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Research article

The general tensor regular splitting iterative method for multilinear PageRank problem

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Abstract: The paper presents an iterative scheme called the general tensor regular splitting iterative (GTRS) method for solving the multilinear PageRank problem, which is based on a (weak) regular splitting technique and further accelerates the iterative process by introducing a parameter. The method yields familiar iterative schemes through the use of specific splitting strategies, including fixed-point, inner-outer, Jacobi, Gauss-Seidel and successive overrelaxation methods. The paper analyzes the convergence of these solvers in detail. Numerical results are provided to demonstrate the effectiveness of the proposed method in solving the multilinear PageRank problem.

Keywords: multilinear PageRank; tensor; regular splitting; convergence
Mathematics Subject Classification: 65F10, 65H10

1. Introduction

As the most crucial algorithm for web ranking in the web search engines, Google's PageRank has been extensively researched over the years [1–8]. Gleich et al. [9] extended PageRank to higher-order Markov chains and proposed the following multilinear PageRank problem:

$$\mathbf{x} = \alpha \mathcal{P} \mathbf{x}^{m-1} + (1-\alpha)\mathbf{v}, \tag{1.1}$$

where $\alpha \in (0,1)$ is a parameter, $\mathbf{v} \in \mathbb{R}^n$ is a stochastic vector, i.e., $\mathbf{v} \geq 0, \|\mathbf{v}\|_1 = 1$, $\mathbb{R}^n$ is the set of all n-dimensional real vectors. The stochastic solution $\mathbf{x} \in \mathbb{R}^n$ is called the multilinear PageRank vector. Here $\mathcal{P} = (p_{i_1,i_2,\cdots,i_m})$ $(\forall i_2,\cdots,i_m \in \langle n \rangle)$ is an m^{th}-order n-dimensional stochastic tensor representing an $(m-1)^{th}$-order Markov chain, namely,

$$p_{i_1,i_2,\cdots,i_m} \geq 0, \sum_{i_1=1}^{n} p_{i_1,i_2,\cdots,i_m} = 1,$$

where $\langle n \rangle = \{1, 2, \cdots, n\}$. The tensor-vector product $\mathcal{P}\mathbf{x}^{m-1}$ is a vector in $\mathbb{R}^n$ and its definition is given in Section 2.

In particular, the multilinear PageRank model (1.1) can be transformed into higher-order Markov chain when we take $\alpha = 1$ (see [10] for the general case). When $m = 2$, (1.1) simplifies to the classical PageRank as described in [11].

The model (1.1) can be transformed into

$$\mathbf{x} = \bar{\mathcal{P}}\mathbf{x}^{m-1},$$

where $\bar{\mathcal{P}} = \alpha\mathcal{P} + (1-\alpha)\mathcal{V}$ and $\mathcal{V}$ is an m^{th}-order n-dimensional tensor with $(\mathcal{V})_{i,i_2,\cdots,i_m} = v_i, \forall i, i_2, \cdots, i_m \in \langle n \rangle$. Clearly, $\bar{\mathcal{P}}$ is also a transition probability tensor.

In recent years, the multilinear PageRank problem has attracted much attention. The theory of existence and uniqueness of solutions for the multilinear PageRank problem is a fundamental basis for analyzing the convergence of algorithms. Gleich et al. [9] pointed to the fact that the multilinear PageRank vector is unique in (1.1) for $\alpha < \frac{1}{m-1}$. The above condition can be easily applied to compute the stochastic solution of (1.1) when $\alpha < \frac{1}{2}$. However, when $\frac{1}{2} < \alpha < 1$, especially when $\alpha \approx 1$, it becomes increasingly challenging to solve (1.1). This challenge inspired researchers to search for more general but stricter conditions, such as those discussed in studies by Li et al. [12, 13], Huang and Wu [14], Fasino and Tudisco [15] and Liu et al. [16]. It should be noted that we cannot theoretically compare the merits of these conditions and there is no optimal condition in numerical experiments.

Gleich et al. first proposed Newton's method to solve this problem in [9]. Subsequently, Meini and Poloni [17] and Guo et al. [18] proposed the Perron Newton iterative algorithm and the multi-step Newton method, respectively. These algorithms mentioned above involve gradient computation, which have shown excellent performance. But when solving asymmetric tensor models, the difficulty of gradient computation can lead to expensive calculations. Unlike gradient algorithms, non-gradient algorithms typically do not require the gradient information of the objective function. Instead, they estimate the optimal solution of the objective function through other methods. Non-gradient algorithms have proven to be effective in solving this problem and have gained a significant amount of attention in recent years. For example, Gleich et al. [9] proposed the fixed-point method, the shifted fixed-point method, the inner-outer method and the inverse method. Liu et al. [19] introduced several relaxation methods. Other novel methods have been proposed by various scholars in an effort to achieve better convergence performances (see References [20–23] and their respective citations). Hence, the problem about how to design high-speed, robust and flexible non-gradient algorithms has become the key challenge in solving the multilinear PageRank problem.

As an important class of non-gradient algorithms, the splitting algorithm has significant applications in tensor computations. For example, Liu et al. [24] proposed a tensor splitting algorithm for solving tensor equations and high-order Markov chain models. Cui et al. [25] proposed an iterative refinement method by using higher-order singular value decomposition to solve general multilinear systems. Jiang and Li [26] proposed a new preconditioner for improving the AOR-type method. Cui and Zhang [27] proposed a new constructive way to estimate bounds on the H-eigenvalues of two kinds of interval tensors. This paper will explore a tensor splitting algorithm for solving the multilinear PageRank problem.

The main contributions of this paper can be outlined as follows:

(1) The general tensor regular splitting (GTRS) iterative method is proposed to address the multilinear PageRank problem.

(2) Additionally, we offer five typical splitting methods of the GTRS iteration.

(3) We give the convergence analysis for the proposed method based on the uniqueness condition provided by Li et al. [13].

(4) Several numerical examples are provided and the outcomes significantly outperform the existing ones in most cases.

The reminder of this work is organized as follows. In Section 2, we present some essential definitions and key properties. By making use of the (weak) regular splitting of the coefficient matrix $I-\alpha\mathcal{P}\mathbf{x}^{m-2}$, we propose the GTRS iterative method in Section 3. Section 4 is devoted to the convergence analysis of the GTRS method. In Section 5, some application experiments are presented to demonstrate the effectiveness of our method. Finally, the conclusion is given in Section 6.

2. Preliminaries

In this section, we present a brief overview of the notations and definitions that are necessary for this paper.

An m^{th}-order n-dimensional tensor $\mathcal{A}$ with n^m entries is defined as

$$\mathcal{A} = (a_{i_1,\cdots,i_m}), a_{i_1,\cdots,i_m} \in \mathbb{R}, i_j \in \langle n\rangle, j = 1, \cdots, m,$$

where $\mathbb{R}$ is the real field. $\mathcal{A}$ is called non-negative (positive) if $a_{i_1,\cdots,i_m} \geq 0$ ($a_{i_1,\cdots,i_m} > 0$).

For any two matrices $A = (a_{ij})_{n\times n}, B = (b_{ij})_{n\times n} \in \mathbb{R}^n$, $A \geq B$ means that $a_{ij} \geq b_{ij}$ for $i, j \in \langle n\rangle$.

Definition 1. *[13] Let $\mathcal{A}$ be an m^{th}-order n-dimensional tensor,* $\mathbf{x}$ *be an n-dimensional vector and* $\mathbf{x}_i$ *represent the i^{th} entry of* $\mathbf{x}$. *$\mathcal{A}\mathbf{x}^{m-r}$ is an r^{th}-order n-dimensional tensor is given by*

$$(\mathcal{A}\mathbf{x}^{m-r})_{i,i_2,\ldots,i_r} = \sum_{i_{r+1},\cdots,i_m\in\langle n\rangle} a_{i,i_2,\cdots,i_r,i_{r+1},\cdots,i_m} x_{i_{r+1}} \cdots x_{i_m}. \tag{2.1}$$

Definition 2. *[13] Let $\mathcal{A}$ be an m^{th}-order n-dimensional tensor and both* $\mathbf{x}$ *and* $\mathbf{y}$ *be n-dimensional tensors. Then we get the following definition*

$$\mathcal{A}(\mathbf{x}^{m-r} - \mathbf{y}^{m-r}) \equiv \mathcal{A}\mathbf{x}^{m-r} - \mathcal{A}\mathbf{y}^{m-r}. \tag{2.2}$$

Particularly, when $r = 1$ and $r = 2$, we obtain $\mathcal{A}\mathbf{x}^{m-1}$ and $\mathcal{A}\mathbf{x}^{m-2}$ from (2.1). Noticed that $\mathcal{A}\mathbf{x}^{m-1} = \mathcal{A}\mathbf{x}^{m-2}\mathbf{x}$. Then, (2.2) can be expressed as

$$\mathcal{A}(\mathbf{x}^{m-1} - \mathbf{y}^{m-1}) \equiv \mathcal{A}\mathbf{x}^{m-1} - \mathcal{A}\mathbf{y}^{m-1},\ \mathcal{A}(\mathbf{x}^{m-2} - \mathbf{y}^{m-2}) \equiv \mathcal{A}\mathbf{x}^{m-2} - \mathcal{A}\mathbf{y}^{m-2}.$$

Definition 3. *[13] Let $\mathcal{A}$ be an m^{th}-order n-dimensional tensor and both* $\mathbf{x}$ *and* $\mathbf{y}$ *be n-dimensional tensors. $\mathcal{A}^{(k)}_{xy}$ is an $n \times n$ matrix whose $(i, j)^{th}$- entry is defined as*

$$(\mathcal{A}^{(k)}_{xy})_{i,j} = \sum_{i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m\in\langle n\rangle} a_{i,i_2,\cdots,i_{k-1},j,i_{k+1},\cdots,i_m} x_{i_2} \cdots x_{i_{k-1}} y_{i_{k+1}} \cdots y_{i_m}.$$

It should be noted that (1.1) can be rewritten as

$$(I - \alpha\mathcal{P}\mathbf{x}^{m-2})\mathbf{x} = (1-\alpha)\mathbf{v}. \tag{2.3}$$

Definition 4. *A real square matrix $\hat{A}$ is defined as an M-matrix when it satisfies the following conditions: $\hat{A}$ is a Z-matrix (i.e., all of its off-diagonal elements $\hat{a}_{ij} \leq 0$) and $\hat{A}$ can be expressed as $hI - \hat{B}$, where $\hat{B}$ is a nonnegative matrix with $\hat{b}_{ij} \geq 0$, and its spectral radius $\rho(\hat{B}) < h$.*

Definition 5. *[28] Let O be a null $n \times n$ matrix and $\check{A}$, $\check{M}$ and $\check{N}$ be $n \times n$ real matrices and $\check{A} = \check{M} - \check{N}$ is a regular splitting of matrix $\check{A}$ if $\check{M}$ is nonsingular with $\check{M}^{-1} \geq O$ and $\check{N} \geq O$. Similarly, $\check{A} = \check{M} - \check{N}$ is a weak regular splitting of matrix $\check{A}$ if $\check{M}$ is nonsingular with $\check{M}^{-1} \geq O$ and $\check{M}^{-1}\check{N} \geq O$. $\check{A} = \check{M} - \check{N}$ is a convergent splitting of matrix $\check{A}$ if $\check{M}$ is nonsingular with $\rho(\check{M}^{-1}\check{N}) < 1$.*

Definition 6. *Let $\mathbf{z} \in \mathbb{R}^n$ be a stochastic vector, $\mathbf{0} \in \mathbb{R}^n$ and $\mathbf{z}_+ = \max(\mathbf{z}, \mathbf{0})$. We define the projection* $\text{proj}(\cdot)$ *as follows:*

$$\text{proj}(\mathbf{z}) = \frac{\mathbf{z}_+}{\|\mathbf{z}_+\|_1}.$$

It is clear that $\text{proj}(\cdot)$ *is a stochastic vector.*

3. The GTRS iterative method

Building upon the success of the general inner-outer iterative method (see [29]) for solving the typical PageRank problem, we introduce the GTRS iterative method as an efficient solution for computing the multilinear PageRank vector.

Let

$$I - \alpha\mathcal{P}\mathbf{x}^{m-2} = \bar{M}_{\mathbf{x}} - \bar{N}_{\mathbf{x}} \tag{3.1}$$

be a (weak) regular splitting. Based on (3.1), the iterative method, i.e., GTRS, for (1.1) is constructed as follows:

$$\left(\bar{M}_{\mathbf{x}_k} - \phi\bar{N}_{\mathbf{x}_k}\right)\mathbf{y}_{k+1} = (1-\phi)\bar{N}_{\mathbf{x}_k}\mathbf{x}_k + \mathbf{b}, \mathbf{x}_{k+1} = \text{proj}(\mathbf{y}_{k+1}), k = 0, 1, 2, \cdots \tag{3.2}$$

with $0 < \phi < 1$ and $\mathbf{b} = (1-\alpha)\mathbf{v}$. The scheme (3.2) is derived from the matrix splitting

$$I - \alpha\mathcal{P}\mathbf{x}_k^{m-2} = \check{M}_{\mathbf{x}_k} - \check{N}_{\mathbf{x}_k},$$

where $\check{M}_{\mathbf{x}_k} = \bar{M}_{\mathbf{x}_k} - \phi\bar{N}_{\mathbf{x}_k}$ and $\check{N}_{\mathbf{x}_k} = (1-\phi)\bar{N}_{\mathbf{x}_k}$.

Let $\check{U}_{\mathbf{x}} \geq 0$ be the strictly upper-triangular matrix of the matrix $\mathcal{P}\mathbf{x}^{m-2}$, $\check{D}_{\mathbf{x}} \geq 0$ be a diagonal matrix and $\check{L}_{\mathbf{x}} \geq 0$ be a lower-triangular matrix such that

$$\mathcal{P}\mathbf{x}^{m-2} = \check{D}_{\mathbf{x}} + \check{U}_{\mathbf{x}} + \check{L}_{\mathbf{x}}. \tag{3.3}$$

Next, we give several typical practical choices of the matrices $\bar{M}_{\mathbf{x}}$ and $\bar{N}_{\mathbf{x}}$. Derived from the above matrix splitting (3.3), those well-known iterative methods for solving (1.1) are presented below:

(1) Let $\bar{M}_{\mathbf{x}_k} = I$ and $\bar{N}_{\mathbf{x}_k} = \alpha\mathcal{P}\mathbf{x}_k^{m-2}$; then, we can obtain the fixed point iteration (denote by GTRS-FP):

$$(I - \phi\alpha\mathcal{P}\mathbf{x}_k^{m-2})\mathbf{y}_{k+1} = (\alpha - \phi\alpha)\mathcal{P}\mathbf{x}_k^{m-2}\mathbf{x}_k + \mathbf{b}. \tag{3.4}$$

(2) Let $\bar{M}_{\mathbf{x}_k} = I - \beta\mathcal{P}\mathbf{x}_k^{m-2}$ and $\bar{N}_{\mathbf{x}_k} = (\alpha - \beta)\mathcal{P}\mathbf{x}_k^{m-2}$; then, we get another method, i.e., the inner-outer iteration (denote by GTRS-IO):

$$(I - \beta\mathcal{P}\mathbf{x}_k^{m-2} - \phi(\alpha - \beta)\mathcal{P}\mathbf{x}_k^{m-2})\mathbf{y}_{k+1} = (1 - \phi)(\alpha - \beta)\mathcal{P}\mathbf{x}_k^{m-2}\mathbf{x}_k + \mathbf{b}, \tag{3.5}$$

where $0 < \beta < \alpha$.

