# Mathematics

*Research article*

# Two-sided jumps risk model with proportional investment and random observation periods

**Chunwei Wang**$^{*,\dagger}$**, Jiaen Xu**$^{\dagger}$**, Naidan Deng and Shujing Wang**

School of Mathematics and Statistics, Henan University of Science and Technology, Luoyang 471023, Henan, China
† These authors contributed equally to this work.

\* **Correspondence:** Email: wangchunwei@haust.edu.cn; Tel: +86037964231482.

**Abstract:** In this paper, we consider a two-sided jumps risk model with proportional investments and random observation periods. The downward jumps represent the claim while the upward jumps represent the random returns. Suppose an insurance company invests all of their surplus in risk-free and risky investments in proportion. In real life, corporate boards regularly review their accounts rather than continuously monitoring them. Therefore, we assume that insurers regularly observe surplus levels to determine whether they will ruin and that the random observation periods are exponentially distributed. Our goal is to study the Gerber-Shiu function (i.e., the expected discounted penalty function) of the two-sided jumps risk model under random observation. First, we derive the integral differential equations (IDEs) satisfied by the Gerber-Shiu function. Due to the difficulty in obtaining explicit solutions for the IDEs, we utilize the sinc approximation method to obtain the approximate solution. Second, we analyze the error between the approximate and explicit solutions and find the upper bound of the error. Finally, we discuss examples of sensitivity analysis.

## 1. Introduction

Risk theory is a significant area of research in financial mathematics and actuarial science and the two-sided jumps risk model can be traced back to Boucherie et al. [1]. Then, it was further studied and promoted by many scholars. For example, Zhang [2] studied a two-sided jumps risk model and derived the integral differential equation for the expected discount penalty function. Additionally, the Gerber-Shiu function is obtained in a special case (claim amount is exponential distribution). Cheung

et al. [3] considered a two-sided jumps renewal risk model. Since ruin may not happen, they studied the joint moments of the total discounted claim costs and benefit costs. In some examples, explicit solutions (ES) are obtained under different cost functions. For further exploration of two-sided jumps, interested readers can refer to relevant literature [4–8].

Based on the above studies on the two-sided jumps risk model, we defined the surplus process $C(t)$ as

$$C(t) = x + ct - \sum_{i=1}^{M_1(t)} Y_i + \sum_{i=1}^{M_2(t)} Z_i, \quad t \geq 0, \tag{1.1}$$

where $x$ ($x \geq 0$) is the initial surplus, $c$ ($c > 0$) represents the constant rate of premium. Note that the two random components $\sum_{i=1}^{M_1(t)} Y_i$ and $\sum_{i=1}^{M_2(t)} Z_i$ in the model are both compound Poisson processes which respectively represent the total claim and aggregate random return at time $t$. $\{M_1(t)\}_{t \geq 0}$ and $\{M_2(t)\}_{t \geq 0}$ are homogeneous Poisson processes with rate $\lambda_1 > 0$ and $\lambda_2 > 0$, respectively. The claim sizes are given by the sequence of independent identically distributed (i.i.d.) positive random variables (r.v.'s) $\{Y_i\}_{i=1}^{\infty}$, the cumulative distribution function (c.d.f.) $F_Y(\cdot)$ and probability density function (p.d.f.) $f_Y(\cdot)$, while the random return amount is given by the sequence of i.i.d. positive r.v.'s $\{Z_i\}_{i=1}^{\infty}$ with c.d.f. $F_Z(\cdot)$ and p.d.f. $f_Z(\cdot)$. $M_1(t) = \sup\{i : T_1 + T_2 + \cdots + T_i \leq t\}$ and $M_2(t) = \sup\{i : K_1 + K_2 + \cdots + K_i \leq t\}$ are defined, where the i.i.d. inter-claim times $\{T_i\}_{i=1}^{\infty}$ and the i.i.d. inter-gain times $\{K_i\}_{i=1}^{\infty}$ have common exponential distributions with intensities $\lambda_1$ and $\lambda_2$ respectively.

To safeguard the interests of all parties involved, especially the insured, insurance companies must make reasonable and effective use of their funds. Insurance companies tend to invest their surplus in a specific portfolio of risk-free and risky asset. As the investment income of insurance companies is becoming more and more important in their total income, it is necessary to consider the risk model of investment factors. For this purpose, we assume that an insurance company allocates a portion of its funds to risk-free investment and another portion to risky investment. Especially, the risk-free investment $\{R_t\}_{t \geq 0}$ is of the form

$$dR_t = rR_t dt, \tag{1.2}$$

where $r$ denotes the interest rate of the risk-free asset, it is easy to know that $r$ is greater than zero. The venture capital $\{Q_t\}_{t \geq 0}$ is defined as follows

$$Q_t = e^{(at + \sigma W_t)}, \tag{1.3}$$

where $a$ and $\sigma$ represent the expected rate of return and volatility of the risky asset respectively and both of them are positive. $\{W_t, t \geq 0\}$ is the standard Brownian motion. The risky asset process $\{Q_t\}_{t \geq 0}$ satisfies

$$\frac{dQ_t}{Q_t} = (a + \frac{1}{2}\sigma^2)dt + \sigma dW_t. \tag{1.4}$$

Let $p$ ($0 < p < 1$) represent the proportion of all surplus invested in the risky asset. The remaining surplus is invested in the risk-free asset. Then, the surplus process of the insurance company can be written as

$$dU(t) = pU(t-)\frac{dQ_t}{Q_t} + (1-p)U(t-)\frac{dR_t}{R_t} + cdt - dS_{1t} + dS_{2t}$$

$$= p\sigma U(t-)dW_t + (\xi U(t-) + c)dt - d\sum_{i=1}^{M_1(t)} Y_i + d\sum_{i=1}^{M_2(t)} Z_i, \tag{1.5}$$

where $\xi = (a + \frac{1}{2}\sigma^2 - r)p + r$ and the security loading condition is $c + \lambda_2 E[Z_1] > \lambda_1 E[Y_1]$.

