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Research article

Traveling wave fronts in a single species model with cannibalism and strongly nonlocal effect

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Abstract: In this paper we studied traveling front solutions of a single species model with cannibalism and nonlocal effect. For a particular class of kernels, the existence of traveling front solutions connecting the extinction state with the positive equilibrium was established for the strongly nonlocal effect case. Our approach was to reformulate it as a singular perturbed problem, and then tackle this problem by using dynamical systems techniques, in particular, geometric singular perturbation theory and Fenichel's invariant manifold theory.

Keywords: traveling wave front; strongly nonlocal effect; cannibalism; geometric singular perturbation theory; invariant manifold theory
Mathematics Subject Classification: 34D15, 34E15, 35C07, 35Q53

1. Introduction

In this paper, we are concerned with the following nonlocal reaction diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + ru\left[1 - \frac{(f * u)(x,t)}{K}\right] - hu^2, \tag{1.1}$$

for $-\infty < x < +\infty$ and $t > 0$, where the parameter $r > 0, K > 0, h > 0$ and the spatiotemporal convolution $f * u$ is defined by

$$(f * u)(x,t) = \int_{-\infty}^{t} \int_{-\infty}^{+\infty} f(x-y, t-s)u(y,s)dyds, \tag{1.2}$$

and the kernel f satisfies the usual normalization assumption, namely

$$\int_0^{+\infty}\int_{-\infty}^{+\infty} f(x,t)dxdt = 1,$$

so that the kernel does not affect the spatially uniform steady states, which in this model will be the extinction state $u \equiv 0$ and the positive equilibrium $u \equiv u^* = \frac{Kr}{r+Kh}$. This type of equation was introduced in [1] to model the dynamical behavior of a single species. The parameters r, K, h represent the intrinsic growth rate of the species, the capacity of the environment, and cannibalism rate, respectively. More specifically, the term hu^2 signifies intraspecific cannibalism, which is a widespread phenomenon in a variety of animals. Not only could it result in an increase in death rate, but it could also possess the potential of regulating population size. The convolution term $\frac{(f*u)(x,t)}{K}$ signifies the nonlocal consumption of the resources. For the specific biological background to the model, please refer to [2, 3].

When the kernel f is taken to be $f(x,t) = \frac{1}{\sqrt{4\pi\rho}}e^{-\frac{x^2}{4\rho}}\delta(t)$, ρ represents the nonlocal effect. By applying super-sub solution method [4–6] as well as Leray-Schauder topological degree theory [7], Zhang and Li [1] proved that there exist traveling wave fronts connecting the equilibrium $u \equiv 0$ to the positive equilibrium $u \equiv u^* = \frac{Kr}{r+Kh}$ when the wave speed $c \geq 2\sqrt{r}$. In the present paper, we are interested in traveling front solutions for another particular class of kernels of the form

$$f(x,t) = \frac{1}{2\rho}e^{-\frac{|x|}{\rho}}\delta(t), \tag{1.3}$$

in which the parameter ρ is representative of the nonlocal effect. For such a type of kernel function (1.3), the nonlinear convolution term $\frac{(f*u)(x,t)}{K}$ implies that the individuals in the population consume the resources not only at the point where they are located but also in some area around this point. We shall consider the traveling wave problem of Eq (1.1) when $\frac{1}{\rho} \ll 1$, i.e., the parameter ρ is sufficiently large, which signifies the strongly nonlocal effect. As pointed out in [6], this can be understood as the limit of a highly mobile resource in which the population is represented by u feeds. Note that this particular choice of kernel like (1.3) is the Green's function for an ordinary differential equation, we can rewrite Eq (1.1) as the coupled reaction-diffusion equations. Thus, the phase space of the system for the traveling wave problem corresponding to Eq (1.1) is four-dimensional. A traveling wave front can be characterized as a heteroclinic connection in this phase space, and then the dynamical systems theory, especially the geometric singular perturbation theory and Fenichel's invariant manifold theory [8–10], can be successfully used to establish the existence of such a connection.