(3) Let $\bar{M}_{\mathbf{x}_k} = I - \alpha\check{D}_{\mathbf{x}_k}$ and $\bar{N}_{\mathbf{x}_k} = \alpha(\check{L}_{\mathbf{x}_k} + \check{U}_{\mathbf{x}_k})$; we derive the Jacobi iteration (denote by GTRS-JCB):

$$(I - \alpha\check{D}_{\mathbf{x}_k} - \phi\alpha(\check{L}_{\mathbf{x}_k} + \check{U}_{\mathbf{x}_k}))\mathbf{y}_{k+1} = \alpha(1 - \phi)(\check{L}_{\mathbf{x}_k} + \check{U}_{\mathbf{x}_k})\mathbf{x}_k + \mathbf{b}. \tag{3.6}$$

(4) Let $\bar{M}_{\mathbf{x}_k} = I - \alpha\check{D}_{\mathbf{x}_k} - \alpha\check{U}_{\mathbf{x}_k}$ and $\bar{N}_{\mathbf{x}_k} = \alpha\check{L}_{\mathbf{x}_k}$; then, we can get the Gauss-Seidel iteration (denote by GTRS-GS):

$$(I - \alpha\check{D}_{\mathbf{x}_k} - \alpha\check{U}_{\mathbf{x}_k} - \phi\alpha\check{L}_{\mathbf{x}_k})\mathbf{y}_{k+1} = \alpha(1 - \phi)\check{L}_{\mathbf{x}_k}\mathbf{x}_k + \mathbf{b}. \tag{3.7}$$

(5) Let $\bar{M}_{\mathbf{x}_k} = \frac{1}{\omega}\left(I - \alpha\check{D}_{\mathbf{x}_k} - \omega\alpha\check{L}_{\mathbf{x}_k}\right)$ and $\bar{N}_{\mathbf{x}_k} = \frac{1}{\omega}\left[(1 - \omega)(I - \alpha\check{D}_{\mathbf{x}_k}) + \omega\alpha\check{U}_{\mathbf{x}_k}\right]$; then, we have the successive overrelaxation iteration (denote by GTRS-SOR):

$$\begin{aligned}(I - \alpha\check{D}_{\mathbf{x}_k} - \omega\alpha\check{L}_{\mathbf{x}_k} - \phi(1 - \omega)(I - \alpha\check{D}_{\mathbf{x}_k}) - \phi\omega\alpha\check{U}_{\mathbf{x}_k})\mathbf{y}_{k+1} \\ = (1 - \phi)\left[(1 - \omega)(I - \alpha\check{D}_{\mathbf{x}_k}) + \omega\alpha\check{U}_{\mathbf{x}_k}\right]\mathbf{x}_k + \omega\mathbf{b}.\end{aligned} \tag{3.8}$$

Finally, the whole algorithm of the GTRS iterative method can be outlined as follows:

Algorithm 1 The GTRS iterative method

Require: $\bar{M}_{\mathbf{x}}, \bar{N}_{\mathbf{x}}, \alpha, \phi, \beta, \omega, \mathbf{v}$, maximum k_{max}, termination tolerance ε, and an initial stochastic vector $\mathbf{x}_0$

Ensure: $\mathbf{x}$

```
k ← 1
while x_k − αP x_k^{m−1} − b < ε do
    if k < k_max then
        M̌_{x_k} ← M̄_{x_k} − φ N̄ x_k
        Ň_{x_k} ← (1 − φ) N̄_{x_k}
        y_{k+1} ← M̌_{x_k}^{−1} Ň_{x_k} x_k + M̌_{x_k}^{−1} b
        x_{k+1} ← proj(y_{k+1})
        k = k + 1
    end if
end while
```

4. Convergence analysis

In this section, we analyze the convergence of the GTRS iterative method. We establish the convergence of the GTRS iteration by using the uniqueness condition of the solution to the multilinear PageRank problem in [13]. At first, we give several lemmas which will be used later.

Lemma 1. *[13] Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be n-dimensional stochastic vectors, and let* $\mathcal{J}^{(k)}(k = 2, 3, \cdots, m)$ *be* m^{th}*-order n-dimensional tensors defined as*

$$\left(\mathcal{J}^{(k)}\right)_{i_1,\cdots,i_k,\cdots,i_m} = \sigma^{(k)}_{i_1,i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m}, \quad \forall i_k \in \langle n \rangle,$$

where $\sigma^{(k)}_{i_1,i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m} \in \mathbb{R}$ *for any* $i_l \in \langle n \rangle$, $l = 1, 2, \cdots, k-1, k+1, \cdots, i_m$ *and* $k = 2, 3, \cdots, m$*; then,*

$$J^{(k)}_{xy} \Delta\mathbf{x} = 0,$$

where $\Delta\mathbf{x} = \mathbf{x} - \mathbf{y}$.

Lemma 2. *[13] Let* $\mathcal{P}$ *be an* m^{th}*-order stochastic tensor and* $\mathbf{v}$ *be a stochastic vector; then, the multilinear PageRank model* (1.1) *has the unique solution if*

$$\alpha < \frac{1}{\min\{\mu, \nu\}},$$

where $\mu = \min_{\mathcal{J}^{(k)},k=2,3,\cdots,m} \mu\left(\mathcal{J}^{(2)}, \cdots, \mathcal{J}^{(m)}\right)$ *and* $\nu = \min_{\mathcal{J}^{(k)},k=2,3,\cdots,m} \nu\left(\mathcal{J}^{(2)}, \cdots, \mathcal{J}^{(m)}\right)$.

Lemma 3. *Let* $\mathcal{P}$ *be an* m^{th}*-order n-dimensional stochastic tensor, and let* $\mathbf{y}$ *and* $\mathbf{z}$ *be n-dimensional stochastic vectors; then, we have*

$$\left\|\mathcal{P}\mathbf{x}^{m-2}(\mathbf{y} - \mathbf{z})\right\|_1 \le \gamma \left\|\mathbf{y} - \mathbf{z}\right\|_1,$$

where $\gamma = \min\{\bar{\gamma}, \check{\gamma}\}$ *with* $\bar{\gamma} = 1 - \min_{i_3,\cdots,i_m \in \langle n \rangle} \sum_{i=1}^{n} \min_{i_2 \in \langle n \rangle} p_{i,i_2,\cdots,i_m}$ *and* $\check{\gamma} = \max_{i_3,\cdots,i_m \in \langle n \rangle} \sum_{i=1}^{n} \max_{i_2 \in \langle n \rangle} p_{i,i_2,\cdots,i_m} - 1$.

Proof. Let $\Delta\mathbf{y} = \mathbf{y} - \mathbf{z}$ and Δy_i be the i-th entry of $\Delta\mathbf{y}$, and let $\mathcal{J}^{(2)}$ be defined in Lemma 1; we have

$$\begin{aligned} \left\|\mathcal{P}\mathbf{x}^{m-2}\Delta\mathbf{y}\right\|_1 &= \sum_{i_1=1}^{n} \left| \sum_{i_2,\cdots,i_m \in \langle n \rangle} \left(p_{i_1,i_2,\cdots,i_m} - \sigma^{(2)}_{i_1,i_3,\cdots,i_m}\right) \Delta y_{i_2} y_{i_3} \cdots y_{i_m} \right| \\ &\le \sum_{i_1=1}^{n} \sum_{i_2,\cdots,i_m \in \langle n \rangle} \left|p_{i_1,i_2,\cdots,i_m} - \sigma^{(2)}_{i_1,i_3,\cdots,i_m}\right| \left|\Delta y_{i_2}\right| y_{i_3} \cdots y_{i_m} \\ &\le \max_{i_2,\cdots,i_m \in \langle n \rangle} \sum_{i_1=1}^{n} \left|p_{i_1,i_2,\cdots,i_m} - \sigma^{(2)}_{i_1,i_3,\cdots,i_m}\right| \left\|\Delta\mathbf{y}\right\|_1. \end{aligned} \tag{4.1}$$

Substituting $\sigma^{(2)}_{i_1,i_3,\cdots,i_m} = \min_{i_2} p_{i_1,i_2,\cdots,i_m}$ and $\sigma^{(2)}_{i_1,i_3\cdots,i_m} = \max_{i_2} p_{i_1,i_2,\cdots,i_m}$ into (4.1) respectively, it follows that

$$\left\|\mathcal{P}\mathbf{x}^{m-2}\Delta\mathbf{y}\right\|_1 \le \bar{\gamma} \left\|\Delta\mathbf{y}\right\|_1$$

and

$$\left\|\mathcal{P}\mathbf{x}^{m-2}\Delta\mathbf{y}\right\|_1 \le \check{\gamma} \left\|\Delta\mathbf{y}\right\|_1,$$

where

$$\bar{\gamma} = 1 - \min_{i_3,\cdots,i_m \in \langle n \rangle} \sum_{i=1}^{n} \min_{i_2 \in \langle n \rangle} p_{i,i_2,\cdots,i_m} \text{ and } \check{\gamma} = \max_{i_3,\cdots,i_m \in \langle n \rangle} \sum_{i=1}^{n} \max_{i_2 \in \langle n \rangle} p_{i,i_2,\cdots,i_m} - 1.$$

Let $\gamma = \min\{\bar{\gamma}, \check{\gamma}\}$; we have

$$\left\|\mathcal{P}\mathbf{x}^{m-2}\Delta\mathbf{y}\right\|_1 \le \gamma \left\|\Delta\mathbf{y}\right\|_1.$$

This completes the proof. □

Lemma 4. *[13] Let $\mathcal{P}$ be an m^{th}-order n-dimensional stochastic tensor, and let* $\mathbf{x}$*,* $\mathbf{y}$ *and* $\mathbf{z}$ *be n-dimensional stochastic vectors. For any set $\mathcal{J}^{(k)}(k = 3, \cdots, m)$, we have*

$$\left\|\mathcal{P}\left(\mathbf{x}^{m-2} - \mathbf{y}^{m-2}\right)\mathbf{z}\right\|_1 \leq \bar{\mu}\left(\mathcal{J}^{(3)}, \cdots, \mathcal{J}^{(m)}\right)\|\Delta\mathbf{x}\|_1,$$

where

$$\begin{aligned}&\bar{\mu}\left(\mathcal{J}^{(3)}, \cdots, \mathcal{J}^{(m)}\right)\\ &= \max_{i_2, i_k \in \langle n\rangle} \sum_{k=3}^{m} \max_{i_3, \cdots, i_{k-1}, i_{k+1}, \cdots, i_m \in \langle n\rangle} \sum_{i_1=1}^{n} \left|p_{i_1, i_2, \cdots, i_k, \cdots, i_m} - \sigma^{(k)}_{i_1, i_2, \cdots, i_{k-1}, i_{k+1}, \cdots, i_m}\right|.\end{aligned}$$

Lemma 5. *[19] Let $\hat{\mathbf{y}} = (\hat{y}_i) \in \mathbb{R}^n$ with $\sum_{i=1}^{n} \hat{y}_i = 1$; $\mathbf{z} = (z_i)$ is a stochastic vector. If $\mathbf{y} = \mathrm{proj}(\hat{\mathbf{y}})$, then* $\|\mathbf{y} - \mathbf{z}\|_1 \leq \|\hat{\mathbf{y}} - \mathbf{z}\|_1$.

Lemma 6. *Let $\hat{\mathbf{y}} = (\hat{y}_i) \in \mathbb{R}^n$; $\mathbf{z} = (z_i)$ is a stochastic vector. If $\mathbf{y} = \mathrm{proj}(\hat{\mathbf{y}})$, then* $\|\mathbf{y} - \mathbf{z}\|_1 \leq 2\|\hat{\mathbf{y}} - \mathbf{z}\|_1$.

Proof. Since $\mathbf{y} = \mathrm{proj}(\hat{\mathbf{y}}) = \frac{\hat{\mathbf{y}}_+}{\|\hat{\mathbf{y}}_+\|_1}$, it follows that

$$\begin{aligned}\|\mathbf{y} - \mathbf{z}\|_1 = \left\|\mathrm{proj}(\hat{\mathbf{y}}) - \mathbf{z}\right\|_1 &= \left\|\frac{\hat{\mathbf{y}}_+}{\|\hat{\mathbf{y}}_+\|_1} - \mathbf{z}\right\|_1\\ &= \frac{\left\|(1 - \|\hat{\mathbf{y}}_+\|_1)\hat{\mathbf{y}}_+ + \|\hat{\mathbf{y}}_+\|_1(\hat{\mathbf{y}}_+ - \mathbf{z})\right\|_1}{\|\hat{\mathbf{y}}_+\|_1}\\ &\leq \|\hat{\mathbf{y}} - \mathbf{z}\|_1 + \left|1 - \|\hat{\mathbf{y}}_+\|_1\right|\\ &= 2\|\hat{\mathbf{y}} - \mathbf{z}\|_1 + \left|1 - \|\hat{\mathbf{y}}_+\|_1\right| - \|\hat{\mathbf{y}} - \mathbf{z}\|_1\\ &\leq 2\|\hat{\mathbf{y}} - \mathbf{z}\|_1 + \left|1 - \|\hat{\mathbf{y}}_+\|_1\right| - \left|\|\hat{\mathbf{y}}_+\|_1 - 1\right|\\ &= 2\|\hat{\mathbf{y}} - \mathbf{z}\|_1.\end{aligned}$$

The proof is completed. □

Theorem 1. *Let $\mathcal{P}$ be an m^{th}-order n-dimensional stochastic tensor, $\alpha < \frac{1}{\min\{\mu,\nu\}}$ and* $\mathbf{x}$ *be the exact solution of* (1.1)*. Then, for an arbitrary initial stochastic guess $\mathbf{x}_0$, the iterative sequence $\{\mathbf{x}_k\}_{k=0}^{\infty}$ generated by* (3.2) *is convergent and*

$$\|\mathbf{x}_k - \mathbf{x}\|_1 \leq \delta_{\mathbf{x}_{k-1}} \cdots \delta_{\mathbf{x}_0}\|\mathbf{x}_0 - \mathbf{x}\|_1,$$

where $\mu_{\mathbf{x}_k} = \frac{(1-\phi)\left\|\bar{M}_{\mathbf{x}_k}^{-1}\bar{N}_{\mathbf{x}_k}\right\|_1 + \alpha\bar{\mu}\left(\mathcal{J}^{(3)}, \cdots, \mathcal{J}^{(m)}\right)\left\|\bar{M}_{\mathbf{x}_k}^{-1}\right\|_1}{\left(1-\phi\left\|\bar{M}_{\mathbf{x}_k}^{-1}\bar{N}_{\mathbf{x}_k}\right\|_1\right)}$ *and* $\delta_{\mathbf{x}_k} = 2\mu_{\mathbf{x}_k}$.

Proof. By (3.2), we have

$$\left(\bar{M}_{\mathbf{x}_k} - \phi\bar{N}_{\mathbf{x}_k}\right)\mathbf{y}_{k+1} = (1 - \phi)\bar{N}_{\mathbf{x}_k}\mathbf{x}_k + \mathbf{b}. \tag{4.2}$$

Note that $\mathbf{x}$ is the solution of (1.1); hence,

$$\left(\bar{M}_{\mathbf{x}} - \phi\bar{N}_{\mathbf{x}}\right)\mathbf{x} = (1 - \phi)\bar{N}_{\mathbf{x}}\mathbf{x} + \mathbf{b}. \tag{4.3}$$

Let $\hat{\mathbf{e}}_k = \mathbf{y}_k - \mathbf{x}$ and $\mathbf{e}_k = \mathbf{x}_k - \mathbf{x}$. Subtracting (4.3) from (4.2) yields

$$\begin{aligned}\bar{M}_{\mathbf{x}_k}\hat{\mathbf{e}}_{k+1} &= \phi\bar{N}_{\mathbf{x}_k}\hat{\mathbf{e}}_{k+1} + (1 - \phi)\bar{N}_{\mathbf{x}_k}\mathbf{e}_k + (\bar{N}_{\mathbf{x}_k} - \bar{M}_{\mathbf{x}_k})\mathbf{x} + (\bar{M}_{\mathbf{x}} - \bar{N}_{\mathbf{x}})\mathbf{x}\\ &= \phi\bar{N}_{\mathbf{x}_k}\hat{\mathbf{e}}_{k+1} + (1 - \phi)\bar{N}_{\mathbf{x}_k}\mathbf{e}_k + (\alpha\mathcal{P}\mathbf{x}_k^{m-2} - I)\mathbf{x} + (I - \alpha\mathcal{P}\mathbf{x}^{m-2})\mathbf{x}\\ &= \phi\bar{N}_{\mathbf{x}_k}\hat{\mathbf{e}}_{k+1} + (1 - \phi)\bar{N}_{\mathbf{x}_k}\mathbf{e}_k + \alpha\mathcal{P}(\mathbf{x}_k^{m-2} - \mathbf{x}^{m-2})\mathbf{x}.\end{aligned}$$

Hence

$$(I - \phi \bar{M}_{\mathbf{x}_k}^{-1} \bar{N}_{\mathbf{x}_k})\hat{\mathbf{e}}_{k+1} = (1-\phi)\bar{M}_{\mathbf{x}_k}^{-1}\bar{N}_{\mathbf{x}_k}\mathbf{e}_k + \alpha \bar{M}_{\mathbf{x}_k}^{-1}\mathcal{P}(\mathbf{x}_k^{m-2} - \mathbf{x}^{m-2})\mathbf{x}. \tag{4.4}$$

