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Research article

Advancements in q-Hermite-Appell polynomials: a three-dimensional exploration

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Abstract: In this research, we leverage various q-calculus identities to introduce the notion of q-Hermite-Appell polynomials involving three variables, elucidating their formalism. We delve into numerous properties and unveil novel findings regarding these q-Hermite-Appell polynomials, encompassing their generating function, series representation, summation equations, recurrence relations, q-differential formula, and operational principles. Our investigation sheds light on the intricate nature of these polynomials, elucidating their behavior and facilitating deeper understanding within the realm of q-calculus.

Keywords: special polynomials; q-calculus; monomiality principle; explicit form; operational connection; determinant form; symmetric identities; summation formulae
Mathematics Subject Classification: 33E20, 33C45, 33B10, 33E30, 11T23

1. Introduction and preliminaries

The Hermite polynomials, revered as one of the oldest and most significant orthogonal special functions dating back to classical mathematics, boast widespread utility. They serve as solutions to differential equations that model the quantum mechanical Schrödinger equation for harmonic oscillators. Moreover, in the realm of classical boundary-value problems within parabolic regions and coordinates, Hermite polynomials assume a pivotal role. Their significance extends to signal processing, where they feature as Hermitian wavelets in wavelet transform analysis and probability studies. Furthermore, Hermite polynomials exhibit relevance in diverse fields such as the Edgeworth series, Brownian motion, combinatorics, and numerical computations, notably in the context of the Appell series and umbral calculus. For a comprehensive understanding of Hermite polynomials and their myriad applications, interested readers are encouraged to explore the referenced research papers [1–6].

In [7, 8], Dattoli and colleagues acknowledged the utility of Hermite polynomials, which have been employed in solving challenges related to optical beam transport and quantum mechanics. In this framework, they presented generalized harmonic oscillator eigenfunctions along with the necessary annihilation-creation operator algebra. Further, in [9], Subuhi and her co-authors recognized Hermite-based Appell polynomials, which represent a fascinating extension of classical Hermite polynomials, offering a versatile framework for addressing a wide array of mathematical problems. These polynomials combine the robustness of Hermite polynomials with the flexibility of the Appell sequence, yielding a powerful toolset for mathematical analysis. They find application in diverse fields such as quantum mechanics, statistical physics, signal processing, and combinatorics. Hermite-based Appell polynomials inherit key properties from both parent families, including orthogonality, recurrence relations, and differential equations, making them invaluable for solving differential equations, generating special functions, and modeling complex phenomena. Moreover, their connections to umbral calculus and other advanced mathematical concepts further enhance their utility in theoretical and applied contexts. The study and exploration of Hermite-based Appell polynomials continue to uncover new insights and applications, enriching our understanding of mathematical structures and their practical implications.

The triadic Hermite-based Appell polynomials, denoted as ${}_{\mathcal{H}}\mathcal{A}_n(u, v, w)$, are elegantly crafted and defined through both a generating function and a series representation. Herein lies the generating function that births the trivariate Hermite-Appell polynomials ${}_{\mathcal{H}}\mathcal{A}_n(u, v, w)$ [9]:

$$\mathcal{A}(\xi)e^{u\xi+v\xi^2+w\xi^3} = \sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{A}_n(u, v, w)\frac{\xi^n}{n!} \tag{1.1}$$

complemented by the elucidation provided by the series definition [9]:

$${}_{\mathcal{H}}\mathcal{A}_n(u, v, w) = n! \sum_{k=0}^{[n/3]} \frac{w^k {}_{\mathcal{H}}\mathcal{A}_{n-k}(u, v)}{k!(n-3k)!}. \tag{1.2}$$

For the HAP ${}_{\mathcal{H}}\mathcal{A}_n(u, v, w)$, the succeeding differential recurrence relationships are furnished [9]:

$$\begin{aligned}
&\frac{\partial}{\partial u} {}_{\mathcal{H}}\mathcal{A}_n(u, v, w) = n {}_{\mathcal{H}}\mathcal{A}_{n-1}(u, v, w), \ \ n \geq 1, \\
&\frac{\partial^2}{\partial u^2} {}_{\mathcal{H}}\mathcal{A}_n(u, v, w) = n(n-1) {}_{\mathcal{H}}\mathcal{A}_{n-2}(u, v, w), \ \ n \geq 2, \\
&\frac{\partial}{\partial v} {}_{\mathcal{H}}\mathcal{A}_n(u, v, w) = n(n-1) {}_{\mathcal{H}}\mathcal{A}_{n-2}(u, v, w), \ \ n \geq 2, \\
&\frac{\partial}{\partial w} {}_{\mathcal{H}}\mathcal{A}_n(u, v, w) = n(n-1)(n-2) {}_{\mathcal{H}}\mathcal{A}_{n-3}(u, v, w), \ \ n \geq 3,
\end{aligned}$$

$$\frac{\partial^2}{\partial u^2}{}_{\mathcal{H}}\mathcal{A}_n(u,v,w) = \frac{\partial}{\partial y}{}_{\mathcal{H}}\mathcal{A}_n(u,v,w),$$
$$\frac{\partial^3}{\partial u^3}{}_{\mathcal{H}}\mathcal{A}_n(u,v,w) = \frac{\partial}{\partial z}{}_{\mathcal{H}}\mathcal{A}_n(u,v,w). \tag{1.3}$$

In reference [9], the differential equation governing the HAP ${}_{\mathcal{H}}\mathcal{A}_n(u,v,w)$ of three variables is presented:

$$\left(3w\frac{\partial^3}{\partial u^3} + 2v\frac{\partial^2}{\partial v^2} + u\frac{\partial}{\partial u} + \frac{\mathcal{A}'(\frac{\partial}{\partial u})}{\mathcal{A}(\frac{\partial}{\partial u})} - n\right){}_{\mathcal{H}}\mathcal{A}_n(u,v,w) = 0. \tag{1.4}$$

Quantum calculus, often abbreviated as q-calculus, stands out as a pivotal extension of traditional calculus, particularly notable for its profound relevance to quantum mechanics and various scientific domains such as mathematical analysis, combinatorics, and the theory of orthogonal polynomials. Initially proposed by Jackson [10], the framework of q-calculus has since been developed and expanded upon by numerous scholars. This mathematical framework facilitates the exploration and analysis of q-analogues, which are counterparts of elementary and special functions under q-transformations. Notably, recent research efforts have focused on investigating specific families of special polynomials within the realm of q-calculus [11, 12], elucidating their properties and applications across diverse disciplines. This ongoing research underscores the enduring relevance and profound impact of q-calculus in shaping contemporary mathematical theory and its interdisciplinary applications.

We revisit essential concepts, symbols, and insights derived from our exploration of quantum mathematics, pivotal for the subsequent discourse in this paper. For any complex number Ω, its q-analogue can be delineated as elucidated in references [1,4,13]:

$$[\Omega]_q = \frac{1-q^{\Omega}}{1-q} = \sum_{k=1}^{\Omega} q^{k-1}, \quad 0 < q < 1. \tag{1.5}$$

The q-factorial is [1,4,13] is defined by:

$$[\Omega]_q! = \prod_{\xi=1}^{k} [\xi]_q, \quad 0 < q < 1, \quad k \geq 1 \tag{1.6}$$

and

$$[\Omega]_q! = 1, \quad k = 0. \tag{1.7}$$

Moreover, the q-binomial value attributed to Gauss, as outlined in references [1,4,13], is defined as:

$$\begin{bmatrix} \xi \\ k \end{bmatrix}_q = \frac{[\xi]_q!}{|\xi-k|_q![k]_q!}, \quad k = 0, 1, \cdots, \xi. \tag{1.8}$$

The definition of the ascending and descending q-powers is provided in reference [1,4,13]:

$$(u \pm h)_q^k = \sum_{u=0}^{k} \begin{bmatrix} k \\ l \end{bmatrix}_q q^{\binom{l}{2}} u^{k-l} (\pm h)^k. \tag{1.9}$$

The expression $\begin{bmatrix} k \\ l \end{bmatrix}_q$ is given by Eq (1.8). The definitions of a set of q-exponential expressions are as delineated in [1, 4, 13]:

$$\epsilon_q(u) = \sum_{n=0}^{\infty} \frac{u^n}{n_q!}, \quad 0 < q < 1 \tag{1.10}$$

and

$$\mathcal{E}_q(u) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{u^n}{n!}, \quad 0 < q < 1. \tag{1.11}$$

The correlation between the preceding two q-exponential functions is expounded upon in [1, 4, 13]:

$$\epsilon_q(u)\mathcal{E}_q(-u) = 1. \tag{1.12}$$

Readers are referred to [1, 4, 13] and the sources cited therein for further details. As stated in [14], the q-derivative of function g with respect to u is defined by the following formula:

$$\mathcal{D}_{q,u}g(u) = \frac{f(qu) - f(u)}{qu - u}, \quad 0 < q < 1, \quad u \neq 0. \tag{1.13}$$

Particularly, it is evident that

$$\mathcal{D}_{q,u}u^n = [n]_q u^{n-1}. \tag{1.14}$$

The subsequent derivatives of the q-exponential functions corresponding to the w^{th} order are detailed in [14]:

$$\mathcal{D}^j_{q,u}\epsilon_q(\delta u) = \delta^j \epsilon_q(\delta u), \quad j \in \mathbb{N} \tag{1.15}$$

and

$$\mathcal{D}^j_{q,u}\epsilon_q(\delta u) = \delta^j q^{\binom{j}{2}} \epsilon_q(\delta q^j u), \quad j \in \mathbb{N}, \tag{1.16}$$

where, $\mathcal{D}^j_{q,u}$ denotes the j^{th} order partial derivative relative to u. Further, it is observed in [15], that:

$$\mathcal{D}_{q,u}(g(u)k(u)) = g(u)\mathcal{D}_{q,u}k(u) + k(qu)\mathcal{D}_{q,u}g(u). \tag{1.17}$$

The q-partial derivative of the exponential $\epsilon_q\left(u\xi^2\right)$ with respect to ξ is provided in [16]:

$$\mathcal{D}_{q,\xi}\epsilon_q\left(u\xi^2\right) = u\xi\epsilon_q\left(u\xi^2\right) + qu\xi\epsilon_q\left(qu\xi^2\right). \tag{1.18}$$