It is important to stress that in the above model, the surplus is considered to be observed continuously. But in real life, corporate boards regularly review the balance of their books to determine whether to pay dividends or whether the surplus is zero or less (e.g., Albrecher et al. [9] and [10], Zhuo et al. [11], Cheung and Zhang [12], Peng et al. [13], Zhang et al. [14]). In other words, dividends and surplus are observed in discrete time. So we introduce the random observation periods into the risk model. We assume that the insurer only observes the surplus process at a series of discrete time points $\{S_k\}_{k=0}^\infty$ where $S_k$ is the $k$-th observation time. In particular, $S_0 = 0$, $S_{k^*}$ is the time of ruin where $k^* = \inf\{k \geq 1 : X(k) < 0\}$. Let $T_k = S_k - S_{k-1}$ be the $k$-th time interval between observations. In addition, $\{T_k\}_{k=0}^\infty$ is assumed to be an i.i.d. sequence where the universal r.v.'s $T$ follows a common exponential distribution with $\gamma > 0$. At the same time, we assume that $\{Y_i\}_{i=1}^\infty$, $\{Z_i\}_{i=1}^\infty$, $\{M_1(t)\}_{t\geq0}$, $\{M_2(t)\}_{t\geq0}$, $\{W_t, t \geq 0\}$ and $\{T_k\}_{k=0}^\infty$ are mutually independent. Let $X(k) = U(S_k)$ be the surplus level at the $k$-th observation. We have

$$X(k) = X(k-1) + \int_{S_{k-1}}^{S_k} p\sigma X(s)dW_s + \int_{S_{k-1}}^{S_k} (\xi X(s) + c)ds$$
$$- \int_{S_{k-1}}^{S_k} d\sum_{i=1}^{M_1(s)} Y_i + \int_{S_{k-1}}^{S_k} d\sum_{i=1}^{M_2(s)} Z_i. \tag{1.6}$$

Gerber and Shiu [15] first proposed a discounted penalty function for ruin in an insurance risk model. This function has become a very important and powerful analytical tool in risk theory. Scholars studied the expected discounted penalty problem in various risk models. For example, Hu et al. [16] considered a dual risk model under a mixed dividends strategy and gave the general result of the Gerber-Shiu function under periodic observation. Yang et al. [17] studied the perturbed compound Poisson model under the constant barrier dividends strategy and used the Fourier-cosine (COS) method to approximate the expected present value of dividends payments before ruin and the Gerber-Shiu function. More relevant literature can be referred to [18–23]. The Gerber-Shiu function of the model (1.5) is defined as follows

$$m(x) = E\left[e^{-\delta S_{k^*}} \omega\left(X(k^*-1), |X(k^*)|\right) I_{(S_{k^*} < \infty)} \Big| X(0) = x\right], \tag{1.7}$$

where $\omega(x_0, y_0)$, ( $x_0 \geq 0$, $y_0 \geq 0$) is a non-negative penalty function, $X(k^*-1)$ represents the instantaneous surplus before ruin, $|X(k^*)|$ represents the deficit at ruin time, $\delta > 0$ is the discounted factor and $I(\cdot)$ is the indicative function.

**Remark 1.1.** *In particular, if $\delta = 0$ and $\omega(x_0, y_0) = 1$, $m(x)$ is converted to the ruin probability $\psi(x) = P(S_{k^*} < \infty | X(0) = x)$. In addition, we assume that $m(x)$ is fully smooth.*

To show the innovativeness of our research more intuitively, we compared our work with some relevant studies (see Table 1).

**Table 1.** Comparison table of relevant literature contributions.

| Research paper | risk model | | | numerical method | random observation |
|---|---|---|---|---|---|
| | two-sided jumps | investment | Gerber-Shiu function | | |
| (2005)Yuen and Wang [18] | | ✓ | ✓ | | |
| (2011)Albrecher et al. [10] | | | ✓ | | ✓ |
| (2013)Chen and Ou [19] | | ✓ | ✓ | ✓ | |
| (2013)Dong and Liu [4] | ✓ | | ✓ | | |
| (2018)Cheung et al. [3] | ✓ | | ✓ | | |
| (2018)Zhuo et al. [11] | | ✓ | ✓ | ✓ | ✓ |
| (2018)Zhang et al. [14] | | | ✓ | | ✓ |
| (2020)Palmowski and Vatamidou [7] | ✓ | | ✓ | | |
| (2021)Zhang [8] | ✓ | | ✓ | | |
| (2022)Yang et al. [17] | | | ✓ | | |
| (2022)Martín-González et al. [6] | ✓ | | ✓ | | |
| (2023)Hu et al. [16] | | | ✓ | | ✓ |
| (2023)Wang et al. [23] | ✓ | ✓ | ✓ | ✓ | |
| Our work | ✓ | ✓ | ✓ | ✓ | ✓ |

From the perspective of the research on two-sided jumps, compared with the literature [3, 4, 6–8] in Table 1, we consider investing the funds of insurance companies to obtain greater returns. From the perspective of investment [14, 18, 19, 23], the model we consider is more realistic, that is, surplus can only be observed at random observation times to determine whether to pay dividends or whether the surplus is zero or less. Most current studies [10, 11, 14, 16] involving random observations do not consider random returns (upward jumps). We consider a more complex two-sided jumps model and solve it numerically using the sinc method.

This article is organized as follows. In Section 2, we derive the IDEs and boundary conditions satisfied by the Gerber-Shiu function. In Section 3, we obtain the approximate solution (AS) of IDEs by the sinc numerical method. In Section 4, we discuss some examples of sensitivity analysis.

## 2. IDEs for the Gerber-Shiu function

It is important to note that if a claim occurs before the observation time, the surplus may be less than zero and cannot be observed. Thus, the domain of $m(x)$ is extended to $\mathbb{R}$ even though time 0 is usually declared as an observation time. In addition, $m(x)$ behaves differently when $x$ is greater than and less than 0. For convenience, we denote

$$m(x) = \begin{cases} m_1(x), & x < 0, \\ m_2(x), & x \geq 0. \end{cases}$$

Then, we get the following.

**Theorem 2.1.** *Assume that $F_Y$ and $F_Z$ have a general continuous cumulative distribution function. Then, for any $-\infty < x < 0$, $m_1(x)$ satisfies the integral differential equation*

$$\frac{1}{2}p^2x^2\sigma^2 m_1''(x) + (\xi x + c)m_1'(x) - (\delta + \gamma + \lambda_1 + \lambda_2)m_1(x) + \gamma\omega_2(-x)$$

$$+ \lambda_2 \left[ \int_0^{-x} m_1(x+z)dF_Z(z) + \int_{-x}^{\infty} m_2(x+z)dF_Z(z) \right]$$

$$+ \lambda_1 \int_0^{\infty} m_1(x-y)dF_Y(y) = 0 \tag{2.1}$$

*and for $0 \le x < \infty$, $m_2(x)$ satisfies the integral differential equation*

$$\frac{1}{2}p^2x^2\sigma^2 m_2''(x) + (\xi x + c)m_2'(x) - (\delta + \lambda_1 + \lambda_2)m_2(x)$$