By taking the 1-norm of both sides of (4.4) and Lemma 4 we get

$$\begin{aligned}&\left(1 - \phi \left\|\bar{M}_{\mathbf{x}_k}^{-1}\bar{N}_{\mathbf{x}_k}\right\|_1\right) \|\hat{\mathbf{e}}_{k+1}\|_1 \\ &\leq [(1-\phi)\left\|\bar{M}_{\mathbf{x}_k}^{-1}\bar{N}_{\mathbf{x}_k}\right\|_1 + \alpha\bar{\mu}\left(\mathcal{J}^{(3)}, \cdots, \mathcal{J}^{(m)}\right)\left\|\bar{M}_{\mathbf{x}_k}^{-1}\right\|_1] \|\mathbf{e}_k\|_1 .\end{aligned}$$

Thus

$$\|\hat{\mathbf{e}}_{k+1}\|_1 \leq \frac{(1-\phi)\left\|\bar{M}_{\mathbf{x}_k}^{-1}\bar{N}_{\mathbf{x}_k}\right\|_1 + \alpha\bar{\mu}\left(\mathcal{J}^{(3)}, \cdots, \mathcal{J}^{(m)}\right)\left\|\bar{M}_{\mathbf{x}_k}^{-1}\right\|_1}{\left(1 - \phi\left\|\bar{M}_{\mathbf{x}_k}^{-1}\bar{N}_{\mathbf{x}_k}\right\|_1\right)} \|\mathbf{e}_k\|_1 = \mu_{\mathbf{x}_k} \|\mathbf{e}_k\|_1 . \tag{4.5}$$

From (4.5) and Lemma 6, we obtain

$$\|\mathbf{e}_{k+1}\|_1 \leq 2\|\hat{\mathbf{e}}_{k+1}\|_1 \leq \delta_{\mathbf{x}_k}\|\mathbf{e}_k\|_1 \leq \delta_{\mathbf{x}_k}\delta_{\mathbf{x}_{k-1}}\cdots\delta_{\mathbf{x}_0}\|\mathbf{e}_0\|_1 . \tag{4.6}$$

Then the GTRS iteration converges linearly with the convergence rate $\delta_{\mathbf{x}_k}$ when $\mu_{\mathbf{x}_k} < \frac{1}{2}$. □

Remark 1. *In Theorem 1, the value of* $\delta_{\mathbf{x}_k}$ *is associated with* $\mathbf{x}_k$*, which is difficult to prove in practical scenarios. Therefore, we will provide convergence analysis for specific splittings of the GTRS iterative method, such as* (3.4)–(3.8).

Theorem 2. *Let* $\mathcal{P}$ *be an* m^{th}*-order n-dimensional stochastic tensor,* $\mathbf{x}$ *be a solution of* (1.1)*, and* $\alpha < \min\{\frac{1}{\phi\bar{\mu}\left(\mathcal{J}^{(3)},\cdots,\mathcal{J}^{(m)}\right)+\gamma}, \frac{1}{\min\{\mu,\nu\}}\}$*; then, for an arbitrary initial stochastic guess* $\mathbf{x}_0$*, the iterative sequence* $\{\mathbf{x}_k\}_{k=0}^{\infty}$ *generated by* (3.4) *is convergent and*

$$\|\mathbf{x}_k - \mathbf{x}\|_1 \leq \varsigma^k \|\mathbf{x}_0 - \mathbf{x}\|_1 ,$$

where $\varsigma = \frac{\phi\alpha\bar{\mu}\left(\mathcal{J}^{(3)},\cdots,\mathcal{J}^{(m)}\right)+(\alpha-\phi\alpha)\gamma}{1-\phi\alpha\gamma}$.

Proof. It is easy to verify that $\alpha < \frac{1}{\min\{\mu,\nu\}}$ when $\alpha < \min\{\frac{1}{\phi\bar{\mu}\left(\mathcal{J}^{(3)},\cdots,\mathcal{J}^{(m)}\right)+\gamma}, \frac{1}{\min\{\mu,\nu\}}\}$. According to Lemma 2, the solution $\mathbf{x}$ is unique.

Set $\hat{\mathbf{e}}_k = \mathbf{y}_k - \mathbf{x}$ and $\mathbf{e}_k = \mathbf{x}_k - \mathbf{x}$. From (3.4), it follows that

$$\mathbf{y}_{k+1} = \phi\alpha\mathcal{P}\mathbf{x}_k^{m-2}\mathbf{y}_{k+1} + (\alpha - \phi\alpha)\mathcal{P}\mathbf{x}_k^{m-2}\mathbf{x}_k + \mathbf{b}. \tag{4.7}$$

Note that $\mathbf{x}$ is the solution of (1.1) yields

$$\mathbf{x} = \phi\alpha\mathcal{P}\mathbf{x}^{m-2}\mathbf{x} + (\alpha - \phi\alpha)\mathcal{P}\mathbf{x}^{m-2}\mathbf{x} + \mathbf{b}. \tag{4.8}$$

Subtracting (4.8) from (4.7), we have

$$\hat{\mathbf{e}}_{k+1} = \phi\alpha\mathcal{P}\mathbf{x}_k^{m-2}\hat{\mathbf{e}}_{k+1} + \alpha\mathcal{P}(\mathbf{x}_k^{m-2} - \mathbf{x}^{m-2})\mathbf{x} + (\alpha - \phi\alpha)\mathcal{P}\mathbf{x}_k^{m-2}\mathbf{e}_k. \tag{4.9}$$

By (4.7), we have

$$\begin{aligned}\sum_{i=1}^{n}((I - \phi\alpha\mathcal{P}\mathbf{x}_k^{m-2})\mathbf{y}_{k+1})_i &= (\alpha - \phi\alpha)\sum_{i=1}^{n}\sum_{i_2,\cdots,i_m=1}^{n} p_{ii_2,\cdots,i_m}x_{k,i_2}\cdots x_{k,i_m} + \sum_{i=1}^{n} b_i \\ &= \alpha - \phi\alpha + 1 - \alpha \\ &= 1 - \phi\alpha.\end{aligned}$$

On the other hand

$$1-\phi\alpha = \sum_{i=1}^{n} y_{k+1,i} - \phi\alpha \sum_{i=1}^{n} \sum_{i_2,i_3,\cdots,i_m\in\langle n\rangle} p_{ii_2,\cdots,i_m} y_{k+1,i_2} x_{k,i_3} \cdots x_{k,i_m}$$
$$= \sum_{i=1}^{n} y_{k+1,i} - \phi\alpha \sum_{i=1}^{n} y_{k+1,i}.$$

Therefore, we get

$$\sum_{i=1}^{n} y_{k+1,i} = 1, \sum_{i=1}^{n} \hat{e}_{k+1,i} = 0. \tag{4.10}$$

Combining (4.9) and Lemmas 3 and 4 together gives

$$\|\hat{\mathbf{e}}_{k+1}\|_1 \le \phi\alpha\gamma \|\hat{\mathbf{e}}_{k+1}\|_1 + \phi\alpha\bar{\mu}\left(\mathcal{J}^{(3)},\cdots,\mathcal{J}^{(m)}\right)\|\mathbf{e}_k\|_1 + (\alpha-\phi\alpha)\gamma\|\mathbf{e}_k\|_1 .$$

It is noted that

$$\|\hat{\mathbf{e}}_{k+1}\|_1 \le \frac{\phi\alpha\bar{\mu}\left(\mathcal{J}^{(3)},\cdots,\mathcal{J}^{(m)}\right) + (\alpha-\phi\alpha)\gamma}{1-\phi\alpha\gamma}\|\mathbf{e}_k\|_1 = \varsigma\|\mathbf{e}_k\|_1 .$$

Then by (4.10) and Lemma 5, we get

$$\|\mathbf{e}_{k+1}\|_1 \le \|\hat{\mathbf{e}}_{k+1}\|_1 \le \varsigma\|\mathbf{e}_k\|_1 \le \varsigma^{k+1}\|\mathbf{e}_0\|_1 .$$

By the above proof, we have that $\varsigma < 1$; then, the proof is completed. □

Theorem 3. *Let $\mathcal{P}$ be an m^{th}-order n-dimensional stochastic tensor, $\mathbf{x}$ be a solution of* (1.1) *and* $\alpha < \min\{\frac{1-(1-\phi)\beta\bar{\mu}(\mathcal{J}^{(3)},\cdots,\mathcal{J}^{(m)})}{\phi\bar{\mu}(\mathcal{J}^{(3)},\cdots,\mathcal{J}^{(m)})+\gamma}, \frac{1}{\min\{\mu,\nu\}}\}$*; then, for an arbitrary initial stochastic guess $\mathbf{x}_0$, the iterative sequence $\{\mathbf{x}_k\}_{k=0}^{\infty}$ generated by* (3.5) *is convergent and*

$$\|\mathbf{x}_k - \mathbf{x}\|_1 \le \zeta^k \|\mathbf{x}_0 - \mathbf{x}\|_1 ,$$

where $\zeta = \frac{(\beta+\phi\alpha-\phi\beta)\bar{\mu}(\mathcal{J}^{(3)},\cdots,\mathcal{J}^{(m)})+(1-\phi)(\alpha-\beta)\gamma}{1-(\beta+\phi\alpha-\phi\beta)\gamma}$.

Proof. Clearly, we have the unique solution $\mathbf{x}$ if $\alpha < \min\{\frac{1-(1-\phi)\beta\bar{\mu}(\mathcal{J}^{(3)},\cdots,\mathcal{J}^{(m)})}{\phi\bar{\mu}(\mathcal{J}^{(3)},\cdots,\mathcal{J}^{(m)})+\gamma}, \frac{1}{\min\{\mu,\nu\}}\}$ from Lemma 2.

By (3.5), we have

$$\mathbf{y}_{k+1} = (\beta+\phi\alpha-\phi\beta))\mathcal{P}\mathbf{x}_k^{m-2}\mathbf{y}_{k+1} + (1-\phi)(\alpha-\beta)\mathcal{P}\mathbf{x}_k^{m-2}\mathbf{x}_k + \mathbf{b}.$$

Let $\hat{\mathbf{e}}_k = \mathbf{y}_k - \mathbf{x}$ and $\mathbf{e}_k = \mathbf{x}_k - \mathbf{x}$. Then, we can get the following equation:

$$\hat{\mathbf{e}}_{k+1} = (\beta+\phi\alpha-\phi\beta)\mathcal{P}\mathbf{x}_k^{m-2}\hat{\mathbf{e}}_{k+1} + \alpha\mathcal{P}(\mathbf{x}_k^{m-2} - \mathbf{x}^{m-2})\mathbf{x} + (1-\phi)(\alpha-\beta)\mathcal{P}\mathbf{x}_k^{m-2}\mathbf{e}_k. \tag{4.11}$$

By the same proof as in Theorem 2, we get

$$\sum_{i=1}^{n}(\mathbf{y}_{k+1} - (\beta+\phi\alpha-\phi\beta)\mathcal{P}\mathbf{x}_k^{m-2}\mathbf{y}_{k+1})_i = (1-\phi)(\alpha-\beta)\sum_{i=1}^{n}\sum_{i_2,\cdots,i_m=1}^{n} p_{ii_2,\cdots,i_m} x_{k,i_2}\cdots x_{k,i_m} + \sum_{i=1}^{n} b_i$$
$$= (1-\phi)(\alpha-\beta) + 1 - \alpha$$
$$= 1 - \beta - \phi\alpha + \phi\beta.$$

Similarly, we have

$$\begin{aligned}1-\beta-\phi\alpha+\phi\beta &= \sum_{i=1}^{n} y_{k+1,i} - (\beta+\phi\alpha-\phi\beta)\sum_{i=1}^{n}\sum_{i_2,i_3,\cdots,i_m\in\langle n\rangle} p_{ii_2,\cdots,i_m} y_{k+1,i_2} x_{k,i_3}\cdots x_{k,i_m}\\ &= \sum_{i=1}^{n} y_{k+1,i} - (\beta+\phi\alpha-\phi\beta)\sum_{i=1}^{n} y_{k+1,i}.\end{aligned}$$

Hence

$$\sum_{i=1}^{n} y_{k,i} = 1, \sum_{i=1}^{n} \hat{e}_{k,i} = 0. \tag{4.12}$$

By combining (4.11) with Lemmas 3 and 4, we get

$$\|\hat{\mathbf{e}}_{k+1}\|_1 \le (\beta+\phi\alpha-\phi\beta)\gamma \|\hat{\mathbf{e}}_{k+1}\|_1 + (\beta+\phi\alpha-\phi\beta)\bar{\mu}\left(\mathcal{J}^{(3)},\cdots,\mathcal{J}^{(m)}\right)\|\mathbf{e}_k\|_1 + (1-\phi)(\alpha-\beta)\gamma\|\mathbf{e}_k\|_1,$$

which implies that

$$\|\hat{\mathbf{e}}_{k+1}\|_1 \le \frac{(\beta+\phi\alpha-\phi\beta)\bar{\mu}\left(\mathcal{J}^{(3)},\cdots,\mathcal{J}^{(m)}\right)+(1-\phi)(\alpha-\beta)\gamma}{1-(\beta+\phi\alpha-\phi\beta)\gamma}\|\mathbf{e}_k\|_1 = \zeta\|\mathbf{e}_k\|_1.$$

By (4.12) and Lemma 5, we have

$$\|\mathbf{e}_{k+1}\|_1 \le \|\hat{\mathbf{e}}_{k+1}\|_1 \le \zeta\|\mathbf{e}_k\|_1 \le \zeta^{k+1}\|\mathbf{e}_0\|_1.$$

It is easy to check that $\zeta < 1$ and the proof is completed. □

Theorem 4. *Let $\mathcal{P}$ be an m^{th}-order n-dimensional stochastic tensor,* $\mathbf{x}$ *be a solution of* (1.1) *and* $\alpha < \min\{\frac{1}{2\gamma(1-\phi)+3\bar{\mu}\left(\mathcal{J}^{(3)},\cdots,\mathcal{J}^{(m)}\right)}, \frac{1}{\min\{\mu,\nu\}}\}$*; then, for an arbitrary initial stochastic guess* $\mathbf{x}_0$*, the iterative sequence* $\{\mathbf{x}_k\}_{k=0}^{\infty}$ *generated by* (3.6) *is convergent and*

$$\|\mathbf{x}_k-\mathbf{x}\|_1 \le \xi^k\|\mathbf{x}_0-\mathbf{x}\|_1,$$

where $\xi = 2\frac{(1-\phi)\alpha\gamma+\alpha\bar{\mu}\left(\mathcal{J}^{(3)},\cdots,\mathcal{J}^{(m)}\right)}{1-\alpha\bar{\mu}\left(\mathcal{J}^{(3)},\cdots,\mathcal{J}^{(m)}\right)}$.

Proof. By Lemma 2 and the condition that $\alpha < \min\{\frac{1}{2\gamma(1-\phi)+3\bar{\mu}\left(\mathcal{J}^{(3)},\cdots,\mathcal{J}^{(m)}\right)}, \frac{1}{\min\{\mu,\nu\}}\}$, we know that x is unique.