In 1880, Appell [17] established a significant foundation in polynomial theory by defining what are now known as Appell polynomials. These polynomials have since become essential tools in various branches of mathematics, finding applications in both theoretical and practical contexts. Building upon this classical framework, Sharma and Chak [18] introduced a groundbreaking extension by incorporating the notion of q-integers, resulting in what they termed the q-harmonic sequence of polynomials. This q-analogue adds a new dimension to the versatility of Appell polynomials, enhancing their utility in fields such as quantum mechanics and statistical physics. Subsequently, in 1967, Al-Salam [19] contributed further to the generalization of Appell polynomials, deepening our understanding of their properties and expanding their applicability in mathematical analysis and beyond. The q-Appell polynomials (abbreviated as qAP $\mathcal{A}_{n,q}(u)$), adopting the subsequent generating function:

$$\mathcal{A}_q(\xi)\epsilon_q(u\xi) = \sum_{n=0}^{\infty} \mathcal{A}_{n,q}(u)\frac{\xi^n}{[n]_q!}, \tag{1.19}$$

with

$$\mathcal{A}_q(\xi) = \sum_{n=0}^{\infty} \mathcal{A}_{n,q}\frac{\xi^n}{[n]_q!}, \tag{1.20}$$

accompanied by the definition of a series:

$$\mathcal{A}_{n,q}(u) = [n]_q! \sum_{k=0}^{n} \mathcal{A}_{k,q}\frac{u^{n-k}}{[k]_q!\,[n-k]_q!}. \tag{1.21}$$

Further, Nusrat and co-authors, introduced 3-VqHP [20], by using the generating relation:

$$\epsilon_q(u\xi)\epsilon_q(v\xi^2)\epsilon_q(w\xi^3) = \sum_{n=0}^{\infty} \mathcal{H}_{n,q}(u,v,w)\frac{\xi^n}{[n]_q!}, \tag{1.22}$$

accompanied by the definition of a series:

$$\mathcal{H}_{n,q}(u,v,w) = [n]_q! \sum_{r=0}^{[\frac{n}{3}]} \sum_{k=0}^{[\frac{n-3r}{2}]} \frac{w^r\,\mathcal{H}_{n-3r,q}(u,v)}{[r]_q!\,[n-3r]_q!} \tag{1.23}$$

and a subsequent operational definition:

$$\mathcal{H}_{n,q}(u,v,w) = \epsilon_q\left(wD_{q,u}^3\right)\mathcal{H}_{n,q}(u,v). \tag{1.24}$$

The q-Hermite-Appell polynomials will hold significant importance across a spectrum of mathematical and scientific disciplines, including “non-commutative probability, quantum physics, and combinatorics”. Stemming from the exploration of their q-analogue, the concept of q-Hermite-Appell polynomials has garnered considerable attention among scholars. The interest in these polynomials has led to a multitude of published research findings, showcasing their relevance in various contexts [16, 21, 22]. Their versatility and applicability make them invaluable tools for modeling complex phenomena and solving a wide range of mathematical problems.

We found inspiration in the diverse applications of Hermite-Appell polynomials across various branches of engineering and science, as highlighted in [2]. Similarly, the frequent utilization of three-variable Hermite-Appell polynomials in addressing challenges related to charged-beam transport in traditional mechanics, as well as in the intricate calculations of quantum-phase-space mechanics, spurred our interest. Umbral techniques have also been extensively employed to scrutinize their properties. Additionally, the seminal work of Dattoli [23] on the characteristics of three-variable Hermite-Appell polynomials and their subsequent generalizations [3, 7, 24] served as a further source of motivation.

Moreover, our interest was piqued by the myriad applications of quantum calculus in modeling quantum computing, non-commutative probability, combinatorics, functional analysis, mathematical physics, and approximation theory. This prompted us to introduce three-variable q-Hermite-Appell polynomials and delve into an exploration of their properties.

Given the expressions (1.19) and (1.22), we proceed to construct the q-Hermite-Appell polynomials of three variables (3VqHAP), denoted by ${}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w)$, with the following generating function:

$$\mathcal{A}_q(\xi)\ \epsilon_q(u\xi)\epsilon_q(v\xi^2)\epsilon_q(w\xi^3) = \sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w)\frac{\xi^n}{[n]_q!}. \tag{1.25}$$

For, $\mathcal{A}_q(\xi) = 1$, these polynomials reduce to the 3-variable q-Hermite polynomials, represented by the generating relation (1.22) and for, $\mathcal{A}_q(\xi) = 1,\ w = 0$, these polynomials reduce to the 2-variable q-Hermite polynomials, represented by the generating relation:

$$\epsilon_q(u\xi)\epsilon_q(v\xi^2) = \sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v)\frac{\xi^n}{[n]_q!}. \tag{1.26}$$

The subsequent sections of the article unfold as follows: In Section 2, we unveil the 3-variable q-Hermite-Appell polynomials through their series representations. Section 3 delves into operational identity, q-differential recurrence relations, and determinant representation for these polynomials. Summation formulae and pure recurrence relations are derived in Section 4, presenting several key findings. Section 5 focuses on exploring select members of q-Appell polynomials, accompanied by the establishment of corresponding results. Finally, concluding remarks are framed.

2. q-Hermite-Appell polynomials with three variables

The q-Hermite-Appell polynomials with three variables ${}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w)$ represent a significant extension of classical Hermite and Appell polynomials, introducing the parameter q from the theory of q-calculus. These polynomials possess remarkable properties that make them indispensable in various areas of mathematics, physics, and engineering. They generalize the classical Hermite and Appell polynomials by incorporating an additional parameter q, allowing for more flexibility in modeling complex phenomena. Their series definition provides a powerful tool for solving differential equations, integral transforms, and studying quantum mechanics, statistical mechanics, and combinatorics. First, we find the series representations of these polynomials using the following results:

Theorem 2.1. *The q-Hermite-Appell polynomials with three variables ${}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w)$ satisfy the following series representations:*

$${}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w) = [n]_q! \sum_{k=0}^{n} \frac{\mathcal{A}_{k,q}\ \mathcal{H}_{n-k,q}(u,v,w)}{[k]_q!\ [n-k]_q!}, \quad 0 < q < 1. \tag{2.1}$$

Proof. Expanding the l.h.s. of expression (1.25) in view of expressions (1.19) and (1.22), it follows that

$$\mathcal{A}_q(\xi)\ \epsilon_q(u\xi)\epsilon_q(v\xi^2)\epsilon_q(w\xi^3) = \sum_{k=0}^{\infty} \mathcal{A}_{n,q}\frac{\xi^n}{[k]_q!} \sum_{n=0}^{\infty} \mathcal{H}_{n,q}(u,v,w)\frac{\xi^n}{[n]_q!}. \tag{2.2}$$

Inserting the r.h.s. of expression (1.25) in the l.h.s. of the previous expression (2.3), we find

$$\sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w)\frac{\xi^n}{[n]_q!} = \sum_{k=0}^{\infty} \mathcal{A}_{n,q}\frac{\xi^n}{[k]_q!} \sum_{n=0}^{\infty} \mathcal{H}_{n,q}(u,v,w)\frac{\xi^n}{[n]_q!}. \tag{2.3}$$

Further, the right-hand aspect of the previous expression on utilizing the subsequent series rearrangement method [1]:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\mathbb{A}(m,n)=\sum_{n=0}^{\infty}\sum_{n=0}^{n}\mathbb{A}(m,n-m). \tag{2.4}$$

We obtain

$$\sum_{n=0}^{\infty}{}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w)\frac{\xi^n}{[n]_q!}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\mathcal{A}_{k,q}[k]_q!\mathcal{H}_{n-k,q}(u,v,w)\frac{\xi^n}{[n-k]_q!}. \tag{2.5}$$

Hence, upon juxtaposing the respective values of ξ from each perspective, we attain the series representation of the 3-variable q-Hermite-Appell polynomials, denoted as 3VqHAP ${}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w)$. This substantiates the claim presented in (2.1). □

Remark 2.1. *For $w=0$, the 3VqHAP reduces to 2VqHAP denoted by ${}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v)$, thus satisfying the series representation:*

$$ {}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v)=[n]_q!\sum_{k=0}^{n}\frac{\mathcal{A}_{k,q}\ \mathcal{H}_{n-k,q}(u,v)}{[k]_q!\ [n-k]_q!},\quad 0<q<1. \tag{2.6}$$

Remark 2.2. *For $v=w=0$, the 3VqHAP reduces to qAP denoted by $\mathcal{A}_{n,q}(u)$, thus satisfying the series representation:*

$$\mathcal{A}_{n,q}(u)=[n]_q!\sum_{k=0}^{n}\frac{\mathcal{A}_{n-k,q}\ u^k}{[k]_q!\ [n-k]_q!},\quad 0<q<1. \tag{2.7}$$

Theorem 2.2. *The q-Hermite-Appell polynomials with three variables ${}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w)$ satisfy the following series of representations:*

$$ {}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w)=[n]_q!\sum_{r=0}^{[\frac{n}{3}]}\frac{w^r\ {}_{\mathcal{H}}\mathcal{A}_{n-3r,q}(u,v)}{[r]_q!\ [n-3r]_q!},\quad 0<q<1. \tag{2.8}$$

Proof. Expanding the l.h.s. of expression (1.25) in view of expression (2.6) and expanding the exponential term $\epsilon_q(w\xi^3)$ in the following manner, we find

$$\mathcal{A}_q(\xi)\ \epsilon_q(u\xi)\epsilon_q(v\xi^2)\epsilon_q(w\xi^3)=\sum_{n=0}^{\infty}{}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v)\frac{\xi^n}{[n]_q!}\sum_{k=0}^{\infty}w^k\frac{\xi^{3k}}{[k]_q!}. \tag{2.9}$$

Inserting the r.h.s. of expression (1.25) in the l.h.s. of the previous expression (2.9), we find

$$\mathcal{A}_q(\xi)\ \epsilon_q(u\xi)\epsilon_q(v\xi^2)\epsilon_q(w\xi^3)=\sum_{n=0}^{\infty}{}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v)\sum_{k=0}^{\infty}w^k\frac{\xi^{n+3k}}{[n]_q!\ [k]_q!}. \tag{2.10}$$

Further, the right-hand aspect of the previous expression on utilizing the subsequent series rearrangement method [1]:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\mathbb{A}(m,n)=\sum_{n=0}^{\infty}\sum_{n=0}^{[\frac{n}{3}]}\mathbb{A}(m,n-3m). \tag{2.11}$$

We obtain

$$\sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w)\frac{\xi^n}{[n]_q!} = \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{3}]} w^k \; {}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v)\frac{\xi^n}{[k]_q! \; [n-k]_q!}. \tag{2.12}$$

Hence, upon juxtaposing the respective values of ξ from each perspective, we attain the series representation of the 3-variable q-Hermite-Appell polynomials, denoted as 3VqHAP ${}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w)$. This substantiates the claim presented in (2.8). □