$$+ \lambda_1 \left[ \int_0^x m_2(x-y)dF_Y(y) + \int_x^{\infty} m_1(x-y)dF_Y(y) \right]$$

$$+ \lambda_2 \int_0^{\infty} m_2(x+z)dF_Z(z) = 0. \tag{2.2}$$

*The following boundary conditions are satisfied*

$$\lim_{u \to -\infty} m_1(x) = \frac{\gamma}{\delta + \gamma}; \tag{2.3}$$

$$\lim_{u \to \infty} m_2(x) = 0. \tag{2.4}$$

*Proof.* In the small interval $(0, dt]$ and discussing whether the first time observation time occurs whether the first time claim occurs or whether the first time random return occurs. For $-\infty < x < 0$, we get

$$m_1(x) = e^{-\delta dt} \Big\{ (1 - \gamma dt)P_0 E[m_1(h_t)] + \gamma dt P_0 E[\omega_2(-h_t)]$$

$$+ (1 - \gamma dt)P_1 E[m_1(h_t - Y_1)] + (1 - \gamma dt)P_2 E\Big[ E[m_1(h_t + Z_1)|Z_1 \in (0, -h_t)]$$

$$+ E[m_2(h_t + Z_1)|Z_1 \in (-h_t, \infty)] \Big] \Big\} \tag{2.5}$$

and for $0 \le x < \infty$ we have

$$m_2(x) = e^{-\delta dt} \Big\{ (1 - \gamma dt)P_0 E[m_2(h_t)] + \gamma dt P_0 E[m_2(h_t)]$$

$$+ (1 - \gamma dt)P_1 \Big[ E[m_2(h_t - Y_1)|Y_1 \in (0, h_t)] + E[m_1(h_t - Y_1)|Y_1 \in (h_t, \infty)] \Big]$$

$$+ (1 - \gamma dt)P_2 E[m_2(h_t + Z_1)] \Big\} \tag{2.6}$$

where

$$h_t = x + px\sigma dW_t + (\xi x + c)dt, \tag{2.7}$$

$$P_0 = P(T_1 > dt, K_1 > dt) = 1 - (\lambda_1 + \lambda_2)dt + o(dt), \tag{2.8}$$

$$P_1 = P(T_1 \le dt, K_1 > dt) = \lambda_1 dt + o(dt), \tag{2.9}$$

$$P_2 = P(T_1 > dt, K_1 \le dt) = \lambda_2 dt + o(dt). \tag{2.10}$$

By *Itô* formula, we have

$$E[m_i(h_t)] = E[m_i(x) + h_t m_i'(x) + \frac{1}{2}(h_t)^2 m_i''(x)] + o(t), \quad (i = 1, 2). \tag{2.11}$$

Substituting Eqs (2.7)–(2.11) into (2.5) and (2.6) respectively and letting $dt$ approach 0. Finally, the IDEs (2.1) and (2.2) are obtained.

If $x \rightarrow -\infty$, the ruin happens at the first observation time $S_1$ and the time interval between observations obeys the exponential distribution of the parameter $\gamma$ then the limit condition (2.3) can be obtained by using the definition of $m(x)$; if $x \rightarrow \infty$, ruin does not happen at all. Thus, condition (2.4) is satisfied. This completes the proof. □

**Remark 2.1.** *Due to the smoothness of $m(x)$, we obtain $m_1(0-) = m_2(0+)$. Refer to the analysis by Albrecher et al. [9], when $x = 0$, simultaneous Eqs (2.3) and (2.4) can obtain $cm'_1(0-) = cm'_2(0+) + \gamma m_1(0-)$. Therefore, $m(x)$ is generally non differentiable.*

## 3. Sinc asymptotic numerical analysis

Since Frank Stenger [24] developed the sinc numerical method, it has been widely concerned and applied in numerical analysis such as [23, 25, 26]. The ES of Eqs (2.1) and (2.2) are difficult to obtain theoretically. So, we change our thinking and find the AS by numerical method. At present, the numerical methods for solving IDEs include the sinc method, finite element method, COS method, finite difference method and so on. In the sinc approximation method, the error between the approximate solution and the exact solution reaches exponential order convergence through exponential transformation (see [24]). At the same time, sinc function approximates the boundary value problem and oscillation problem well, see [27]. So, for our study we adopted the sinc method.

### 3.1. Numerical approximate solution of $m(x)$

Define one-to-one mapping of $\mathbb{R} \rightarrow \mathbb{R}$, let $\varsigma(z) = \log(z + \sqrt{1 + z^2})$ where $z \in \mathbb{R}$. For $h > 0$, the sinc grid points $z_k$ ($k = 0, \pm 1, \pm 2, \dots$) are denoted by

$$z_k = \varsigma^{-1}(kh) = \frac{e^{kh} - e^{-kh}}{2}, \ (k = 0, \pm 1, \pm 2, \dots).$$

We apply the sinc method step to rearrange the Eqs (2.1) and (2.2) into

$$\frac{1}{2}p^2 x^2 \sigma^2 m''(x) + (\xi x + c)m'(x) - (\delta + \gamma I_{(x<0)} + \lambda_1 + \lambda_2)m(x) + \gamma \omega(-x)I_{(x<0)}$$
$$+ \lambda_1 \int_0^\infty m(x - y)dF_Y(y) + \lambda_2 \int_0^\infty m(x + z)dF_Z(z) = 0. \tag{3.1}$$

Using the property of convolution, the Eq (3.1) can be further expressed as

$$\frac{1}{2}p^2 x^2 \sigma^2 m''(x) + (\xi x + c)m'(x) - (\delta + \gamma I_{(x<0)} + \lambda_1 + \lambda_2)m(x) + \gamma \omega(-x)I_{(x<0)}$$
$$+ \lambda_1 \int_{-\infty}^x m(y)f_Y(x - y)dy + \lambda_2 \int_x^\infty m(z)f_Z(z - x)dz = 0. \tag{3.2}$$

Furthermore,

$$\lim_{x \to -\infty} m(x) = \frac{\gamma}{\delta + \gamma};$$

$$\lim_{x \to \infty} m(x) = 0.$$

According to Definition 1.5.2 in reference [25] or Definition 1 in reference [19], it can be seen that

$$g(x) = \frac{m(t_1) + \zeta(x)m(t_2)}{1 + \zeta(x)}$$

where $\zeta(x) = e^{S(x)} = x + \sqrt{1 + x^2}$. When $t_1 = -\infty$, $t_2 \to \infty$, set

$$H(x) = m(x) - g(x) = m(x) - \frac{\gamma}{(\delta + \gamma)[1 + x + \sqrt{1 + x^2}]}. \tag{3.3}$$

