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Research article

Some new identities for colored partitions with parts in multiples of 4

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Abstract: To start, we provided the deformations of certain specific modular equations derived from Ramanujan's notebook, Part III, by using the complete square formula. Subsequently, we established fifteen identities for partitions with distinct colors that arose from these deformed modular equations in the theory of modular equations. Throughout our investigation, we found that the fifteen identities have parts that were in multiples of 4.

Keywords: colored partitions; modular equations; theta function identities
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1. Introduction

Farkas and Kra [13] initiated a pioneering investigation into partition identities derived from theta-function identities and established an elegant theorem about colored partitions. Hirschhorn [15] provided a basic q-series proof of the theorem in Farkas and Kra's paper, while the reviewer of [15] noted that the partition identity was essentially equivalent to a modular equation of degree 7, as demonstrated by Guetzlaff [14].

Berndt [9] observed that a variety of intriguing partition identities could be derived from modular equations of Russell-type, which were initially extensively studied by Schröter [23], and later by Ramanujan and Russell in their respective works [18–20]. Baruah and Berndt furthered this study [4,5] by inspecting additional modular equations of Ramanujan from his notebooks [18] and displayed how these identities can be reformulated into intriguing partition identities. The search for combinatorial proofs of these identities naturally arises as a significant question. For further insights, readers are encouraged to refer to the works of Warnaar [24] and Kim [16], where they offer combinatorial methods for classes of these partition identities.

Sandon and Zanello [21, 22], building upon Kim's perspective, uncovered a variety of intriguing partition identities, although they did not formally prove them. These conjectures of Sandon and Zanello have been proven using modular equations by Berndt and Zhou in the works [10, 11]. Almost simultaneously, many of the conjectures of Sandon and Zanello have been established by Baruah and Boruah [6]. Soon after, Zhou [25] considered the method of reciprocation to discover modular equations and establish some more complicated colored partition identities. In 2021, Kim [17] expanded upon Warnaar's general identity, which implies the modular equations of degrees 3 and 7, and derived many partition identities from the generalization. The paper [26] presents several novel modular equations of composite degrees and degree 7, along with new partition identities.

We start by giving the definition of a modular equation. The complete elliptic integral of the first kind is defined for $|k| < 1$ by

$$K := K(k) := \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}.$$

The number k is called the modulus. The complementary modulus k' is defined by $k' = \sqrt{1 - k^2}$. Set $K' = K(k')$. Expanding the integrand to a binomial series and integrating term-wise, one finds that

$$K = \frac{\pi}{2} {}_2F_1\Big(\frac{1}{2}, \frac{1}{2}; 1; k^2\Big),$$

where ${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right)$ denotes the ordinary hypergeometric function. Let K, K', L, and L' denote the complete elliptic integrals of the first kind associated with the moduli k, k', ℓ, and $\ell' := \sqrt{1 - \ell^2}$, respectively. Suppose that the equality

$$n\frac{K'}{K} = \frac{L'}{L} \tag{1.1}$$

holds for some positive integer n. A relation between k and ℓ induced by (1.1) is referred to as a modular equation of degree n. Following Ramanujan, we define α as k^2 and β as ℓ^2. It is conventional to state that β has a degree of n over α.

If

$$q := \exp\left(-\pi \frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1 - \alpha)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; \alpha)}\right) = \exp\left(-\pi \frac{K'}{K}\right), \tag{1.2}$$

then, one of the fundamental theorems in the theory of elliptic functions [7, p. 101, Entry 6] states that

$$\varphi^2(q) = {}_2F_1\Big(\frac{1}{2}, \frac{1}{2}; 1; \alpha\Big) =: z, \tag{1.3}$$

where

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}, \quad |q| < 1.$$

If we further set $z_n := \varphi^2(q^n)$, then the multiplier m of degree n is defined by

$$m := \frac{z_1}{z_n}. \tag{1.4}$$

We are now ready to introduce the purpose of our paper to the readers. In Ramanujan's notebooks, parts III [7], Berndt proved many modular equations of various degrees. Ramanujan focused on the algebraic expression $\{\frac{1}{2}(1+(\alpha\beta)^{1/2}+((1-\alpha)(1-\beta))^{1/2})\}^{1/2}$ which is very useful in simplifying modular equations. In this paper, we concern the modular equations with $\{\frac{1}{2}(1+(\alpha\beta)^{1/2}+((1-\alpha)(1-\beta))^{1/2})\}^{1/2}$ terms. We convert the expression $\{\frac{1}{2}(1+(\alpha\beta)^{1/2}+((1-\alpha)(1-\beta))^{1/2})\}^{1/2}$ into $\frac{1}{2}\{(1+\sqrt{1-\alpha})^{1/2}(1+\sqrt{1-\beta})^{1/2}+(1-\sqrt{1-\alpha})^{1/2}(1-\sqrt{1-\beta})^{1/2}\}$, employing the complete square formula, reformulate the modular equation with regard to q-products, and then interpret the identity of q-products through the utilization of colored partitions.

For any complex number a and $|q|<1$, define

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n).$$

Recall that Ramanujan's theta function $f(-q)$ and his function $\chi(q)$ are defined by

$$f(-q) := (q;q)_\infty, \tag{1.5}$$

$$\chi(q) := (-q;q^2)_\infty. \tag{1.6}$$

The following evaluations of Ramanujan may be found in [7, p. 124] or [8, p. 123].

Lemma 1.1. *If α, q, and z are related by* (1.2) *and* (1.3)*, then*

$$f(-q^2) = 2^{-1/3}\sqrt{z}\{\alpha(1-\alpha)/q\}^{1/12}, \tag{1.7}$$

$$f(-q^4) = 2^{-2/3}\sqrt{z}(\alpha/q)^{1/6}(1-\alpha)^{1/24}, \tag{1.8}$$

$$\chi(q) = 2^{1/6}\{\alpha(1-\alpha)/q\}^{-1/24}, \tag{1.9}$$

$$\chi(-q) = 2^{1/6}(1-\alpha)^{1/12}(\alpha/q)^{-1/24}, \tag{1.10}$$

$$\chi(-q^2) = 2^{1/3}(1-\alpha)^{1/24}(\alpha/q)^{-1/12}. \tag{1.11}$$

For the purpose of proving the following theorems, we obtain the following two useful identities by using Lemma 1.1, namely,

$$\chi(q)\chi(-q^2) = 2^{1/2}\{\alpha/q\}^{-1/8}, \tag{1.12}$$

$$\chi(-q)/\chi(q) = (1-\alpha)^{1/8}. \tag{1.13}$$

Lemma 1.2. *If α, q, and z are related by* (1.2) *and* (1.3)*, then*

$$\chi(q^2) = 2^{1/12}(1+\sqrt{1-\alpha})^{1/4}(\alpha/q)^{-1/12}(1-\alpha)^{-1/48}, \tag{1.14}$$

$$\chi(-q^4) = 2^{5/12}(1+\sqrt{1-\alpha})^{1/4}(\alpha/q)^{-1/6}(1-\alpha)^{1/48}. \tag{1.15}$$

Proof. Applying the principle of duplication to (1.9) and (1.11), we finish the proof. □

Assume that β has degree n over α. If we substitute q^n for q above, then the same evaluations remain valid: substituting β for α and z_n for z.

In the subsequent proofs, we frequently make use of Euler's famous identity (see [1, p. 5], [3, p. 3], and [12, p. 15])

$$\frac{1}{(q;q^2)_\infty} = (-q;q)_\infty. \tag{1.16}$$

i.e., the number of partitions of a positive integer n into odd parts equals to the number of partitions of n into distinct parts.

2. Colored partition identities by modular equations with degree 3

Theorem 2.1. *Let S denote the set of partitions whose even parts occur at most once in four different colors such that none of the parts are congruent to* $0 \pmod 3$*, and odd parts occur at most once in six different colors of which only five colors appear with parts congruent to* ± 1 *modulo* 6*. Let T denote the set of partitions whose odd parts occur in three different colors such that none of the red parts are congruent to* $0 \pmod 3$*, and even parts occur at most once in each of six different colors such that the red and blue parts are in multiples of* 4 *but not multiples of* 3*, and the remaining four colors appear with parts congruent to* 2 *modulo* 4*. Let U denote the set of partitions whose odd parts occur in three different colors such that none of the red parts are congruent to* $0 \pmod 3$*, and even parts occur at most once in each of six different colors such that the red and blue parts are congruent to* 2 *modulo* 4 *but not multiples of* 3*, and the remaining four colors appear with parts in multiples of* 4*. Let $P_1(N)$, $P_2(N)$, and $P_3(N)$ be the number of partitions of N in S, T, and U, respectively. Then, for $N \geq 3$, $P_1(N) = P_2(N) + 4P_3(N-2)$.*

Proof. Consider the following modular equation for degree 3 (see [7, p. 231, Entry 5 (viii)] and [8, p. 146, Theorem 6.3.4 (vii)]),

$$1 - \Big(\frac{\beta^3(1-\alpha)^3}{\alpha(1-\beta)}\Big)^{1/8} = \Big\{\frac{1}{2}(1 + (\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2})\Big\}^{1/2}.$$

By employing elementary algebraic methods, we can readily verify that

$$1 - \Big(\frac{\beta^3(1-\alpha)^3}{\alpha(1-\beta)}\Big)^{1/8} = \frac{1}{2}\Big\{(1+\sqrt{1-\alpha})^{1/2}(1+\sqrt{1-\beta})^{1/2} + (1-\sqrt{1-\alpha})^{1/2}(1-\sqrt{1-\beta})^{1/2}\Big\}. \tag{2.1}$$