Theorem 2.3. *The q-Hermite-Appell polynomials with three variables* ${}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w)$ *satisfy the following series representations:*

$$ {}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w) = [n]_q! \sum_{k=0}^{[\frac{n-2l}{3}]}\sum_{l=0}^{[\frac{n}{2}]} \frac{\mathcal{A}_{n-3k-2l,q}(u)\; v^l \; w^k}{[l]_q! \; [k]_q! \; [n-3k-2l]_q!}, \quad 0<q<1. \tag{2.13}$$

Proof. Expanding the l.h.s. of expression (1.25) in view of expressions (1.19) and expanding the terms $\epsilon_q(v\xi^2)\;\epsilon_q(w\xi^3)$ in the following manner, we find

$$\mathcal{A}_q(\xi)\;\epsilon_q(u\xi)\epsilon_q(v\xi^2)\epsilon_q(w\xi^3) = \sum_{n=0}^{\infty}\mathcal{A}_{n,q}(u)\frac{\xi^n}{[n]_q!}\;\sum_{l=0}^{\infty} v^l\frac{\xi^{2l}}{[l]_q!}\sum_{k=0}^{\infty} w^k\frac{\xi^{3k}}{[k]_q!}. \tag{2.14}$$

Inserting the r.h.s. of expression (1.25) in the l.h.s. of the previous expression (2.14), we find

$$\sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w)\frac{\xi^n}{[n]_q!} = \sum_{n=0}^{\infty}\mathcal{A}_{n,q}(u)\;\sum_{l=0}^{\infty} v^l\frac{\xi^{2l}}{[l]_q!}\sum_{k=0}^{\infty} w^k\frac{\xi^{n+3k}}{[n]_q![k]_q!}. \tag{2.15}$$

Further, the right-hand aspect of the previous expression on utilizing the subsequent series rearrangement method [1]:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\mathbb{A}(m,n) = \sum_{n=0}^{\infty}\sum_{n=0}^{[\frac{n}{3}]}\mathbb{A}(m,n-3m). \tag{2.16}$$

We obtain

$$\sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w)\frac{\xi^n}{[n]_q!} = \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{3}]}\sum_{l=0}^{\infty} v^l\; w^k\;\mathcal{A}_{n-3k,q}(u)\frac{\xi^{n+2l}}{[l]_q!\;[k]_q!\;[n-3k]_q!}. \tag{2.17}$$

Again, the right-hand aspect of the previous expression on utilizing the subsequent series rearrangement method [1]:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\mathbb{A}(m,n) = \sum_{n=0}^{\infty}\sum_{n=0}^{[\frac{n}{2}]}\mathbb{A}(m,n-2m). \tag{2.18}$$

We obtain

$$\sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w)\frac{\xi^n}{[n]_q!} = \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n-2l}{3}]}\;\sum_{l=0}^{[\frac{n}{2}]} v^l\; w^k\;\mathcal{A}_{n-3k-2l,q}\frac{\xi^{n}}{[l]_q!\;[k]_q!\;[n-3k-2l]_q!}. \tag{2.19}$$

Hence, upon juxtaposing the respective values of ξ from each perspective, we attain the series representation of the 3-variable q-Hermite-Appell polynomials, denoted as 3VqHAP ${}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w)$. This substantiates the claim presented in (2.13). □

Theorem 2.4. *The q-Hermite-Appell polynomials with three variables* ${}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w)$ *satisfy the following series representations:*

$$ {}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w) = [n]_q! \sum_{k=0}^{[\frac{n-3k-2l}{3}]} \sum_{l=0}^{[\frac{n}{2}]} \sum_{p=0}^{n} \frac{\mathcal{A}_{n-3k-2l-p,q}\, u^p\, v^l\, w^k}{[p]_q![l]_q!\,[k]_q!\,[n-3k-2l-p]_q!}, \quad 0 < q < 1. \tag{2.20} $$

Proof. Expanding the l.h.s. of expression (1.25) in view of expressions (1.20) and expanding the terms $\epsilon_q(u\xi)\epsilon_q(v\xi^2)\ \epsilon_q(w\xi^3)$ in the following manner, we find, we find

$$ \mathcal{A}_q(\xi)\ \epsilon_q(u\xi)\epsilon_q(v\xi^2)\epsilon_q(w\xi^3) = \sum_{n=0}^{\infty} \mathcal{A}_{n,q} \frac{\xi^n}{[n]_q!} \sum_{p=0}^{\infty} u^p \frac{\xi^p}{[p]_q!} \sum_{l=0}^{\infty} v^l \frac{\xi^{2l}}{[l]_q!} \sum_{k=0}^{\infty} w^k \frac{\xi^{3k}}{[k]_q!}. \tag{2.21} $$

Inserting the r.h.s. of expression (1.25) in the l.h.s. of the previous expression (2.14), we find

$$ \sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w) \frac{\xi^n}{[n]_q!} = \sum_{n=0}^{\infty} \mathcal{A}_{n,q} \sum_{p=0}^{\infty} u^p \frac{\xi^p}{[p]_q!} \sum_{l=0}^{\infty} v^l \frac{\xi^{2l}}{[l]_q!} \sum_{k=0}^{\infty} w^k \frac{\xi^{n+3k}}{[n]_q![k]_q!}. \tag{2.22} $$

Further, the right-hand aspect of the previous expression on utilizing the subsequent series rearrangement method [1]:

$$ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \mathbb{A}(m,n) = \sum_{n=0}^{\infty} \sum_{n=0}^{[\frac{n}{3}]} \mathbb{A}(m, n-3m). \tag{2.23} $$

We obtain

$$ \sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w) \frac{\xi^n}{[n]_q!} = \sum_{n=0}^{\infty} \sum_{k=0}^{[\frac{n}{3}]} \sum_{l=0}^{\infty} \sum_{p=0}^{\infty} u^p \frac{\xi^p}{[p]_q!}\ v^l\ w^k\ \mathcal{A}_{n-3k,q} \frac{\xi^{n+2l}}{[l]_q!\,[k]_q!\,[n-3k]_q!}. \tag{2.24} $$

Again, the right-hand aspect of the previous expression on utilizing the subsequent series rearrangement method [1]:

$$ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \mathbb{A}(m,n) = \sum_{n=0}^{\infty} \sum_{n=0}^{[\frac{n}{2}]} \mathbb{A}(m, n-2m), \tag{2.25} $$

we obtain

$$ \sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w) \frac{\xi^n}{[n]_q!} = \sum_{n=0}^{\infty} \sum_{k=0}^{[\frac{n-2l}{3}]} \sum_{l=0}^{[\frac{n}{2}]} \sum_{p=0}^{\infty} u^p\ v^l\ w^k\ \mathcal{A}_{n-3k-2l,q} \frac{\xi^n}{[l]_q!\,[k]_q!\,[n-3k-2l]_q!}. \tag{2.26} $$

Finally, the right-hand aspect of the previous expression on utilizing the subsequent series rearrangement method [1]:

$$ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \mathbb{A}(m,n) = \sum_{n=0}^{\infty} \sum_{n=0}^{n} \mathbb{A}(m, n-m). \tag{2.27} $$

We obtain

$$\sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w)\frac{\xi^n}{[n]_q!} = \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n-3k-2l}{3}]}\sum_{l=0}^{[\frac{n}{2}]}\sum_{p=0}^{n} u^p\, v^l\, w^k\, \mathcal{A}_{n-3k-2l-p,q}\frac{\xi^n}{[l]_q!\,[k]_q!\,[n-3k-2l-p]_q!}. \tag{2.28}$$

Hence, upon juxtaposing the respective values of ξ from each perspective, we attain the series representation of the 3-variable q-Hermite-Appell polynomials, denoted as 3VqHAP ${}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w)$. This substantiates the claim presented in (2.28). □

3. Operational formalism and determinant form

Operational formalism is pivotal in elucidating the significance of q-special polynomials, particularly in the realm of quantum mechanics. This formalism provides a framework for interpreting mathematical expressions in terms of physical operations or measurements, facilitating a deeper understanding of the physical implications of q-special polynomials. These polynomials often arise as solutions to difference equations with quantum group symmetries, and operational formalism aids in comprehending these symmetries and their ramifications in physical systems. Moreover, in the context of integrable quantum systems, operational methods are essential for studying quantum integrability and analyzing system behaviors. The operational interpretation of q-special polynomials also finds applications in quantum algorithms, quantum statistical mechanics, and non-commutative geometry, enabling insights into quantum phenomena, quantum information processing, the statistical properties of quantum systems, and the geometric properties of non-commutative spaces. Thus, operational formalism serves as a crucial bridge between mathematical formalism and physical intuition, facilitating a comprehensive understanding of q-special polynomials and their role in quantum physics.

Differentiating (1.25) w.r.t. u, v, w, we find the following q-partial differential recurrence relations satisfied by 3VqHAP ${}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w)$:

$$\mathcal{D}_{q,u}\Big[{}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w)\Big] = [n]_q\, {}_{\mathcal{H}}\mathcal{A}_{n-1,q}(u,v,w), \quad n \geq 1, \tag{3.1}$$

$$\mathcal{D}_{q,v}\Big[{}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w)\Big] = [n]_q[n-1]_q\, {}_{\mathcal{H}}\mathcal{A}_{n-2,q}(u,v,w), \quad n \geq 2, \tag{3.2}$$

$$\mathcal{D}_{q,w}\Big[{}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w)\Big] = [n]_q[n-1]_q[n-2]_q\, {}_{\mathcal{H}}\mathcal{A}_{n-3,q}(u,v,w), \quad n \geq 3. \tag{3.3}$$

In view of the above expressions, it is evident that ${}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w)$ are solutions to the expressions:

$$\begin{aligned}\mathcal{D}_{q,v}\Big[{}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w)\Big] &= \mathcal{D}^2_{q,u}\Big[{}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w)\Big],\\ \mathcal{D}_{q,w}\Big[{}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w)\Big] &= \mathcal{D}^3_{q,u}\Big[{}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w)\Big],\end{aligned} \tag{3.4}$$

with subject to initial constraints:

$${}_{\mathcal{H}}\mathcal{A}_{n,q}(u,0,0) = \mathcal{A}_{n,q}(u). \tag{3.5}$$

Therefore from preceding expressions (3.4) and (3.5), it is evident that

$$ {}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w) = \Big(\epsilon_q(v\mathcal{D}^2_{q,u})\epsilon_q(w\mathcal{D}^3_{q,u})\Big)\Big\{\mathcal{A}_{n,q}(u)\Big\}. \tag{3.6} $$

The determinant form plays a crucial role in the study of q-special polynomials by providing a compact and elegant representation of these polynomials. This form encapsulates essential properties such as orthogonality, recurrence relations, and generating functions, facilitating their manipulation and analysis in various mathematical contexts. Moreover, the determinant form serves as a foundation for exploring connections with other mathematical structures, enabling researchers to uncover deeper insights into the underlying principles governing q-special polynomials. Thus, its importance lies in both its practical utility and its role in advancing theoretical understanding within the realm of special function theory.