The remaining part of this paper is organized as follows. In Section 2, we formulate the traveling wave problem of system (1.1) with the kernel (1.3) from the viewpoint of the dynamical system, which can be viewed as a singular perturbation problem when ρ is taken to be a sufficiently large perturbed parameter. In Section 3, by analyzing the dynamics of limiting slow and limiting fast systems for the singular perturbation problem, we give a singular heteroclinic orbit in the phase space of the traveling wave system of Eq (2.4), which is composed of the solutions of limiting slow and fast systems. In Section 4, we employ geometric singular perturbation theory and Fenichel's invariant manifold theory to show that the above singular heteroclinic orbit persists if the parameter ρ is taken to be sufficiently large. Finally, we summarize our results in Section 5.

2. Geometric singular perturbation formulation for traveling wave problem

In this section, we will formulate the traveling wave problem of system (1.1) as a geometric singular perturbation problem.

First, if we define $w = f * u$, namely,

$$w(x,t) = \int_{-\infty}^{t}\int_{-\infty}^{+\infty} \frac{1}{2\rho} e^{-\frac{|x-y|}{\rho}} \delta(t-s)u(y,s)dyds, \tag{2.1}$$

it is straightforward to see that w satisfies

$$\frac{\partial^2 w}{\partial x^2} + \frac{1}{\rho^2}(u-w) = 0, \tag{2.2}$$

and, thus the integrodifferential equation (1.1) can be rewritten as the following coupled reaction-diffusion system

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + ru\left(1 - \dfrac{w}{K}\right) - hu^2, \\ \dfrac{\partial^2 w}{\partial x^2} + \dfrac{1}{\rho^2}(u-w) = 0. \end{cases} \tag{2.3}$$

Let $\varepsilon = \frac{1}{\rho}$, then ε is sufficiently small if ρ is sufficiently large. Our objective now is to establish the existence of traveling wavefront solutions of (2.3) connecting the two uniform steady-states $(u,w) = (0,0)$ and (u^*,u^*), for sufficiently small ε. Converting to traveling wave form, by setting

$$u(x,t) = U(z), \quad w(x,t) = W(z), \quad z = x - ct,$$

we have

$$\begin{cases} \dfrac{d^2U}{dz^2} + c\dfrac{dU}{dz} + rU(1 - aU - bW) = 0, \\ \dfrac{d^2W}{dz^2} + \varepsilon^2(U - W) = 0, \end{cases} \tag{2.4}$$

where

$$a := \frac{h}{r}, \quad b := \frac{1}{K}. \tag{2.5}$$

Note that system (2.4) is invariant under the transformation $(c,z) \mapsto (-c;-z)$ and thus we may assume, without loss of generality, that $c > 0$. Upon introducing the two new variables $V := \frac{dU}{dz}$ and $Y := \varepsilon^{-1}\frac{dW}{dz}$, system (2.4) can be reformulated as

$$\begin{cases} \dfrac{dU}{dz} = V, \\ \dfrac{dV}{dz} = -cV - rU(1 - aU - bW), \\ \dfrac{dW}{dz} = \varepsilon Y, \\ \dfrac{dY}{dz} = \varepsilon(W - U), \end{cases} \tag{2.6}$$

which is called the fast system provided that ε is sufficiently small. In terms of the slow scale $\xi := \varepsilon z$, the corresponding slow system of (2.6) becomes

$$\begin{cases} \varepsilon \dfrac{dU}{d\xi} = V, \\ \varepsilon \dfrac{dV}{d\xi} = -cV - rU(1 - aU - bW), \\ \dfrac{dW}{d\xi} = Y, \\ \dfrac{dY}{d\xi} = W - U. \end{cases} \tag{2.7}$$

Thus, traveling wave fronts of (1.1) correspond to heteroclinic orbits of the fast system (2.6) or the slow system (2.7) connecting its two equilibrium points, that is,

$$\begin{cases} \lim\limits_{z\to-\infty} (U, V, W, Y) = (u^*, 0, u^*, 0) := A^-, \\ \lim\limits_{z\to+\infty} (U, V, W, Y) = (0, 0, 0, 0) := A^+. \end{cases} \tag{2.8}$$

3. Properties of limiting systems

In this section, we consider the fast and slow systems (2.6) and (2.7) from the geometric singular perturbation point of view. When $\varepsilon = 0$, we have the following limiting fast and limiting slow systems

$$\begin{cases} \dfrac{dU}{dz} = V, \\ \dfrac{dV}{dz} = -cV - rU(1 - aU - bW), \\ \dfrac{dW}{dz} = 0, \\ \dfrac{dY}{dz} = 0, \end{cases} \tag{3.1}$$

and

$$\begin{cases} 0 = V, \\ 0 = -cV - rU(1 - aU - bW), \\ \dfrac{dW}{d\xi} = Y, \\ \dfrac{dY}{d\xi} = W - U. \end{cases} \tag{3.2}$$