Multiplying both sides of the identity (2.1) by $2(\alpha/q)^{\frac{-1}{6}}(1-\alpha)^{\frac{-1}{24}}(\beta/q^3)^{\frac{-1}{3}}(1-\beta)^{\frac{1}{24}}$, we arrive at

$$\begin{aligned}
&\frac{1}{2}\{2^{\frac{1}{6}}\{\alpha(1-\alpha)/q\}^{\frac{-1}{24}}\}^2 2^{\frac{1}{3}}(1-\alpha)^{\frac{1}{24}}(\alpha/q)^{\frac{-1}{12}}\{2^{\frac{1}{6}}\{\beta(1-\beta)/q^3\}^{\frac{-1}{24}}\}^2\{2^{\frac{1}{3}}(1-\beta)^{\frac{1}{24}}(\beta/q^3)^{\frac{-1}{12}}\}^3\\
&- q\frac{\{2^{\frac{1}{3}}(1-\alpha)^{\frac{1}{24}}(\alpha/q)^{\frac{-1}{12}}\}^2\{2^{\frac{1}{6}}(1-\alpha)^{\frac{1}{12}}(\alpha/q)^{\frac{-1}{24}}\}^3}{2^{\frac{1}{6}}(1-\beta)^{\frac{1}{12}}(\beta/q^3)^{\frac{-1}{24}}}\\
&= \frac{1}{2}\{2^{\frac{1}{12}}(1+\sqrt{1-\alpha})^{\frac{1}{4}}(\alpha/q)^{\frac{-1}{12}}(1-\alpha)^{\frac{-1}{48}}\}^2\{2^{\frac{5}{12}}(1+\sqrt{1-\beta})^{\frac{1}{4}}(\beta/q^3)^{\frac{-1}{6}}(1-\beta)^{\frac{1}{48}}\}^2\\
&+ \frac{2q^2}{\{2^{\frac{5}{12}}(1+\sqrt{1-\alpha})^{\frac{1}{4}}(\alpha/q)^{\frac{-1}{6}}(1-\alpha)^{\frac{1}{48}}\}^2\{2^{\frac{1}{12}}(1+\sqrt{1-\beta})^{\frac{1}{4}}(\beta/q^3)^{\frac{-1}{12}}(1-\beta)^{\frac{-1}{48}}\}^2}.
\end{aligned}$$

Employing the evaluations (1.9)–(1.11) in Lemma 1.1 and utilizing (1.14) and (1.15), we have

$$\frac{1}{2}\chi^2(q)\chi(-q^2)\chi^2(q^3)\chi^3(-q^6) - q\frac{\chi^2(-q^2)\chi^3(-q)}{\chi(-q^3)} = \frac{1}{2}\chi^2(q^2)\chi^2(-q^{12}) + \frac{2q^2}{\chi^2(-q^4)\chi^2(q^6)},$$

which can be transformed into

$$\frac{1}{2}\frac{(-q;q^2)^2_\infty(-q^3;q^6)^2_\infty}{(-q^2;q^2)_\infty(-q^6;q^6)^3_\infty} - q\frac{(q;q^2)^3_\infty}{(-q^2;q^2)^2_\infty(q^3;q^6)_\infty} = \frac{1}{2}\frac{(-q^2;q^4)^2_\infty}{(-q^{12};q^{12})^2_\infty} + 2q^2\frac{(-q^4;q^4)^2_\infty}{(-q^6;q^{12})^2_\infty},$$

by the definition of χ in (1.6) and Euler's identity (1.16). Multiply both sides of the last identity by $\frac{2(-q^2;q^2)_\infty^2(q^3;q^6)_\infty}{(q;q^2)_\infty^3}$ and use Euler's identity (1.16), then

$$\begin{aligned}&\frac{(-q^2;q^2)_\infty^4(-q;q^2)_\infty^5(-q^3;q^6)_\infty}{(-q^6;q^6)_\infty^4}-2q\\&=\frac{(q^3;q^6)_\infty}{(q;q^2)_\infty^3}\left\{\frac{(-q^2;q^4)_\infty^4(-q^4;q^4)_\infty^2}{(-q^{12};q^{12})_\infty^2}+4q^2\frac{(-q^2;q^4)_\infty^2(-q^4;q^4)_\infty^4}{(-q^6;q^{12})_\infty^2}\right\}.\end{aligned}$$

Extracting the coefficients of q^N on both sides of the equation, we finish the proof. □

Theorem 2.2. *Let S denote the set of partitions whose even parts occur at most once in eight different colors such that the red and blue parts are congruent to ± 2 modulo 6 and the remaining six colors with parts in multiples of 6, and odd parts not in multiples of 3 occur at most once and the parts congruent to 3 modulo 6 occur in six different colors. Let T denote the set of partitions whose odd parts being multiples of 3 occur in eight different colors, and even parts occur at most once in each of six different colors such that the red and blue parts are in multiples of 4 but not multiples of 3, and the remaining four colors appear with parts congruent to 2 modulo 4. Let U denote the set of partitions whose odd parts in multiples of 3 occur in eight different colors, and even parts occur at most once in each of six different colors such that the red and blue parts are congruent to 2 modulo 4 but not multiples of 3, and the remaining four colors appear with parts in multiples of 4. Let $P_1(N)$, $P_2(N)$, and $P_3(N)$ be the number of partitions of N in S, T, and U, respectively. Then, for $N \geq 3$,*

$$8P_1(N-2) = P_2(N) + 4P_3(N-2).$$

Proof. Recall the modular equation for degree 3 (see [7, p. 231, Entry 5 (viii)]),

$$(\alpha\beta^5)^{1/8}+\left\{(1-\alpha)(1-\beta)^5\right\}^{1/8}=\left\{\frac{1}{2}(1+(\alpha\beta)^{1/2}+\{(1-\alpha)(1-\beta)\}^{1/2})\right\}^{1/2}.$$

According to elementary algebraic methods, we get

$$\begin{aligned}&(\alpha\beta^5)^{1/8}+\left\{(1-\alpha)(1-\beta)^5\right\}^{1/8}\\&=\frac{1}{2}\left\{(1+\sqrt{1-\alpha})^{1/2}(1+\sqrt{1-\beta})^{1/2}+(1-\sqrt{1-\alpha})^{1/2}(1-\sqrt{1-\beta})^{1/2}\right\}.\end{aligned}\tag{2.2}$$

Multiplying both sides of the identity (2.2) by $2(\alpha/q)^{-1/6}(1-\alpha)^{-1/24}(\beta/q^3)^{-1/3}(1-\beta)^{1/24}$, and applying (1.9)–(1.11) in Lemma 1.1 and (1.14) and (1.15) in Lemma 1.2, we obtain

$$4q^2\frac{\chi(q)}{\chi^2(-q^6)\chi^3(q^3)}+\frac{1}{2}\chi^2(-q^2)\chi^8(-q^3)=\frac{1}{2}\chi^2(q^2)\chi^2(-q^{12})+\frac{2q^2}{\chi^2(-q^4)\chi^2(q^6)}.$$

Making use of the definition of χ in (1.6) and employing Euler's identity (1.16), we deduce that

$$4q^2\frac{(-q;q^2)_\infty(-q^6;q^6)_\infty^2}{(-q^3;q^6)_\infty^3}+\frac{1}{2}\frac{(q^3;q^6)_\infty^8}{(-q^2;q^2)_\infty^2}=\frac{1}{2}\frac{(-q^2;q^4)_\infty^2}{(-q^{12};q^{12})_\infty^2}+2q^2\frac{(-q^4;q^4)_\infty^2}{(-q^6;q^{12})_\infty^2}.$$

Multiplying both sides of the last identity by $2\frac{(-q^2;q^2)_\infty^2}{(q^3;q^6)_\infty^8}$ and applying Euler's identity (1.16), we obtain that

$$
\begin{aligned}
&8q^2\frac{(-q;q^2)_\infty(-q^6;q^6)_\infty^4(-q^2;q^2)_\infty^2}{(-q^3;q^6)_\infty(q^3;q^6)_\infty^6}+1\\
=&\frac{(-q^4;q^4)_\infty^2(-q^2;q^4)_\infty^4}{(q^3;q^6)_\infty^8(-q^{12};q^{12})_\infty^2}+4q^2\frac{(-q^2;q^4)_\infty^2(-q^4;q^4)_\infty^4}{(q^3;q^6)_\infty^8(-q^6;q^{12})_\infty^2}.
\end{aligned}
$$

Equating the coefficients of q^N on both sides of the identity, we complete the proof. □

Theorem 2.3. *Let S denote the set of partitions whose even parts occur at most once in eleven different colors such that the red, blue, green, pink, and yellow parts are not in multiples of* 3 *and the remaining six colors with parts in multiples of* 6*, and odd parts occur in eleven different colors such that the red, blue, green, pink, and yellow parts are congruent to* ± 1 *modulo* 6 *and the remaining six colors with parts congruent to* 3 *modulo* 6*. Let T denote the set of partitions whose odd parts occur in eight different colors such that the red and blue parts appear at most once, and even parts occur at most once in each of ten different colors such that the red and blue parts are in multiples of* 12*, the green and pink parts are congruent to* ± 2 *modulo* 12*, and the remaining six colors appear with parts congruent to* 6 *modulo* 12*. Let U denote the set of partitions whose odd parts occur in eight different colors such that the red and blue parts appear at most once, and even parts occur at most once in each of ten different colors such that the red and blue parts are congruent to* 6 *modulo* 12*, the green and pink parts are congruent to* ± 4 *modulo* 12*, and the remaining six colors appear with parts in multiples of* 12*. Let $P_1(N)$, $P_2(N)$, and $P_3(N)$ be the number of partitions of N in S, T, and U, respectively. Then, for $N \geq 3$,*

$$8P_1(N-1) = P_2(N) + 4P_3(N-2).$$

Proof. Consider the modular equation for degree 3 (see [7, p. 231, Entry 5 (viii)]),

$$(\alpha^5\beta)^{1/8}+\left\{(1-\alpha)^5(1-\beta)\right\}^{1/8}=\left\{\frac{1}{2}(1+(\alpha\beta)^{1/2}+\{(1-\alpha)(1-\beta)\}^{1/2})\right\}^{1/2}.$$