Let $\hat{\mathbf{e}}_k = \mathbf{y}_k - \mathbf{x}$ and $\mathbf{e}_k = \mathbf{x}_k - \mathbf{x}$. By (3.6), we obtain

$$\mathbf{y}_{k+1} = \alpha\check{D}_{\mathbf{x}_k}\mathbf{y}_{k+1} + \phi\alpha(\check{L}_{\mathbf{x}_k}+\check{U}_{\mathbf{x}_k})\mathbf{y}_{k+1} + \alpha(1-\phi)(\check{L}_{\mathbf{x}_k}+\check{U}_{\mathbf{x}_k})\mathbf{x}_k + \mathbf{b};$$

then we obtain

$$\begin{aligned}\hat{\mathbf{e}}_{k+1} &= (\alpha\check{D}_{\mathbf{x}_k} + \phi\alpha(\check{L}_{\mathbf{x}_k}+\check{U}_{\mathbf{x}_k}))\hat{\mathbf{e}}_{k+1} + \alpha(1-\phi)(\check{L}_{\mathbf{x}_k}+\check{U}_{\mathbf{x}_k})\mathbf{e}_k + \alpha\mathcal{P}(\mathbf{x}_k^{m-2}-\mathbf{x}^{m-2})\mathbf{x}\\ &\le \alpha\mathcal{P}\mathbf{x}_k^{m-2}\hat{\mathbf{e}}_{k+1} + \alpha(1-\phi)\mathcal{P}\mathbf{x}_k^{m-2}\mathbf{e}_k + \alpha\mathcal{P}(\mathbf{x}_k^{m-2}-\mathbf{x}^{m-2})\mathbf{x}.\end{aligned}$$

Combining this with Lemmas 3 and 4 leads to

$$\|\hat{\mathbf{e}}_{k+1}\|_1 \le \alpha\gamma\|\hat{\mathbf{e}}_{k+1}\|_1 + (1-\phi)\alpha\gamma\|\mathbf{e}_k\|_1 + \alpha\bar{\mu}\left(\mathcal{J}^{(3)},\cdots,\mathcal{J}^{(m)}\right)\|\mathbf{e}_k\|_1.$$

Thus

$$\|\hat{\mathbf{e}}_{k+1}\|_1 \leq \frac{(1-\phi)\alpha\gamma + \alpha\bar{\mu}\left(\mathcal{J}^{(3)}, \cdots, \mathcal{J}^{(m)}\right)}{1-\alpha\bar{\mu}\left(\mathcal{J}^{(3)}, \cdots, \mathcal{J}^{(m)}\right)} \|\mathbf{e}_k\|_1 = \frac{1}{2}\xi \|\mathbf{e}_k\|_1 . \tag{4.13}$$

By (4.13) and Lemma 6, we have

$$\|\mathbf{e}_{k+1}\|_1 \leq 2\|\hat{\mathbf{e}}_{k+1}\|_1 \leq \xi \|\mathbf{e}_k\|_1 \leq \xi^{k+1} \|\mathbf{e}_0\|_1 .$$

Notice that $\xi < 1$. This completes the proof of the theorem. □

Theorem 5. *Let $\mathcal{P}$ be an m^{th}-order n-dimensional stochastic tensor,* $\mathbf{x}$ *be a solution of* (1.1) *and* $\alpha < \min\{\frac{1}{2\gamma(1-\phi)+3\bar{\mu}\left(\mathcal{J}^{(3)},\cdots,\mathcal{J}^{(m)}\right)}, \frac{1}{\min\{\mu,\nu\}}\}$*; then, for an arbitrary initial stochastic guess* $\mathbf{x}_0$*, the iterative sequence* $\{\mathbf{x}_k\}_{k=0}^{\infty}$ *generated by* (3.7) *is convergent and*

$$\|\mathbf{x}_k - \mathbf{x}\|_1 \leq \xi^k \|\mathbf{x}_0 - \mathbf{x}\|_1 .$$

Proof. It is similar to the proof of Theorem 4. □

Theorem 6. *Let $\mathcal{P}$ be an m^{th}-order n-dimensional stochastic tensor,* $\mathbf{x}$ *be a solution of* (1.1)*,* $\alpha < \min\{\frac{2\omega+(1-\omega)\phi}{3(1-\phi)+\omega\gamma+2\omega\bar{\mu}+\left(\mathcal{J}^{(3)},\cdots,\mathcal{J}^{(m)}\right)}, \frac{1}{\min\{\mu,\nu\}}\}$ *and* $\omega \in (\frac{1-\phi}{2-\phi}, 1)$*; then, for an arbitrary initial stochastic guess* $\mathbf{x}_0$*, the iterative sequence* $\{\mathbf{x}_k\}_{k=0}^{\infty}$ *generated by* (3.8) *is convergent and*

$$\|\mathbf{x}_k - \mathbf{x}\|_1 \leq \vartheta^k \|\mathbf{x}_0 - \mathbf{x}\|_1 ,$$

where $\vartheta = 2\frac{\omega\alpha\bar{\mu}\left(\mathcal{J}^{(3)},\cdots,\mathcal{J}^{(m)}\right)+(1-\phi)(1-\omega)(1+\alpha)+\omega\alpha(1-\phi)}{1-\phi(1-\omega)-\alpha(1-\phi)-\omega\alpha\gamma}$.

Proof. According to Lemma 2, it is obvious that the solution $\mathbf{x}$ is unique when $\alpha < \min\{\frac{2\omega+(1-\omega)\phi}{3(1-\phi)+\omega\gamma+2\omega\bar{\mu}+\left(\mathcal{J}^{(3)},\cdots,\mathcal{J}^{(m)}\right)}, \frac{1}{\min\{\mu,\nu\}}\}$ and $\omega \in (\frac{1-\phi}{2-\phi}, 1)$.

By (3.8), we have

$$\begin{aligned}\mathbf{y}_{k+1} =& \left(\alpha\check{D}_{\mathbf{x}_k} + \omega\alpha\check{L}_{\mathbf{x}_k} + \phi(1-\omega)(I-\alpha\check{D}_{\mathbf{x}_k}) + \phi\omega\alpha\check{U}_{\mathbf{x}_k}\right)\mathbf{y}_{k+1} \\ &+ (1-\phi)\left[(1-\omega)(I-\alpha\check{D}_{\mathbf{x}_k}) + \omega\alpha\check{U}_{\mathbf{x}_k}\right]\mathbf{x}_k + \omega\mathbf{b}.\end{aligned}$$

Taking $\hat{\mathbf{e}}_k = \mathbf{y}_k - \mathbf{x}$ and $\mathbf{e}_k = \mathbf{x}_k - \mathbf{x}$, by a proof that is analogous to that for the above theorems, we get

$$\begin{aligned}\hat{\mathbf{e}}_{k+1} =& \left(\alpha\check{D}_{\mathbf{x}_k} + \omega\alpha\check{L}_{\mathbf{x}_k} + \phi(1-\omega)(I-\alpha\check{D}_{\mathbf{x}_k}) + \phi\omega\alpha\check{U}_{\mathbf{x}_k}\right)\hat{\mathbf{e}}_{k+1} \\ &+ (1-\phi)\left[(1-\omega)(I-\alpha\check{D}_{\mathbf{x}_k}) + \omega\alpha\check{U}_{\mathbf{x}_k}\right]\mathbf{e}_k + \omega\alpha\mathcal{P}(\mathbf{x}_k^{m-2} - \mathbf{x}^{m-2})\mathbf{x} \\ =& [\phi(1-\omega)I + \alpha(1-\phi)\check{D}_{\mathbf{x}_k} + \omega\alpha\check{L}_{\mathbf{x}_k} + \omega\alpha\phi(\check{D}_{\mathbf{x}_k} + \check{U}_{\mathbf{x}_k})]\hat{\mathbf{e}}_{k+1} \\ &+ (1-\phi)\left[(1-\omega)(I-\alpha\check{D}_{\mathbf{x}_k}) + \omega\alpha\check{U}_{\mathbf{x}_k}\right]\mathbf{e}_k + \omega\alpha\mathcal{P}(\mathbf{x}_k^{m-2} - \mathbf{x}^{m-2})\mathbf{x} \\ \leq& [\phi(1-\omega)I + \alpha(1-\phi)\check{D}_{\mathbf{x}_k} + \omega\alpha\mathcal{P}\mathbf{x}_k^{m-2}]\hat{\mathbf{e}}_{k+1} + \omega\alpha\mathcal{P}(\mathbf{x}_k^{m-2} - \mathbf{x}^{m-2})\mathbf{x} \\ &+ (1-\phi)\left[(1-\omega)(I-\alpha\check{D}_{\mathbf{x}_k}) + \omega\alpha\check{U}_{\mathbf{x}_k}\right]\mathbf{e}_k.\end{aligned} \tag{4.14}$$

Due to Lemma 4, the above inequality has the following estimation

$$\begin{aligned}\|\hat{\mathbf{e}}_{k+1}\|_1 \leq& [\phi(1-\omega) + \alpha(1-\phi) + \omega\alpha\gamma]\|\hat{\mathbf{e}}_{k+1}\|_1 + \omega\alpha\bar{\mu}\left(\mathcal{J}^{(3)}, \cdots, \mathcal{J}^{(m)}\right)\|\mathbf{e}_{k+1}\|_1 \\ &+ [(1-\phi)(1-\omega)(1+\alpha) + \omega\alpha(1-\phi)]\|\mathbf{e}_{k+1}\|_1 .\end{aligned}$$

Hence

$$\|\hat{\mathbf{e}}_{k+1}\|_1 \leq \frac{\omega\alpha\bar{\mu}\left(\mathcal{J}^{(3)},\cdots,\mathcal{J}^{(m)}\right)+(1-\phi)(1-\omega)(1+\alpha)+\omega\alpha(1-\phi)}{1-\phi(1-\omega)-\alpha(1-\phi)-\omega\alpha\gamma}\|\hat{\mathbf{e}}_{k+1}\|_1 = \frac{1}{2}\vartheta\|\mathbf{e}_k\|_1 . \tag{4.15}$$

According to (4.15) and Lemma 6, we have

$$\|\mathbf{e}_{k+1}\|_1 \leq 2\|\hat{\mathbf{e}}_{k+1}\|_1 \leq \vartheta\|\mathbf{e}_k\|_1 \leq \vartheta^{k+1}\|\mathbf{e}_0\|_1 .$$

and $\vartheta < 1$. This completes the proof. □

Remark 2. *For the uniqueness conditions stated in other papers, such as those presented in [12, 13], we can also offer corresponding convergence analysis for our proposed algorithm. Here, we omit the detailed description.*

5. Numerical experiments

In this section, we present numerical experiments to exhibit the asset of the algorithm we are proposing.

All experiments were performed by using MATLAB R2016a on a Windows 10 64 bit computer equipped with a 1.00 GHz Intel ® Core ™ i4-29210M CPU processor and 8.00 GB of RAM. We chose to use three parameters to evaluate the effectiveness of the proposed method, which are the iteration number (denoted by IT), the computing time in seconds (denoted by CPU) and the error rate (denoted by *err*), respectively. The *err* parameter is defined by

$$err = \left\|\mathbf{x}_k - \alpha\mathcal{P}\mathbf{x}_k^{m-1} - \mathbf{b}\right\|_1 .$$

We have opted to employ large damping factors during the computation process, and in this case, iterative methods typically face major convergence problems when solving multilinear PageRank problems. Therefore, we tested the following values for the damping parameter α=0.95, 0.99, 0.995, 0.999, respectively. We selected that the maximum iterative number to be 1,000, the initial value $\mathbf{x}_0 = \mathbf{v} = \frac{1}{n}\mathbf{e}$, where $\mathbf{e}$ is an n-dimensional vector with all entries being 1, and the termination tolerance as $\varepsilon = 10^{-10}$. The stopping criterion for all test methods was set as $\left\|\mathbf{x}_k - \alpha\mathcal{P}\mathbf{x}_k^{m-1} - \mathbf{b}\right\|_1 < \varepsilon$. We searched the parameter ϕ (or ω) from 0.1 to 1 with the step size 0.01 and $\beta \in (0, \alpha)$ from 0.1 to α with the interval of 0.01; also, we set

$$\check{D}_{\mathbf{x}} = \frac{1}{2}\operatorname{diag}(\mathcal{P}\mathbf{x}^{m-2}), \check{L}_{\mathbf{x}} = \operatorname{tril}(\mathcal{P}\mathbf{x}^{m-2}, -1) + \check{D}_{\mathbf{x}}, \check{U}_{\mathbf{x}} = \operatorname{triu}(\mathcal{P}\mathbf{x}^{m-2}, 1).$$

Next, we provide the numerical analysis for the GTRS iteration discussed in Section 3, using four examples. We compare our approach with the relaxation methods presented by Liu et al. [19] and the TARS method. The TARS iterative method is described in Algorithm 2. To clarify, we denote Algorithms 1–4 in the work of Liu et al. as Al_1, Al_2, Al_3 and Al_4 respectively.

Example 1. *In this example, we considered three tensors from the practical problems in [30, 31]. Example (i) involves DNA sequence data, while Example (ii) applies interpersonal relationship data, and Example (iii) involves physicists' occupational mobility data. The numerical results are listed in Tables 1–3, where the minimum CPU times in each row are indicated in bold font.*

Algorithm 2 The TARS iterative method

Require: $\bar{M}_{\mathbf{x}}, \bar{N}_{\mathbf{x}}, \alpha, \beta, \omega, \mathbf{v}$, relaxation parameter $\hat{\gamma}$, maximun k_{max}, termination tolerance ε, and an initial stochastic vector $\mathbf{x}_0$

Ensure: $\mathbf{x}$

$k \leftarrow 1$
while $\mathbf{x}_k - \alpha\mathcal{P}\mathbf{x}_k^{m-1} - \mathbf{b} < \varepsilon$ **do**
  if $k < k_{max}$ **then**
    $\mathbf{y}_{k+1} \leftarrow \bar{M}_{\mathbf{x}_k}^{-1}\bar{N}_{\mathbf{x}_k}\mathbf{x}_k + \bar{M}_{\mathbf{x}_k}^{-1}\mathbf{b}$
    $\hat{\mathbf{x}}_{k+1} \leftarrow \hat{\gamma}\mathbf{y}_{k+1} + (1-\hat{\gamma})\mathbf{x}_k$
    $\mathbf{x}_{k+1} \leftarrow \mathrm{proj}(\hat{\mathbf{x}}_{k+1})$
    $k = k + 1$
  end if
end while

(i)

$$\mathcal{P}(:,:,1) = \begin{pmatrix} 0.6000 & 0.4083 & 0.4935 \\ 0.2000 & 0.2568 & 0.2426 \\ 0.2000 & 0.3349 & 0.2639 \end{pmatrix}, \mathcal{P}(:,:,2) = \begin{pmatrix} 0.5217 & 0.3300 & 0.4152 \\ 0.2232 & 0.2800 & 0.2658 \\ 0.2551 & 0.3900 & 0.3190 \end{pmatrix},$$

$$\mathcal{P}(:,:,3) = \begin{pmatrix} 0.5565 & 0.3648 & 0.4500 \\ 0.2174 & 0.2742 & 0.2600 \\ 0.2261 & 0.3610 & 0.2900 \end{pmatrix};$$

(ii)

$$\mathcal{P}(:,:,1) = \begin{pmatrix} 0.5810 & 0.2432 & 0.1429 \\ 0 & 0.4109 & 0.0701 \\ 0.4190 & 0.3459 & 0.7870 \end{pmatrix}, \mathcal{P}(:,:,2) = \begin{pmatrix} 0.4708 & 0.1330 & 0.0327 \\ 0.1341 & 0.5450 & 0.2042 \\ 0.3951 & 0.3220 & 0.7631 \end{pmatrix},$$

$$\mathcal{P}(:,:,3) = \begin{pmatrix} 0.4381 & 0.1003 & 0 \\ 0.0229 & 0.4338 & 0.0930 \\ 0.5390 & 0.4659 & 0.9070 \end{pmatrix};$$

(iii)

$$\mathcal{P}(:,:,1) = \begin{pmatrix} 0.9000 & 0.3340 & 0.3106 \\ 0.0690 & 0.6108 & 0.0754 \\ 0.0310 & 0.0552 & 0.6140 \end{pmatrix}, \mathcal{P}(:,:,2) = \begin{pmatrix} 0.6700 & 0.1040 & 0.0805 \\ 0.2892 & 0.8310 & 0.2956 \\ 0.0408 & 0.0650 & 0.6239 \end{pmatrix},$$

$$\mathcal{P}(:,:,3) = \begin{pmatrix} 0.6604 & 0.0945 & 0.0710 \\ 0.0716 & 0.6133 & 0.0780 \\ 0.2680 & 0.2922 & 0.8501 \end{pmatrix}.$$

From Tables 1–3, we have the following observations and remarks:

(1) Although our algorithms require more iterative steps, the computational times are typically shorter than those of Al_1, Al_2, Al_3 and Al_4 and TARS, provided that the parameters are chosen appropriately.

Table 1. Comparison of GTRS iterative method with Al_1–Al_4 and TARS for Example 1(i).