Thus, the critical manifold S is given by

$$S := \left\{(U, V, W, Y) \in \mathbb{R}^4 \mid V = 0,\ U(1 - aU - bW) = 0\right\}, \tag{3.3}$$

which is the set of equilibria of the limiting fast system (3.1). This critical manifold S consists of the two two-dimensional manifolds S_1, S_2, which can be parameterized by the slow variables W and given by

$$S_1 := \left\{(U, V, W, Y) \in \mathbb{R}^4 \mid V = 0,\ U = 0\right\}, \tag{3.4}$$

and

$$S_2 := \left\{ (U, V, W, Y) \in \mathbb{R}^4 \middle| V = 0, \;\; U = \frac{1}{a}(1 - bW) \right\}. \tag{3.5}$$

Moreover, the manifolds S_1 and S_2 intersect along the line $W = \frac{1}{b}$. See Figure 1 for a schematic depiction of the two manifolds S_1, S_2 and the heteroclinic orbit associated to the traveling wave front.

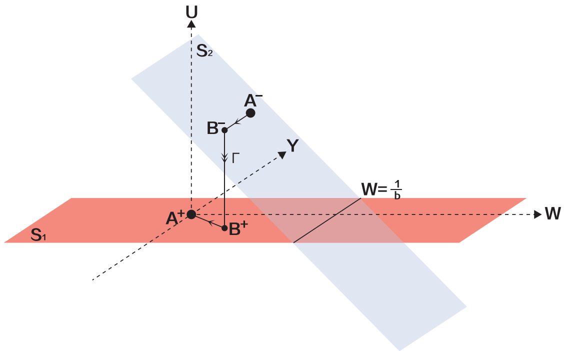

Figure 1. Schematic depiction of the two manifolds S_1, S_2 and the heteroclinic orbit associated to the traveling wave front. The big black dots indicate the equilibrium points $A^{\pm}$ that determine the asymptotic boundary condition (2.8). The black line at $W = \frac{1}{b}$ indicates the location where the manifolds S_1 and S_2 coincide.

3.1. limiting slow system

Next, we study the reduced dynamics on the critical manifold S. It follows from the limiting slow system (3.2) that the reduced dynamics on the manifold S_1 is determined by the linear system

$$\begin{cases} \dfrac{dW}{d\xi} = Y, \\ \dfrac{dY}{d\xi} = W, \end{cases} \tag{3.6}$$

since $U = 0$ on S_1. The solutions of system (3.6) can be directly solved by

$$\begin{cases} W(\xi) = C_1 e^{\xi} + C_2 e^{-\xi}, \\ Y(\xi) = C_1 e^{\xi} - C_2 e^{-\xi}, \end{cases} \tag{3.7}$$

for arbitrary constants C_1, C_2. Similarly, the reduced dynamics on the manifold S_2 are determined by the linear system

$$\begin{cases} \dfrac{dW}{d\xi} = Y, \\ \dfrac{dY}{d\xi} = (1 + \dfrac{b}{a})W - \dfrac{1}{a}, \end{cases} \tag{3.8}$$

since $U = \frac{1}{a}(1 - bW)$ on S_2. The solutions of system (3.8) can be directly solved by

$$\begin{cases} W(\xi) = \dfrac{1}{a+b} + C_3 e^{\sqrt{\frac{a+b}{a}}\xi} + C_4 e^{-\sqrt{\frac{a+b}{a}}\xi}, \\ Y(\xi) = C_3 \sqrt{\dfrac{a+b}{a}} e^{\sqrt{\frac{a+b}{a}}\xi} - C_4 \sqrt{\dfrac{a+b}{a}} e^{-\sqrt{\frac{a+b}{a}}\xi}, \end{cases} \tag{3.9}$$

for arbitrary constants C_3, C_4. These constants $C_i (i = 1, 2, 3, 4)$ are determined by the asymptotic boundary conditions (2.8) and by the dynamics of layer problem (3.1). Following the ideas used in [11], we divide our spatial domain into three fields (with respect to the slow variable ξ): two slow fields I_s^-, I_s^+ which are away from the layer dynamics and one fast field I_f which is near the layer dynamics. Without loss of generality, we assume here that the layer dynamics are centered around zero. Thus, these fast and slow fields can be chosen as follows