Using elementary algebra, we have

$$
\begin{aligned}
&(\alpha^5\beta)^{1/8}+\left\{(1-\alpha)^5(1-\beta)\right\}^{1/8}\\
=&\frac{1}{2}\left\{(1+\sqrt{1-\alpha})^{1/2}(1+\sqrt{1-\beta})^{1/2}+(1-\sqrt{1-\alpha})^{1/2}(1-\sqrt{1-\beta})^{1/2}\right\}.
\end{aligned}
\tag{2.3}
$$

Multiplying both sides of the identity (2.3) by $2(\alpha/q)^{-1/3}(1-\alpha)^{1/24}(\beta/q^3)^{-1/6}(1-\beta)^{-1/24}$, we find that

$$4q\frac{\chi(q^3)}{\chi^2(-q^2)\chi^3(q)}+\frac{1}{2}\chi^8(-q)\chi^2(-q^6)=\frac{1}{2}\chi^2(-q^4)\chi^2(q^6)+\frac{2q^2}{\chi^2(q^2)\chi^2(-q^{12})}$$

by the identities (1.9)–(1.11) in Lemma 1.1 and (1.14) and (1.15) in Lemma 1.2. Applying (1.6) and Euler's identity (1.16), we see that

$$4q\frac{(-q^3;q^6)_\infty(-q^2;q^2)_\infty^2}{(-q;q^2)_\infty^3}+\frac{1}{2}\frac{(q;q^2)_\infty^8}{(-q^6;q^6)_\infty^2}=\frac{1}{2}\frac{(-q^6;q^{12})_\infty^2}{(-q^4;q^4)_\infty^2}+2q^2\frac{(-q^{12};q^{12})_\infty^2}{(-q^2;q^4)_\infty^2}.$$

Multiplying both sides of the last identity by $2\frac{(-q^6;q^6)_\infty^2}{(q;q^2)_\infty^8}$ and utilizing Euler's identity (1.16), we have

$$\frac{8q(-q^2;q^2)_\infty^5(-q^6;q^6)_\infty}{(q;q^2)_\infty^5(q^3;q^6)_\infty}+1$$
$$=\frac{(-q;q^2)_\infty^2(-q^6;q^6)_\infty^2}{(q;q^2)_\infty^6}\{(-q^2;q^4)_\infty^2(-q^6;q^{12})_\infty^2+4q^2(-q^4;q^4)_\infty^2(-q^{12};q^{12})_\infty^2\}.$$

Equating the coefficients of q^N on both sides of the foregoing identity, we finish the proof. □

Theorem 2.4. *Let S denote the set of partitions whose odd parts not in multiples of 3 occur in one color, odd parts being multiples of 3 occur in two different colors, and even parts occur no more than once in each of four different colors such that the red and blue parts are in multiples of 4 and the green and pink parts are congruent to 6 modulo 12. Let T denote the set of partitions whose odd parts not in multiples of 3 occur in one color, odd parts in multiples of 3 occur in two different colors, and even parts occur at most once in each of four different colors such that the red and blue parts are congruent to 2 modulo 4 and the green and pink parts are in multiples of 12. Let $P_1(N)$ be the number of partitions of N in S. Let $P_2(N)$ be the number of partitions of N in T. Then, for $N \geq 2$,*

$$P_1(N) = P_2(N-1).$$

Proof. Rewrite the modular equation for degree 3 (see [2, p. 387, Entry 17.3.10] and [7, p. 230, Entry 5 (ii)]),

$$(\alpha\beta)^{1/4}+\Big\{(1-\alpha)(1-\beta)\Big\}^{1/4}=1.$$

Square the last identity and arrange terms of the resulting identity to obtain

$$\Big\{\alpha\beta(1-\alpha)(1-\beta)\Big\}^{1/4}=\frac{1}{2}\Big\{1-\sqrt{\alpha\beta}-\sqrt{(1-\alpha)(1-\beta)}\Big\}.$$

The square root of the above equation yields

$$\Big\{\alpha\beta(1-\alpha)(1-\beta)\Big\}^{1/8}$$
$$=\frac{1}{2}\Big\{(1-\sqrt{1-\alpha})^{1/2}(1+\sqrt{1-\beta})^{1/2}-(1+\sqrt{1-\alpha})^{1/2}(1-\sqrt{1-\beta})^{1/2}\Big\}. \tag{2.4}$$

It is also possible that there is a minus sign on the right side of the last equation, but this will not really affect the subsequent proof. Multiply both sides of the identity (2.4) by $2^{1/3}q^{-1/2}(\alpha/q)^{-1/6}(1-\alpha)^{-1/24}(\beta/q^3)^{-1/6}(1-\beta)^{-1/24}$. Hence,

$$\chi(-q)\chi(-q^3)=\frac{\chi^2(q^6)}{\chi^2(-q^4)}-q\frac{\chi^2(q^2)}{\chi^2(-q^{12})},$$

by the identities (1.10), (1.14), and (1.15). Applying (1.6) and Euler's identity (1.16), we arrive at

$$(q;q^2)_\infty(q^3;q^6)_\infty=(-q^4;q^4)_\infty^2(-q^6;q^{12})_\infty^2-q(-q^2;q^4)_\infty^2(-q^{12};q^{12})_\infty^2.$$

Divide both sides of the last identity by $(q;q^2)_\infty(q^3;q^6)_\infty$. Then,

$$1=\frac{(-q^4;q^4)_\infty^2(-q^6;q^{12})_\infty^2}{(q;q^2)_\infty(q^3;q^6)_\infty}-q\frac{(-q^2;q^4)_\infty^2(-q^{12};q^{12})_\infty^2}{(q;q^2)_\infty(q^3;q^6)_\infty}.$$

Extracting the coefficients of q^N on both sides of the equation, we complete the proof. □

Theorem 2.5. *Let S denote the set of partitions whose odd parts can occur at most once in eight distinct colors with the red and blue colors not appearing in multiples of* 3*, and even parts not in multiples of* 3 *can occur at most once in one color. Let T denote the set of partitions whose even parts not in multiples of* 3 *occur in five different colors such that the red parts appear at most once, and odd parts in multiples of* 3 *occur at most once in six different colors. Let U denote the set of partitions whose odd parts occur in two different colors, and even parts occur in each of eight different colors such that the orange and white parts are not in multiples of* 3*, the red and blue parts appearing at most once are congruent to* ± 2 *modulo* 12*, and the remaining four colors appear at most once with parts congruent to* 6 *modulo* 12*. Let V denote the set of partitions whose odd parts occur in two different colors, and even parts occur in each of eight different colors such that the orange and white parts are not in multiples of* 3*, the red and blue parts appearing at most once are congruent to* ± 4 *modulo* 12*, and the remaining four colors appear at most once with parts in multiples of* 12*. Let $P_1(N)$, $P_2(N)$, $P_3(N)$, and $P_4(N)$ be the number of partitions of N in S, T, U, and V, respectively. Then, for $N \geq 3$,*

$$P_1(N) + 3P_2(N) = 4P_3(N) + 16P_4(N-2).$$

Proof. Recall the formula for the multiplier of degree 3 [7, p. 231, Entry 5 (xi)],

$$m + \frac{3}{m} = 4\Big\{\frac{1}{2}(1 + (\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2})\Big\}^{1/2},$$

where m is defined by (1.4). Using elementary algebra, we deduce that

$$m + \frac{3}{m} = 2\Big\{(1 + \sqrt{1-\alpha})^{1/2}(1 + \sqrt{1-\beta})^{1/2} + (1 - \sqrt{1-\alpha})^{1/2}(1 - \sqrt{1-\beta})^{1/2}\Big\}. \tag{2.5}$$

Multiply both sides of the identity (2.5) by $2^{8/3}(\alpha/q)^{-1/3}(1-\alpha)^{1/24}(\beta/q^3)^{-1/3}(1-\beta)^{1/24}$, and apply (1.7)–(1.9) in Lemma 1.1, and (1.12)–(1.15). Hence,

$$\begin{aligned}&\frac{f^2(-q^2)}{f^2(-q^{12})}\chi^6(q)\chi^3(-q^2)\frac{\chi(-q^3)}{\chi(q^3)} + \frac{3f^2(-q^6)}{f^2(-q^4)}\chi^6(q^3)\chi^3(-q^6)\frac{\chi(-q)}{\chi(q)}\\ =&4\chi^2(-q^4)\chi^2(-q^{12}) + \frac{16q^2}{\chi^2(q^2)\chi^2(q^6)}.\end{aligned}$$