Keleshteri and Mahmudov [25] delve into the analysis of the determinant representation of q-Appell polynomials. Recognizing the significance of determinant forms in computational and applied contexts, the determinant formulations of the q-special polynomials outlined earlier are rigorously established. Specifically, the determinant definition of the 3VqHAP, denoted as ${}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w)$, is derived through the proof of the following theorem:

Theorem 3.1. *The following determinant form for the 3VqHAP* ${}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w)$ *of degree n holds true:*

$$ {}_{\mathcal{H}}\mathcal{A}_{0,q}(u,v,w) = 1, $$

$$ {}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w) = \frac{(-1)^n}{(\delta_{0,q})^{n+1}} \begin{vmatrix} 1 & \mathcal{H}_{1,q}(u,v,w) & \mathcal{H}_{2,q}(u,v,w) & .... & \mathcal{H}_{n-1,q}(u,v,w) & \mathcal{H}_{n,q}(u,v,w) \\ \delta_{o,q} & \delta_{1,q} & \delta_{2,q} & .... & \delta_{n-1,q} & \delta_{n,q} \\ 0 & \delta_{0,q} & \binom{2}{1}_q \delta_{1,q} & .... & \binom{n-1}{1}_q \delta_{n-2,q} & \binom{n}{1}_q \delta_{n-1,q} \\ 0 & 0 & \delta_{0,q} & .... & \binom{n-1}{2}_q \delta_{n-3,q} & \binom{n}{2}_q \delta_{n-2,q} \\ . & . & . & .... & . & . \\ . & . & . & .... & . & . \\ . & . & . & .... & . & . \\ 0 & 0 & 0 & .... & \delta_{0,q} & \binom{n}{n-1}_q \delta_{1,q} \end{vmatrix}, \tag{3.7} $$

where $n = 1, 2, ...$ *and* $\mathcal{H}_{n,q}(u,v,w)$ $(n = 0, 1, 2, ...)$ *are the q-Hermite polynomials of degree n;* $\delta_{0,q} \neq 0$ *and*

$$ \begin{aligned} \delta_{0,q} &= \frac{1}{\mathcal{A}_{0,q}}, \\ \delta_{n,q} &= -\frac{1}{\mathcal{A}_{0,q}}\Big(\sum_{k=1}^{n} \begin{bmatrix} \xi \\ k \end{bmatrix}_q \mathcal{A}_{k,q}\, \delta_{n-k,q}\Big), \qquad n = 1, 2, ... \end{aligned} \tag{3.8} $$

Proof. Taking $n = 0$ in series representation (2.1) of the 3VqHAP ${}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w)$, we find

$$ {}_{\mathcal{H}}\mathcal{A}_{0,q}(u,v,w) = 1. $$

Since, the determinant form of the q-Appell polynomials $\{\mathcal{A}_{n,q}(u)\}_{n=0}^{\infty}$ [25] is given as:

$$\mathcal{A}_{n,q}(u)=\frac{(-1)^n}{(\delta_{0,q})^{n+1}}\begin{vmatrix}1 & u & u^2 & .... & u^{n-1} & u^n\\ \delta_{o,q} & \delta_{1,q} & \delta_{2,q} & .... & \delta_{n-1,q} & \delta_{n,q}\\ 0 & \delta_{0,q} & \binom{2}{1}_q\delta_{1,q} & .... & \binom{n-1}{1}_q\delta_{n-2,q} & \binom{n}{1}_q\delta_{n-1,q}\\ 0 & 0 & \delta_{0,q} & .... & \binom{n-1}{2}_q\delta_{n-3,q} & \binom{n}{2}_q\delta_{n-2,q}\\ . & . & . & .... & . & .\\ . & . & . & .... & . & .\\ . & . & . & .... & . & .\\ 0 & 0 & 0 & .... & \delta_{0,q} & \binom{n}{n-1}_q\delta_{1,q}\end{vmatrix},$$

where $\delta_{0,q},\delta_{1,q},\delta_{2,q},...,\delta_{n,q}\in\mathbb{R}$, $\delta_{0,q}\neq 0$ and $n=1,2,3,....$

Therefore, expanding the above determinant along the first row, it follows that

$$\mathcal{A}_{n,q}(u)=\frac{(-1)^n}{(\delta_{0,q})^{n+1}}\begin{vmatrix}\delta_{1,q} & \delta_{2,q} & .... & \delta_{n-1,q} & \delta_{n,q}\\ \delta_{0,q} & \binom{2}{1}_q\delta_{1,q} & .... & \binom{n-1}{1}_q\delta_{n-2,q} & \binom{n}{1}_q\delta_{n-1,q}\\ 0 & \delta_{0,q} & .... & \binom{n-1}{2}_q\delta_{n-3,q} & \binom{n}{2}_q\delta_{n-2,q}\\ . & . & .... & . & .\\ . & . & .... & . & .\\ . & . & .... & . & .\\ 0 & 0 & .... & \delta_{0,q} & \binom{n}{n-1}_q\delta_{1,q}\end{vmatrix}$$

$$+\frac{(-1)^{n+1}u}{(\delta_{0,q})^{n+1}}\begin{vmatrix}\delta_{o,q} & \delta_{2,q} & .... & \delta_{n-1,q} & \delta_{n,q}\\ 0 & \binom{2}{1}_q\delta_{1,q} & .... & \binom{n-1}{1}_q\delta_{n-2,q} & \binom{n}{1}_q\delta_{n-1,q}\\ 0 & \delta_{0,q} & .... & \binom{n-1}{2}_q\delta_{n-3,q} & \binom{n}{2}_q\delta_{n-2,q}\\ . & . & .... & . & .\\ . & . & .... & . & .\\ . & . & .... & . & .\\ 0 & 0 & .... & \delta_{0,q} & \binom{n}{n-1}_q\delta_{1,q}\end{vmatrix}$$

$$+\frac{(-1)^{n+2}u^2}{(\delta_{0,q})^{n+1}}\begin{vmatrix} \delta_{o,q} & \delta_{1,q} & .... & \delta_{n-1,q} & \delta_{n,q} \\ 0 & \delta_{0,q} & .... & \binom{n-1}{1}_q\delta_{n-2,q} & \binom{n}{1}_q\delta_{n-1,q} \\ 0 & 0 & .... & \binom{n-1}{2}_q\delta_{n-3,q} & \binom{n}{2}_q\delta_{n-2,q} \\ . & . & .... & . & . \\ . & . & .... & . & . \\ . & . & .... & . & . \\ 0 & 0 & .... & \delta_{0,q} & \binom{n}{n-1}_q\delta_{1,q} \end{vmatrix}$$

$$+...+\frac{(-1)^{2n-1}u^{n-1}}{(\delta_{0,q})^{n+1}}\begin{vmatrix} \delta_{o,q} & \delta_{1,q} & \delta_{2,q} & .... & \delta_{n-2,q} & \delta_{n,q} \\ 0 & \delta_{0,q} & \binom{2}{1}_q\delta_{1,q} & .... & \binom{n-2}{1}_q\delta_{n-3,q} & \binom{n}{1}_q\delta_{n-1,q} \\ 0 & 0 & \delta_{0,q} & .... & \binom{n-2}{2}_q\delta_{n-4,q} & \binom{n}{2}_q\delta_{n-2,q} \\ . & . & . & .... & . & . \\ . & . & . & .... & . & . \\ . & . & . & .... & . & . \\ 0 & 0 & 0 & .... & 0 & \binom{n}{n-1}_q\delta_{1,q} \end{vmatrix}$$

$$+\frac{(-1)^{2n}u^{n}}{(\delta_{0,q})^{n+1}}\begin{vmatrix} \delta_{o,q} & \delta_{1,q} & \delta_{2,q} & .... & \delta_{n-2,q} & \delta_{n-1,q} \\ 0 & \delta_{0,q} & \binom{2}{1}_q\delta_{1,q} & .... & \binom{n-2}{1}_q\delta_{n-3,q} & \binom{n-1}{1}_q\delta_{n-2,q} \\ 0 & 0 & \delta_{0,q} & .... & \binom{n-2}{2}_q\delta_{n-4,q} & \binom{n-1}{2}_q\delta_{n-3,q} \\ . & . & . & .... & . & . \\ . & . & . & .... & . & . \\ . & . & . & .... & . & . \\ 0 & 0 & 0 & .... & 0 & \delta_{0,q} \end{vmatrix}.$$

Again, since each minor is independent of u, therefore replacing $u^1, u^2, ..., u^n$ by $\mathcal{H}_{1,q}(u,v,w), \mathcal{H}_{2,q}(u,v,w), \cdots, \mathcal{H}_{n,q}(u,v,w)$, respectively, and using operational relation (3.6) in the l.h.s. and then combining the terms in the r.h.s., we are led to assertion (3.7). □

4. Recurrence relation and summation formulae

Recurrence relations and summation formulae are fundamental tools in understanding the significance of q-special polynomials. These mathematical relationships provide a systematic way to generate the polynomials, establish their properties, and derive useful identities. Recurrence relations describe how a polynomial of a certain degree relates to those of lower degrees, facilitating efficient computation and recursion algorithms. Meanwhile, summation formulae offer compact representations of q-special polynomials, allowing for simplified expressions and efficient calculations of sums involving these polynomials. Moreover, recurrence relations and summation formulae often embody the underlying symmetries and algebraic structures associated with q-special polynomials, providing insights into their properties and connections to other mathematical frameworks, such as quantum groups and non-commutative geometry. Overall, the study of recurrence relations and summation formulae is crucial for uncovering the rich mathematical structure and applications of q-special polynomials across various fields, including quantum mechanics, statistical physics, and combinatorics.