$$I_s^- := (-\infty, -\varepsilon^{\frac{1}{2}}), \quad I_f := [-\varepsilon^{\frac{1}{2}}, \varepsilon^{\frac{1}{2}}], \quad I_s^+ := (\varepsilon^{\frac{1}{2}}, +\infty), \tag{3.10}$$

where I_f corresponds to the layer dynamics from S_2 to S_1, while I_s^- and I_s^+ correspond to the reduced dynamics on S_2 and S_1, respectively. With the asymptotic scaling $\varepsilon^{\frac{1}{2}}$, we choose to ensure that it is asymptotically small with respect to the slow variable ξ and asymptotically large with respect to the fast variable $z := \varepsilon^{-1}\xi$. In fact, it is not hard to find that $\varepsilon^{\frac{1}{2}} \ll 1$ and $\varepsilon^{\frac{1}{2}-1} \gg 1$.

According to the asymptotic boundary conditions (2.8), the heteroclinic orbit associated to the traveling front solution should approach A^- as $\xi \to -\infty$. So, the critical manifold of interest is S_2 for $\xi \in I_s^-$ (see the top frame of Figure 2). Thus, the slow variable W and Y are determined by (3.9). Note that $W(-\infty) = u^*, Y(-\infty) = 0$, then we can derive that $C_4 = 0$. Similarly, for $\xi \in I_s^+$, the critical manifold of interest is S_1 (see the bottom frame of Figure 2), and the slow variables W and Y are determined by (3.7). The boundary condition $W(+\infty) = Y(+\infty) = 0$ yields that $C_1 = 0$. Consequently, the solutions (3.7) and (3.9) become

$$\begin{cases} W(\xi) = C_2 e^{-\xi}, \\ Y(\xi) = -C_2 e^{-\xi}, \end{cases} \tag{3.11}$$

and

$$\begin{cases} W(\xi) = \dfrac{1}{a+b} + C_3 e^{\sqrt{\frac{a+b}{a}}\xi}, \\ Y(\xi) = C_3 \sqrt{\dfrac{a+b}{a}} e^{\sqrt{\frac{a+b}{a}}\xi}. \end{cases} \tag{3.12}$$

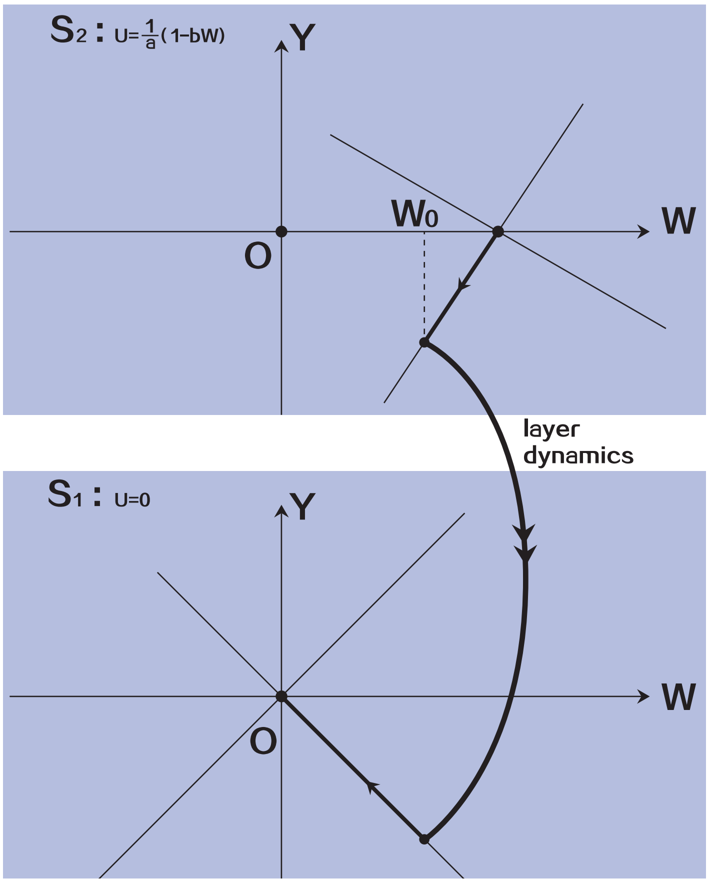

Figure 2. Schematic depiction of the slow flow on the different two branches of the critical manifold for the heteroclinic orbit associated to the traveling wave front, and the jump between the branches of the critical manifold (i.e., the fast transition) occurs at $W = W_0$.