The last identity can be converted into the q-product identity by the definitions of f and χ in (1.5) and (1.6), respectively, and Euler's identity (1.16),

$$\begin{aligned}&\frac{(q^2;q^2)_\infty^2(-q;q^2)_\infty^6(q^3;q^6)_\infty}{(q^{12};q^{12})_\infty^2(-q^2;q^2)_\infty^3(-q^3;q^6)_\infty} + 3\frac{(q^6;q^6)_\infty^2(-q^3;q^6)_\infty^6(q;q^2)_\infty}{(q^4;q^4)_\infty^2(-q^6;q^6)_\infty^3(-q;q^2)_\infty}\\ =&\frac{4}{(-q^4;q^4)_\infty^2(-q^{12};q^{12})_\infty^2} + \frac{16q^2}{(-q^2;q^4)_\infty^2(-q^6;q^{12})_\infty^2}.\end{aligned}$$

Multiplying both sides of the last identity by $\frac{(-q;q^2)_\infty(-q^2;q^2)_\infty^3(q^{12};q^{12})_\infty^2}{(q;q^2)_\infty(q^2;q^2)_\infty^2}$ and using Euler's identity (1.16), we find that

$$\frac{(-q^2;q^2)_\infty(-q;q^2)_\infty^8}{(-q^6;q^6)_\infty(-q^3;q^6)_\infty^2}+3\frac{(q^6;q^6)_\infty^4(-q^2;q^2)_\infty(-q^3;q^6)_\infty^6}{(q^2;q^2)_\infty^4(-q^6;q^6)_\infty}$$
$$=\frac{4(q^6;q^6)_\infty^2(-q^2;q^4)_\infty^2(-q^6;q^{12})_\infty^2}{(q;q)_\infty^2}+\frac{16q^2(q^6;q^6)_\infty^2(-q^4;q^4)_\infty^2(-q^{12};q^{12})_\infty^2}{(q;q)_\infty^2}.$$

Equating the coefficients of q^N on both sides of the equation, we finish the proof. □

Theorem 2.6. *Let S denote the set of partitions whose even parts not in multiples of 3 can occur in two distinct colors, and odd parts in multiples of 3 appear at most once in six distinct colors. Let T denote the set of partitions whose even parts not in multiples of 3 can occur in two distinct colors, and odd parts not in multiples of 3 can occur at most once in three different colors. Let U denote the set of partitions whose odd parts occur in two different colors such that the red parts appear at most once, and even parts occur at most once in each of eight different colors such that the blue parts are in multiples of 12, the green and pink parts are congruent to ±2 modulo 12, and the remaining five colors appear with parts congruent to 6 modulo 12. Let V denote the set of partitions whose odd parts occur in two different colors such that the red parts appear at most once, and even parts occur at most once in each of eight different colors such that the blue parts are congruent to 6 modulo 12, the green and pink parts are congruent to ±4 modulo 12, and the remaining five colors appear with parts in multiples of 12. Let $P_1(N)$, $P_2(N)$, $P_3(N)$, and $P_4(N)$ be the number of partitions of N in S, T, U, and V, respectively. Then, for $N \geq 3$,*

$$P_1(N)+2P_2(N-1)=P_3(N)+4P_4(N-2).$$

Proof. Consider the formula for the multiplier of degree 3 [7, p. 352, Entry 3 (viii)],

$$1+\left\{\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}\right\}^{1/8}=m\left\{\frac{1}{2}(1+(\alpha\beta)^{1/2}+\{(1-\alpha)(1-\beta)\}^{1/2})\right\}^{1/2},$$

where m is defined by (1.4). Employing elementary algebra, we have

$$1+\left\{\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}\right\}^{\frac{1}{8}}=\frac{z_1}{2z_3}\left\{(1+\sqrt{1-\alpha})^{\frac{1}{2}}(1+\sqrt{1-\beta})^{\frac{1}{2}}+(1-\sqrt{1-\alpha})^{\frac{1}{2}}(1-\sqrt{1-\beta})^{\frac{1}{2}}\right\}. \tag{2.6}$$

Multiply both sides of the identity (2.6) by $2^{\frac{1}{3}}\frac{z_3}{z_1}(\alpha/q)^{\frac{-1}{6}}(1-\alpha)^{\frac{-1}{24}}(\beta/q^3)^{\frac{-1}{6}}(1-\beta)^{\frac{-1}{24}}$, and make use of (1.7), (1.9), (1.11) in Lemma 1.1, and (1.12)–(1.15) to deduce

$$\frac{f^2(-q^6)}{2f^2(-q^2)}\frac{\chi(-q)}{\chi(q)}\frac{\chi^4(-q^6)\chi^3(q^3)}{\chi^3(-q^3)}+q\frac{f^2(-q^6)}{f^2(-q^2)}\frac{\chi(q)\chi(-q^2)\chi(-q^3)}{\chi^2(q^3)}=\frac{1}{2}\chi^2(q^2)\chi^2(q^6)+\frac{2q^2}{\chi^2(-q^4)\chi^2(-q^{12})}.$$

Utilizing the definitions of f and χ in (1.5) and (1.6), respectively, and Euler's identity (1.16), we obtain

$$\frac{(q^6;q^6)_\infty^2(q;q^2)_\infty(-q^3;q^6)_\infty^6}{2(q^2;q^2)_\infty^2(-q;q^2)_\infty(-q^6;q^6)_\infty}+q\frac{(q^6;q^6)_\infty^2(-q;q^2)_\infty(q^3;q^6)_\infty}{(q^2;q^2)_\infty^2(-q^2;q^2)_\infty(-q^3;q^6)_\infty^2}$$
$$=\frac{1}{2}(-q^2;q^4)_\infty^2(-q^6;q^{12})_\infty^2+2q^2(-q^4;q^4)_\infty^2(-q^{12};q^{12})_\infty^2.$$

Multiplying both sides of the last identity by $2\frac{(-q;q^2)_\infty(-q^6;q^6)_\infty}{(q;q^2)_\infty}$ and using Euler's identity (1.16), we see that

$$\frac{(q^6;q^6)_\infty^2}{(q^2;q^2)_\infty^2}\Big\{(-q^3;q^6)_\infty^6+2q\frac{(-q;q^2)_\infty^3}{(-q^3;q^6)_\infty^3}\Big\}$$
$$=\frac{(-q;q^2)_\infty(-q^6;q^6)_\infty}{(q;q^2)_\infty}\{(-q^2;q^4)_\infty^2(-q^6;q^{12})_\infty^2+4q^2(-q^4;q^4)_\infty^2(-q^{12};q^{12})_\infty^2\}.$$

Equating the coefficients of q^N on both sides of the equation, we finish the proof. □

Theorem 2.7. *Let S denote the set of partitions whose odd parts can occur at most once in nine distinct colors of which three colors do not appear in multiples of* 3*. Let T denote the set of partitions whose odd parts occur in three different colors with the red color not appearing in multiples of* 3*, and even parts occur in each of* 8 *different colors such that the orange and white parts are in multiples of* 4 *but not multiples of* 3*, the red and blue parts appearing at most once are congruent to* ± 2 *modulo* 12*, and the remaining four colors appear at most once with parts congruent to* 6 *modulo* 12*. Let U denote the set of partitions whose odd parts occur in three different colors with the red color not appearing in multiples of* 3*, and even parts occur in each of eight different colors such that the orange and white parts are in multiples of* 4 *but not multiples of* 3*, the red and blue parts appearing at most once are congruent to* ± 4 *modulo* 12*, and the remaining four colors appear at most once with parts in multiples of* 12*. Let $P_1(N)$, $P_2(N)$, and $P_3(N)$ be the number of partitions of N in S, T, and U, respectively. Then, for $N \geq 3$,*

$$P_1(N) = 3P_2(N) + 12P_3(N-2).$$

Proof. Recall the formula for the multiplier of degree 3 [7, p. 352, Entry 3 (ix)],

$$1+\Big\{\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}\Big\}^{1/8}=\frac{3}{m}\Big\{\frac{1}{2}(1+(\alpha\beta)^{1/2}+\{(1-\alpha)(1-\beta)\}^{1/2})\Big\}^{1/2},$$

where m is defined by (1.4). Using elementary algebra, we have

$$\frac{z_1}{z_3}\Big\{1+\Big\{\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}\Big\}^{\frac{1}{8}}\Big\}=\frac{3}{2}\Big\{(1+\sqrt{1-\alpha})^{\frac{1}{2}}(1+\sqrt{1-\beta})^{\frac{1}{2}}+(1-\sqrt{1-\alpha})^{\frac{1}{2}}(1-\sqrt{1-\beta})^{\frac{1}{2}}\Big\}. \quad (2.7)$$

Multiplying both sides of the identity (2.7) by $2^{\frac{5}{3}}(\alpha/q)^{\frac{-1}{3}}(1-\alpha)^{\frac{1}{24}}(\beta/q^3)^{\frac{-1}{3}}(1-\beta)^{\frac{1}{24}}$, and applying (1.7), (1.9), (1.11) in Lemma 1.1, and (1.12)–(1.15), we have

$$\frac{f^2(-q^2)}{2f^2(-q^6)}\frac{\chi^5(q)\chi^4(-q^2)\chi^2(-q^6)\chi(-q^3)}{\chi(-q)\chi(q^3)}+\frac{f^2(-q^2)}{f^2(-q^6)}\chi^3(-q)\chi(-q^3)\chi^2(q^3)\chi^2(-q^6)$$
$$=\frac{3}{2}\{\chi^2(-q^4)\chi^2(-q^{12})+\frac{2q^2}{\chi^2(q^2)\chi^2(q^6)}\}.$$

