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*Research article*

# On simultaneous characterizations of partner-ruled surfaces in Minkowski 3-space

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**Abstract:** In this study, the partner-ruled surfaces in Minkowski 3-space, which are defined according to the Frenet vectors of non-null space curves, are introduced with extra conditions that guarantee the existence of definite surface normals. First, the requirements of each pair of partner-ruled surfaces to be simultaneously developable and minimal (or maximal for spacelike surfaces) are investigated. The surfaces also simultaneously characterize the asymptotic, geodesic and curvature lines of the parameter curves of these surfaces. Finally, the study provides examples of timelike and spacelike partner-ruled surfaces and includes their graphs.

**Keywords:** partner-ruled surface; developable and minimal surface; geodesic curve; asymptotic curve
**Mathematics Subject Classification:** 53A04, 53A05

## 1. Introduction

The surface concept has been researched by many mathematicians, philosophers and scientists for thousands of years over the course of history. In the process, the theory of surfaces has been greatly consolidated through the development of differential geometry. As well as Gauss, Riemann and Poincaré being the pioneers in this research area, Monge also made some significant contributions to the study of surfaces. Based on Monge's approach, surfaces are represented as graphs of functions of two variables. This approach has deeply influenced the progress of the theory of surfaces and their application areas in the 19th and 20th centuries and is still popular. Guggenheimer (1963) and Hoschek (1971) examined the ruled surfaces from different perspectives with some significant contributions to differential geometry. A ruled surface is a surface that can be generated by moving a straight line along a curve in space [1,2]. Ruled surfaces are preferred to study since they have relatively simple structures and allow us to interpret more complex surfaces. The classification of ruled surfaces, properties related to the base curve, geodesics, shape operators of surfaces and the study of developable and non-developable ruled surfaces are among the major areas of research on ruled surfaces. The survey of ruled surfaces in Minkowski space shows similar characteristics in Euclidean space, but there are exciting differences due to the structure of Minkowski space. Since the characterization of ruled surfaces depends on the base curve and the direction, the geometry of ruled surfaces in Minkowski space is more complex than that in Euclidean space. As it is known, the ruled surfaces can be classified as developable and non-developable ones. The developable ruled surfaces are ruled surfaces whose tangent planes are the same along the main lines. A classic result in differential geometry states that the elements of developable ruled surfaces can be expressed as cylinders, cones and tangent surfaces. This is valid for both Euclidean and Minkowski spaces. Naturally, degenerate tangent planes are excluded from this rule. Generally, the first fundamental form must be non-degenerate for a surface in Minkowski space. A spacelike surface is obtained if the first fundamental form is positively defined. If the first fundamental form is indefinite, a timelike surface is constructed. The surfaces that fit into the curvature situations where the Gaussian curvature and the mean curvature are constant, or one of them is constant, have been studied in different studies [3–6]. Rich data on ruled surfaces can be found in detail in [7–15]. Recently, Li et al. investigated partner-ruled surfaces formed from polynomial curves with the Flc frame [16], and Soukaina also studied the developability of partner-ruled surfaces using the Darboux frame simultaneously [17].

In this study, partner-ruled surfaces generated by the vectors of the Frenet frame of non-null space curves in Minkowski 3-space are introduced. Then, conditions are simultaneously provided for each partner-ruled surface to be developable or minimal (or maximal for spacelike surfaces), depending on the curvatures of the base curve. These conditions are also associated with the characterizations of parametric curves such as asymptotic, geodesic or curvature lines. At the end of the study, examples related to partner-ruled surfaces are provided, and the graphics of the surfaces are presented using the MATLAB R2023a program.

## 2. Preliminaries

The Minkowski 3-space $\mathbb{R}^3_1$ is given by the Lorentzian inner product

$$\langle x, y\rangle = -x_1y_1 + x_2y_2 + x_3y_3,$$

where $x = (x_1, x_2, x_3),\ y = (y_1, y_2, y_3) \in \mathbb{R}^3$. The norm of arbitrary vector $x \in \mathbb{R}^3_1$ is $\|x\| = \sqrt{|\langle x, x\rangle|}$. Also, the vector product of any vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in $\mathbb{R}^3_1$ is defined by

$$x \times y = -\begin{vmatrix} e_1 & e_2 & -e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = (x_3y_2 - x_2y_3, x_1y_3 - x_3y_1, x_1y_2 - x_2y_1),$$

where $e_1 \times e_2 = e_3, \quad e_2 \times e_3 = -e_1, \quad e_3 \times e_1 = -e_2$. The character of an arbitrary vector $x \in \mathbb{R}^3_1$ is defined as follows:

(i) if $\langle x, x\rangle > 0$ or $x = 0$ then $x$ is a spacelike vector,
(ii) if $\langle x, x\rangle < 0$, then $x$ is a timelike vector,
(iii) if $\langle x, x\rangle = 0$, $x \neq 0$, then $x$ is a lightlike (or null) vector.