	GTRS-FP			GTRS-IO				GTRS-JCB			GTRS-GS			GTRS-SOR				TARS-IO		TARS-JCB		TARS-GS		TARS-SOR		AL_1		AL_2		AL_3		AL_4	
α	ϕ	CPU	IT	β	ϕ	CPU	IT	ϕ	CPU	IT	ϕ	CPU	IT	ω	ϕ	CPU	IT	CPU	IT	CPU	IT	CPU	IT	CPU	IT	CPU	IT	CPU	IT	CPU	IT	CPU	IT
0.9	0.49	0.000067	12	0.3	0.41	**0.000059**	12	0.94	0.000066	11	0.92	0.000066	11	0.6	1	0.000067	11	0.000073	10	0.000097	9	0.000093	10	0.000156	8	0.000069	14	0.000097	14	0.004579	13	0.000422	12
0.95	0.34	0.000067	13	0.7	0.51	**0.000059**	12	0.29	0.000066	10	0.96	0.000066	12	0.7	0.78	0.000065	15	0.000067	10	0.000068	9	0.000065	11	0.000091	8	0.000070	9	0.000095	12	0.000135	13	0.000410	13
0.99	0.39	0.000067	13	0.9	0.19	**0.000062**	12	0.97	0.000066	12	1	0.000066	12	0.3	0.7	0.000064	29	0.000066	10	0.000073	9	0.000071	11	0.000115	8	0.000069	11	0.000095	12	0.000077	13	0.000376	13
0.999	0.86	0.000067	12	0.8	0.75	**0.000063**	12	0.58	0.000065	9	0.76	0.000066	13	0.6	1	0.000067	12	0.000074	10	0.000070	9	0.000072	11	0.000075	8	0.000069	14	0.000096	16	0.000076	13	0.000386	13

Table 2. Comparison of GTRS iterative method with Al_1–Al_4 and TARS for Example 1(ii).

	GTRS-FP			GTRS-IO				GTRS-JCB			GTRS-GS			GTRS-SOR				TARS-IO		TARS-JCB		TARS-GS		TARS-SOR		AL_1		AL_2		AL_3		AL_4	
α	ϕ	CPU	IT	β	ϕ	CPU	IT	ϕ	CPU	IT	ϕ	CPU	IT	ω	ϕ	CPU	IT	CPU	IT	CPU	IT	CPU	IT	CPU	IT	CPU	IT	CPU	IT	CPU	IT	CPU	IT
0.9	0.92	0.000068	25	0.8	0.13	**0.000060**	25	0.24	0.000064	24	0.26	0.000063	26	0.8	0.36	0.000068	32	0.000077	17	0.000063	17	0.000062	20	0.000134	24	0.000065	20	0.000096	23	0.000124	29	0.000309	28
0.95	0.29	0.000069	31	0.9	0.99	**0.000057**	27	0.35	0.000066	26	0.19	0.000066	29	0.8	0.18	0.000065	37	0.000066	18	0.000061	19	0.000062	22	0.000061	26	0.000065	26	0.000097	30	0.000165	31	0.000247	30
0.99	0.93	0.000070	29	0.9	0.29	**0.000058**	28	0.6	0.000067	28	0.32	0.000068	31	1	0.9	0.000068	30	0.000071	19	0.000072	21	0.000072	24	0.000064	27	0.000066	29	0.000098	36	0.000075	33	0.000253	32
0.999	0.72	0.000069	30	0.9	0.87	**0.000059**	29	0.52	0.000069	28	0.17	0.000068	31	0.3	0.32	0.000067	94	0.000257	20	0.000072	21	0.000343	24	6.11E-05	28	0.000068	28	0.000099	28	0.000076	33	0.000275	32

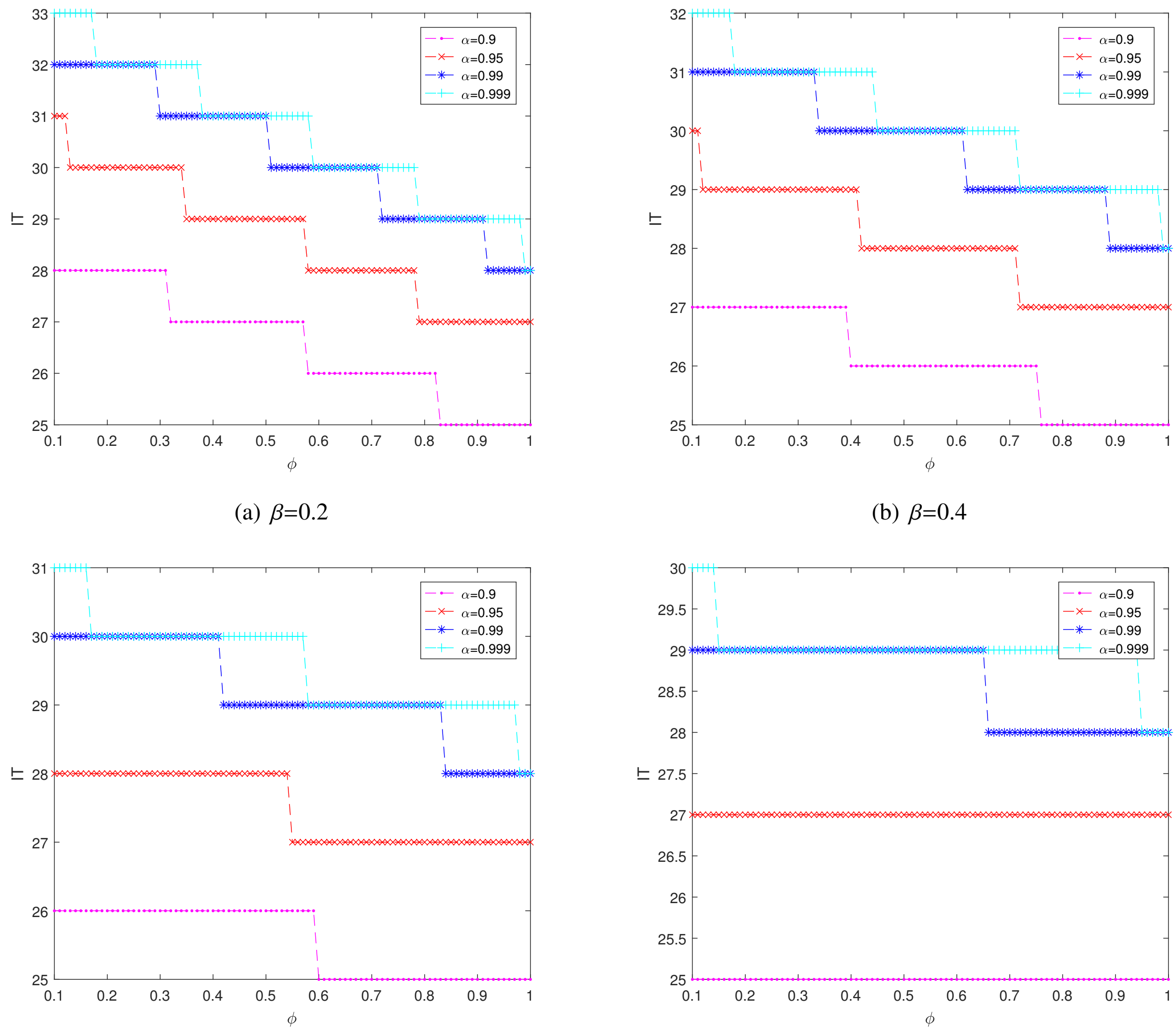
Table 3. Comparison of GTRS iterative method with Al_1–Al_4 and TARS for Example 1(iii).

	GTRS-FP			GTRS-IO				GTRS-JCB			GTRS-GS			GTRS-SOR				TARS-IO		TARS-JCB		TARS-GS		TARS-SOR		AL_1		AL_2		AL_3		AL_4	
α	ϕ	CPU	IT	β	ϕ	CPU	IT	ϕ	CPU	IT	ϕ	CPU	IT	ω	ϕ	CPU	IT	CPU	IT	CPU	IT	CPU	IT	CPU	IT	CPU	IT	CPU	IT	CPU	IT	CPU	IT
0.9	0.98	0.000071	51	0.6	0.12	**0.000061**	55	0.98	0.000072	50	0.88	0.000071	52	0.9	0.9	0.000066	52	0.000082	39	0.000093	40	0.000108	51	0.002955	51	0.000070	47	0.000099	60	0.002206	64	0.000249	61
0.95	0.55	0.000069	71	0.7	1	**0.000061**	61	0.96	0.000068	61	0.96	0.000067	62	0.7	0.34	0.000065	90	0.000084	45	0.000068	48	0.000069	63	0.000095	61	0.000070	48	0.000102	92	0.000103	78	0.000203	75
0.99	0.98	0.000067	74	0.6	0.54	0.000079	78	0.33	0.000067	72	0.28	0.000066	88	0.5	0.7	**0.000066**	82	0.000068	56	0.000101	58	0.000131	76	0.000075	74	0.000068	57	0.000103	88	0.000074	94	0.000180	90
0.999	0.45	0.000066	91	0.2	0.25	**0.000060**	93	0.94	0.000067	76	0.35	0.000066	90	0.3	0.74	0.000065	132	0.000070	58	0.000063	61	0.000081	80	0.000082	77	0.000066	57	0.000099	94	0.000072	98	0.000181	94

(2) Considering the least CPU times obtained from GTRS-FP, GTRS-IO, GTRS-JCB, GTRS-GS and GTRS-SOR, we observe that they account for approximately 0%, 91.7%, 0%, 0% and 8.3%, respectively. Therefore, in this example, GTRS-IO appears to be the most efficient algorithm among the five tested ones.

(3) Among the five types of splitting in the GTRS iteration, GTRS-JCB may result in higher CPU time usage compared to other methods, but it exhibits fewer iterative steps. This indicates that in cases with fewer iterations, GTRS-JCB can converge faster, while other methods may require more iterations to achieve the same convergence performance.

Next, we demonstrate the relationship between the iteration count and different values of the parameters α and ϕ in the GTRS iteration based on the inner-outer splitting approach, using Example 1(ii). By choosing β= {0.2, 0.4, 0.6,0.8} and ϕ from 0.1 to 1 in the interval of 0.01, we obtain Figure 1. It is shown that the GTRS iteration requires fewer iteration steps for a larger ϕ, which is more evident for a larger α, such as when α = 0.999. It is clear that the iteration number changes significantly depending on the chosen β, especially with β = 0.2.



(a) β=0.2 (b) β=0.4

(c) β=0.6 (d) β=0.8

Figure 1. The iterative steps for Example 1(ii) with different values of α and ϕ.

Example 2. *[12, 19] Let Γ be a directed graph with node set $\mathbb{V} = \langle n \rangle = \mathbb{D} \cup \mathbb{P}$, where $\mathbb{D}$ and $\mathbb{P}$ denote the sets of dangling nodes and pairwise connected ones, respectively. Denote n_p ($n_p \geq 2$) as the number of nodes in $\mathbb{P}$, that is, $\mathbb{P} = \{1, 2, ..., n_p\}$. A nonnegative tensor $\mathcal{A}$ is constructed as follows:*

$$a_{i_1 i_2 \cdots i_l} = \begin{cases} a_{i_1 i_2 \cdots i_l}, & i_k \neq i_{k+1}, i_1, i_k, i_l \in \mathbb{P}, k = 1, \cdots, l-1, \\ 0, & i_1 = i_2 \,(\, or\; i_1 \in \mathbb{D}), i_k \neq i_{k+1}, i_k \in \mathbb{P}, k = 2, \cdots, l-1, \\ \frac{1}{n}, & else, \end{cases}$$

where $a_{i_1 i_2 \cdots i_l}$ is taken randomly in (0, 1). Normalizing the entries with $p_{i_1 i_2 \cdots i_l} = \frac{a_{i_1 i_2 \cdots i_l}}{\sum_{i_1=1}^{n} a_{i_1 i_2 \cdots i_l}}$ yields a stochastic tensor $\mathcal{P} = p_{i_1 i_2 \cdots i_l}$.

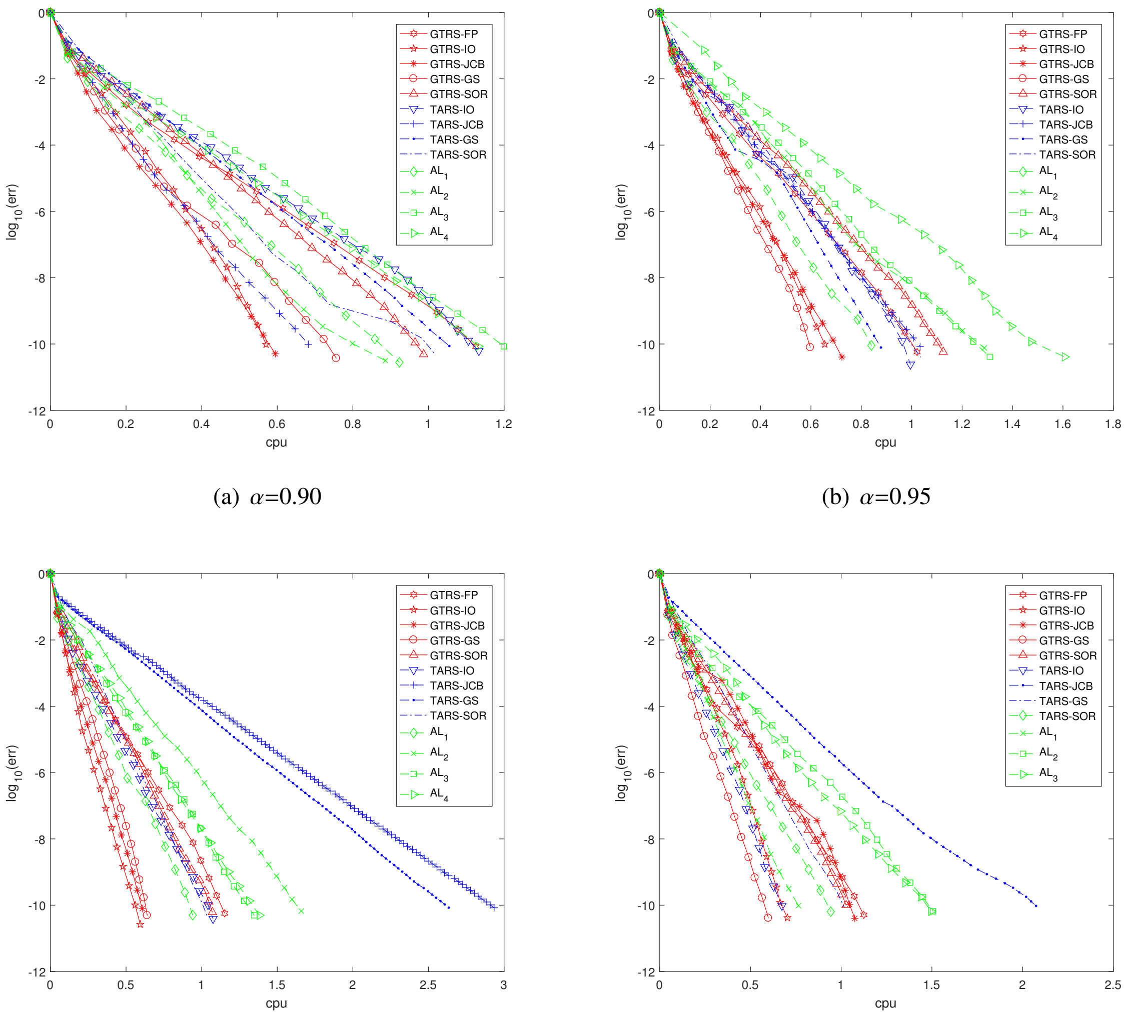
Based on the data presented in Table 4, there are observations that can be made as follows:

(1) The GTRS iteration demonstrates significantly shorter CPU time than those for Al_1–Al_4 and TARS. This suggests that the GTRS iterative method demonstrates higher efficiency in terms of computational time when appropriate parameters are chosen.

(2) In the case of all tested algorithms for a larger n_p, such as the case of $n_p = 120$, GTRS-SOR exhibits shorter CPU times than other methods.

(3) For $n = 60$, in terms of the CPU times, GTRS-IO achieves the best performance when α=0.9 or 0.95, while GTRS-FP, GTRS-JCB,GTRS-GS and GTRS-SOR outperform GTRS-IO when α=0.99 or 0.999. This implies that the behavior and reliability of these methods do not exhibit a monotonic relationship with α.

Furthermore, we display the test results of the relationship between CPU time and *err* at each step in Figure 2 with $n = 150$ and $n_p = 120$. The values of *err* are taken as the logarithm of base-10. It is known from Figure 2 that the GTRS iterative method performs better than Al_1–Al_4 and TARS in terms of both CPU time and error measures.



(a) α=0.90 (b) α=0.95

(c) α=0.99 (d) α=0.999

Figure 2. The relationship between CPU time and *err* for different values of α in Example 2.

Example 3. *[13] We generated third-order stochastic tensors with dimension n of 100, 150 and 200 using the function rand of MATLAB. The results are reported in Table 5.*

From Table 5, we observe the following:

(1) If the parameters are chosen suitably, the GTRS iterative method can achieve shorter CPU times than Al_1–Al_4 and TARS.