Then, $H(x) \in L_{\hat{\alpha}, \tau, \hat{d}}(\varsigma)$ so

$$m(x) = H(x) + g(x) = H(x) + \frac{\gamma}{(\delta + \gamma)[1 + x + \sqrt{1 + x^2}]}, \tag{3.4}$$

$$m'(x) = H'(x) + g'(x) = H'(x) - \frac{\gamma\left[1 + \frac{x}{\sqrt{1+x^2}}\right]}{(\delta + \gamma)[1 + x + \sqrt{1 + x^2}]^2}, \tag{3.5}$$

$$m''(x) = H''(x) + g''(x) = H''(x) - \frac{\gamma\left[\frac{1 + x + \sqrt{1+x^2}}{(1+x^2)^{\frac{3}{2}}} - 2\left(1 + \frac{x}{\sqrt{1+x^2}}\right)^2\right]}{(\delta + \gamma)[1 + x + \sqrt{1 + x^2}]^3}. \tag{3.6}$$

Substituting (3.4)–(3.6) into (3.2), we have

$$\beta_0(x)H''(x) + \beta_1(x)H'(x) + \beta_2(x)H(x) + \lambda_1 \int_{-\infty}^{x} f_Y(x - y)H(y)dy$$

$$+ \lambda_2 \int_{x}^{\infty} f_Z(z - x)H(z)dz + R(x) = 0 \tag{3.7}$$

where $\beta_0(x) = \frac{p^2 x^2 \sigma^2}{2}$, $\beta_1(x) = (\xi x + c)$, $\beta_2(x) = -(\delta + \gamma I_{(x<0)} + \lambda_1 + \lambda_2)$,

$$R(x) = \beta_0(x)g''(x) + \beta_1(x)g'(x) + \beta_2(x)g(x) + \gamma\omega(-x)I_{(x<0)}$$

$$+ \lambda_1 \int_{-\infty}^{x} g(y)f_Y(x - y)dy + \lambda_2 \int_{x}^{\infty} g(z)f_Z(z - x)dz. \tag{3.8}$$

When $x$ goes to $-\infty$ or $\infty$, the limit of $H(x)$ is zero.

Then, according to the Theorems 1.5.13, 1.5.14 and 1.5.20 in reference [25] we can get

$$\int_{-\infty}^{x} f_Y(x - y)H(y)dy \approx \sum_{j=-M}^{N} \sum_{i=-M}^{N} \omega_i A_{ij} U_j, \tag{3.9}$$

$$\int_{x}^{\infty} f_Z(z - x)H(z)dz \approx \sum_{j=-M}^{N} \sum_{i=-M}^{N} \omega_i B_{ij} U_j, \tag{3.10}$$

$$H(x) \approx \tilde{H}(x) = \sum_{j=-M}^{N} U_j S(j, h) \circ \varsigma(x), \tag{3.11}$$

where $A_{ij}$ and $B_{ij}$ are $(i, j)$ elements of the matrix $A = XF(S)X^{-1}$ and $B = YF(S)Y^{-1}$, respectively, and $S$ is a diagonal matrix. $U_j$ represents an approximate estimate of $H(u_j)$.

Substituting (3.9), (3.10) and (3.11) into Eq (3.7) then using the sinc grid points $x_k$ tends to $x$, we get

$$\sum_{j=-M}^{N} \left\{ \beta_0(x_k)\varsigma''(x_k)\frac{\delta_{jk}^{(1)}}{h} + \beta_0(x_k)(\varsigma'(x_k))^2\frac{\delta_{jk}^{(2)}}{h^2} + \beta_1(x_k)\varsigma'(x_k)\frac{\delta_{jk}^{(1)}}{h} + \beta_2(x_k)\delta_{jk}^{(0)} \right.$$
$$\left. + \lambda_1 \sum_{i=-M}^{N} \omega_i(x_k)A_{ij} + \lambda_2 \sum_{i=-M}^{N} \omega_i(x_k)B_{ij} \right\} U_j = -R(x_k), \quad k = -M, \ldots, N \tag{3.12}$$

where

$$\delta_{jk}^{(0)} = [S(j, h) \circ \varsigma(z)]|_{z=z_k} = \begin{cases} 0, & k \neq j, \\ 1, & k = j, \end{cases}$$

$$\delta_{jk}^{(1)} = h\frac{d}{d\varsigma}[S(j, h) \circ \varsigma(z)]|_{z=z_k} = \begin{cases} \frac{(-1)^{k-j}}{k-j}, & k \neq j, \\ 0, & k = j, \end{cases}$$

$$\delta_{jk}^{(2)} = h^2\frac{d^2}{d\varsigma^2}[S(j, h) \circ \varsigma(z)]|_{z=z_k} = \begin{cases} \frac{-2(-1)^{k-j}}{(k-j)^2}, & k \neq j, \\ -\frac{\pi^2}{3}, & k = j. \end{cases}$$

Multiply the Eq (3.12) by $\frac{h^2}{(\varsigma'(x_k))^2}$, we have

$$\sum_{j=-M}^{N} \left\{ \beta_0(x_k)\delta_{jk}^{(2)} + h\left[\frac{\beta_0(x_k)\varsigma''(x_k)}{(\varsigma'(x_k))^2} + \frac{\beta_1(x_k)}{\varsigma'(x_k)}\right]\delta_{jk}^{(1)} + h^2\frac{\beta_2(x_k)}{(\varsigma'(x_k))^2}\delta_{jk}^{(0)} \right.$$
$$\left. + \frac{h^2}{(\varsigma'(x_k))^2}\left[\lambda_1 \sum_{i=-M}^{N} \omega_i(x_k)A_{ij} + \lambda_2 \sum_{i=-M}^{N} \omega_i(x_k)B_{ij}\right] \right\} U_j = -\frac{h^2 R(x_k)}{(\varsigma'(x_k))^2}. \tag{3.13}$$

Since

$$\delta_{jk}^{(0)} = \delta_{kj}^{(0)}, \quad \delta_{jk}^{(1)} = -\delta_{kj}^{(1)}, \quad \delta_{jk}^{(2)} = \delta_{kj}^{(2)}, \quad \frac{\varsigma''(x_k)}{(\varsigma'(x_k))^2} = -\left(\frac{1}{\varsigma'(x_k)}\right)',$$

the formula (3.13) can be turned into

$$\sum_{j=-M}^{N} \left\{ \beta_0(x_k)\delta_{kj}^{(2)} + h\left[\beta_0(x_k)\left(\frac{1}{\varsigma'(x_k)}\right)' - \frac{\beta_1(x_k)}{\varsigma'(x_k)}\right]\delta_{kj}^{(1)} + h^2\frac{\beta_2(x_k)}{(\varsigma'(x_k))^2}\delta_{kj}^{(0)} \right.$$
$$\left. + \frac{h^2}{(\varsigma'(x_k))^2}\left[\lambda_1 \sum_{i=-M}^{N} \omega_i(x_k)A_{ij} + \lambda_2 \sum_{i=-M}^{N} \omega_i(x_k)B_{ij}\right] \right\} U_j = -\frac{h^2 R(x_k)}{(\varsigma'(x_k))^2}. \tag{3.14}$$