During the transition through the fast field I_f, the evolution equations for the slow variables W and Y are given by

$$\begin{cases} \dfrac{dW}{dz} = \varepsilon Y, \\ \dfrac{dY}{dz} = \varepsilon(W - U). \end{cases} \tag{3.13}$$

Note that $\varepsilon \ll \varepsilon^{\frac{1}{2}}$, hence the changes of both W and Y are, to leading order, constant during the transition through the fast field I_f. In other words, both W and Y should match to leading order at zero, i.e., $W(0^-) = W(0^+), Y(0^-) = Y(0^+)$. By substituting this into (3.11) and (3.12), the two remaining constants C_2 and C_3 can be determined and given by

$$C_2 = \frac{\sqrt{1+\frac{b}{a}}}{(a+b)\left(1+\sqrt{1+\frac{b}{a}}\right)}, \quad C_3 = -\frac{1}{(a+b)\left(1+\sqrt{1+\frac{b}{a}}\right)}.$$

Therefore, we have

$$W(\xi) = \begin{cases} u^* - (u^* - W_0)e^{\sqrt{1+\frac{b}{a}}\xi}, & \xi \in I_s^-, \\ W_0 e^{-\xi}, & \xi \in I_s^+, \end{cases} \tag{3.14}$$

and

$$Y(\xi) = \begin{cases} -W_0 e^{\sqrt{1+\frac{b}{a}}\xi}, & \xi \in I_s^-, \\ -W_0 e^{-\xi}, & \xi \in I_s^+, \end{cases} \tag{3.15}$$

where

$$W_0 = \frac{\sqrt{1+\frac{b}{a}}}{(a+b)\left(1+\sqrt{1+\frac{b}{a}}\right)}. \tag{3.16}$$

It is easily seen that the fast transition always occurs at $W = W_0$. Furthermore, by combining (3.4)–(3.5) with (3.14)–(3.15), the leading order profiles for other variables in the slow fields can now be successfully obtained. In particular, we have

$$aU(\xi) = \begin{cases} 1 - bu^* + b(u^* - W_0)e^{\sqrt{1+\frac{b}{a}}\xi}, & \xi \in I_s^-, \\ 0, & \xi \in I_s^+, \end{cases}$$

and

$$V(\xi) = 0, \quad \xi \in I_s^- \cup I_s^+.$$

3.2. *Limiting fast system*

Now, we turn to study the layer dynamics in the fast field I_f. In fact, the dynamics of the heteroclinic orbit are, to leading order, determined by the limiting fast system (3.1), and the orbit has to transit from S_2 to S_1. Note that W is to leading order constant in the fast field, i.e., $W = W_0$ in I_f. Consequently, the two-components (U, V) equations in the limiting fast system with $W = W_0$ becomes

$$\begin{cases} \dfrac{dU}{dz} = V, \\ \dfrac{dV}{dz} = -cV - rU(L - aU), \end{cases} \tag{3.17}$$

where

$$L = 1 - bW_0. \tag{3.18}$$

Obviously, system (3.17) can be rewritten as the following two-order ODE

$$\frac{d^2U}{dz^2} + c\frac{dU}{dz} + rU(L - aU) = 0, \tag{3.19}$$

which is exactly the traveling wave equation for the classical Fisher-KPP equation. It is well-known that system (3.17) admits a heteroclinic orbit connecting its two uniform steady-state $(U, V) = (0, 0)$ and $(U, V) = (\frac{L}{a}, 0)$. Moreover, system (3.19) has a unique monotonic decreasing solution if and only if,

$$c \geq c_m := 2\sqrt{rL}, \tag{3.20}$$

where c_m is the so-called minimum wave speed for the associated Fisher-KPP equation. This restriction on c arises from the fact that the equilibrium point $(U, V) = (0, 0)$ in the planar system (3.17) will change from a stable node to a stable focus as c decreases past c_m, so that U becomes negative for sufficiently large z when $c < c_m$. In particular, for sufficiently large $z \gg 1$, we have

$$U = \begin{cases} O(ze^{-\frac{1}{2}c_m z}), & c = c_m, \\ O(e^{\mu_+ z}), & c > c_m, \end{cases}$$

where

$$\mu_\pm = \frac{-c \pm \sqrt{c^2 - 4rL}}{2}.$$

For more details, please refer to [12] and references therein.