Appealing to (1.5), (1.6), and Euler's identity (1.16) and simplifying, we deduce that

$$\frac{(q^2;q^2)_\infty^2(-q;q^2)_\infty^6(q^3;q^6)_\infty^2}{2(q^6;q^6)_\infty^2(-q^2;q^2)_\infty^3(-q^6;q^6)_\infty}+\frac{(q^2;q^2)_\infty^2(q;q^2)_\infty^3(q^3;q^6)_\infty(-q^3;q^6)_\infty^2}{(q^6;q^6)_\infty^2(-q^6;q^6)_\infty^2}$$

$$=\frac{3}{2}\Big\{\frac{1}{(-q^4;q^4)^2_\infty(-q^{12};q^{12})^2_\infty}+\frac{4q^2}{(-q^2;q^4)^2_\infty(-q^6;q^{12})^2_\infty}\Big\}.$$

Multiplying both sides of the last identity by $2\frac{(-q^6;q^6)^4_\infty(q^6;q^6)^2_\infty(q^3;q^6)_\infty}{(q;q^2)^3_\infty(q^2;q^2)^2_\infty}$ and applying Euler's identity (1.16), we arrive at

$$\frac{(-q;q^2)^9_\infty}{(-q^3;q^6)^3_\infty}+2=\frac{3(q^{12};q^{12})^2_\infty(q^3;q^6)_\infty}{(q^4;q^4)^2_\infty(q;q^2)^3_\infty}\{(-q^2;q^4)^2_\infty(-q^6;q^{12})^2_\infty+4q^2(-q^4;q^4)^2_\infty(-q^{12};q^{12})^2_\infty\}.$$

Equating the coefficients of q^N on both sides of the identity above, we can complete the proof. □

3. Colored partition identities by modular equations with degree 5

Theorem 3.1. *Let S denote the set of partitions whose even parts occur at most once in two different colors such that none of the parts are in multiples of* 5*, and odd parts occur at most once in seven different colors such that the red, blue and green parts are not in multiples of* 5*, and the remaining four colors appear with parts congruent to* 5 *modulo* 10*. Let T denote the set of partitions whose odd parts occur in one color such that none of the red parts are congruent to* $0\ (\mathrm{mod}\ 5)$*, and even parts occur at most once in each of twelve different colors such that the red and blue parts are in multiples of* 4*, the green, pink, yellow, and orange parts are congruent to* 2 *modulo* 4 *but not multiples of* 5*, and the remaining six colors appear with parts congruent to* 10 *modulo* 20*. Let U denote the set of partitions whose odd parts occur in one color such that none of the red parts are congruent to* $0\ (\mathrm{mod}\ 5)$*, and even parts occur at most once in each of twelve different colors such that the red and blue parts are congruent to* 2 *modulo* 4*, the green, pink, yellow, and orange parts are in multiples of* 4 *but not multiples of* 5*, and the remaining six colors appear with parts in multiples of* 20*. Let $P_1(N)$, $P_2(N)$, and $P_3(N)$ be the number of partitions of N in S, T, and U, respectively. Then, for $N\geq 4$,*

$$P_1(N)=P_2(N)+4P_3(N-3).$$

Proof. Recall the modular equation for degree 5 [7, p. 281, Entry 13 (vii)],

$$1-2^{1/3}\Big(\frac{\beta^5(1-\alpha)^5}{\alpha(1-\beta)}\Big)^{1/24}=\Big\{\frac{1}{2}(1+(\alpha\beta)^{1/2}+\{(1-\alpha)(1-\beta)\}^{1/2})\Big\}^{1/2}.$$

Applying elementary algebra, we can easily check that

$$1-2^{\frac{1}{3}}\Big(\frac{\beta^5(1-\alpha)^5}{\alpha(1-\beta)}\Big)^{\frac{1}{24}}=\frac{1}{2}\Big\{(1+\sqrt{1-\alpha})^{\frac{1}{2}}(1+\sqrt{1-\beta})^{\frac{1}{2}}+(1-\sqrt{1-\alpha})^{\frac{1}{2}}(1-\sqrt{1-\beta})^{\frac{1}{2}}\Big\}. \tag{3.1}$$

Multiply both sides of the identity (3.1) by $4(\alpha/q)^{-1/6}(1-\alpha)^{-1/24}(\beta/q^5)^{-1/3}(1-\beta)^{1/24}$. Thus,

$$\chi^2(q)\chi(-q^2)\chi^2(q^5)\chi^3(-q^{10})-2q\chi^2(-q)\chi(q)\chi(-q^2)\chi(q^5)\chi(-q^{10})=\chi^2(q^2)\chi^2(-q^{20})+\frac{4q^3}{\chi^2(-q^4)\chi^2(q^{10})},$$

by (1.9)–(1.11) in Lemma 1.1 and (1.14) and (1.15) in Lemma 1.2. Making use of the definition of χ in (1.6), and invoking Euler's identity (1.16), we deduce that

$$\frac{(-q;q^2)^2_\infty(-q^5;q^{10})^2_\infty}{(-q^2;q^2)_\infty(-q^{10};q^{10})^3_\infty}-2q\frac{(-q^5;q^{10})_\infty(q;q^2)_\infty}{(-q^{10};q^{10})_\infty(-q^2;q^2)^2_\infty}=\frac{(-q^2;q^4)^2_\infty}{(-q^{20};q^{20})^2_\infty}+\frac{4q^3(-q^4;q^4)^2_\infty}{(-q^{10};q^{20})^2_\infty}.$$

Multiplying the last q-product identity by $\frac{(-q^2;q^2)_\infty^2(-q^{10};q^{10})_\infty}{(q;q^2)_\infty(-q^5;q^{10})_\infty}$ and using Euler's identity (1.16), we arrive at

$$\frac{(-q^2;q^2)_\infty^2(-q;q^2)_\infty^3(-q^5;q^{10})_\infty}{(-q^{10};q^{10})_\infty^2}-2q$$
$$=\frac{(q^5;q^{10})_\infty(-q^2;q^2)_\infty^2}{(q;q^2)_\infty}\{(-q^{10};q^{20})_\infty^2(-q^2;q^4)_\infty^2+4q^3(-q^4;q^4)_\infty^2(-q^{20};q^{20})_\infty^2\}.$$

Extracting the coefficients of q^N on both sides of the equation, we complete the proof. □

Theorem 3.2. *Let S denote the set of partitions whose even parts occur at most once in six different colors such that the red and blue parts are not in multiples of 5, the remaining four colors with parts in multiples of 10, and odd parts in not multiples of 5 occur at most once, and the parts congruent to 5 modulo 10 occur in four different colors. Let T denote the set of partitions whose odd parts being multiples of 5 occur in four different colors, and even parts occur at most once in each of twelve different colors such that the red and blue parts are in multiples of 4, the green, pink, yellow, and orange parts are congruent to 2 modulo 4 but not multiples of 5, and the remaining six colors appear with parts congruent to 10 modulo 20. Let U denote the set of partitions whose odd parts in multiples of 5 occur in four different colors, and even parts occur at most once in each of twelve different colors such that the red and blue parts are congruent to 2 modulo 4, the green, pink, yellow, and orange parts are in multiples of 4 but not multiples of 5, and the remaining six colors appear with parts in multiples of 20. Let $P_1(N)$, $P_2(N)$, and $P_3(N)$ be the number of partitions of N in S, T, and U, respectively. Then, for $N \geq 4$,*

$$4P_1(N-2)=P_2(N)+4P_3(N-3).$$

Proof. Rewrite the modular equation for degree 5 [7, p. 281, Entry 13 (vii)],

$$(\alpha\beta^3)^{1/8}+\left\{(1-\alpha)(1-\beta)^3\right\}^{1/8}=\left\{\frac{1}{2}(1+(\alpha\beta)^{1/2}+\{(1-\alpha)(1-\beta)\}^{1/2})\right\}^{1/2}.$$

Using elementary algebra, we can easily check that

$$(\alpha\beta^3)^{\frac{1}{8}}+\left\{(1-\alpha)(1-\beta)^3\right\}^{\frac{1}{8}}=\frac{1}{2}\left\{(1+\sqrt{1-\alpha})^{\frac{1}{2}}(1+\sqrt{1-\beta})^{\frac{1}{2}}+(1-\sqrt{1-\alpha})^{\frac{1}{2}}(1-\sqrt{1-\beta})^{\frac{1}{2}}\right\}. \quad (3.2)$$

Multiply both sides of the identity (3.2) by $2(\alpha/q)^{-1/6}(1-\alpha)^{-1/24}(\beta/q^5)^{-1/3}(1-\beta)^{1/24}$. Hence,

$$2q^2\frac{\chi(q)}{\chi(q^5)}+\chi^2(-q^2)\chi^4(-q^{10})\frac{\chi^2(-q^5)}{2\chi^2(q^5)}=\frac{1}{2}\chi^2(q^2)\chi^2(-q^{20})+\frac{2q^3}{\chi^2(-q^4)\chi^2(q^{10})}$$

by the use of the identities (1.9)–(1.11) in Lemma 1.1 and (1.14) and (1.15) in Lemma 1.2. Applying the definition of χ in (1.6) and Euler's identity (1.16), we can convert the last equation into q-products, namely,

$$\frac{2q^2(-q;q^2)_\infty}{(-q^5;q^{10})_\infty}+\frac{(q^5;q^{10})_\infty^4}{2(-q^2;q^2)_\infty^2(-q^{10};q^{10})_\infty^2}=\frac{(-q^2;q^4)_\infty^2}{2(-q^{20};q^{20})_\infty^2}+\frac{2q^3(-q^4;q^4)_\infty^2}{(-q^{10};q^{20})_\infty^2}.$$