Let $\alpha : I \to \mathbb{R}$ be a regular unit speed non-null curve parametrized by arc-length $s$ in Minkowski 3-space. If the vectors $T$, $N$ and $B$ denote the tangent, principal normal and binormal unit vectors at any point $\alpha(s)$ of the non-null curve $\alpha$, respectively. Then the Frenet formulas are given

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix}_s = \begin{bmatrix} 0 & \kappa & 0 \\ -\varepsilon_1\varepsilon_2\kappa & 0 & \tau \\ 0 & -\varepsilon_2\varepsilon_3\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{2.1}$$

where $\langle T, T\rangle = \varepsilon_1$, $\langle N, N\rangle = \varepsilon_2$ and $\langle B, B\rangle = \varepsilon_3$. Also, $N \times T = \varepsilon_3 B$, $B \times N = \varepsilon_1 T$, $T \times B = \varepsilon_2 N$ and $\varepsilon_1\varepsilon_2\varepsilon_3 = -1$. Here $\kappa(s)$ and $\tau(s)$ are the curvature and the torsion of the curve $\alpha$, respectively, $s$ is the arc-length of the non-null curve [18, 19]. Let $\{T, N, B\}$ be the moving frame of $\alpha$ satisfying the following conditions:

(i) $\varepsilon_1 = -1,\ \varepsilon_2 = 1,\ \varepsilon_3 = 1$ for the timelike curve,
(ii) $\varepsilon_1 = 1,\ \varepsilon_2 = -1,\ \varepsilon_3 = 1$ for the spacelike curve with timelike normal,
(iii) $\varepsilon_1 = 1,\ \varepsilon_2 = 1,\ \varepsilon_3 = -1$ for the spacelike curve with timelike binormal.

In Minkowski 3-space $\mathbb{R}^3_1$, a ruled surface $M$ is a regular surface that is parameterized as:

$$\begin{aligned} &\varphi : I \times \mathbb{R} \to \mathbb{R}^3_1 \\ &\ (s, v) \to \varphi(s, v) = \alpha(s) + v r(s), \end{aligned}$$

where $\alpha(s)$ and $r(s)$ are known as base and director curves of a ruled surface, respectively. By restricting ourselves to the non-null cases, classification of the character of a ruled surface $\varphi(s, v)$ can be formed according to whether the base curve $\alpha$ and the director curve $r$ are timelike or spacelike curves [8, 9];

(i) if the curve $\alpha$ is timelike and the curve $r$ is spacelike, the ruled surface $\varphi(s, v)$ indicates a timelike surface,
(ii) if the curve $\alpha$ is spacelike and the curve $r$ is spacelike, the ruled surface $\varphi(s, v)$ indicates a spacelike surface,
(iii) if the curve $\alpha$ is spacelike and the curve $r$ is timelike, the ruled surface $\varphi(s, v)$ indicates a timelike surface.

Let $\varphi(s, v)$ be a ruled surface in $\mathbb{R}^3_1$, then the various quantities associated with the ruled surface are given as follows:

(i) The unit normal vector field: $U = \frac{\varphi_s \times \varphi_v}{\|\varphi_s \times \varphi_v\|}$, where $\varphi_s = \frac{\partial\varphi}{\partial s}$ and $\varphi_v = \frac{\partial\varphi}{\partial v}$.
(ii) First fundamental form: $I = Eds^2 + 2Fdsdv + Gdv^2$, where the coefficients of $I$ are

$$E = \langle\varphi_s, \varphi_s\rangle,\ F = \langle\varphi_s, \varphi_v\rangle,\ G = \langle\varphi_v, \varphi_v\rangle. \tag{2.2}$$

(iii) Second fundamental form: $II = eds^2 + 2fdsdv + gdv^2$, where the coefficients of $II$ are

$$e = \langle\varphi_{ss}, U\rangle,\ f = \langle\varphi_{sv}, U\rangle,\ g = \langle\varphi_{vv}, U\rangle. \tag{2.3}$$

Moreover, the Gaussian curvature and the mean curvature of the surface $\varphi(s, v)$ are defined by

$$K = \varepsilon\frac{eg - f^2}{EG - F^2} \text{ and } H = \varepsilon\frac{Eg - 2Ef + Ge}{2(EG - F^2)}, \tag{2.4}$$

respectively, and $\varepsilon = 1 (= -1)$ for timelike (spacelike) surfaces. Also, the surfaces with vanishing Gaussian curvature are called developable and any surfaces with vanishing mean curvature are called minimal (or maximal for spacelike surfaces) [8, 9, 19].