First, we derive the pure recurrence relation satisfied by the 3VqHAP denoted as ${}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w)$, through the proof of the following theorem:

Theorem 4.1. *The recurrence relation for 3VqHAP ${}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w)$, with $n \geq 2$, is represented in the form:*

$$\begin{aligned}{}_{\mathcal{H}}\mathcal{A}_{n+1,q}(u,v,w) &= \Big(u+\delta_{0,q}\Big){}_{\mathcal{H}}\mathcal{A}_{n,q}(uq,qv,qw)\\ &+\sum_{k=1}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q \delta_{k,q}{}_{\mathcal{H}}\mathcal{A}_{n,q}(uq,qv,qw)+v[n]_q({}_{\mathcal{H}}\mathcal{A}_{n-1,q}(u,v,qw)\\ &+q{}_{\mathcal{H}}\mathcal{A}_{n-1,q}(u,qv,qw))+w[n]_q[n-1]_q({}_{\mathcal{H}}\mathcal{A}_{n-2,q}(u,v,w)\\ &+q{}_{\mathcal{H}}\mathcal{A}_{n-2,q}(u,v,qw)+q^2{}_{\mathcal{H}}\mathcal{A}_{n-2,q}(u,v,q^2w)),\end{aligned} \tag{4.1}$$

where

$$\frac{\mathcal{A}'_q(\xi)}{\mathcal{A}_q(\xi)}=\sum_{k=0}^{\infty}\delta_{k,q}\frac{\xi^k}{[k]_q!}. \tag{4.2}$$

Proof. In consideration of expression (1.17), for q-derivatives, and leveraging the q-derivative of both components of expression (1.23) concerning the parameter ξ, we readily obtain:

$$\sum_{n=0}^{\infty}\mathcal{D}_{q,t}\left[{}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w)\frac{\xi^n}{[n]_q!}\right]=\mathcal{D}_{q,t}\left[\mathcal{A}_q(\xi)\ \epsilon_q(u\xi)\epsilon_q(v\xi^2)\epsilon_q(w\xi^3)\right], \tag{4.3}$$

which can further be expressed as

$$\begin{aligned}\sum_{n=0}^{\infty}\left[n{}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w)\frac{\xi^{n-1}}{[n]_q!}\right] &= \left[\mathcal{A}_q(\xi)\mathcal{D}_{q,t}\Big(\epsilon_q(u\xi)\epsilon_q(v\xi^2)\epsilon_q(w\xi^3)\Big)+\epsilon_q(uq\xi)\epsilon_q(vq\xi^2)\epsilon_q(wq\xi^3)\mathcal{D}_{q,t}\Big(\mathcal{A}_q(\xi)\Big)\right]\\ &= \mathcal{A}_q(\xi)\left[\left(\mathcal{D}_{q,t}\epsilon_q(u\xi)\epsilon_q(vq\xi^2)+\epsilon_q(u\xi)\mathcal{D}_{q,t}\epsilon_q(v\xi^2)\right)\epsilon_q(wq\xi^3)\right.\\ &\left.+\epsilon_q(u\xi)\epsilon_q(v\xi^2)\mathcal{D}_{q,t}\Big(\epsilon_q(w\xi^3)\Big)\right]+\frac{\mathcal{A}'_q(\xi)}{\mathcal{A}_q(\xi)}\Big(\mathcal{A}_q(\xi)\epsilon_q(qu\xi)\epsilon_q(vq\xi^2)\epsilon_q(wq\xi^3)\Big).\end{aligned} \tag{4.4}$$

Thus, in consideration of expression (1.18) in the r.h.s. of the preceding expression and expression (1.13) in the l.h.s. of the preceding expression, it follows that

$$\sum_{n=0}^{\infty}\left[{}_{\mathcal{H}}\mathcal{A}_{n+1,q}(u,v,w)\frac{\xi^n}{[n]_q!}\right]=\Big[u\ \mathcal{A}_q(\xi)\ \epsilon_q(qu\xi)\epsilon_q(qv\xi^2)\epsilon_q(qw\xi^3)+v\xi\ \mathcal{A}_q(\xi)\ \epsilon_q(u\xi)\epsilon_q(v\xi^2)\epsilon_q(qw\xi^3)$$
$$+qv\xi\ \mathcal{A}_q(\xi)\ \epsilon_q(u\xi)\epsilon_q(qv\xi^2)\epsilon_q(qw\xi^3)+w\xi^2\mathcal{A}_q(\xi)\ \epsilon_q(u\xi)\epsilon_q(v\xi^2)\epsilon_q(w\xi^3)$$
$$+qw\xi^2\mathcal{A}_q(\xi)\ \epsilon_q(u\xi)\epsilon_q(v\xi^2)\epsilon_q(qw\xi^3)+q^2w\xi^2\mathcal{A}_q(\xi)\ \epsilon_q(u\xi)\epsilon_q(v\xi^2)\epsilon_q(wq^2\xi^3)$$
$$+\frac{\mathcal{A}'_q(\xi)}{\mathcal{A}_q(\xi)}\mathcal{A}_q(\xi)\epsilon_q(qu\xi)\epsilon_q(vq\xi^2)\epsilon_q(wq\xi^3)\Big],\tag{4.5}$$

and further can be simplified as

$$\sum_{n=0}^{\infty}\left[{}_{\mathcal{H}}\mathcal{A}_{n+1,q}(u,v,w)\frac{\xi^n}{[n]_q!}\right]=\sum_{n=0}^{\infty}\Big[u\ {}_{\mathcal{H}}\mathcal{A}_{n,q}(qu,qv,qw)+v\xi\ {}_{\mathcal{H}}\mathcal{A}_{n+1,q}(u,v,qw)$$
$$+qv\xi\ {}_{\mathcal{H}}\mathcal{A}_{n,q}(u,qv,qw)+w\xi^2{}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w)+qw\xi^2{}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,qw)$$
$$+q^2w\xi^2{}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,q^2w)+\sum_{k=0}^{\infty}\delta_{k,q}{}_{\mathcal{H}}\mathcal{A}_{n,q}(qu,qv,qw)\frac{\xi^k}{[k]_q!}\Big]\frac{\xi^n}{[n]_q!}.\tag{4.6}$$

Comparing the coefficients of like powers of ξ on both aspects of the above expression, the proof of the theorem is asserted. □

Further, differentiating the expression (1.25) partially k-times w.r.t. u,v,w, we propose the succeeding q-partial differential recurrence relations for 3VqHAP ${}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w)$:

$$\mathcal{D}^k_{q,u}\Big[{}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w)\Big]=\frac{[n]_q!}{[n-k]_q!}\ {}_{\mathcal{H}}\mathcal{A}_{n-k,q}(u,v,w),\quad 0\le k\le n,\tag{4.7}$$
$$\mathcal{D}^k_{q,v}\Big[{}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w)\Big]=\frac{[n]_q!}{[n-2k]_q!}\ {}_{\mathcal{H}}\mathcal{A}_{n-2k,q}(u,v,w),\quad 0\le k\le \frac{n}{2},\tag{4.8}$$
$$\mathcal{D}^k_{q,w}\Big[{}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w)\Big]=\frac{[n]_q!}{[n-3k]_q!}\ {}_{\mathcal{H}}\mathcal{A}_{n-3k,q}(u,v,w),\quad 0\le k\le \frac{n}{3}.\tag{4.9}$$

Next, we establish a few summation formulae satisfied by 3VqHAP ${}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w)$ in the form of succeeding demonstrations:

Theorem 4.2. *For 3VqHAP ${}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w)$, the succeeding summation formulae holds true:*

$${}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v)=[n]_q!\ \sum_{k=0}^{[\frac{n}{3}]}\frac{{}_{\mathcal{H}}\mathcal{A}_{n-3k,q}(u,v,w)q^{\binom{k}{2}}(-w)^k}{[n-3k]_q![k]_q!}.\tag{4.10}$$

Proof. In consideration of expression (1.11), it is evident that

$$\mathcal{A}_q(\xi)\ \epsilon_q(u\xi)\epsilon_q(v\xi^2)=\mathcal{A}_q(\xi)\ \epsilon_q(u\xi)\epsilon_q(v\xi^2)\epsilon_q(w\xi^3)\mathcal{E}_q(-w\xi^3),\tag{4.11}$$

thus, on utilizing expressions (1.11), (1.25), and (1.26), it follows that

$$\sum_{n=0}^{\infty}{}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v)\frac{\xi^n}{[n]_q!}=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{{}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w)q^{\binom{k}{2}}(-w)^k\xi^{n+3k}}{[n]_q![k]_q!},\tag{4.12}$$

which, in view of expression (2.24) in the r.h.s. of thepreceding, expression gives

$$\sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v)\frac{\xi^n}{[n]_q!} = \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{3}]} \frac{{}_{\mathcal{H}}\mathcal{A}_{n-3k,q}(u,v,w)q^{\binom{k}{2}}(-w)^k\xi^n}{[n-3k]_q![k]_q!}. \tag{4.13}$$

Thus, on comparison of like exponents of ξ on both sides of the preceding expression, assertion (4.10) is established. □

Theorem 4.3. *For 3VqHAP ${}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w)$, the succeeding summation formulae holds true:*

$$ {}_{\mathcal{H}}\mathcal{A}_{3p,q}(u,v) = [3p]_q! \sum_{k=0}^{[\frac{3p}{2}]} \frac{{}_{\mathcal{H}}\mathcal{A}_{3p-2k,q}(u,v,w)q^{\binom{k}{2}}(-v)^k}{[3p-2k]_q![k]_q!}. \tag{4.14}$$

Proof. In consideration of expression (1.11), it is evident that

$$\mathcal{A}_q(\xi)\ \epsilon_q(u\xi)\epsilon_q(w\xi^3) = \mathcal{A}_q(\xi)\ \epsilon_q(u\xi)\epsilon_q(v\xi^2)\epsilon_q(w\xi^3)\mathcal{E}_q(-y\xi^2), \tag{4.15}$$

thus, on utilizing expressions (1.11), (1.25), and (1.26), it follows that

$$\sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{A}_{n,q}(u,w)\frac{\xi^n}{[n]_q!} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{{}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w)q^{\binom{k}{2}}(-v)^k\xi^{n+2k}}{[n]_q![k]_q!}, \tag{4.16}$$

which, in view of expression (2.24) in the r.h.s. of the preceding expression gives

$$\sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v)\frac{\xi^n}{[n]_q!} = \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{2}]} \frac{{}_{\mathcal{H}}\mathcal{A}_{n-2k,q}(u,v,w)q^{\binom{k}{2}}(-v)^k\xi^n}{[n-2k]_q![k]_q!}. \tag{4.17}$$

Thus, on comparison of like exponents of ξ on both sides of the preceding expression, assertion (4.14) is established. □