3.3. Singular heteroclinic orbit in the singular limit $\varepsilon \to 0$

Based on the above analyses on limiting slow and limiting fast systems, we are now able to construct singular heteroclinic orbit in the singular limit $\varepsilon \to 0$.

Let's denote by $U(A^-)$ (respectively, $S(A^+)$) the unstable (respectively, stable) manifold of $A^- = (u^*, 0, u^*, 0)$ (respectively, $A^+ = (0, 0, 0, 0)$) on S_2 (respectively, S_1). It follows from (3.14)–(3.15) that $U(A^-)$ and $S(A^+)$ can be explicitly represented as

$$U(A^-) := \left\{ (U, V, W, Y) \in S_2 \,\middle|\, Y = \sqrt{1 + \frac{b}{a}}(W - u^*) \right\},$$

and

$$S(A^+) := \{(U, V, W, Y) \in S_1 \,|\, Y = -W\}.$$

It is easy to see that both $U(A^-)$ and $S(A^+)$ are straight lines. Let Λ_- be the limiting slow orbit from A^- to $B^- := (\frac{L}{a}, 0, W_0, -W_0) \in S_2$, and Λ_+ be the limiting slow orbit from $B^+ := (0, 0, W_0, -W_0) \in S_1$ to A^+. Then, we have

$$\Lambda_- := \{(U(\xi), V(\xi), W(\xi), Y(\xi)) \in U(A^-) \,|\, -\infty < \xi < 0\},$$

and

$$\Lambda_+ := \{(U(\xi), V(\xi), W(\xi), Y(\xi)) \in S(A^+) \,|\, 0 < \xi < +\infty\}.$$

Let Γ be the limiting fast orbit from B^- to B^+, which is determined by system (3.17). In fact, we can find that Γ is a curve locating at the two-dimensional plane $\pi := \left\{(U, V, W, Y) \in \mathbb{R}^4 \,|\, W = W_0, Y = -W_0\right\}$. Thus, the curve segment

$$\Lambda := \Lambda_- \cup \Gamma \cup \Lambda_+$$

is the singular heteroclinic orbit from A^- to A^+ in the singular limit $\varepsilon \to 0$. See also Figure 1 for a schematic depiction.

4. Persistence of singular heteroclinic orbit for $0 < \varepsilon \ll 1$

In this section, we show the persistence of singular heteroclinic orbit for sufficiently $0 < \varepsilon \ll 1$ in system (2.6) or (2.7) and thus the existence of traveling wave fronts in Eq (1.1). We summarize the main results of this paper as follows.

Theorem 4.1. *For any fixed $c \geq 2\sqrt{rL}$, where $L = \sqrt{\frac{Kh}{r+Kh}}$, and for the case when the kernel f is given by* (1.3)*, Eq* (1.1) *possesses a traveling front solution $u(x,t) = U(x-ct)$ satisfying $U(-\infty) = u^*$ and $U(+\infty) = 0$, provided that the nonlocal parameter ρ is sufficiently large.*

Proof. Notice that W is given by (3.14), then we have that $W \neq \frac{1}{b}$ along the singular heteroclinic orbit. Thus, both the manifold S_1 and S_2 are normally hyperbolic along the singular orbit and this singular orbit is a heteroclinic connection between S_2 and S_1. It follows from Fenichel's invariant manifold theory [9] that, for ε sufficiently small and after appropriately compactifying S_1 and S_2, there exist locally invariant slow manifold $S_{1,\varepsilon}$ and $S_{2,\varepsilon}$ in the system (2.6) or (2.7) that are $O(\varepsilon)-$ close to S_1 and S_2, respectively. Moreover, system (2.6) or (2.7) also admits locally invariant stable and unstable manifolds $\mathcal{W}^s(S_{1,\varepsilon})$ and $\mathcal{W}^u(S_{2,\varepsilon})$ which are $O(\varepsilon)-$ close to $\mathcal{W}^s(S_1)$ and $\mathcal{W}^u(S_2)$, respectively. Notice that $\Lambda_- \in S_2, \Lambda_+ \in S_1$, then $\mathcal{W}^s(\Lambda_+)$ and $\mathcal{W}^u(\Lambda_-)$ possess the similar properties as $\mathcal{W}^s(S_1)$ and $\mathcal{W}^u(S_2)$, respectively. Note that the singular orbit $\Lambda := \Lambda_- \cup \Gamma \cup \Lambda_+$ is contained in the intersection $\mathcal{W}^s(\Lambda_+) \bigcap \mathcal{W}^u(\Lambda_-)$, and it follows that this singular orbit will persist for sufficiently small $0 < \varepsilon \ll 1$ if the intersection $\mathcal{W}^s(\Lambda_+) \bigcap \mathcal{W}^u(\Lambda_-)$ is transversal along the limiting fast orbit Γ. In fact, first we can derive from the signs of eigenvalues presented in Subsection 3.1 that $dim(\mathcal{W}^s(\Lambda_+)) = 2 + 1 = 3$ and $dim(\mathcal{W}^u(\Lambda_-)) = 1 + 1 = 2$, then it implies that they might intersect along a one-dimensional curve in four-dimensional phase space $\mathbb{R}^4$. Moreover, we can observe that the tangent space $T\mathcal{W}^s(\Lambda_+)$ along Γ is given by