(2) For GTRS-JCB, the optimal performance is achieved when $\alpha = 0.99$ or 0.999, and $n = 100$ or 150. Similarly, GTRS-GS performs best when $n = 150$, while GTRS-SOR achieves the best results when $n = 200$. Notably, GTRS-IO shows the best performance when $\alpha = 0.95$. GTRS-FP requires the least iterative steps.

(3) In terms of CPU times, GTRS-IO outperforms GTRS-JCB when $\alpha = 0.9$ or 0.95. However, GTRS-JCB , GTRS-GS and GTRS-SOR outperform GTRS-IO when $\alpha = 0.99$ or 0.999.

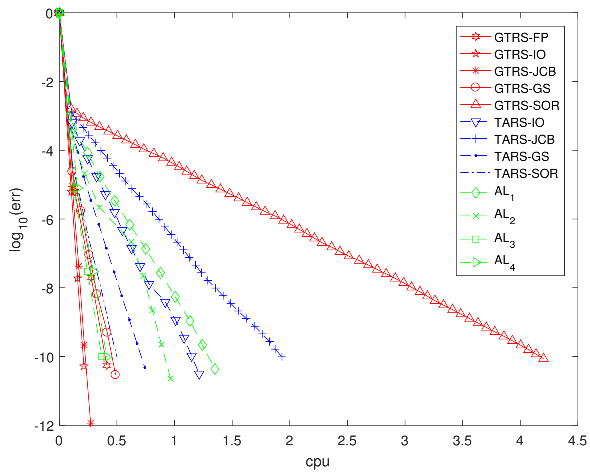
Table 4. Comparison of GTRS iterative method with Al_1–Al_4 and TARS for Example 2.

		GTRS-FP			GTRS-IO				GTRS-JCB			GTRS-GS			GTRS-SOR				TARS-IO		TARS-JCB		TARS-GS		TARS-SOR		AL_1		AL_2		AL_3		AL_4	
n	α	ϕ	CPU	IT	β	ϕ	CPU	IT	ϕ	CPU	IT	ϕ	CPU	IT	ω	ϕ	CPU	IT	CPU	IT	CPU	IT	CPU	IT	CPU	IT	CPU	IT	CPU	IT	CPU	IT	CPU	IT
60	0.9	0.78	0.067781	30	0.2	0.25	**0.049511**	36	0.48	0.057342	35	0.41	0.06151	32	0.7	0.6	0.05134	40	0.07825	20	0.07809	33	0.06519	28	0.06048	36	0.08123	24	0.07108	43	0.11308	39	0.19676	39
	0.95	0.1	0.068289	51	0.3	0.34	**0.048066**	42	0.47	0.057637	43	0.47	0.05796	39	0.2	0.62	0.05124	134	0.05745	22	0.06632	42	0.07181	35	0.07352	45	0.08710	33	0.07085	50	0.09384	49	0.19562	49
	0.99	0.72	0.067574	48	0.3	0.2	0.073720	56	0.7	0.056396	48	0.55	0.05525	48	0.3	1	**0.05087**	40	0.06472	30	0.07392	55	0.07376	46	0.07325	57	0.09754	41	0.06903	77	0.09697	63	0.18844	63
	0.999	0.96	0.065590	44	0.2	0.93	0.078802	44	0.86	0.055738	47	0.96	0.05820	43	0.2	0.98	**0.05130**	50	0.08935	32	0.06949	60	0.05869	49	0.05847	61	0.08533	29	0.07103	74	0.08697	68	0.20649	68
90	0.9	0.67	0.100722	33	0.4	0.88	0.071830	28	0.65	0.070657	33	0.88	0.07013	28	0.2	0.88	0.07801	55	0.09061	20	0.06785	35	0.06886	29	0.07486	37	0.08308	20	0.08977	46	0.09231	41	0.20913	41
	0.95	0.33	0.093380	49	0.9	0.51	**0.073547**	34	0.99	0.078598	33	0.93	0.07651	34	0.9	0.94	0.08669	35	0.14558	23	0.07550	45	0.08165	36	0.07369	46	0.08449	25	0.08936	58	0.11726	52	0.20301	52
	0.99	0.33	0.101617	64	0.2	0.22	0.078264	62	0.52	0.074716	57	0.33	**0.07038**	53	1	0.18	0.07446	56	0.07229	32	0.07778	59	0.07402	47	0.07806	59	0.08448	42	0.08910	61	0.10506	68	0.21183	68
	0.999	0.55	0.103049	61	0.2	0.52	0.079088	59	0.56	0.071014	61	0.11	**0.06921**	61	0.7	0.84	0.07499	55	0.10306	35	0.07600	65	0.07530	51	0.07626	63	0.07921	41	0.08972	75	0.10494	73	0.20750	73
120	0.9	0.7	0.157773	16	0.7	0.16	0.105258	15	0.76	0.080931	16	1	0.07766	14	0.6	0.34	**0.07067**	32	0.0742168	31	0.08554	20	0.07240	39	0.11592	19	0.09979	14	0.08058	24	0.12620	19	0.22842	19
	0.95	0.96	0.176812	15	0.9	0.24	0.085191	15	0.12	0.075656	21	0.44	0.07943	17	0.2	0.26	**0.06796**	120	0.0933628	14	0.07782	37	0.07571	27	0.12220	22	0.09910	16	0.08187	24	0.13799	21	0.20813	21
	0.99	0.75	0.145725	18	0.4	0.96	0.092986	16	0.22	0.074801	22	0.95	0.07781	16	0.7	0.46	**0.07029**	27	0.0891279	23	0.07859	107	0.07475	96	0.12151	22	0.08999	17	0.13877	26	0.11978	22	0.23414	22
	0.999	0.48	0.142763	20	0.8	0.69	0.090735	16	0.34	0.080148	21	0.72	0.07928	17	0.6	0.54	**0.07111**	30	0.0984028	16	0.08016	20	0.07543	69	0.11979	22	0.10156	15	0.08310	21	0.13226	22	0.26749	22

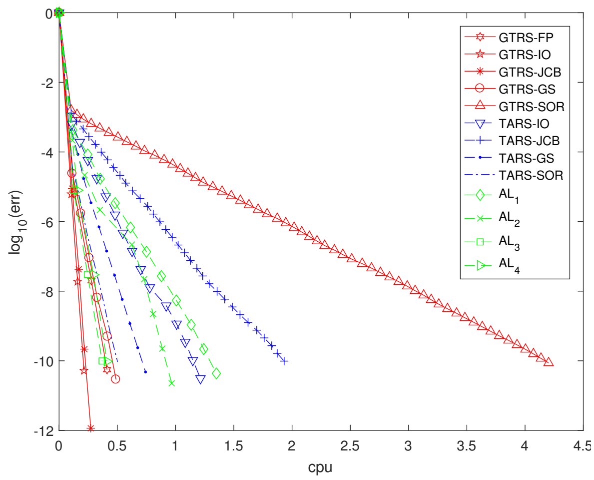
Table 5. Comparison of GTRS iterative method with Al_1–Al_4 and TARS for Example 3.

		GTRS-FP			GTRS-IO				GTRS-JCB			GTRS-GS			GTRS-SOR				TARS-IO		TARS-JCB		TARS-GS		TARS-SOR		AL_1		AL_2		AL_3		AL_4	
n	α	ϕ	CPU	IT	β	ϕ	CPU	IT	ϕ	CPU	IT	ϕ	CPU	IT	ω	ϕ	CPU	IT	CPU	IT	CPU	IT	CPU	IT	CPU	IT	CPU	IT	CPU	IT	CPU	IT	CPU	IT
100	0.9	0.23	0.01863	4	0.8	0.91	**0.01419**	4	0.26	0.01441	4	0.69	0.01437	6	0.7	0.48	0.01442	11	0.01466	11	0.01677	20	0.01463	19	0.01488	8	0.02349	6	0.01810	6	0.03137	4	0.06729	4
	0.95	0.66	0.01887	4	0.4	0.75	**0.01420**	4	0.83	0.01441	4	0.48	0.01441	7	1	0.26	0.01434	8	0.01432	15	0.01448	36	0.01468	12	0.01935	7	0.02341	9	0.01848	9	0.03054	4	0.07217	4
	0.99	0.91	0.01868	4	0.6	0.89	0.01527	4	0.49	**0.01434**	4	0.93	0.01451	4	0.9	1	0.01436	4	0.01448	12	0.01446	8	0.02156	15	0.01472	8	0.02324	9	0.01792	9	0.02348	4	0.07210	4
	0.999	0.17	0.01875	4	0.7	0.57	0.01459	4	0.96	**0.01417**	4	0.7	0.01438	6	0.6	0.24	0.01432	17	0.01456	15	0.01655	13	0.01435	15	0.02128	8	0.02339	9	0.01829	6	0.03042	4	0.06519	4
150	0.9	1	0.09606	4	0.5	0.25	0.07084	4	0.94	0.07177	4	0.87	**0.07008**	4	0.9	0.14	0.07094	9	0.09640	15	0.07460	19	0.07584	9	0.08288	8	0.10726	7	0.07295	8	0.15959	4	0.22549	4
	0.95	0.54	0.10796	4	0.4	0.67	**0.06927**	4	0.2	0.07429	4	0.77	0.07638	5	0.4	0.28	0.07452	27	0.08699	4	0.07486	78	0.08091	79	0.07491	7	0.10241	4	0.07412	8	0.14460	4	0.23545	4
	0.99	0.62	0.10090	4	0.5	0.53	0.09392	4	0.21	**0.07004**	4	0.76	0.07156	5	0.5	0.22	0.08315	21	0.08172	4	0.07614	8	0.07600	8	0.07606	4	0.10547	9	0.07243	7	0.15219	4	0.22312	4
	0.999	0.23	0.10765	4	0.4	1	0.09054	4	0.67	0.07137	4	1	**0.06991**	4	0.5	0.96	0.07547	6	0.07734	4	0.07529	11	0.07539	79	0.07722	4	0.10334	4	0.07598	8	0.12394	4	0.21592	4
200	0.9	0.88	0.26805	3	0.2	0.7	0.16128	3	0.96	0.16500	3	0.86	0.17025	4	0.4	0.78	**0.15421**	13	0.19223	14	0.17553	8	0.16749	48	0.17695	5	0.21955	9	0.23488	7	0.28720	3	0.56819	3
	0.95	0.64	0.26859	3	0.6	0.16	**0.15979**	3	0.19	0.16674	4	0.47	0.18597	6	0.2	0.28	0.16859	57	0.17618	15	0.16278	33	0.17163	11	0.17772	7	0.22631	11	0.19430	8	0.25026	3	0.54380	3
	0.99	1	0.26398	3	0.3	0.57	0.28735	3	0.6	0.18075	3	0.15	0.17505	7	0.4	0.74	**0.15166**	14	0.18111	8	0.19766	19	0.16150	15	0.17193	6	0.23041	10	0.18182	6	0.34902	4	0.51196	4
	0.999	1	0.24523	3	0.9	0.16	0.20416	3	0.44	0.17493	4	0.96	0.17321	3	0.3	0.44	**0.16722**	30	0.17762	15	0.19128	25	0.19562	26	0.17200	8	0.22839	10	0.20650	10	0.26499	4	0.51895	4

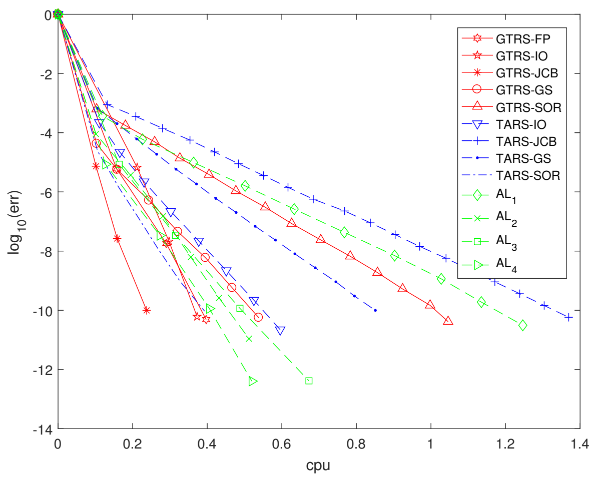
Similarly, we present the test results for the correlation between CPU times and *err* at each iterative step for $n = 200$ in Figure 3.



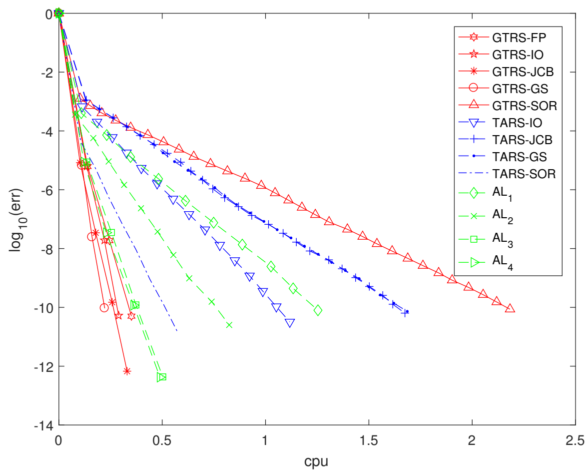
(a) α=0.90



(b) α=0.95



(c) α=0.99



(d) α=0.999

Figure 3. The relationship between CPU time and *err* for different values of α in Example 3.

As depicted in Figure 3, it is evident that the GTRS iterative method consistently outperforms Al_1–Al_4 and TARS in terms of both CPU time and error measurement.

Example 4. *[13] Let t denote the density of zero entries in a tensor.* $\mathcal{P}$ *is a positive tensor or a zero tensor when* $t = 0$ *or* $t = 1$ *respectively. Test sparse tensors with different t are generated by the function randsample of MATLAB.*

In Example 4, we generated third-order tensors of dimension 200 with densities ranging from 0.1 to 0.9 in intervals of 0.1. The numerical results are presented in Table 6. The conclusions drawn from Table 6 are as follows:

(1) When suitable parameters are found, our proposed algorithm consistently outperforms the methods presented in [19] and the TARS method.

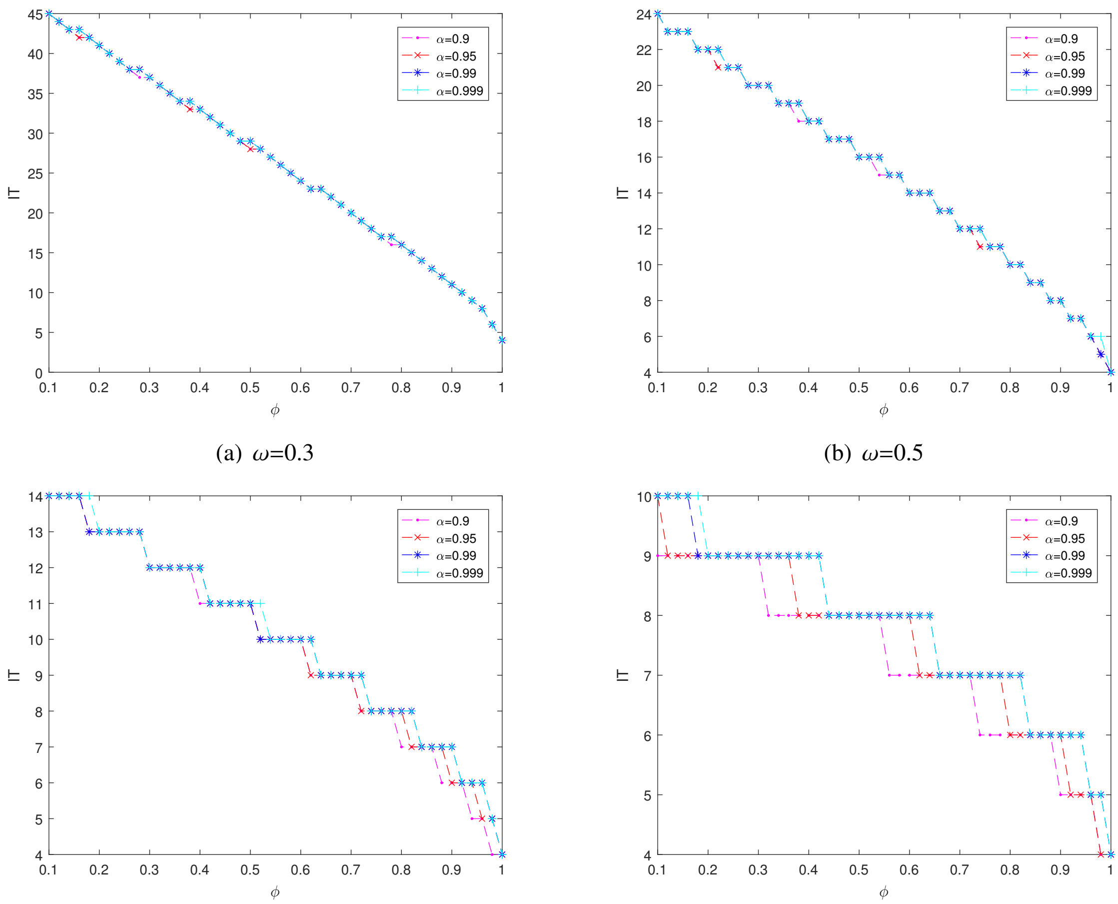
Table 6. Comparison of GTRS iterative method with Al_1–Al_4 and TARS for Example 4.