Multiplying the last q-product identity by $2\frac{(-q^2;q^2)_\infty^2(-q^{10};q^{10})_\infty^2}{(q^5;q^{10})_\infty^4}$ and using Euler's identity (1.16), we arrive at

$$4q^2\frac{(-q;q^2)_\infty(-q^2;q^2)_\infty^2(-q^{10};q^{10})_\infty^2}{(-q^5;q^{10})_\infty(q^5;q^{10})_\infty^4}+1$$
$$=\frac{(-q^2;q^2)_\infty^2}{(q^5;q^{10})_\infty^4}\{(-q^{10};q^{20})_\infty^2(-q^2;q^4)_\infty^2+4q^3(-q^4;q^4)_\infty^2(-q^{20};q^{20})_\infty^2\}.$$

Equating the coefficients of q^N on both sides of the equation, we complete the proof. □

Theorem 3.3. *Let S denote the set of partitions whose even parts occur at most once in seven different colors such that the red, blue, and green parts are not in multiples of* 5 *and the remaining four colors with parts in multiples of* 10*, and odd parts occur in seven different colors such that the red, blue and green parts are not in multiples of* 5 *and the remaining four colors with parts congruent to* 5 *modulo* 10*. Let T denote the set of partitions whose odd parts occur in four different colors, and even parts occur at most once in each of ten different colors such that the red and blue parts are in multiples of* 20*, the green and pink parts are congruent to* 2 *modulo* 4 *but not multiples of* 5*, and the remaining six colors appear with parts congruent to* 10 *modulo* 20*. Let U denote the set of partitions whose odd parts occur in four different colors, and even parts occur at most once in each of ten different colors such that the red and blue parts are congruent to* 10 *modulo* 20*, the green and pink parts are in multiples of* 4 *but not multiples of* 5*, and the remaining six colors appear with parts in multiples of* 20*. Let $P_1(N)$, $P_2(N)$, and $P_3(N)$ be the number of partitions of N in S, T, and U, respectively. Then, for $N \geq 4$,*

$$4P_1(N-1) = P_2(N) + 4P_3(N-3).$$

Proof. Recall the modular equation for degree 5 [7, p. 281, Entry 13 (vii)],

$$(\alpha^3\beta)^{1/8}+\left\{(1-\alpha)^3(1-\beta)\right\}^{1/8}=\left\{\frac{1}{2}(1+(\alpha\beta)^{1/2}+\{(1-\alpha)(1-\beta)\}^{1/2})\right\}^{1/2}.$$

Applying elementary algebra, we can easily check that

$$(\alpha^3\beta)^{\frac18}+\left\{(1-\alpha)^3(1-\beta)\right\}^{\frac18}=\frac12\left\{(1+\sqrt{1-\alpha})^{\frac12}(1+\sqrt{1-\beta})^{\frac12}+(1-\sqrt{1-\alpha})^{\frac12}(1-\sqrt{1-\beta})^{\frac12}\right\}. \quad (3.3)$$

Multiply both sides of the identity (3.3) by $2(\alpha/q)^{-1/3}(1-\alpha)^{1/24}(\beta/q^5)^{-1/6}(1-\beta)^{-1/24}$. Thus,

$$2q\frac{\chi(q^5)}{\chi(q)}+\frac12\chi^2(-q^2)\chi^2(-q^{10})\chi^4(-q)=\frac12\chi^2(-q^4)\chi^2(q^{10})+\frac{2q^3}{\chi^2(q^2)\chi^2(-q^{20})},$$

by (1.9)–(1.11) in Lemma 1.1 and (1.14), (1.15). Making use of the definition of χ in (1.6) and Euler's identity (1.16), we deduce that

$$\frac{2q(-q^5;q^{10})_\infty}{(-q;q^2)_\infty}+\frac{(q;q^2)_\infty^4}{2(-q^2;q^2)_\infty^2(-q^{10};q^{10})_\infty^2}=\frac{(-q^{10};q^{20})_\infty^2}{2(-q^4;q^4)_\infty^2}+\frac{2q^3(-q^{20};q^{20})_\infty^2}{(-q^2;q^4)_\infty^2}.$$

Multiplying the last q-product identity by $2\frac{(-q^2;q^2)_\infty^2(-q^{10};q^{10})_\infty^2}{(q;q^2)_\infty^4}$ and employing Euler's identity (1.16), we have

$$4q\frac{(-q^2;q^2)_\infty^3(-q^{10};q^{10})_\infty}{(q;q^2)_\infty^3(q^5;q^{10})_\infty}+1=\frac{(-q^{10};q^{10})_\infty^2}{(q;q^2)_\infty^4}\{(-q^{10};q^{20})_\infty^2(-q^2;q^4)_\infty^2+4q^3(-q^4;q^4)_\infty^2(-q^{20};q^{20})_\infty^2\}.$$

Equating the coefficients of q^N on both sides of the equation, we finish the proof. □

Theorem 3.4. *Let S denote the set of partitions whose parts in multiples of 4, not multiples of 5, can occur in two distinct colors, parts in multiples of 5, not multiples of 4, can occur in four distinct colors. Let T denote the set of partitions whose even parts not in multiples of 5 can occur in two distinct colors, and odd parts not in multiples of 5 appear at most once in one color. Let U denote the set of partitions whose odd parts not in multiples of 5 occur at most once and the parts congruent to 5 modulo 10 occur in four different colors, and even parts occur at most once in each of ten different colors such that the red and blue parts are in multiples of 20, the green and pink parts are congruent to 2 modulo 4 but not multiples of 5, and the remaining six colors appear with parts congruent to 10 modulo 20. Let V denote the set of partitions whose odd parts not in multiples of 5 occur at most once, and the parts congruent to 5 modulo 10 occur in four different colors, and even parts occur at most once in each of ten different colors such that the red and blue parts are congruent to 10 modulo 20, the green and pink parts are in multiples of 4 but not multiples of 5, and the remaining six colors appear with parts in multiples of 20. Let $P_1(N)$, $P_2(N)$, $P_3(N)$, and $P_4(N)$ be the number of partitions of N in S, T, U, and V, respectively. Then, for $N \geq 4$,*

$$4P_1(N-3) + P_2(N) = P_3(N) + 4P_4(N-3).$$

Proof. Consider the modular equation for degree 5 [7, p. 281, Entry 13 (xi)],

$$\Big(\frac{\beta^5}{\alpha}\Big)^{1/8} + \Big\{\frac{(1-\beta)^5}{(1-\alpha)}\Big\}^{1/8} = m\Big\{\frac{1}{2}(1 + (\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2})\Big\}^{1/2}.$$

where m is defined by (1.4). By simple elementary algebra, we can find that

$$\frac{z_5}{z_1}\Big(\frac{\beta^5}{\alpha}\Big)^{1/8} + \frac{z_5}{z_1}\Big\{\frac{(1-\beta)^5}{(1-\alpha)}\Big\}^{1/8} = \frac{1}{2}\Big\{(1+\sqrt{1-\alpha})^{\frac{1}{2}}(1+\sqrt{1-\beta})^{\frac{1}{2}} + (1-\sqrt{1-\alpha})^{\frac{1}{2}}(1-\sqrt{1-\beta})^{\frac{1}{2}}\Big\}. \quad (3.4)$$

Multiply both sides of the identity (3.4) by $2(\alpha/q)^{-1/6}(1-\alpha)^{-1/24}(\beta/q^5)^{-1/3}(1-\beta)^{1/24}$ to obtain

$$\frac{2q^3 f^2(-q^{20})\chi(q^5)}{f^2(-q^4)\chi(q)} + \frac{1}{2}\frac{f^2(-q^{10})}{f^2(-q^2)}\chi^4(-q^5)\chi^4(-q^{10}) = \frac{1}{2}\chi^2(q^2)\chi^2(-q^{20}) + \frac{2q^3}{\chi^2(-q^4)\chi^2(q^{10})},$$

by the use of the identities (1.7)–(1.11) in Lemma 1.1 and (1.14) and (1.15) in Lemma 1.2. Making use of the definitions of f and χ in (1.5) and (1.6), respectively, and employing Euler's identity (1.16), we can rewrite the last equation in terms of q-products, namely,

$$\frac{2q^3(q^{20};q^{20})_\infty^2(-q^5;q^{10})_\infty}{(q^4;q^4)_\infty^2(-q;q^2)_\infty} + \frac{(q^{10};q^{10})_\infty^2(q^5;q^{10})_\infty^4}{2(q^2;q^2)_\infty^2(-q^{10};q^{10})_\infty^4} = \frac{(-q^2;q^4)_\infty^2}{2(-q^{20};q^{20})_\infty^2} + \frac{2q^3(-q^4;q^4)_\infty^2}{(-q^{10};q^{20})_\infty^2}.$$

Multiplying the last q-product identity by $2\frac{(-q;q^2)_\infty(-q^{10};q^{10})_\infty^4}{(-q^5;q^{10})_\infty(q^5;q^{10})_\infty^4}$, and using Euler's identity (1.16), we see that

$$\begin{aligned}&4q^3\frac{(q^{20};q^{20})_\infty^6}{(q^4;q^4)_\infty^2(q^5;q^5)_\infty^4} + \frac{(-q;q^2)_\infty(q^{10};q^{10})_\infty^2}{(-q^5;q^{10})_\infty(q^2;q^2)_\infty^2}\\ =&\frac{(-q;q^2)_\infty(-q^{10};q^{10})_\infty^2}{(-q^5;q^{10})_\infty(q^5;q^{10})_\infty^4}\{(-q^{10};q^{20})_\infty^2(-q^2;q^4)_\infty^2 + 4q^3(-q^4;q^4)_\infty^2(-q^{20};q^{20})_\infty^2\}.\end{aligned}$$