Set $I^{(m)} = [\delta_{kj}^{(m)}]_{(M+N+1)\times(M+N+1)}$, $m = 0, 1, 2$. We rewrite Eq (3.14) as

$$CU = R \tag{3.15}$$

where

$$U = (U_{-M}, \ldots, U_N)^T,$$

$$R = \left( -h^2 \frac{R(x_{-M})}{(\varsigma'(x_{-M}))^2}, \ldots, -h^2 \frac{R(x_N)}{(\varsigma'(x_N))^2} \right),$$

$$C = \beta_0 I^{(2)} + h D_m \left( \beta_0 \left( \frac{1}{\varsigma'} \right)' - \frac{\beta_1}{\varsigma'} \right) I^{(1)} + h^2 D_m \left( \frac{\beta_2}{(\varsigma')^2} \right) I^{(0)} + h^2 \lambda_1 D_m \left( \frac{1}{(\varsigma')^2} \right) \Omega_m^* A$$

$$+ h^2 \lambda_2 D_m \left( \frac{1}{(\varsigma')^2} \right) \Omega_m^* B.$$

We continue to use the definition of $\Omega_m^*$ in Chen and Ou [19].

Solving Eq (3.15), we get the expression of the approximate solution of $m(x)$:

$$m(x) \approx \tilde{m}(x) = \tilde{H}(x) + \frac{\gamma}{(\delta + \gamma)[1 + x + \sqrt{1 + x^2}]}$$

$$= \sum_{l=-M}^{N} U_l S(j, h) \circ \varsigma(x) + \frac{\gamma}{(\delta + \gamma)[1 + x + \sqrt{1 + x^2}]}. \tag{3.16}$$

### 3.2. Error analysis

In this subsection, since what we obtain is the sinc approximation solution (AS) of the IDEs. It is necessary to analyze the error between the AS and the ES by referring to relevant literature (refer to [24] and [28]). We finally found the upper limit of the error. In addition, depending on the actual situation the initial surplus $x$ is often greater than zero. So, in this section we will proceed under the condition of $x > 0$. Multiply the Eq (3.7) by $\frac{1}{\beta_0(x)}$, we set

$$G(x) = -\frac{\lambda_1}{\beta_0(x)} \int_{-\infty}^{x} f_Y(x - y)H(y)dy - \frac{\lambda_2}{\beta_0(x)} \int_{x}^{\infty} f_Z(z - x)H(z)dz - \frac{R(x)}{\beta_0(x)},$$

so we have

$$H''(x) + \tilde{\beta}_1(x)H'(x) + \tilde{\beta}_2(x)H(x) = G(x) \tag{3.17}$$

where $\tilde{\beta}_1(x) = \frac{\beta_1(x)}{\beta_0(x)}, \tilde{\beta}_2(x) = \frac{\beta_2(x)}{\beta_0(x)}$.

**Assumption 3.1.** *We set $\tilde{\beta}_1/\varsigma'$, $1/((\varsigma)')'$ and $\tilde{\beta}_2/(\varsigma')^2$ belong to $\mathcal{H}^\infty(\mathcal{D})$, that $G/(\varsigma')^2 \in \mathcal{L}_{\hat{\alpha}(\mathcal{D})}$ and the Eq (3.17) has a unique solution $H \in \mathcal{L}_{\hat{\alpha}(\mathcal{D})}$.*

**Theorem 3.2.** *Let Assumption 3.1 be satisfied, $H$ represent the ES of (3.17), $\tilde{H}$ denote the AS satisfying (3.16) and $U = (U_{-M}, \cdots, U_N))^T$ represent the ES of Eq (3.15). There exists a constant $\tilde{c}$ ($\tilde{c} > 0$) independent of $N$, such that*

$$\sup_{x \in \Gamma} |H(x) - \tilde{H}(x)| \le \tilde{c} N^{5/2} e^{-\sqrt{(\pi d \hat{\alpha} N)}}. \tag{3.18}$$

*Proof.* Let

$$\vartheta_N(x) = \sum_{k=-M}^{N} H(x_k) S(k, h) \circ \varsigma(x), \tag{3.19}$$

using the triangle inequality, it is easy to get

$$|H(x) - \tilde{H}(x)| \le |H(x) - \vartheta_N(x)| + |\vartheta_N(x) - \tilde{H}(x)|. \tag{3.20}$$

Based on the Theorem 4.2.5 of [24], there exists a constant $c^*$ that is greater than zero and independent of $N$ and then by Assumption 3.1, $H \in \mathcal{L}_{\hat{\alpha}(\mathscr{D})}$, we can get

$$\sup_{x \in \Gamma} |H(x) - \vartheta_N(x)| \le c^* N^{1/2} e^{-\sqrt{(\pi d \hat{\alpha} N)}}. \tag{3.21}$$

In inequality (3.20), $|\vartheta_N(x) - \tilde{H}(x)|$ satisfies the following relation

$$
\begin{aligned}
|\vartheta_N(x) - \tilde{H}(x)| &= \left| \sum_{j=-M}^{N} [H(x_j) - U_j] S(j,h) \circ \varsigma(x) - \frac{\gamma}{(\delta + \gamma)[1 + x + \sqrt{1 + x^2}]} \right| \\
&\le \sum_{j=-M}^{N} |H(x_j) - U_j| |S(j,h) \circ \varsigma(x)| \\
&\le \sqrt{\left( \sum_{j=-M}^{N} |H(x_j) - U_j|^2 \right) \left( \sum_{j=-M}^{N} |S(j,h) \circ \varsigma(x)|^2 \right)} \\
&\le \sqrt{\left( \sum_{j=-M}^{N} |H(x_j) - U_j|^2 \right)} = \|\mathbf{H} - \mathbf{U}\|.
\end{aligned}
\tag{3.22}
$$