Corollary 4.1. *On usage of expression* (3.1) *in expressions* (4.10) *and* (4.14)*, the 3VqHAP ${}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w)$, holds the succeeding summation formulae:*

$$ {}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v) = [n]_q! \sum_{k=0}^{[\frac{n}{3}]} \frac{\mathcal{D}_{q,u}{}_{\mathcal{H}}\mathcal{A}_{n+1-3k,q}(u,v,w)q^{\binom{k}{2}}(-w)^k}{[n+1-3k]_q![k]_q!} \tag{4.18}$$

and

$$ {}_{\mathcal{H}}\mathcal{A}_{3p,q}(u,v) = [3p]_q! \sum_{k=0}^{[\frac{3p}{2}]} \frac{\mathcal{D}_{q,u}{}_{\mathcal{H}}\mathcal{A}_{3p+1-2k,q}(u,v,w)q^{\binom{k}{2}}(-v)^k}{[3p+1-2k]_q![k]_q!}. \tag{4.19}$$

Corollary 4.2. *On usage of expression* (3.2) *in expressions* (4.10) *and* (4.14)*, the 3VqHAP ${}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w)$, holds the succeeding summation formulae:*

$$ {}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v) = [n]_q! \sum_{k=0}^{[\frac{n}{3}]} \frac{\mathcal{D}_{q,v}{}_{\mathcal{H}}\mathcal{A}_{n+2-3k,q}(u,v,w)q^{\binom{k}{2}}(-w)^k}{[n+2-3k]_q![k]_q!} \tag{4.20}$$

and

$$ {}_{\mathcal{H}}\mathcal{A}_{3p,q}(u,v) = [3p]_q! \sum_{k=0}^{[\frac{3p}{2}]} \frac{\mathcal{D}_{q,v}{}_{\mathcal{H}}\mathcal{A}_{3p+2-2k,q}(u,v,w)q^{\binom{k}{2}}(-v)^k}{[3p+2-2k]_q![k]_q!}. \tag{4.21}$$

Corollary 4.3. *On usage of expression* (3.3) *in expressions* (4.10) *and* (4.14)*, the 3VqHAP* ${}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w)$*, holds the succeeding summation formulae:*

$$
{}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v) = [n]_q! \sum_{k=0}^{[\frac{n}{3}]} \frac{\mathcal{D}_{q,w}{}_{\mathcal{H}}\mathcal{A}_{n+3-3k,q}(u,v,w) q^{\binom{k}{2}} (-w)^k}{[n+3-3k]_q! [k]_q!} \tag{4.22}
$$

and

$$
{}_{\mathcal{H}}\mathcal{A}_{3p,q}(u,v) = [3p]_q! \sum_{k=0}^{[\frac{3p}{2}]} \frac{\mathcal{D}_{q,w}{}_{\mathcal{H}}\mathcal{A}_{3p+3-2k,q}(u,v,w) q^{\binom{k}{2}} (-v)^k}{[3p+3-2k]_q! [k]_q!}. \tag{4.23}
$$

Theorem 4.4. *For* $u \to u_1 + u_2$ *in expression* (1.25)*, the 3VqHAP* ${}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w)$ *fulfills the succeeding formulae:*

$$
{}_{\mathcal{H}}\mathcal{A}_{n,q}(u_1+u_2,v,w) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q {}_{\mathcal{H}}\mathcal{A}_{k,q}(u_1,v,w) u_2^{n-k}. \tag{4.24}
$$

Proof. Substituting $u \to u_1 + u_2$ in expression (1.25), it follows that

$$
\mathcal{A}_q(\xi)\ \epsilon_q((u_1+u_2)\xi)\epsilon_q(v\xi^2)\epsilon_q(w\xi^3) = \epsilon_q(u_2\xi)\mathcal{A}_q(\xi)\ \epsilon_q(u_1\xi)\epsilon_q(v\xi^2)\epsilon_q(w\xi^3),
$$

which further can be expressed as

$$
\sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{A}_{n,q}(u_1+u_2,v,w) \frac{\xi^n}{[n]_q!} = \sum_{n=0}^{\infty} u_2^n \frac{\xi^n}{[n]_q!} \sum_{k=0}^{\infty} {}_{\mathcal{H}}\mathcal{A}_{k,q}(u_1,v,w) \frac{\xi^k}{[k]_q!}.
$$

Therefore, using the Cauchy product rule in the r.h.s. of the preceding expression, we have

$$
\sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{A}_{n,q}(u_1+u_2,v,w) \frac{\xi^n}{[n]_q!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q {}_{\mathcal{H}}\mathcal{A}_{k,q}(u_1,v,w) u_2^{n-k} \frac{\xi^n}{[n]_q!}.
$$

Thus, on comparison of coefficients of the same powers of $\frac{\xi^n}{[n]_q!}$ on both sides of the preceding expression, assertion (4.24) is established. □

Theorem 4.5. *For* $v \to v_1 + v_2$ *in expression* (1.25)*, the 3VqHAP* ${}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w)$ *fulfills the succeeding formulae:*

$$
{}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v_1+v_2,w) = [n]_q! \sum_{k=0}^{[\frac{n}{2}]} \frac{{}_{\mathcal{H}}\mathcal{A}_{n-2k,q}(u,v_1,w) v_2^k}{[k]_q! [n-2k]_q!}. \tag{4.25}
$$

Proof. Substituting $v \to v_1 + v_2$ in expression (1.25), it follows that

$$
\mathcal{A}_q(\xi)\ \epsilon_q(u\xi)\epsilon_q((v_1+v_2)\xi^2)\epsilon_q(w\xi^3) = \epsilon_q(v_2\xi^2)\mathcal{A}_q(\xi)\ \epsilon_q(u\xi)\epsilon_q(v_2\xi^2)\epsilon_q(w\xi^3),
$$

which further can be expressed as

$$
\sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v_1+v_2,w) \frac{\xi^n}{[n]_q!} = \sum_{k=0}^{\infty} v_2^k \frac{\xi^{2k}}{[k]_q!} \sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v_1,w) \frac{\xi^n}{[n]_q!}.
$$

Therefore, using the Cauchy product rule in the r.h.s. of the preceding expression, we have

$$\sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{A}_{n,q}(u, v_1 + v_2, w)\frac{\xi^n}{[n]_q!} = \sum_{n=0}^{\infty}[n]_q! \sum_{k=0}^{[\frac{n}{2}]} {}_{\mathcal{H}}\mathcal{A}_{n-2k,q}(u, v_1, w)v_2^k \frac{\xi^n}{[k]_q![n-2k]_q![n]_q!}.$$

Thus, on comparison of coefficients of the same powers of $\frac{\xi^n}{[n]_q!}$ on both sides of the preceding expression, assertion (4.25) is established. □

Theorem 4.6. *For $w \to w_1 + w_2$ in expression* (1.25)*, the 3VqHAP ${}_{\mathcal{H}}\mathcal{A}_{n,q}(u, v, w)$ fulfills the succeeding formulae:*

$$ {}_{\mathcal{H}}\mathcal{A}_{n,q}(u, v, w_1 + w_2) = [n]_q! \sum_{k=0}^{[\frac{n}{3}]} \frac{{}_{\mathcal{H}}\mathcal{A}_{n-3k,q}(u, v, w_1)w_2^k}{[k]_q![n-3k]_q!}. \tag{4.26}$$

Proof. Substituting $w \to w_1 + w_2$ in expression (1.25), it follows that

$$\mathcal{A}_q(\xi)\ \epsilon_q(u\xi)\epsilon_q(v\xi^2)\epsilon_q((w_1 + w_2)\xi^3) = \epsilon_q(w_2\xi^3)\mathcal{A}_q(\xi)\ \epsilon_q(u\xi)\epsilon_q(v\xi^2)\epsilon_q(w_1\xi^3),$$

which can further be expressed as

$$\sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{A}_{n,q}(u, v, w_1 + w_2)\frac{\xi^n}{[n]_q!} = \sum_{k=0}^{\infty} w_2^k \frac{\xi^{3k}}{[k]_q!} \sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{A}_{n,q}(u, v, w_1)\frac{\xi^n}{[n]_q!}.$$

Therefore, using the Cauchy product rule in the r.h.s. of the preceding expression, we have

$$\sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{A}_{n,q}(u, v, w_1 + w_2)\frac{\xi^n}{[n]_q!} = \sum_{n=0}^{\infty}[n]_q! \sum_{k=0}^{[\frac{n}{3}]} {}_{\mathcal{H}}\mathcal{A}_{n-3k,q}(u, v, w_1)w_2^k \frac{\xi^n}{[k]_q![n-3k]_q![n]_q!}.$$

Thus, on comparison of coefficients of the same powers of $\frac{\xi^n}{[n]_q!}$ on both sides of the preceding expression, assertion (4.26) is established. □

5. Examples

Below are some of the members of the q-Appell family provided in Table 1. For every member of the q-Appell family, there exists a unique special polynomial within the 3VqHAP family. The derivation of the generating function and series definition for these members of the 3VqHAP family entails selecting an appropriate generating function, $\mathcal{A}_q(\xi)$, as detailed in expression (1.25). In Table 2, we showcase these members alongside their respective notations, names, generating functions, and series definitions.

Furthermore, the parameters linked with the members of the q-Hermite Appell polynomials family are established. Setting $u = v = w = 0$ in Eq (1.25) yields the resulting series definition for the 3VqHA numbers: ${}_{\mathcal{H}}\mathcal{A}_{n,q} := {}_{\mathcal{H}}\mathcal{A}_{n,q}(0, 0, 0)$ is obtained:

$$ {}_{\mathcal{H}}\mathcal{A}_{n,q} = \sum_{k=0}^{n} \binom{n}{k}_q \mathcal{A}_{k,q}\mathcal{H}_{n-k,q}, \tag{5.1}$$

where $\mathcal{H}_{n,q}$ denotes the q-Hermite numbers.

Table 1. Certain members belonging to the q-Appell family.