Equating the coefficients of q^N on both sides of the last equation, we finish the proof. □

Theorem 3.5. *Let S denote the set of partitions whose odd parts can occur in five distinct colors with the red color not appearing in multiples of* 5*, and even parts can occur at most once in five distinct colors with the red color not appearing in multiples of* 5*. Let T denote the set of partitions whose odd parts occur in four different colors, and even parts occur in each of* 14 *different colors such that the orange and white parts are not in multiples of* 5*, the red and blue parts appearing at most once are in multiples of* 4*, the green, pink, yellow, and orange parts appearing at most once are congruent to* ± 2 *and* ± 6 *modulo* 20*, and the remaining six colors appear at most once with parts congruent to* 10 *modulo* 20*. Let U denote the set of partitions whose odd parts occur in four different colors, and even parts occur in each of fourteen different colors such that the black and white parts are not in multiples of* 5*, the red and blue parts appearing at most once are congruent to* 2 *modulo* 4*, the green, pink, yellow and orange parts appearing at most once are in multiples of* 4 *but not multiples of* 5*, and the remaining six colors appear at most once with parts in multiples of* 20*. Let $P_1(N)$, $P_2(N)$, and $P_3(N)$ be the number of partitions of N in S, T, and U, respectively. Then, for $N \geq 4$,*

$$4P_1(N) = 5P_2(N) + 20P_3(N-3).$$

Proof. Recall the formula for the multiplier of degree 5 (see [7, p. 281, Entry 13 (xi)]),

$$\Big(\frac{\alpha^5}{\beta}\Big)^{1/8} + \Big\{\frac{(1-\alpha)^5}{(1-\beta)}\Big\}^{1/8} = \frac{5}{m}\Big\{\frac{1}{2}(1 + (\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2})\Big\}^{1/2},$$

where m is defined by (1.4). By simple elementary algebra, we can easily check that

$$\frac{z_1}{z_5}\Big(\frac{\alpha^5}{\beta}\Big)^{\frac{1}{8}} + \frac{z_1}{z_5}\Big\{\frac{(1-\alpha)^5}{(1-\beta)}\Big\}^{\frac{1}{8}} = \frac{5}{2}\Big\{(1+\sqrt{1-\alpha})^{\frac{1}{2}}(1+\sqrt{1-\beta})^{\frac{1}{2}} + (1-\sqrt{1-\alpha})^{\frac{1}{2}}(1-\sqrt{1-\beta})^{\frac{1}{2}}\Big\}. \quad (3.5)$$

Multiply both sides of the identity (3.5) by $2(\alpha/q)^{-1/3}(1-\alpha)^{1/24}(\beta/q^5)^{-1/6}(1-\beta)^{-1/24}$. Hence, from (1.7), (1.10), (1.11), (1.14), and (1.15), it suffices to prove that

$$2\frac{f^2(-q^2)\chi(-q^5)\chi(-q^{10})}{f^2(-q^{10})\chi(-q)\chi(-q^2)} + \frac{1}{2}\frac{f^2(-q^2)}{f^2(-q^{10})}\chi^4(-q^2)\chi^4(-q) = \frac{5}{2}\chi^2(-q^4)\chi^2(q^{10}) + \frac{10q^3}{\chi^2(q^2)\chi^2(-q^{20})}.$$

Using the aforementioned representations of f and χ in (1.5) and (1.6), respectively, and Euler's identity (1.16), we find that

$$\frac{2(q^2;q^2)_\infty^2(-q^2;q^2)_\infty(q^5;q^{10})_\infty}{(q^{10};q^{10})_\infty^2(-q^{10};q^{10})_\infty(q;q^2)_\infty} + \frac{(q^2;q^2)_\infty^2(q;q^2)_\infty^4}{2(q^{10};q^{10})_\infty^2(-q^2;q^2)_\infty^4} = \frac{5(-q^{10};q^{20})_\infty^2}{2(-q^4;q^4)_\infty^2} + \frac{10q^3(-q^{20};q^{20})_\infty^2}{(-q^2;q^4)_\infty^2}.$$

Multiplying the last q-product identity by $2\frac{(-q^2;q^2)_\infty^4(q^{10};q^{10})_\infty^2}{(q;q^2)_\infty^4(q^2;q^2)_\infty^2}$ and applying Euler's identity (1.16), we deduce that

$$\begin{aligned}&4\frac{(-q^2;q^2)_\infty^5(q^5;q^{10})_\infty}{(-q^{10};q^{10})_\infty(q;q^2)_\infty^5} + 1\\ =&\frac{5(q^{10};q^{10})_\infty^2(-q^2;q^2)_\infty^2}{(q^2;q^2)_\infty^2(q;q^2)_\infty^4}\{(-q^{10};q^{20})_\infty^2(-q^2;q^4)_\infty^2 + 4q^3(-q^4;q^4)_\infty^2(-q^{20};q^{20})_\infty^2\}.\end{aligned}$$

Equating the coefficients of q^N on both sides of the equation, we complete the proof. □

4. Colored partition identities by modular equations with degree 7

Theorem 4.1. *Let S denote the set of partitions whose odd parts occur at most once in nine different colors such that the red, blue, and green parts are not in multiples of 7, and the remaining six colors appear with parts in multiples of 7. Let T denote the set of partitions whose odd parts not in multiples of 7 occur in one color, odd parts being multiples of 7 occur in two different colors, and even parts occur at most once in each of six different colors such that the red and blue parts are congruent to 2 modulo 4 but not multiples of 7, the remaining four colors appear with parts congruent to 14 modulo 28. Let U denote the set of partitions whose odd parts not in multiples of 7 occur in one color, odd parts in multiples of 7 occur in two different colors, and even parts occur at most once in each of six different colors such that the red and blue parts are in multiples of 4 but not multiples of 7, and the remaining four colors appear with parts in multiples of 28. Let $P_1(N)$, $P_2(N)$, and $P_3(N)$ be the number of partitions of N in S, T, and U, respectively. Then, for $N \geq 5$,*

$$P_1(N) = P_2(N) + 4P_3(N-4).$$

Proof. Recall the formula for degree 7 [7, p. 314, Entry 19 (i)],

$$1 - \left\{\alpha\beta(1-\alpha)(1-\beta)\right\}^{1/8} = \left\{\frac{1}{2}(1 + (\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2})\right\}^{1/2}. \tag{4.1}$$

Utilizing simple elementary algebra, we can find that

$$1 - \left\{\alpha\beta(1-\alpha)(1-\beta)\right\}^{1/8} = \frac{1}{2}\left\{(1+\sqrt{1-\alpha})^{\frac{1}{2}}(1+\sqrt{1-\beta})^{\frac{1}{2}} + (1-\sqrt{1-\alpha})^{\frac{1}{2}}(1-\sqrt{1-\beta})^{\frac{1}{2}}\right\}. \tag{4.2}$$

Multiply both sides of the identity (4.2) by $2^{1/3}(\alpha/q)^{-1/6}(1-\alpha)^{-1/24}(\beta/q^7)^{-1/6}(1-\beta)^{-1/24}$. Hence,

$$\frac{1}{2}\chi^2(q)\chi(-q^2)\chi^2(q^7)\chi(-q^{14}) - q\chi(-q)\chi(-q^7) = \frac{1}{2}\chi^2(q^2)\chi^2(q^{14}) + \frac{2q^4}{\chi^2(-q^4)\chi^2(-q^{28})},$$

by (1.9)–(1.11) in Lemma 1.1 and (1.14) and (1.15) in Lemma 1.2. Because of the definition of χ in (1.6) and Euler's identity (1.16), we can rewrite the formula above in the form

$$\begin{aligned}&\frac{(-q;q^2)^2_\infty(-q^7;q^{14})^2_\infty}{2(-q^2;q^2)_\infty(-q^{14};q^{14})_\infty} - q(q;q^2)_\infty(q^7;q^{14})_\infty \\ =&\frac{1}{2}(-q^2;q^4)^2_\infty(-q^{14};q^{28})^2_\infty + 2q^4(-q^4;q^4)^2_\infty(-q^{28};q^{28})^2_\infty.\end{aligned}$$

Multiplying both sides of the last identity by $\frac{2}{(q;q^2)_\infty(q^7;q^{14})_\infty}$ and using Euler's identity (1.16), we have

$$(-q;q^2)^3_\infty(-q^7;q^{14})^3_\infty - 2q = \frac{1}{(q;q^2)_\infty(q^7;q^{14})_\infty}\{(-q^2;q^4)^2_\infty(-q^{14};q^{28})^2_\infty + 4q^4(-q^4;q^4)^2_\infty(-q^{28};q^{28})^2_\infty\}.$$

Extract the coefficients of q^N on both sides of the equation to finish the proof. □