Similar to Theorem 7.2.6 of [24], if $x \in \Gamma$ then $\sum_{k \in \mathbb{Z}} |S(k,h) \circ \varsigma(x)|^2 = 1$, we have

$$
\begin{aligned}
\|\mathbf{H} - \mathbf{U}\| &= \|C^{-1}C(\mathbf{H} - \mathbf{U})\| \\
&= \|C^{-1}[C\mathbf{H} - R]\| \\
&\le \left\| C^{-1} \right\| \|C\mathbf{H} - R\| \\
&\le c^* N^{5/2} e^{-\sqrt{(\pi d \hat{\alpha} N)}}
\end{aligned}
\tag{3.23}
$$

where $c^*$ is independent of $N$. Therefore, the inequality (3.18) is finally obtained by combining formulas (3.19)–(3.23). □

Through formulas (3.4), (3.16) and (3.18), we get

$$\sup_{x \in \Gamma} |m(x) - \tilde{m}(x)| \le \tilde{c} N^{5/2} e^{-\sqrt{(\pi d \hat{\alpha} N)}}. \tag{3.24}$$

## 4. Numerical illustrations

In this section, we assume that the p.d.f. $f_Z(z)$ of the random return is given by $f_Z(z) = \mu_2 e^{-\mu_2 y} I_{(y>0)}$. While the p.d.f. of the claim amount $f_Y(y)$ follows an exponential or lognormal distribution commonly used in actuarial research [29, 30].

## 4.1. The exponential distribution case

For this part, $f_Y(y)$ is defined as

$$f_Y(y) = \begin{cases} \mu_1 e^{-\mu_1 y}, & 0 < y < \infty, \\ 0, & -\infty < y \le 0. \end{cases}$$

The formula (3.8) is converted to

$$
\begin{aligned}
R(x) = & \beta_0(x)g''(x) + \beta_1(x)g'(x) + \beta_2(x)g(x) + \gamma\omega(-x)I_{(x<0)} \\
& + \lambda_1 \int_{-\infty}^{x} g(y)\mu_1 e^{-\mu_1(x-y)}dy + \lambda_2 \int_{x}^{\infty} g(z)\mu_2 e^{-\mu_2(z-x)}dz.
\end{aligned}
\tag{4.1}
$$

Then, we describe the effects of parameters $p$, $\sigma$ and $\gamma$ on the ruin probability $\psi(x)$. For investigation purposes, the basic parameters are set as follows in the following examples unless otherwise specified: $\delta = 0$, $a = 0.5$, $c = 0.4$, $r = 0.06$, $\lambda_1 = 5$, $\lambda_2 = 1$, $\mu_1 = 5$, $\mu_2 = 1$.

**Example 4.1.** *In the case that the amount of claim follows the exponential distribution, we consider the influence of the investment ratio $p$ on the ruin probability. Set parameters $\gamma = 3$, $\sigma = 0.8$. As shown in Figure 1, it is easy to see that when insurance companies put more of their surplus in risky asset, their ruin probability will fluctuate greatly. Some data are shown in Table 2.*
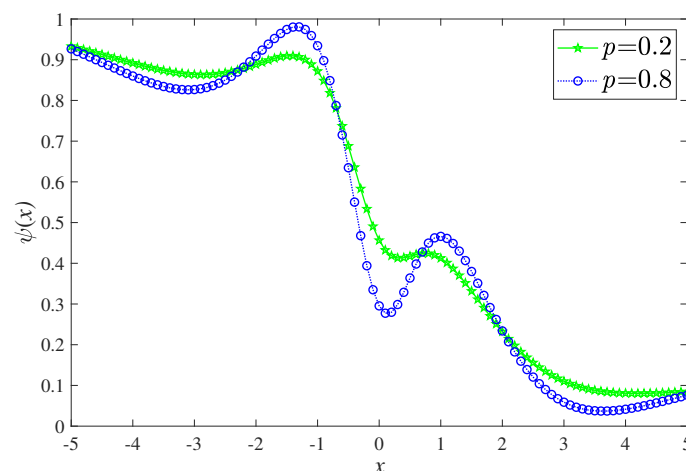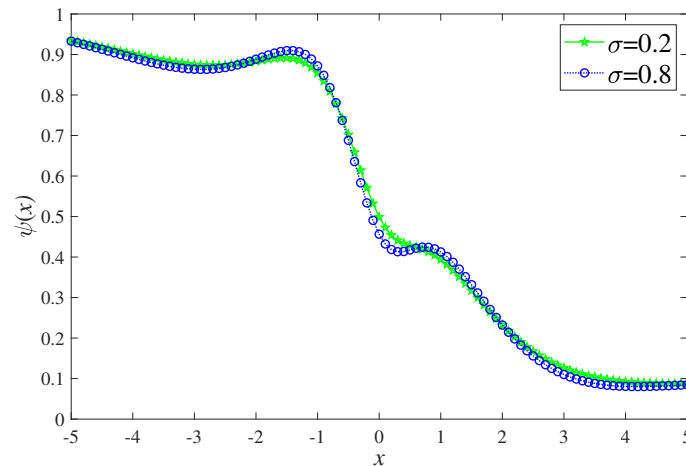


**Figure 1.** Ruin probability $\psi(x)$ when $\hat{d} = \hat{\alpha} = \frac{\pi}{4}$, $\tau = \frac{\pi}{8}$, $N = 10$, $\gamma = 3$ and $\sigma = 0.8$.

**Table 2.** When $p = 0.2$ and $p = 0.8$, ruin probability $\psi(x)$ under different $x$.

|  | $x=-5$ | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $p=0.2$ | 0.93 | 0.89 | 0.86 | 0.89 | 0.87 | 0.46 | 0.41 | 0.23 | 0.11 | 0.08 | 0.08 |
| $p=0.8$ | 0.93 | 0.86 | 0.83 | 0.91 | 0.93 | 0.30 | 0.47 | 0.23 | 0.06 | 0.04 | 0.08 |

**Example 4.2.** *In the case that the amount of claim follows the exponential distribution, we consider the effect of the parameter $\sigma$ on $\psi(x)$. Set parameters $\gamma = 3$, $p = 0.2$. As can be seen from Figure 2, the greater the volatility $\sigma$ of risky asset the larger the fluctuation of the ruin probability curve. Some data are shown in Table 3.*
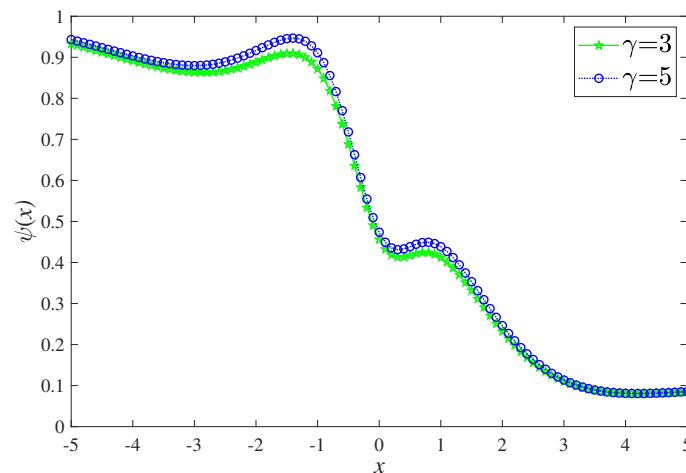
**Figure 2.** Ruin probability $\psi(x)$ when $\hat{d} = \hat{\alpha} = \frac{\pi}{4}$, $\tau = \frac{\pi}{8}$, $N = 10$, $\gamma = 3$ and $p = 0.2$.