S. No.	$\mathcal{A}_q(\xi)$	Name of the q-special polynomial and related number	Generating function	Series definition
I.	$\left(\frac{\xi}{\epsilon_q(\xi)-1}\right)$	q-Bernoulli polynomials and numbers [26, 27]	$\left(\frac{\xi}{\epsilon_q(\xi)-1}\right)\epsilon_q(x\xi)=\sum_{n=0}^{\infty}\mathcal{B}_{n,q}(u,v,w)\frac{\xi^n}{[n]_q!}$ $\left(\frac{\xi}{\epsilon_q(\xi)-1}\right)=\sum_{n=0}^{\infty}\mathcal{B}_{n,q}\frac{\xi^n}{[n]_q!}$ $\mathcal{B}_{n,q}:=\mathcal{B}_{n,q}(0)$	$\mathcal{B}_{n,q}(u,v,w)=\sum_{k=0}^{n}\binom{n}{k}_q\mathcal{B}_{k,q}\ x^{n-k}$
II.	$\left(\frac{2}{\epsilon_q(\xi)+1}\right)$	q-Euler polynomials and numbers [26, 28]	$\left(\frac{2}{\epsilon_q(\xi)+1}\right)\epsilon_q(x\xi)=\sum_{n=0}^{\infty}\epsilon_{n,q}(u,v,w)\frac{\xi^n}{[n]_q!}$ $\left(\frac{2}{\epsilon_q(\xi)+1}\right)=\sum_{n=0}^{\infty}\epsilon_{n,q}\frac{\xi^n}{[n]_q!}$ $\epsilon_{n,q}:=\epsilon_{n,q}(0)$	$\epsilon_{n,q}(u,v,w)=\sum_{k=0}^{n}\binom{n}{k}_q\epsilon_{k,q}x^{n-k}$
III.	$\left(\frac{2\xi}{\epsilon_q(\xi)+1}\right)$	q-Genocchi polynomials and numbers [15, 28]	$\left(\frac{2\xi}{\epsilon_q(\xi)+1}\right)\epsilon_q(x\xi)=\sum_{n=0}^{\infty}\mathcal{G}_{n,q}(u,v,w)\frac{\xi^n}{[n]_q!}$ $\left(\frac{2\xi}{\epsilon_q(\xi)+1}\right)=\sum_{n=0}^{\infty}\mathcal{G}_{n,q}\frac{\xi^n}{[n]_q!}$ $\mathcal{G}_{n,q}:=\mathcal{G}_{n,q}(0)$	$\mathcal{G}_{n,q}(u,v,w)=\sum_{k=0}^{n}\binom{n}{k}_q\mathcal{G}_{k,q}x^{n-k}$

Table 2. Specific members within the 3VqHAP ${}_{\mathcal{H}}\mathcal{A}_{n,q}(u,v,w)$ family.

S. No.	$\mathcal{A}_q(\xi)$	Notation and name of the resultant member	Generating function	Series definition
I.	$\left(\frac{\xi}{\epsilon_q(\xi)-1}\right)$	${}_{\mathcal{H}}\mathcal{B}_{n,q}(u,v,w):=$ 3VqH Bernoulli polynomials (3VqHBP)	$\left(\frac{\xi}{\epsilon_q(\xi)-1}\right)\epsilon_q(u\xi)\epsilon_q(v\xi^2)\epsilon_q(w\xi^3)$ $=\sum_{n=0}^{\infty}{}_{\mathcal{H}}\mathcal{B}_{n,q}(u,v,w)\frac{\xi^n}{[n]_q!}$	${}_{\mathcal{H}}\mathcal{B}_{n,q}(u,v,w)=\sum_{k=0}^{n}\binom{n}{k}_q\mathcal{B}_{k,q}\mathcal{H}_{n-k,q}(u,v,w)$
II.	$\left(\frac{2}{\epsilon_q(\xi)+1}\right)$	${}_{\mathcal{H}}\mathcal{E}_{n,q}(u,v,w):=$ 3VqH Euler polynomials (3VqHEP)	$\left(\frac{2}{\epsilon_q(\xi)+1}\right)\epsilon_q(u\xi)\epsilon_q(v\xi^2)\epsilon_q(w\xi^3)$ $=\sum_{n=0}^{\infty}{}_{\mathcal{H}}\mathcal{E}_{n,q}(u,v,w)\frac{\xi^n}{[n]_q!}$	${}_{\mathcal{H}}\mathcal{E}_{n,q}(u,v,w)=\sum_{k=0}^{n}\binom{n}{k}_q\mathcal{E}_{k,q}\mathcal{H}_{n,q}(u,v,w)$
III.	$\left(\frac{2\xi}{\epsilon_q(\xi)+1}\right)$	${}_{\mathcal{H}}\mathcal{G}_{n,q}(u,v,w):=$ 3VqH Genocchi polynomials (3VqHGP)	$\left(\frac{2\xi}{\epsilon_q(\xi)+1}\right)\epsilon_q(u\xi)\epsilon_q(v\xi^2)\epsilon_q(w\xi^3)$ $=\sum_{n=0}^{\infty}{}_{\mathcal{H}}\mathcal{G}_{n,q}(u,v,w)\frac{\xi^n}{[n]_q!}$	${}_{\mathcal{H}}\mathcal{G}_{n,q}(u,v,w)=\sum_{k=0}^{n}\binom{n}{k}_q\mathcal{G}_{k,q}{}_{\mathcal{H}}\mathcal{G}_{n-k,q}(u,v,w)$

Next, the numbers related to the 3VqHBP ${}_{\mathcal{H}}\mathcal{B}_{n,q}(u,v,w)$, 3VqHEP ${}_{\mathcal{H}}\epsilon_{n,k}(u,v,w)$ and 3VqHGP ${}_{\mathcal{H}}\mathcal{G}_{n,q}(u,v,w)$ are obtained. Taking $u = v = w = 0$ in the series definitions of the 3VqHBP ${}_{\mathcal{H}}\mathcal{B}_{n,q}(u,v,w)$, 3VqHEP ${}_{\mathcal{H}}\mathcal{E}_{n,k}(u,v,w)$, and 3VqHGP ${}_{\mathcal{H}}\mathcal{G}_{n,q}(u,v,w)$ provided in Table 2 and concerning the notations outlined in Table 1, the $3Vq$-Hermite Bernoulli, $3Vq$-Hermite Euler, and $3Vq$-Hermite Genocchi numbers are derived. The tabulated values for these numbers are presented in Table 3.

Table 3. Numbers associated with specific members of the $3Vq$-Hermite Appell family.

S. No.	Notation and name of the resultant numbers	Series definition
I.	${}_{\mathcal{H}}\mathcal{B}_{n,q} := {}_{\mathcal{H}}\mathcal{B}_{n,q}(0,0,0)$ $3Vq$-Hermite Bernoulli numbers	${}_{\mathcal{H}}\mathcal{B}_{n,q} = \sum_{k=0}^{n} \binom{n}{k}_q \mathcal{B}_{k,q}\mathcal{H}_{n-k,q}$
II.	${}_{\mathcal{H}}\mathcal{E}_{n,q} := {}_{\mathcal{H}}\mathcal{E}_{n,q}(0,0,0)$ $3Vq$-Hermite Euler numbers	${}_{\mathcal{H}}\mathcal{E}_{n,q} = \sum_{k=0}^{n} \binom{n}{k}_q \mathcal{E}_{k,q}\mathcal{H}_{n-k,q}$
III.	${}_{\mathcal{H}}\mathcal{G}_{n,q} := {}_{\mathcal{H}}\mathcal{G}_{n,q}(0,0,0)$ $3Vq$-Hermite Genocchi numbers	${}_{\mathcal{H}}\mathcal{G}_{n,q} = \sum_{k=0}^{n} \binom{n}{k}_q \mathcal{G}_{k,q}\mathcal{H}_{n-k,q}$

The following are some illustrative examples showing that there exist $3VqHBP$, $3VqHEP$ and $3VqHGP$ and their respective graphs, see Figures 1–3.

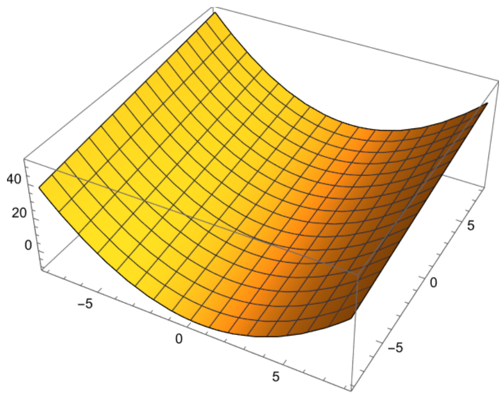

Figure 1. ${}_{\mathcal{H}}\mathcal{B}_{2,0.5}(u,v,w)$.

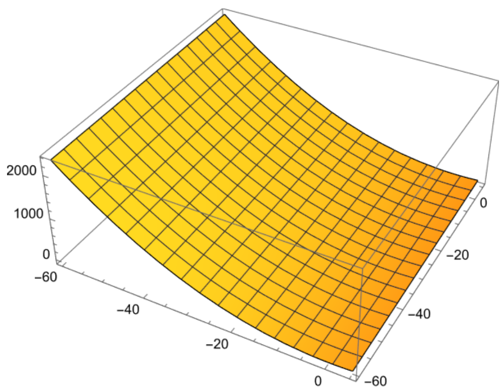

Figure 2. ${}_{\mathcal{H}}\mathcal{E}_{2,0.5}(u,v,w)$.

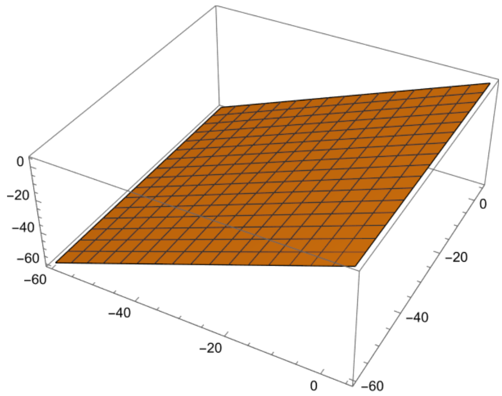

Figure 3. ${}_{\mathcal{H}}\mathcal{G}_{2,0.5}(u, v, w)$.