		GTRS-FP			GTRS-IO				GTRS-JCB			GTRS-GS			GTRS-SOR				TARS-IO		TARS-JCB		TARS-GS		TARS-SOR		AL_1		AL_2		AL_3		AL_4	
n	α	ϕ	CPU	IT	β	ϕ	CPU	IT	ϕ	CPU	IT	ϕ	CPU	IT	ω	ϕ	CPU	IT	CPU	IT	CPU	IT	CPU	IT	CPU	IT	CPU	IT	CPU	IT	CPU	IT	CPU	IT
0.1	0.9	0.95	0.19642	3	0.4	0.57	0.14413	3	0.64	**0.14412**	3	0.62	0.14913	5	0.5	0.36	0.14571	18	0.14552	3	0.14759	73	0.15818	74	0.15566	4	0.21365	10	0.15553	8	0.22266	3	0.47466	3
	0.95	0.17	0.20225	3	0.5	0.49	**0.14305**	3	0.66	0.14367	3	0.97	0.15258	3	0.2	0.32	0.14551	53	0.15240	8	0.15832	74	0.14742	47	0.15609	7	0.21183	5	0.15498	10	0.23578	3	0.48953	3
	0.99	0.38	0.20092	3	0.4	0.97	0.18022	3	0.8	**0.14429**	3	0.64	0.14984	5	0.8	0.3	0.14972	9	0.15950	14	0.15275	3	0.15652	7	0.15783	6	0.21065	6	0.15460	9	0.22561	3	0.48589	3
	0.999	0.75	0.20141	3	0.5	0.88	0.19840	3	0.65	**0.14604**	3	0.37	0.15242	6	0.5	0.24	0.14740	20	0.14490	3	0.15765	11	0.15297	8	0.15753	6	0.21166	10	0.15471	4	0.23385	3	0.49783	3
0.2	0.9	0.99	0.21186	3	0.4	0.18	**0.14552**	3	0.42	0.14559	3	0.93	0.14556	3	0.3	0.68	0.14469	20	0.15173	11	0.15827	11	0.15538	47	0.15922	6	0.21481	4	0.15672	4	0.25215	3	0.50077	3
	0.95	0.98	0.21072	3	0.2	0.37	**0.14374**	3	0.41	0.14435	3	0.93	0.15314	4	0.2	0.68	0.14729	30	0.15974	8	0.14836	19	0.15982	48	0.16212	6	0.21048	8	0.15427	8	0.24552	3	0.52172	3
	0.99	0.27	0.20794	3	0.5	0.99	0.19411	3	0.51	**0.14580**	3	0.76	0.14726	4	0.6	0.8	0.15218	8	0.15927	14	0.16016	19	0.14984	19	0.15956	7	0.21239	10	0.15655	10	0.23365	3	0.52896	3
	0.999	0.77	0.20039	3	0.2	0.53	0.19909	3	0.81	**0.14613**	3	0.96	0.14667	3	0.6	0.3	0.14740	15	0.15399	8	0.15940	8	0.15988	8	0.16108	7	0.21393	10	0.15664	10	0.21781	3	0.50479	3
0.3	0.9	0.79	0.19314	3	0.2	0.14	0.14622	4	0.63	0.14673	3	0.94	0.14635	4	0.5	0.56	**0.14573**	15	0.14919	11	0.14642	8	0.14712	11	0.14793	6	0.20942	7	0.15784	9	0.21804	4	0.51359	4
	0.95	0.65	0.19430	3	0.9	0.83	**0.14552**	3	0.67	0.14724	4	0.56	0.14653	6	0.4	0.84	0.14557	11	0.14696	11	0.14669	4	0.14686	77	0.14577	8	0.21063	11	0.15655	5	0.24682	4	0.55810	4
	0.99	0.71	0.19434	4	0.6	0.53	0.16199	3	0.32	0.14628	4	0.85	0.14620	4	1	0.18	**0.14589**	8	0.14617	4	0.14661	15	0.14721	11	0.14645	8	0.20728	4	0.15907	7	0.34163	4	0.53196	4
	0.999	0.28	0.19469	4	0.4	0.15	0.14798	4	0.85	**0.14557**	3	0.9	0.14611	4	0.7	0.38	0.14578	11	0.14708	11	0.14806	11	0.14698	20	0.14590	4	0.20649	5	0.15717	7	0.23544	4	0.49171	4
0.4	0.9	0.79	0.19226	4	0.5	0.83	0.15152	4	0.45	0.15379	4	0.24	0.15332	7	1	0.3	**0.15064**	8	0.15737	11	0.15677	4	0.15806	49	0.15708	7	0.20765	11	0.15374	6	0.21360	4	0.50308	4
	0.95	0.55	0.19221	4	0.6	0.22	0.15075	4	0.85	0.15183	4	0.61	0.15352	6	0.6	0.18	**0.15139**	17	0.15763	11	0.15817	4	0.15651	9	0.15663	7	0.20661	5	0.15555	7	0.21131	4	0.50354	4
	0.99	0.37	0.19165	4	0.3	0.37	0.15614	4	0.99	0.15367	4	0.28	0.15352	7	0.7	0.98	**0.15089**	5	0.15691	11	0.15760	19	0.15867	8	0.15661	6	0.20801	5	0.15476	6	0.21246	4	0.50895	4
	0.999	0.57	0.20102	4	0.7	0.18	0.16410	4	0.98	0.15379	4	0.91	0.15317	4	0.6	0.26	**0.15453**	16	0.15598	8	0.15457	49	0.15615	35	0.15697	8	0.20817	11	0.15621	8	0.20973	4	0.50399	4
0.5	0.9	0.26	0.21041	4	0.3	0.11	0.14663	4	0.87	0.14536	4	0.95	0.14570	4	0.6	0.1	**0.14514**	18	0.17206	4	0.17689	4	0.17628	8	0.17007	4	0.21408	11	0.15898	4	0.21593	4	0.52894	4
	0.95	0.53	0.20991	4	0.4	0.77	**0.14426**	4	0.21	0.15410	4	0.17	0.15057	8	0.6	0.24	0.14663	16	0.17090	15	0.16328	11	0.16553	8	0.16872	5	0.21353	5	0.15637	6	0.21681	4	0.52522	4
	0.99	0.45	0.20541	4	0.5	0.44	0.20040	4	0.79	0.15076	4	0.96	0.16091	4	0.2	0.2	**0.14966**	64	0.17478	15	0.16647	78	0.16376	13	0.16593	6	0.21397	6	0.15915	8	0.24374	4	0.50062	4
	0.999	0.2	0.20789	4	0.3	0.35	0.19865	4	0.3	**0.14774**	4	0.72	0.14974	5	0.6	0.32	0.15732	16	0.17555	11	0.16837	19	0.17116	9	0.16124	7	0.21352	7	0.15881	5	0.21705	4	0.52109	4
0.6	0.9	0.76	0.23882	4	0.3	0.66	**0.15600**	4	0.1	0.17507	4	0.89	0.17825	4	0.5	0.44	0.16122	18	0.16261	4	0.16398	8	0.16587	12	0.15299	8	0.24702	6	0.16154	11	0.31112	4	0.64738	4
	0.95	0.89	0.23388	4	0.3	0.55	0.17275	4	0.97	0.16780	4	0.97	0.17455	4	0.6	0.94	**0.14888**	6	0.17159	4	0.15328	8	0.16860	9	0.16323	7	0.24929	5	0.15324	10	0.26301	4	0.79157	4
	0.99	0.61	0.23289	4	0.4	0.66	0.19417	4	0.17	0.15638	4	0.26	0.16145	7	0.5	0.5	**0.14717**	17	0.16949	8	0.18164	12	0.18022	12	0.14839	7	0.24588	4	0.15480	10	0.25240	4	0.64654	4
	0.999	0.64	0.23399	4	0.2	0.68	0.25873	4	0.2	0.16938	4	0.23	0.17446	7	0.4	0.72	**0.14695**	15	0.14754	15	0.16349	26	0.18106	36	0.16140	8	0.24889	7	0.15461	10	0.32796	4	0.72918	4
0.7	0.9	0.22	0.21431	4	0.6	0.37	0.14613	4	0.63	0.14752	4	0.89	0.14829	4	0.4	0.66	**0.14590**	17	0.15861	4	0.17224	12	0.16393	12	0.16407	6	0.21340	10	0.15355	12	0.24924	4	0.52420	4
	0.95	0.44	0.21411	4	0.8	0.81	**0.14487**	4	0.61	0.14708	4	0.69	0.15214	6	0.3	0.16	0.14504	44	0.16918	4	0.14568	15	0.18287	21	0.16241	8	0.21441	6	0.15324	10	0.22028	4	0.53816	4
	0.99	0.3	0.21431	4	0.2	0.47	0.20147	4	0.3	0.14656	4	0.98	0.15153	4	0.5	0.48	**0.14550**	17	0.14720	4	0.16119	15	0.17006	16	0.16168	5	0.20845	9	0.15480	10	0.22898	4	0.53813	4
	0.999	0.13	0.21612	4	0.2	0.41	0.20116	4	0.67	0.14666	4	0.9	0.14663	4	0.3	0.44	**0.14571**	32	0.15500	4	0.15566	26	0.16399	10	0.14667	6	0.21200	10	0.15461	10	0.23828	4	0.53417	4
0.8	0.9	0.24	0.19462	5	0.7	0.7	**0.14510**	4	0.43	0.14746	4	0.81	0.14714	5	0.2	0.94	0.14707	12	0.14881	4	0.14945	12	0.14781	9	0.14904	7	0.20982	11	0.15282	9	0.21831	5	0.48351	5
	0.95	0.35	0.19580	5	0.7	0.86	**0.14473**	4	0.94	0.14740	4	0.22	0.14791	7	0.3	0.64	0.14697	24	0.14761	12	0.14849	16	0.14806	12	0.14723	5	0.21069	7	0.15620	8	0.20987	5	0.47670	5
	0.99	0.38	0.19481	5	0.9	0.89	**0.14006**	4	0.5	0.14658	5	0.2	0.14782	7	0.4	0.84	0.15179	12	0.14879	9	0.14686	5	0.14823	9	0.14921	8	0.20863	7	0.15378	8	0.21490	5	0.47093	5
	0.999	0.84	0.19513	4	0.8	0.1	**0.14124**	4	0.13	0.14748	5	0.38	0.14756	6	1	0.8	0.14788	6	0.14897	12	0.14952	20	0.14993	16	0.14838	7	0.21093	11	0.15511	9	0.21042	5	0.47182	5
0.9	0.9	0.49	0.19820	5	0.3	0.48	0.14732	5	0.7	0.15031	5	0.52	0.14959	6	0.8	0.88	**0.13937**	6	0.15959	5	0.17585	9	0.15832	13	0.15139	6	0.21135	12	0.15462	7	0.22199	5	0.48839	5
	0.95	0.34	0.19624	5	0.9	1	**0.14110**	5	0.35	0.14901	5	0.52	0.14994	6	0.5	0.4	0.14768	19	0.15854	12	0.15560	12	0.15924	10	0.14824	5	0.21440	10	0.15743	9	0.21881	5	0.48272	5
	0.99	0.97	0.19606	5	0.2	0.46	**0.14481**	5	0.38	0.14962	5	0.92	0.14993	5	0.4	0.56	0.14877	21	0.14086	16	0.15999	9	0.19237	13	0.16005	5	0.21269	12	0.15400	6	0.22300	5	0.48468	5
	0.999	0.26	0.19794	5	0.3	0.7	0.15251	5	0.37	0.14963	5	0.51	0.14918	7	0.4	0.1	**0.14264**	34	0.14147	16	0.15843	16	0.15982	12	0.15774	7	0.21210	8	0.15444	9	0.22177	5	0.48756	5

(2) Out of the 36 test cases, in terms of the minimum CPU times, GTRS-SOR, GTRS-GS, GTRS-IO, GTRS-JCB and GTRS-FP occupy about 36.1%, 0%, 44.5%, 19.4% and 0%, respectively. Based on these results, GTRS-SOR appears to be the most competitive algorithm, particularly when t=0.4 or 0.6.

(3) When t is less than 0.5, GTRS-JCB has better convergence performance than GTRS-GS and GTRS-FP. In most cases, GTRS-IO demonstrates a shorter CPU time than the other splitting methods of the GTRS iterative type, especially when t=0.8.

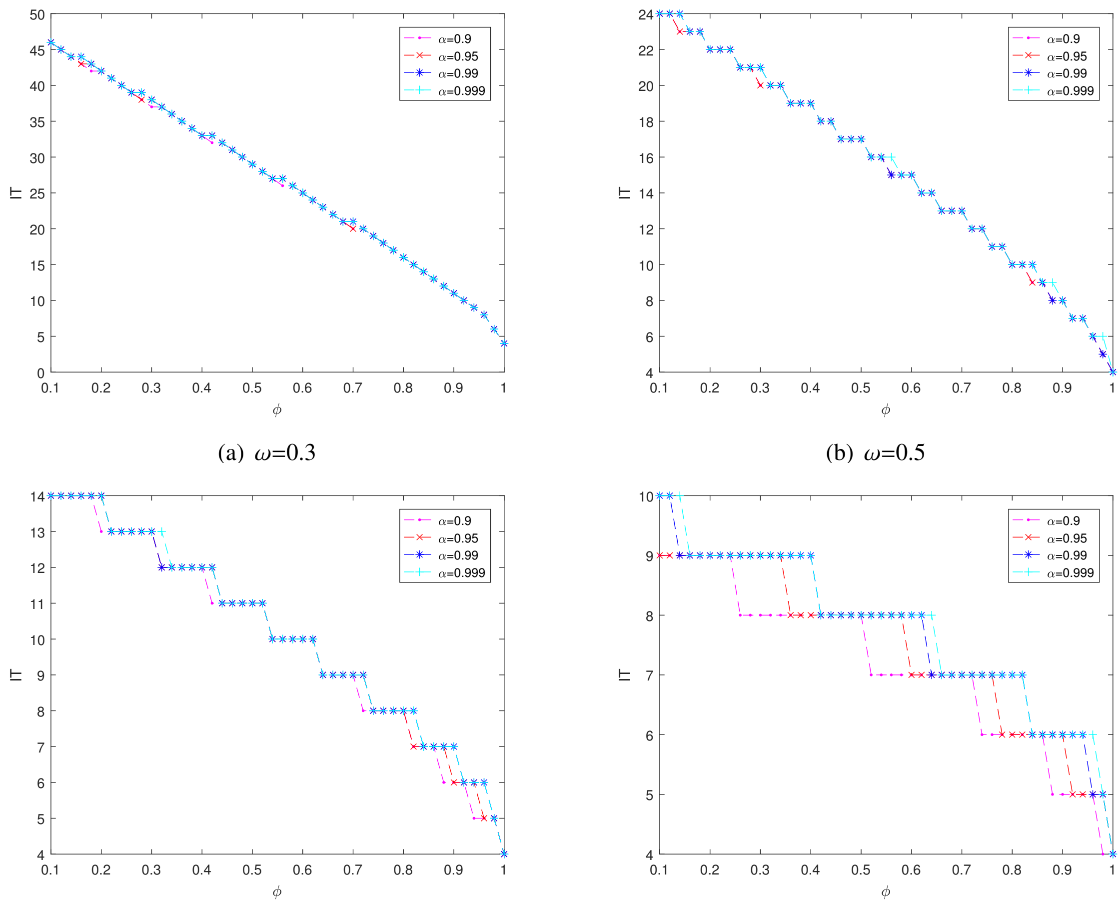
Next, we discuss the relationship between the iterative steps and ϕ with different values of α. Given the superior convergence properties of the GTRS iterative method with successive overrelaxation splitting, and the significant variation in the number of iterations in Example 4, we selected this splitting for the experiment. Specifically, we conducted the experiment by using $\omega = \{0.3, 0.5, 0.7, 0.9\}$, $t = \{0.4, 0.6\}$ and varying ϕ from 0.1 to 1 in increments of 0.01. We have plotted the associated comparison results in Figures 4 and 5. The results indicate that increasing ϕ leads to better performance of the GTRS iterative method, as evidenced by the reduction in the number of iterations required.