**Table 3.** When $\sigma = 0.2$ and $\sigma = 0.8$, ruin probability $\psi(x)$ under different $x$.

| | $x=-5$ | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\sigma=0.2$ | 0.93 | 0.90 | 0.88 | 0.88 | 0.85 | 0.50 | 0.39 | 0.23 | 0.13 | 0.09 | 0.09 |
| $\sigma=0.8$ | 0.93 | 0.89 | 0.86 | 0.89 | 0.87 | 0.46 | 0.41 | 0.23 | 0.11 | 0.08 | 0.08 |

**Example 4.3.** *In the case that the amount of claim follows the exponential distribution, we consider the influence of the random observation parameter $\gamma$ on the ruin probability. Set parameters $\sigma = 0.8$, $p = 0.2$. As can be seen from Figure 3, when the initial surplus is small the parameter $\gamma$ has a significant effect on the ruin probability. When $x$ is large enough, this effect is obviously weakened. Some data are shown in Table 4.*



**Figure 3.** Ruin probability $\psi(x)$ when $\hat{d} = \hat{\alpha} = \frac{\pi}{4}$, $\tau = \frac{\pi}{8}$, $N = 10$, $\sigma = 0.8$ and $p = 0.2$.

**Table 4.** When $\gamma = 3$ and $\gamma = 5$, ruin probability $\psi(x)$ under different $x$.

| $x=$-5 | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\gamma=3$ 0.93 | 0.89 | 0.86 | 0.89 | 0.87 | 0.46 | 0.41 | 0.23 | 0.11 | 0.08 | 0.08 |
| $\gamma=5$ 0.94 | 0.90 | 0.88 | 0.92 | 0.91 | 0.47 | 0.44 | 0.25 | 0.11 | 0.08 | 0.09 |

As can be seen from Examples 4.1–4.3, the effects of several factors on the ruin probability is considered: the investment ratio $p$, the volatility of risky assets $\sigma$, and the random observation parameter $\gamma$. First, when insurance companies allocate a larger portion of their surplus to risky assets, it implies the possibility of achieving higher returns. However, this also leads to significant fluctuations in the ruin probability. This demonstrates the coexistence of danger and opportunity. Furthermore, an increase in the volatility of risky assets indicates the potential for greater profits (resulting in a smaller ruin probability). Conversely, it also implies the risk of greater losses (leading to a larger ruin probability). This is also realistic. Last, a shorter random observation interval (larger $\gamma$) corresponds to a higher ruin probability, whereas a longer random observation interval (smaller $\gamma$) corresponds to a lower ruin probability.

### 4.2. The lognormal distribution case

In this section, $f_Y(y)$ is assumed to follow the lognormal distribution of the parameter $(\mu_3, 2v^2)$, where $\mu_3$ is the mean value of $\ln Y$ and $2v^2$ is the variance of $\ln Y$. So $f_Y(y)$ is defined as

$$f_Y(y) = \begin{cases} \frac{1}{2\pi vy}e^{-\frac{(\ln y - \mu_3)^2}{4v^2}}, & 0 < y < \infty, \\ 0, & -\infty < y \le 0. \end{cases}$$

Then,

$$f_Y(u-y) = \begin{cases} \frac{1}{2\pi v(u-y)}e^{-\frac{[\ln(u-y)-\mu_3]^2}{4v^2}}, & -\infty < y < u, \\ 0, & u \le y < \infty. \end{cases}$$

At this point, the Eq (3.8) is converted to

$$\begin{aligned} R(x) = & \beta_0(x)g''(x) + \beta_1(x)g'(x) + \beta_2(x)g(x) + \gamma\omega(-x)I_{(x<0)} \\ & + \lambda_1 \int_{-\infty}^x g(y)\frac{1}{2\pi v(x-y)}e^{-\frac{[\ln(x-y)-\mu_3]^2}{4v^2}}dy + \lambda_2 \int_x^\infty g(z)\mu_2 e^{-\mu_2(z-x)}dz. \end{aligned} \tag{4.2}$$

The following examples are discussed under $\delta = 0$, $a = 0.5$, $c = 0.4$, $r = 0.06$, $\lambda_1 = 5$, $\lambda_2 = 1$, $\mu_1 = 5$, $\mu_2 = 1$, $\mu_3 = 0.08$, $v = 0.03$, $\gamma = 3$.

**Example 4.4.** *In the case that the amount of claim follows the lognormal distribution, we consider the effect of the investment ratio $p$ on $\psi(x)$. Set parameters $\gamma = 3$, $\sigma = 0.8$. As shown in Figure 4, it is not difficult to see that when insurance companies put more of their surplus in risky asset their ruin probability curve fluctuates greatly. Some data are shown in Table 5.*
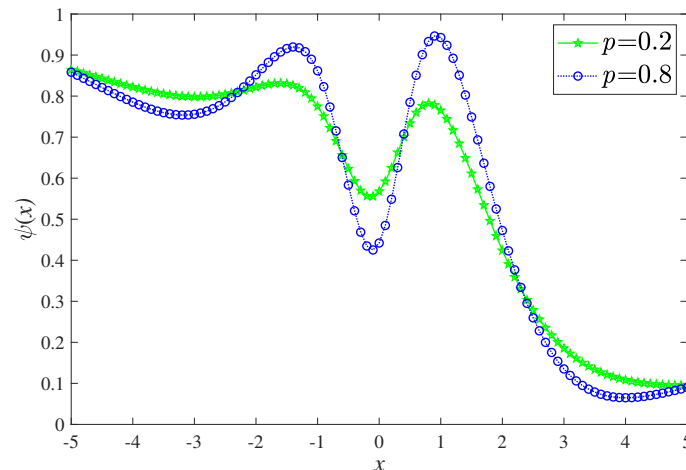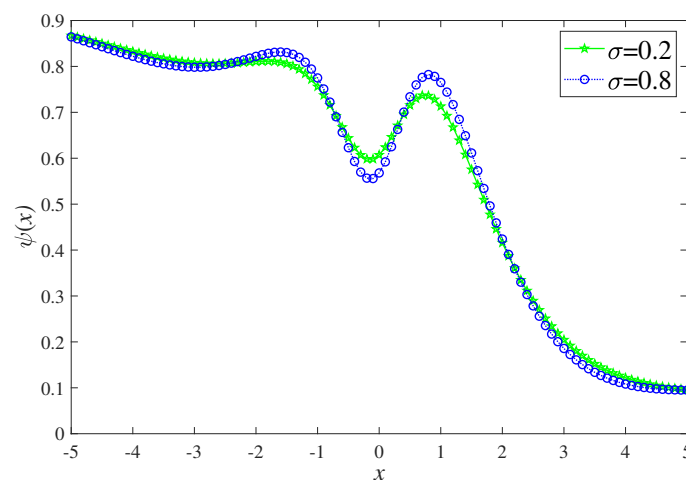
**Figure 4.** Ruin probability $\psi(x)$ when $\hat{d} = \hat{\alpha} = \frac{\pi}{4}$, $\tau = \frac{\pi}{8}$, $N = 10$, $\gamma = 3$ and $\sigma = 0.8$.