## 3. Simultaneous characterizations of partner-ruled surfaces

Two ruling lines generate the partner-ruled surfaces if they simultaneously move along their respective curves. On the other hand, it is a usual approach to examine the Frenet vectors and their relationships in the field of differential geometry since the Frenet vectors provide a framework for deep insight into the geometry of curves. In these regards, by considering the tangent, principal normal and binormal vectors of the Frenet frame along a differentiable unit speed non-null space curve parametrized by arc-length as ruling lines of partner-ruled surfaces, we study the following surfaces couples in Minkowski 3-space. These surfaces can be classified according to the causal characters of the non-null base curve, as shown in Table 1.

**Table 1.** Surface classification based on causal features of non zero basis curve.

| Base curve $\alpha$ | $TN$-partner-ruled surface | $TB$-partner-ruled surface | $NB$-partner-ruled surface |
|---|---|---|---|
| Timelike | Timelike | Timelike | Spacelike |
| Spacelike with timelike normal | Timelike | Spacelike | Timelike |
| Spacelike with timelike binormal | Spacelike | Timelike | Timelike |

**Definition 3.1.** *Let $\alpha : I \to \mathbb{R}$ be a differentiable unit speed non-null space curve parametrized by arc-length $s$ in $\mathbb{R}^3_1$ with Frenet elements $\{T, N, B, \kappa, \tau\}$ such that $\tau(s) \neq \mp\kappa(s)$ and $\tau(s) \neq \mp v\kappa(s)$ for all $s \in I$. The two ruled surfaces represented by*

$$\begin{cases} \varphi_N^T(s, v) = T(s) + vN(s), \\ \varphi_T^N(s, v) = N(s) + vT(s), \end{cases} \tag{3.1}$$

*are called $TN$-partner-ruled surfaces with respect to the Frenet frame of the space curve in $\mathbb{R}^3_1$.*

**Theorem 3.1.** *Let $\varphi_N^T$ and $\varphi_T^N$ be a pair of the $TN$-partner-ruled surfaces in $\mathbb{R}^3_1$, then the $TN$-partner-ruled surfaces are simultaneously developable and minimal (maximal) surfaces if and only if the curve $\alpha$ is a non-null planar curve.*

*Proof.* By differentiating the first equation in equation set Eq (3.1) in terms of $s$ and $v$, respectively and applying the Frenet formulas given by Eq (2.1), we obtain

$$\left(\varphi_N^T\right)_s = \varepsilon_3 v\kappa T + \kappa N + v\tau B, \ \left(\varphi_N^T\right)_v = N. \tag{3.2}$$

By the cross product of the vectors $\left(\varphi_N^T\right)_s$ and $\left(\varphi_N^T\right)_v$ described in Eq (3.2), we determine the normal vector field of the surface $\varphi_N^T$ as follows:

$$U_N^T = \frac{\left(\varphi_N^T\right)_s \times \left(\varphi_N^T\right)_v}{\left\|\left(\varphi_N^T\right)_s \times \left(\varphi_N^T\right)_v\right\|} = \frac{\varepsilon_1 \tau T - \kappa B}{\sqrt{\left|\varepsilon_1 \tau^2 + \varepsilon_3 \kappa^2\right|}}. \tag{3.3}$$

Here the condition $\tau \neq \mp\kappa$ guarantees $\varepsilon_1\tau^2 + \varepsilon_3\kappa^2 \neq 0$. By taking the scalar product of both vectors in Eq (3.2) using Eq (2.2), we derive the components of the first fundamental form of the ruled surface $\varphi_N^T$ as follows:

$$E_N^T = \varepsilon_2\kappa^2 + v^2\left(\varepsilon_1\kappa^2 + \varepsilon_2\tau^2\right),\ F_N^T = \varepsilon_2\kappa,\ G_N^T = \varepsilon_2. \tag{3.4}$$