$$T\mathcal{W}^s(\Lambda_+) = span\{(1, \lambda_1^1, 0, 0)^T, (1, \lambda_2^1, 0, 0)^T, (0, 0, 1, -1)^T\},$$

where the two vectors $(1, \lambda_1^1, 0, 0)^T, (1, \lambda_2^1, 0, 0)^T$ are composed of the two stable eigenvectors respectively, appended with two 0 components representing W, Y components which remain constants through the fast transition; while the latter one vector $(0, 0, 1, -1)^T$ represents the direction of Λ_+. Also, we can observe that the tangent space $T\mathcal{W}^u(\Lambda_-)$ along Γ is given by

$$T\mathcal{W}^u(\Lambda_-) = span\{(1, \lambda_1^2, 0, 0)^T, (\frac{L}{a} - u^*, 0, W_0 - u^*, -W_0)^T\},$$

where the vector $(1, \lambda_1^2, 0, 0)^T$ is composed of the unstable eigenvectors appended with two 0 components representing W, Y components which remain constants through the fast transition; while the latter one vector $(\frac{L}{a} - u^*, 0, W_0 - u^*, -W_0)^T$ represents the direction of Λ_-. One can easily verify that the vector $(\frac{L}{a} - u^*, 0, W_0 - u^*, -W_0)^T$ is linearly independent to the three vectors that span $T\mathcal{W}^s(\Lambda_+)$. Hence, at any points along the limiting fast orbit Γ, the combined tangent spaces $T\mathcal{W}^s(\Lambda_+)$ and $T\mathcal{W}^u(\Lambda_-)$ contain the full tangent space $T(\mathbb{R}^4)$ to the phase space $\mathbb{R}^4$. Thus, it shows the transversality of the intersection $\mathcal{W}^s(\Lambda_+) \bigcap \mathcal{W}^u(\Lambda_-)$, which can ensure the persistence of heteroclinic connection for $0 < \varepsilon \ll 1$. More precisely, for $0 < \varepsilon \ll 1$, $\Lambda_-, \Lambda_+, \Gamma$ persist. Denote by $\Lambda_-^\varepsilon, \Lambda_+^\varepsilon, \Gamma^\varepsilon$ the perturbed objects, respectively. Thus, the orbit $\Lambda^\varepsilon := \Lambda_-^\varepsilon \cup \Gamma^\varepsilon \cup \Lambda_+^\varepsilon$ corresponds to the singular orbit connecting $A^- = (u^*, 0, u^*, 0)$ to $A^+ = (0, 0, 0, 0)$ with $\Lambda^\varepsilon \to \Lambda$ as $\varepsilon \to 0$. In view of (2.5), (3.16), (3.18) and (3.20), we can see that, for given parameter conditions presented in Theorem 4.1, Eq (1.1) possesses a traveling front solution $u(x,t) = U(x-ct)$ connecting its extinction state $u = 0$ with the positive equilibrium $u = u^*$ when ρ is sufficiently large. The proof is completed. □