Theorem 4.2. *Let S denote the set of partitions whose even parts not in multiples of 7 can occur in two distinct colors, and odd parts not in multiples of 7 appear at most once. Let T denote the set of* *partitions whose odd parts in multiples of* 7 *can occur in six distinct colors, even parts can occur in eight distinct colors such that the red and blue parts are in multiples of* 4 *and not multiples of* 7*, and the parts of the remaining six colors are congruent to* 14 *modulo* 28*. Let U denote the set of partitions into distinct colors, where odd parts in multiples of* 7 *have eight colors, odd parts not in multiples of* 7 *are distinct, the green and pink parts are distinct parts congruent to* 2 *modulo* 4 *not in multiples of* 7*, and the parts of the other four colors are distinct parts congruent to* 14 *modulo* 28*. Let V denote the set of partitions into distinct colors, where odd parts in multiples of* 7 *have eight colors, odd parts not in multiples of* 7 *are distinct, the red and blue parts are distinct parts in multiples of* 4 *but not multiples of* 7*, and the parts of the other four colors are distinct parts congruent to* 0 *modulo* 28*. Let $P_1(N)$, $P_2(N)$, $P_3(N)$, and $P_4(N)$ be the number of partitions of N in S, T, U, and V, respectively. Then, for $N \geq 7$,*

$$P_1(N) - 8P_2(N-6) = P_3(N) + 4P_4(N-4).$$

Proof. Consider the modular equation of degree 7 [7, p. 314, Entry 19 (iii)],

$$\left\{\frac{(1-\beta)^7}{1-\alpha}\right\}^{1/8} - \left(\frac{\beta^7}{\alpha}\right)^{1/8} = m\left\{\frac{1}{2}(1+(\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2})\right\}^{1/2},$$

where m is defined by (1.4). Applying elementary algebra, we have

$$\frac{z_7}{z_1}\left\{\frac{(1-\beta)^7}{1-\alpha}\right\}^{1/8} - \frac{z_7}{z_1}\left(\frac{\beta^7}{\alpha}\right)^{1/8} = \frac{1}{2}\left\{(1+\sqrt{1-\alpha})^{\frac{1}{2}}(1+\sqrt{1-\beta})^{\frac{1}{2}} + (1-\sqrt{1-\alpha})^{\frac{1}{2}}(1-\sqrt{1-\beta})^{\frac{1}{2}}\right\}. \quad (4.3)$$

Multiply both sides of the identity (4.3) by $2(\alpha/q)^{-1/6}(1-\alpha)^{-1/24}(\beta/q^7)^{-1/3}(1-\beta)^{1/24}$. Examining both sides of the resulting formula, we call upon (1.7)–(1.11), (1.14), and (1.15) to find that

$$\frac{f^2(-q^{14})\chi^8(-q^7)\chi^2(-q^{14})}{2f^2(-q^2)} - \frac{4q^6 f^2(-q^{28})}{f^2(-q^4)\chi(q)\chi(q^7)\chi^2(-q^{14})} = \frac{1}{2}\chi^2(q^2)\chi^2(-q^{28}) + \frac{2q^4}{\chi^2(-q^4)\chi^2(q^{14})}.$$

Employing the definitions of f and χ in (1.5) and (1.6), respectively, and Euler's identity (1.16), we deduce that

$$\frac{(q^{14};q^{14})_\infty^2(q^7;q^{14})_\infty^8}{2(q^2;q^2)_\infty^2(-q^{14};q^{14})_\infty^2} - \frac{4q^6(q^{28};q^{28})_\infty^2(-q^{14};q^{14})_\infty^2}{(q^4;q^4)_\infty^2(-q;q^2)_\infty(-q^7;q^{14})_\infty} = \frac{(-q^2;q^4)_\infty^2}{2(-q^{28};q^{28})_\infty^2} + \frac{2q^4(-q^4;q^4)_\infty^2}{(-q^{14};q^{28})_\infty^2}.$$

Multiply the last q-product identity by $2\frac{(-q;q^2)_\infty(-q^{14};q^{14})_\infty^2}{(q^7;q^{14})_\infty^8(-q^7;q^{14})_\infty}$ and use Euler's identity (1.16) to obtain

$$\frac{(-q;q^2)_\infty(q^{14};q^{14})_\infty^2}{(-q^7;q^{14})_\infty(q^2;q^2)_\infty^2} - 8q^6\frac{(q^{28};q^{28})_\infty^2}{(q^4;q^4)_\infty^2(q^{14};q^{28})_\infty^6(q^7;q^{14})_\infty^6}$$
$$= \frac{(-q;q^2)_\infty}{(-q^7;q^{14})_\infty(q^7;q^{14})_\infty^8}\{(-q^{14};q^{28})_\infty^2(-q^2;q^4)_\infty^2 + 4q^4(-q^4;q^4)_\infty^2(-q^{28};q^{28})_\infty^2\}.$$

Equating the coefficients of q^N on both sides of the equation, we complete the proof. □

Theorem 4.3. *Let S denote the set of partitions whose odd parts can occur in seven distinct colors with the red color not appearing in multiples of* 7*, and even parts appearing at most once can occur in seven distinct colors with the red color not appearing in multiples of* 7*. Let T denote the set of* *partitions whose odd parts can occur in eight distinct colors, and even parts can occur in eight distinct colors, with the red and blue even parts appearing not in multiples of* 7*, the green and pink colors appearing at most once with parts congruent to* 2 *modulo* 4 *not multiples of* 7*, and the remaining four colors appearing at most once with parts congruent to* 4 *modulo* 28*. Let U denote the set of partitions into* 8 *distinct colors, where the odd parts have eight colors, the red and blue parts are even parts not in multiples of* 7*, the green and pink parts are distinct parts in multiples of* 4 *but not multiples of* 7*, and the remaining four colors are distinct parts congruent to* 0 *modulo* 28*. Let $P_1(N)$, $P_2(N)$, and $P_3(N)$ be the number of partitions of N in S, T, and U, respectively. Then, for $N \geq 3$,*

$$8P_1(N) = 7P_2(N) + 28P_3(N-4).$$

Proof. Recall the modular equation for degree 7 [7, p. 314, Entry 19 (iii)],

$$\left(\frac{\alpha^7}{\beta}\right)^{1/8} - \left\{\frac{(1-\alpha)^7}{1-\beta}\right\}^{1/8} = \frac{7}{m}\left\{\frac{1}{2}(1+(\alpha\beta)^{1/2}+\{(1-\alpha)(1-\beta)\}^{1/2})\right\}^{1/2},$$

where m is defined by (1.4). By utilizing elementary algebraic methods, we can easily confirm that

$$\begin{aligned}&\frac{z_1}{z_7}\left(\frac{\alpha^7}{\beta}\right)^{1/8} - \frac{z_1}{z_7}\left\{\frac{(1-\alpha)^7}{1-\beta}\right\}^{1/8}\\ =&\frac{7}{2}\left\{(1+\sqrt{1-\alpha})^{\frac{1}{2}}(1+\sqrt{1-\beta})^{\frac{1}{2}} + (1-\sqrt{1-\alpha})^{\frac{1}{2}}(1-\sqrt{1-\beta})^{\frac{1}{2}}\right\}.\end{aligned} \tag{4.4}$$

Multiply both sides of the identity (4.4) by $2(\alpha/q)^{-1/3}(1-\alpha)^{1/24}(\beta/q^7)^{-1/6}(1-\beta)^{-1/24}$. Thus,

$$\begin{aligned}&\frac{4f^2(-q^2)\chi(-q^7)\chi(-q^{14})}{f^2(-q^{14})\chi(-q)\chi^3(-q^2)\chi^2(q)} - \frac{1}{2}\frac{f^2(-q^2)}{f^2(-q^{14})}\chi^4(-q^2)\chi^4(-q)\frac{\chi^2(-q)}{\chi^2(q)}\\ =&\frac{7}{2}\chi^2(-q^4)\chi^2(q^{14}) + \frac{14q^4}{\chi^2(q^2)\chi^2(-q^{28})},\end{aligned}$$

by (1.7), (1.9)–(1.11) in Lemma 1.1, and (1.14), (1.15). Making use of the definitions of f and χ in (1.5) and (1.6), respectively, and employing Euler's identity (1.16), we get

$$\frac{4(q^2;q^2)^2_\infty(-q^2;q^2)_\infty(q^7;q^{14})_\infty(-q^2;q^2)^2_\infty}{(q^{14};q^{14})^2_\infty(-q^{14};q^{14})_\infty(q;q^2)_\infty(-q;q^2)^2_\infty} - \frac{(q^2;q^2)^2_\infty(q;q^2)^8_\infty}{2(q^{14};q^{14})^2_\infty(-q^2;q^2)^2_\infty} = \frac{7(-q^{14};q^{28})^2_\infty}{2(-q^4;q^4)^2_\infty} + \frac{14q^4(-q^{28};q^{28})^2_\infty}{(-q^2;q^4)^2_\infty}.$$

Multiplying the last equation by $2\frac{(-q^2;q^2)^2_\infty(q^{14};q^{14})^2_\infty}{(q;q^2)^8_\infty(q^2;q^2)^2_\infty}$ and using Euler's identity (1.16), we arrive at

$$\frac{8(-q^2;q^2)^7_\infty(q^7;q^{14})_\infty}{(-q^{14};q^{14})_\infty(q;q^2)^7_\infty} - 1 = \frac{7(q^{14};q^{14})^2_\infty}{(q^2;q^2)^2_\infty(q;q^2)^8_\infty}\{(-q^{14};q^{28})^2_\infty(-q^2;q^4)^2_\infty + 4q^4(-q^4;q^4)^2_\infty(-q^{28};q^{28})^2_\infty\}.$$

Equating the coefficients of q^N on both sides of the equation, we complete the proof. □

5. Conclusions

Using the theory of modular equations, we have proven 15 partition identities, each of which has parts that are in multiples of 4.

Author contributions

Roberta R. Zhou: Writing -original draft, Methodology; F. Ren: Software, Writing- Reviewing and Editing. Both authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

The authors have no relevant financial or non-financial interests to disclose.

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