** The Euclidean norm of the matrix $({}^{m_2}F_{Le_5})$ can be computed as

$$\left\|{}^{m_2}F_{Le_5}\right\|_E^2 = 2(1^2 + 3^2 + 5^2 + 9^2) + 2(4.18^2 + 3.10^2 + 2.6^2 + 1.2^2) + 1.$$
$$\left\|{}^{m_2}F_{Le_5}\right\|_E = \sqrt{3577} = 59,8080.$$

**Example 3.5.** We can obtain the upper bounds for the spectral norm of $({}^{m_2}F_{Le_4})$ as

$$\left\|{}^{m_2}F_{Le_5}\right\|_2 \leq 9\sqrt{5(5-1)|2|^2 + 5},$$
$$\left\|{}^{m_2}F_{Le_5}\right\|_2 = 23,0052 \leq 82,9759.$$

## 4. Concluding remarks

In this paper, we looked into the generalized max $r$-Frank matrix, deduced some of its linear algebraic properties, and obtained certain conclusions. This generalization is defined in accordance with the previous relationship between certain special type matrices. In computational and applied mathematics, these matrices are especially crucial. Therefore, we hope that these new matrices and properties that we have found will offer a new perspective to the researchers. For our future studies, we plan to study whether Sturm's theorem and bounds for eigenvalues can be applied to the matrix discussed in this study.

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## Conflict of interest

The authors declare that they have no conflicts of interest.

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