5. Conclusions and discussion

In this work we deal with the traveling wave problem for a single species model with cannibalism and nonlocal effect. By employing geometric singular perturbation theory and Fenichel's invariant manifold theory, we have proved that, for the case of strongly nonlocal effect, this model admits a traveling front solution going from the extinction state to the positive equilibrium state. It should be remarked here that, for the case of weak nonlocal effect (i.e., ρ is sufficiently small), the traveling wave problem for Eq (1.1) can also be reformulated as a singular perturbation problem. In fact, our fast and slow systems (2.6) and (2.7) can be rewritten as

$$\begin{cases} \dfrac{dU}{dz} = V, \\ \dfrac{dV}{dz} = -cV - rU\left(1 - aU - bW\right), \\ \rho\dfrac{dW}{dz} = Y, \\ \rho\dfrac{dY}{dz} = W - U, \end{cases} \tag{5.1}$$

and

$$\begin{cases} \dfrac{dU}{d\xi} = \rho V, \\ \dfrac{dV}{d\xi} = \rho[-cV - rU\left(1 - aU - bW\right)], \\ \dfrac{dW}{d\xi} = Y, \\ \dfrac{dY}{d\xi} = W - U, \end{cases} \tag{5.2}$$

respectively. It is easily seen that if the parameter ρ is taken to be sufficiently small, systems (5.1) and (5.2) become the singular perturbed slow and fast system, respectively. Following the ideas in [13, 14], in which the traveling waves for the similar models as Eq (1.1) were studied, we can also establish the existence of traveling wave front of Eq (1.1) connecting its two uniform steady states for sufficiently small ρ. However, for the model (1.1), the wave speed of the traveling wave front has to satisfy the condition $c \geq 2\sqrt{r}$ if ρ is taken to be sufficiently small, in contrast to the case that ρ is taken to be sufficiently large, in which the wave speed satisfies the condition $c \geq 2\sqrt{rL}$. Note that $L < 1$, i.e., $2\sqrt{rL} < 2\sqrt{r}$, so we believe that Eq (1.1) admits traveling wave front connecting its two uniform steady states for all $\rho > 0$, provided that the wave speed satisfies the condition $c \geq 2\sqrt{r}$.

Furthermore, the methodology of embedding the traveling wave problem into a slow-fast structure and subsequently studying the corresponding dynamics of the limiting slow and limiting fast systems can also be extended to study the higher dimensional traveling wave problems. For instance, one can extend this method to investigate the existence problem of heteroclinic traveling wave connecting two stable rest states for the following singularly perturbed reaction-diffusion equations modeling the evolution of three competing species

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = \dfrac{\partial^2 u_1}{\partial x^2} + u_1 g_1(u_1, u_2, u_3), \\ \dfrac{\partial u_2}{\partial t} = \epsilon^2 \dfrac{\partial^2 u_2}{\partial x^2} + r_2 u_2 g_2(u_1, u_2, u_3), \\ \dfrac{\partial u_3}{\partial t} = \epsilon^2 d \dfrac{\partial^2 u_3}{\partial x^2} + r_3 u_3 g_3(u_1, u_2, u_3), \end{cases} \tag{5.3}$$

where the constants r_2, r_3, d are strictly positive and $0 < \epsilon \ll 1$ (i.e., the species u_2 and u_3 diffuse very slowly relative to u_1), while the two rest states are defined by

$$\begin{cases} P_2 : g_1(u_1, u_2, 0) = g_2(u_1, u_2, 0) = 0, \quad u_3 = 0, \\ P_3 : g_1(u_1, 0, u_3) = g_3(u_1, 0, u_3) = 0, \quad u_2 = 0. \end{cases} \tag{5.4}$$

We should mention that the existence of a non-monotone traveling wave connecting the rest states P_2 and P_3 for (5.3) has been established by applying the Conley index theory in [15]. However, it is believed that the same results can be demonstrated rigorously by using geometric singular perturbation theory and Fenichel's invariant manifold theory.

Finally, in this work we only established the existence of the traveling wave front solution for Eq (1.1), and the stability of this traveling wave front solution is not considered. A natural question arises regarding how to study the stability properties of this traveling wave front. We think that a potential approach is to combine the singular limit eigenvalue problem (SLEP) method used in [16–18] with the Evans function method developed in [19–21] to compute eigenvalues. We leave these extensions for future analysis.

Author contributions

Xijun Deng: Writing-original draft preparation, methodology; Aiyong Chen: Writing-review and editing. All authors have read and agreed to the published version of the manuscript.

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Conflict of interest

The authors declare no conflicts of interest regarding the publication of this paper.

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