(a) ω=0.3 (b) ω=0.5

(c) ω=0.7 (d) ω=0.9

Figure 4. Taking t=0.4, the iterative steps with different values of α and ϕ for Example 4.



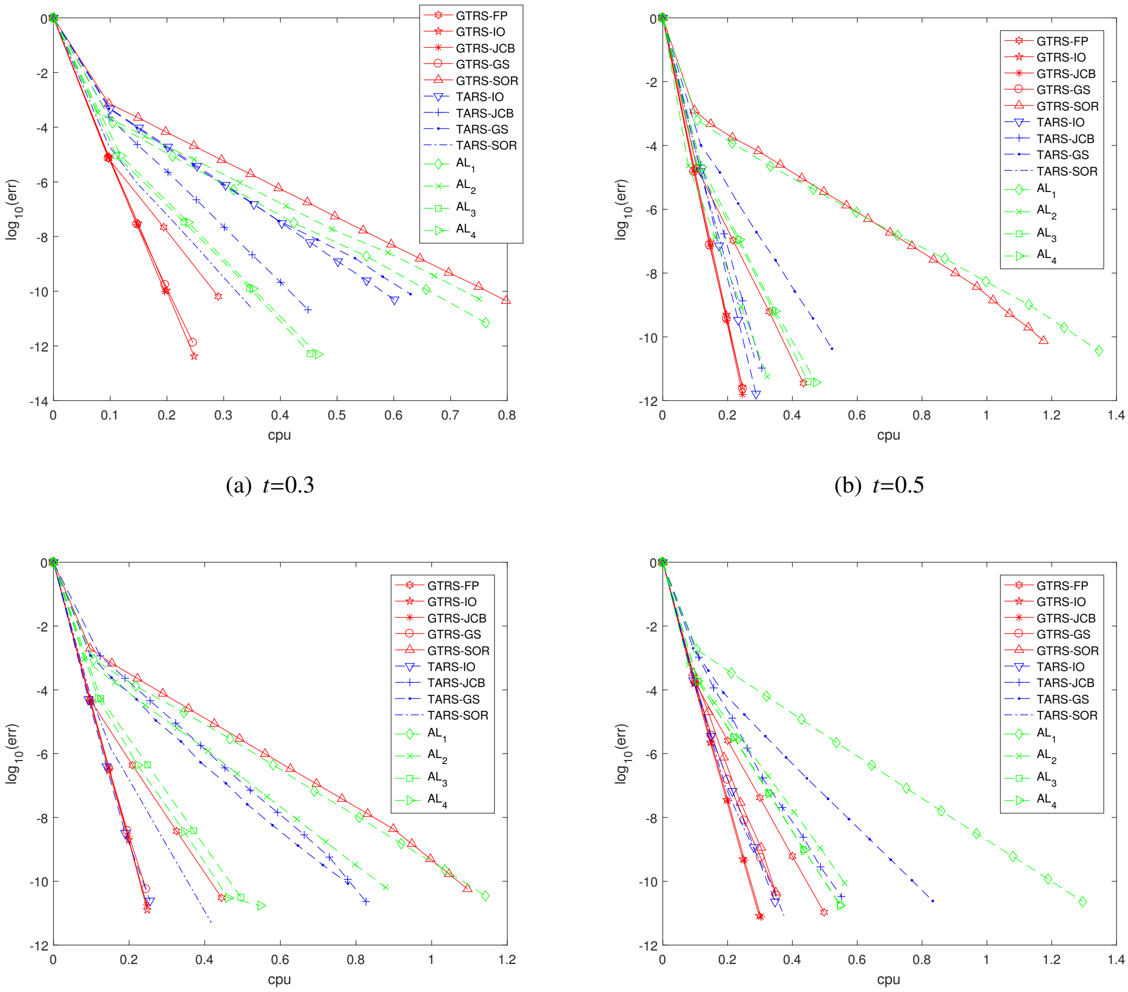
(a) ω=0.3 (b) ω=0.5

(c) ω=0.7 (d) ω=0.9

Figure 5. Taking t=0.6, the iterative steps with different values of α and ϕ for Example 4.

Figures 6–9 show the convergence history of the GTRS iteration under different densities of tensors and values of α. From Figures 6–9, we find that the proposed algorithms converge faster than the existing methods in the most cases, while GTRS-IO performs the best. It is noticed that GTRS-FP, GTRS-IO and GTRS-JCB always require less computing time than the existing methods. This is due to the fact that they compute smaller errors, which reduce the number of iterations required for convergence.

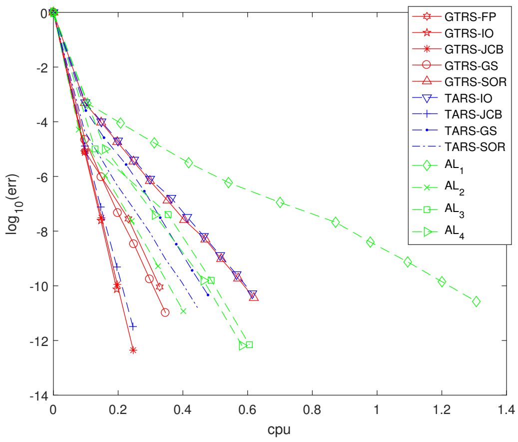
Different parameter settings can lead to optimal performance for each GTRS iteration. By selecting the appropriate values for α and n, the computational efficiency and accuracy of the GTRS iteration can be improved. It is important to note that these optimal parameter settings may vary depending on the specific problem and dataset being analyzed. In general, the GTRS iteration can achieve better convergence performance than Al_1–Al_4 and TARS in terms of solving the multilinear PageRank problem.



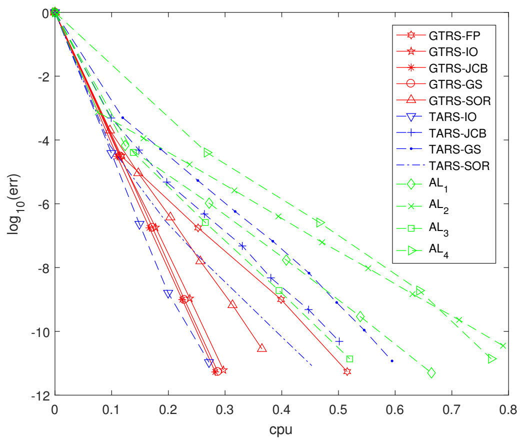
(a) t=0.3 (b) t=0.5

(c) t=0.7 (d) t=0.9

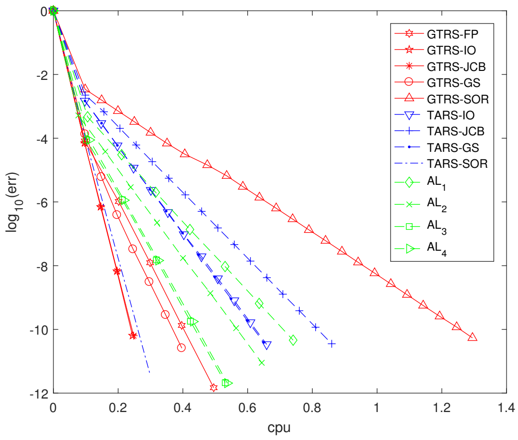
Figure 6. The relationship between CPU time and *err* for different values of t with $\alpha = 0.90$ for Example 4.



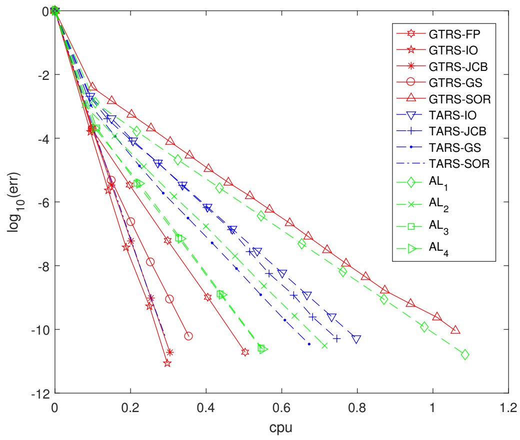
(a) t=0.3



(b) t=0.6

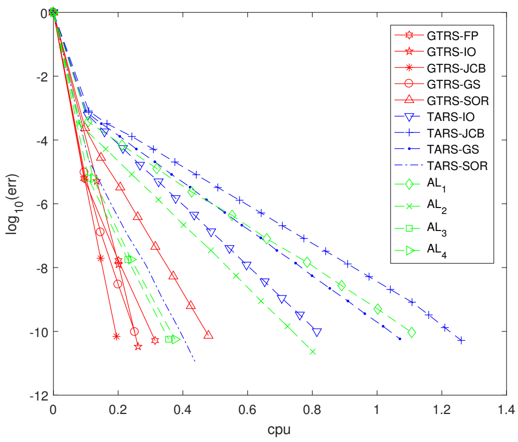


(c) t=0.8

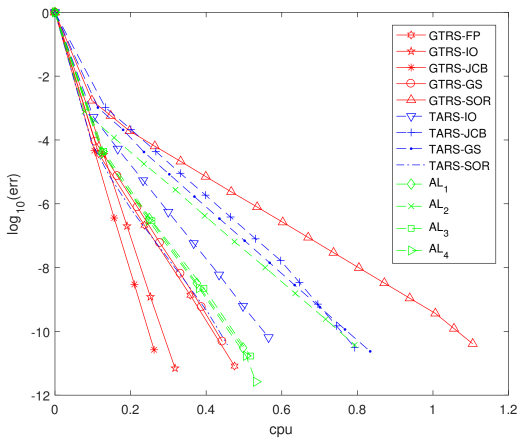


(d) t=0.9

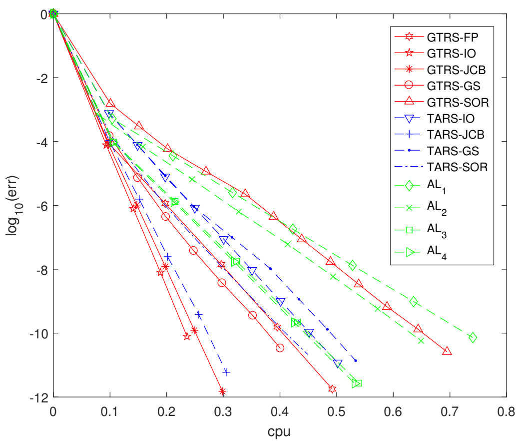
Figure 7. The relationship between CPU time and err for different values of t with $\alpha = 0.95$ for Example 4.



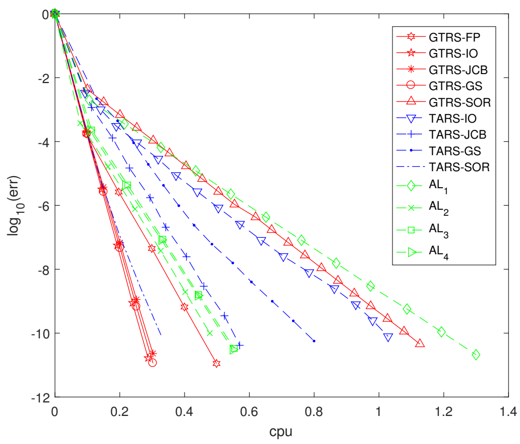
(a) t=0.2



(b) t=0.6

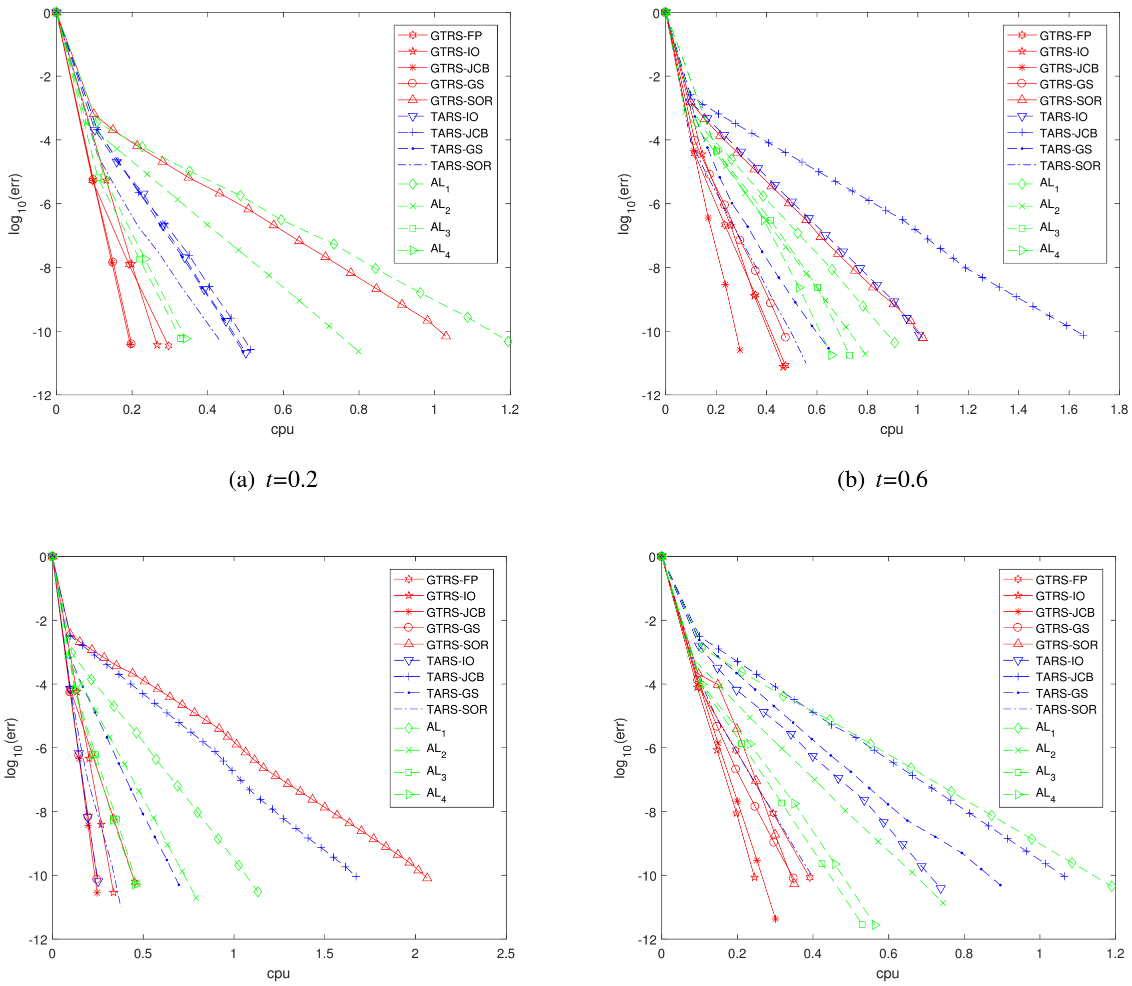


(c) t=0.8



(d) t=0.9

Figure 8. The relationship between CPU time and *err* for different values of t with $\alpha = 0.99$ for Example 4.



(a) t=0.2 (b) t=0.6

(c) t=0.7 (d) t=0.8

Figure 9. The relationship between CPU time and *err* for different values of t with $\alpha = 0.999$ for Example 4.

6. Conclusions

In this paper, based on the (weak) regular splitting of the matrix $I - \alpha\mathcal{P}\mathbf{x}^{m-2}$, we presented the GTRS iterative method for solving (1.1) from the perspective of the multilinear PageRank problem, and we have proved its overall convergence. In addition, we provided several splits of the GTRS iteration and constructed the corresponding convergence theorems. Numerical experiments indicated that the GTRS iterative method is superior to the existing ones in [19] and the TARS method as a result of choosing appropriate parameters. Overall, this research contributes to the advancement of non-gradient algorithms and their practical implications in tensor computations. In future work, we plan to investigate the latest uniqueness conditions to establish better convergence properties.

Use of AI tools declaration

The authors declare that they have not used artificial intelligence tools in the creation of this paper.

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Conflict of interest

The authors declare no conflict of interest.

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