Further, we will show the first few polynomials of $3VqHBP$ ${}_{\mathcal{H}}\mathcal{B}_{n,q}(u, v, w)$ as:

$$\begin{aligned}
{}_{\mathcal{H}}\mathcal{B}_{0,q}(u, v, w) &= 1,\\
{}_{\mathcal{H}}\mathcal{B}_{1,q}(u, v, w) &= u - \frac{q-1}{q^2-1},\\
{}_{\mathcal{H}}\mathcal{B}_{2,q}(u, v, w) &= v + \frac{(q-1)^2}{(q^2-1)^2} - \frac{u(q-1)}{(q^2-1)} + \frac{u^2(q-1)}{(q^2-1)} - \frac{(q-1)^2}{(q^2-1)(q^3-1)}.
\end{aligned}$$

3VqHEP ${}_{\mathcal{H}}\mathcal{E}_{n,q}(u, v, w)$ as:

$$\begin{aligned}
{}_{\mathcal{H}}\mathcal{E}_{0,q}(u, v, w) &= 1,\\
{}_{\mathcal{H}}\mathcal{E}_{1,q}(u, v, w) &= u - \frac{1}{2},\\
{}_{\mathcal{H}}\mathcal{E}_{2,q}(u, v, w) &= v - \frac{u}{2} - \frac{1}{2(q+1)} + \frac{u^2}{(q+1)} + \frac{1}{4},\\
{}_{\mathcal{H}}\mathcal{E}_{3,q}(u, v, w) &= w - \frac{v}{2} + \frac{1}{q+1} + 2u\left(\frac{v}{2} - \frac{1}{4(q+1)} + \frac{1}{8}\right) - \frac{u^2}{2(q+1)} - \frac{1+u^3}{(q+1)(q^2+q+1)} - \frac{1}{8},
\end{aligned}$$

and for the 3VqHGP ${}_{\mathcal{H}}\mathcal{G}_{n,q}(u, v, w)$ as:

$$\begin{aligned}
{}_{\mathcal{H}}\mathcal{G}_{0,q}(u, v, w) &= 0,\\
{}_{\mathcal{H}}\mathcal{G}_{1,q}(u, v, w) &= 1,\\
{}_{\mathcal{H}}\mathcal{G}_{2,q}(u, v, w) &= u - \frac{1}{2},\\
{}_{\mathcal{H}}\mathcal{G}_{3,q}(u, v, w) &= v - \frac{u}{2} - \frac{q-1}{2(q^2-1)} + \frac{u^2(q-1)}{(q^2-1)} + \frac{1}{4}.
\end{aligned}$$

Next, the determinant forms for 3VqHBP, 3VqHEP, and 3VqHGP are established by using the determinant form of 3VqHAP given by expression (3.7):

Remark 5.1. *Given that the polynomials ${}_{\mathcal{H}}\mathcal{B}_{n,q}(u, v, w)$, ${}_{\mathcal{H}}\mathcal{E}_{n,q}(u, v, w)$, and ${}_{\mathcal{H}}\mathcal{G}_{n,q}(u, v, w)$ delineated in Table 2 are specific members of the 3VqHAP family ${}_{\mathcal{H}}\mathcal{A}_{n,q}(u, v, w)$, it follows that by appropriately* *selecting coefficients $\delta_{0,q}$ and $\delta_{i,q}$ (where $i = 1, 2, ..., n$) in the determinant definition of the 3VqHAP ${}_{\mathcal{H}}\mathcal{A}_{n,q}(u, v, w)$, we can derive the determinant definitions of the 3VqHBP ${}_{\mathcal{H}}\mathcal{B}_{n,q}(u, v, w)$, 3VqHEP ${}_{\mathcal{H}}\mathcal{E}_{n,q}(u, v, w)$, and 3VqHGP ${}_{\mathcal{H}}\mathcal{G}_{n,q}(u, v, w)$. Setting $\delta_{0,q} = 1$ and $\delta_{i,q} = \frac{1}{[i+1]_q}$ (for $i = 1, 2, ..., n$) in expression* (3.7)*, we arrive at the subsequent determinant definition of the 3VqHBP ${}_{\mathcal{H}}\mathcal{B}_{n,q}(u, v, w)$:*

Definition 3.1. *The 3VqHBP ${}_{\mathcal{H}}\mathcal{B}_{n,q}(u, v, w)$ of degree n are defined by:*

$$
\begin{aligned}
&{}_{\mathcal{H}}\mathcal{B}_{0,q}(u, v, w) = 1,\\
&{}_{\mathcal{H}}\mathcal{B}_{n,q}(u, v, w) = (-1)^n \begin{vmatrix}
1 & \mathcal{H}_{1,q}(u, v, w) & \mathcal{H}_{2,q}(u, v, w) & ... & \mathcal{H}_{n-1,q}(u, v, w) & \mathcal{H}_{n,q}(u, v, w)\\
1 & \frac{1}{[2]_q} & \frac{1}{[3]_q} & ... & \frac{1}{[n]_q} & \frac{1}{[n+1]_q}\\
0 & 1 & \binom{2}{1}_q \frac{1}{[2]_q} & ... & \binom{n-1}{1}_q \frac{1}{[n-1]_q} & \binom{n}{1}_q \frac{1}{[n]_q}\\
0 & 0 & 1 & ... & \binom{n-1}{2}_q \frac{1}{[n-2]_q} & \binom{n}{2}_q \frac{1}{[n-1]_q}\\
. & . & . & ... & . & .\\
. & . & . & ... & . & .\\
. & . & . & ... & . & .\\
0 & 0 & 0 & ... & 1 & \binom{n}{n-1}_q \frac{1}{[2]_q}
\end{vmatrix}, \qquad (5.2)\\
&n = 1, 2, ...,
\end{aligned}
$$

where $\mathcal{H}_{n,q}(u, v, w)$ $(n = 0, 1, 2, ...)$ are the 3VqHP of degree n.

Next, taking $\delta_{0,q} = 1$ and $\delta_{i,q} = \frac{1}{2} (i = 1, 2, ..., n)$ in expression (3.7), the following determinant definition of the 3VqHEP ${}_{\mathcal{H}}\mathcal{E}_{n,q}(u, v, w)$ is obtained:

Definition 3.2. *The 3VqHEP ${}_{\mathcal{H}}\mathcal{E}_{n,q}(u, v, w)$ of degree n are defined by:*

$$
\begin{aligned}
&{}_{\mathcal{H}}\mathcal{E}_{0,q}(u, v, w) = 1,\\
&{}_{\mathcal{H}}\mathcal{E}_{n,q}(u, v, w) = (-1)^n \begin{vmatrix}
1 & \mathcal{H}_{1,q}(u, v, w) & \mathcal{H}_{2,q}(u, v, w) & ... & \mathcal{H}_{n-1,q}(u, v, w) & \mathcal{H}_{n,q}(u, v, w)\\
1 & \frac{1}{2} & \frac{1}{2} & ... & \frac{1}{2} & \frac{1}{2}\\
0 & 1 & \binom{2}{1}_q \frac{1}{2} & ... & \binom{n-1}{1}_q \frac{1}{2} & \binom{n}{1}_q \frac{1}{2}\\
0 & 0 & 1 & ... & \binom{n-1}{2}_q \frac{1}{2} & \binom{n}{2}_q \frac{1}{2}\\
. & . & . & ... & . & .\\
. & . & . & ... & . & .\\
. & . & . & ... & . & .\\
0 & 0 & 0 & ... & 1 & \binom{n}{n-1}_q \frac{1}{2}
\end{vmatrix}, \qquad (5.3)\\
&n = 1, 2, ...,
\end{aligned}
$$

where $\mathcal{H}_{n,q}(u, v, w)$ $(n = 0, 1, 2, ...)$ are the 3VqHP of degree n.

Further, taking $\delta_{0,q} = 1$ and $\delta_{i,q} = \frac{1}{2[i+1]_q} (i = 1, 2, ..., n)$ in expression (3.7), the following determinant definition of the 3VqHGP ${}_{\mathcal{H}}\mathcal{G}_{n,q}(u, v, w)$ is obtained:

Definition 3.3. *The 3VqHGP ${}_{\mathcal{H}}\mathcal{G}_{n,q}(u,v,w)$ of degree n are defined by:*

$$
{}_{\mathcal{H}}\mathcal{G}_{0,q}(u,v,w) = 1,
$$

$$
{}_{\mathcal{H}}\mathcal{G}_{n,q}(u,v,w) = (-1)^n \begin{vmatrix}
1 & \mathcal{H}_{1,q}(u,v,w) & \mathcal{H}_{2,q}(u,v,w) & \dots & \mathcal{H}_{n-1,q}(u,v,w) & \mathcal{H}_{n,q}(u,v,w) \\
1 & \frac{1}{2[2]_q} & \frac{1}{2[3]_q} & \dots & \frac{1}{2[n]_q} & \frac{1}{2[n+1]_q} \\
0 & 1 & \binom{2}{1}_q \frac{1}{2[2]_q} & \dots & \binom{n-1}{1}_q \frac{1}{2[n-1]_q} & \binom{n}{1}_q \frac{1}{2[n]_q} \\
0 & 0 & 1 & \dots & \binom{n-1}{2}_q \frac{1}{2[n-2]_q} & \binom{n}{2}_q \frac{1}{2[n-1]_q} \\
. & . & . & \dots & . & . \\
. & . & . & \dots & . & . \\
. & . & . & \dots & . & . \\
0 & 0 & 0 & \dots & 1 & \binom{n}{n-1}_q \frac{1}{2[2]_q}
\end{vmatrix}, \tag{5.4}
$$

$$
n = 1, 2, \dots,
$$

where $\mathcal{H}_{n,q}(u,v,w)$ $(n = 0, 1, 2, \dots)$ are the 3VqHP of degree n.

Remark 5.2. *The determinant definitions of the 3VqHBN $\mathcal{HB}_{n,q}$, 3VqHEN $\mathcal{HE}_{n,q}$, and 3VqHGN $\mathcal{HG}_{n,q}$ can be obtained by setting $u = v = w = 0$ in expressions* (3.7) *and using the respective definitions provided in Definitions (3.1)–(3.3), along with appropriate notations from Table 3 (I–III).*

Similarly, additional results such as recurrence relations, operational formalism, and summation formulae for these members of 3VqHAP and their corresponding numbers can be established.

6. Conclusions

In the realm of specialized functions, the allure of q-calculus beckons to many scholars, captivating them with its prowess in shaping models of "quantum computing, non-commutative probability, combinatorics, functional analysis, mathematical physics, approximation theory, and beyond". Moreover, the recent revelation of the q-Hermite polynomials' profound utility in realms such as non-commutative probability, quantum mechanics, and combinatorial domains has illuminated new paths of inquiry and application.

The classical three-variable Hermite polynomials, renowned for their properties, have long been stalwarts in navigating the challenges of charged-beam transport in traditional mechanics. Likewise, their role in the intricate calculations of quantum-phase-space mechanics is unmistakable, further underscored by the extensive utilization of umbral techniques to unravel their intricacies.

In this exposition, we unveil a tapestry of novel features pertaining to the three-variable q-Hermite polynomials. Through meticulous exploration, we unveil their generating function, series definitions, and recurrence relations, along with delving into the realm of q-differential equations, summation techniques, and operational formalisms.

As our narrative unfolds, we present a panorama of applications in Sections 2–4, each unveiling new facets of the 3VqHAP. These revelations promise to pave the way for groundbreaking expressions intertwined with q-special functions and their methodologies, alongside the emergence of hybrid polynomials from diverse classes, igniting the trajectory of future inquiries.

Author contributions

All authors of this article have been contributed equally. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

Prof. Clemente Cesarano is the Guest Editor of Special Issue "Special functions and related applications" for AIMS Mathematics. Prof. Clemente Cesarano was not involved in the editorial review and the decision to publish this article.

The authors declare no competing interests.

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