**Table 5.** When $p = 0.2$ and $p = 0.8$, ruin probability $\psi(x)$ under different $x$.

|         | $x=-5$ | -4   | -3   | -2   | -1   | 0    | 1    | 2    | 3    | 4    | 5    |
|---------|--------|------|------|------|------|------|------|------|------|------|------|
| $p=0.2$ | 0.86   | 0.82 | 0.80 | 0.82 | 0.77 | 0.57 | 0.77 | 0.42 | 0.19 | 0.11 | 0.09 |
| $p=0.8$ | 0.86   | 0.79 | 0.76 | 0.85 | 0.86 | 0.44 | 0.94 | 0.47 | 0.14 | 0.06 | 0.09 |

**Example 4.5.** *In the case that the amount of claim follows the lognormal distribution, we consider the effect of the parameter $\sigma$ on $\psi(x)$. Set parameters $\gamma = 3$, $p = 0.2$. As can be seen from Figure 5, the larger the parameter $\sigma$ of risky asset the greater the fluctuation of the ruin probability will be. Some data are shown in Table 6.*
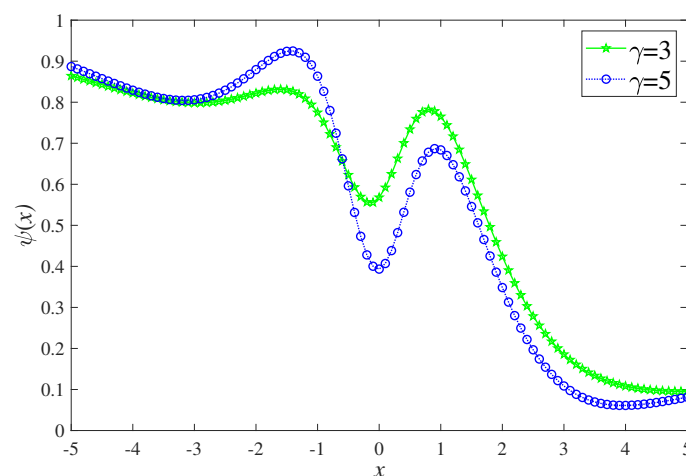


**Figure 5.** Ruin probability $\psi(x)$ when $\hat{d} = \hat{\alpha} = \frac{\pi}{4}$, $\tau = \frac{\pi}{8}$, $N = 10$, $\gamma = 3$ and $p = 0.2$.

**Table 6.** When $\sigma = 0.2$ and $\sigma = 0.8$, ruin probability $\psi(x)$ under different $x$.

| | x=-5 | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\sigma=0.2$ | 0.87 | 0.83 | 0.81 | 0.81 | 0.76 | 0.61 | 0.71 | 0.42 | 0.20 | 0.12 | 0.10 |
| $\sigma=0.8$ | 0.86 | 0.82 | 0.80 | 0.82 | 0.77 | 0.57 | 0.77 | 0.42 | 0.19 | 0.11 | 0.09 |

**Example 4.6.** *In the case that the amount of claim follows the lognormal distribution, we consider the influence of the random observation parameter $\gamma$ on the ruin probability. Set parameters $\sigma = 0.8$, $p = 0.2$. As can be seen from Figure 6, the different values of parameter $\gamma$ have obvious influence on the ruin probability. Some data are shown in Table 7.*



**Figure 6.** Ruin probability $\psi(x)$ when $\hat{d} = \hat{\alpha} = \frac{\pi}{4}$, $\tau = \frac{\pi}{8}$, $N = 10$, $\sigma = 0.8$ and $p = 0.2$.

**Table 7.** When $\gamma = 3$ and $\gamma = 5$, ruin probability $\psi(x)$ under different $x$.

| | x=-5 | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\gamma=3$ | 0.86 | 0.82 | 0.80 | 0.82 | 0.77 | 0.57 | 0.77 | 0.42 | 0.19 | 0.11 | 0.09 |
| $\gamma=5$ | 0.89 | 0.83 | 0.81 | 0.88 | 0.86 | 0.39 | 0.68 | 0.35 | 0.11 | 0.06 | 0.08 |

From the Examples 4.4–4.6, the impact of parameters $p, \sigma$ and $\gamma$ on the ruin probability is comparable to that in the case of exponential distribution. Nevertheless, it is noteworthy that when the claim amount follows a lognormal distribution the ruin probability exhibits higher sensitivity to variations in the aforementioned parameters.

## 5. Conclusions

This paper considers a two-sided jumps risk model with random observation periods and proportional investment. Through the review of the existing literature, we observe that most studies by scholars focus on classical or dual risk models. We wondered if we could combine the random jump components in the two models, that is, consider both upward and downward jumps in one

model. Following this idea, we find that the two-sided jumps risk model not only has important practical significance but also has been studied by more and more scholars. In addition to the studies already mentioned in the paper, readers can also refer to the literature [31–36]. However, to the best of our knowledge no one has ever introduced a random observation period and invested the surplus of an insurance company under this model (two-sided jumps) and the complexity of the model greatly increases the difficulty of processing. To solve this problem, we use the sinc numerical method to find its SA. While there is inevitably some error between the SA and the ES, we provide an upper bound for the error ensuring that it remains within an acceptable range. Perhaps in further research in the future, the ES of the model will be solved.

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

All authors declare no conflicts of interest in this paper.

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