By differentiating Eq (3.2) in terms of $s$ and $v$, we get

$$\begin{aligned}\left(\varphi_N^T\right)_{ss} &= \varepsilon_3\left(\kappa^2 + v\kappa'\right)T + \left(\varepsilon_3 v\kappa^2 + \varepsilon_1 v\tau^2 + \kappa'\right)N + \left(\kappa\tau + v\tau'\right)B,\\ \left(\varphi_N^T\right)_{sv} &= \varepsilon_3\kappa T + \tau B,\ \left(\varphi_N^T\right)_{vv} = 0.\end{aligned}$$

By taking the scalar product of the last equation derived in the previous step with the normal vector field given in Eq (3.3) using Eq (2.3), we can determine the components of the second fundamental form of the ruled surface $\varphi_N^T$ as follows:

$$e_N^T = \frac{v\varepsilon_3\left(\tau\kappa' - \kappa\tau'\right)}{\sqrt{\left|\varepsilon_1\tau^2 + \varepsilon_3\kappa^2\right|}},\ f_N^T = 0,\ g_N^T = 0. \tag{3.5}$$

The Gaussian curvature and the mean curvature of the ruled surface are found by substituting Eqs (3.4) and (3.5) into Eq (2.4) and evaluating the resulting expression. These give us the following expressions for the Gaussian curvature and the mean curvature of the ruled surface $\varphi_N^T$:

$$K_N^T = 0,\ H_N^T = \varepsilon\frac{\varepsilon_3\left(\kappa\tau' - \tau\kappa'\right)}{2v\left(\varepsilon_1\kappa^2 + \varepsilon_3\tau^2\right)\sqrt{\left|\varepsilon_1\tau^2 + \varepsilon_3\kappa^2\right|}}. \tag{3.6}$$

□

On the other hand, by differentiating the second equation in equation set Eq (3.1) with respect to $s$ and $v$, respectively, and applying the Frenet frame derivative formulas, we get

$$\left(\varphi_T^N\right)_s = \varepsilon_3\kappa T + v\kappa N + \tau B,\ \left(\varphi_T^N\right)_v = T. \tag{3.7}$$

By determining the cross-product of the partial derivatives of the surface described in Eq (3.7), we determine the normal vector field of the surface $\varphi_T^N$ as follows:

$$U_T^N = \frac{\left(\varphi_T^N\right)_s \times \left(\varphi_T^N\right)_v}{\left\|\left(\varphi_T^N\right)_s \times \left(\varphi_T^N\right)_v\right\|} = \frac{-\varepsilon_2\tau N + v\varepsilon_3\kappa B}{\sqrt{\left|\varepsilon_2\tau^2 + \varepsilon_3 v^2\kappa^2\right|}}. \tag{3.8}$$

Here the condition $\tau \neq \mp v\kappa$ requires $\varepsilon_2\tau^2 + \varepsilon_3 v^2\kappa^2 \neq 0$. By applying the scalar product for both vectors in Eq (3.8), we have the components of the first fundamental form of the ruled surface $\varphi_T^N$ as follows:

$$E_T^N = \left(v^2\varepsilon_2 + \varepsilon_1\right)\kappa^2 + \varepsilon_3\tau^2, F_T^N = -\varepsilon_2\kappa,\ G_T^N = \varepsilon_1. \tag{3.9}$$

By differentiating Eq (3.7) with respect to $s$ and $v$, we have

$$\begin{aligned}\left(\varphi_T^N\right)_{ss} &= \varepsilon_3\left(v\kappa^2 + \kappa'\right)T + \left(\varepsilon_3\kappa^2 + \varepsilon_1\tau^2 + v\kappa'\right)N + \left(v\kappa\tau + \tau'\right)B,\\ \left(\varphi_T^N\right)_{sv} &= \kappa N,\ \left(\varphi_T^N\right)_{vv} = 0.\end{aligned}$$

We find the components of the second fundamental form of the ruled surface $\varphi_T^N$ by taking the scalar product of the last equation obtained in the previous step with the normal vector field given in Eq (3.8). This yields the following expression for the components of the second fundamental form:

$$e_T^N = \frac{-\tau\left(\varepsilon_3\kappa^2 + \varepsilon_1\tau^2 + v\kappa'\right) + v\kappa\left(v\kappa\tau + \tau'\right)}{\sqrt{\left|\varepsilon_2\tau^2 + \varepsilon_3 v^2\kappa^2\right|}},\ f_T^N = \frac{-\kappa\tau}{\sqrt{\left|\varepsilon_2\tau^2 + \varepsilon_3 v^2\kappa^2\right|}},\ g_T^N = 0. \tag{3.10}$$

Thus, by substituting Eqs (3.9) and (3.10) into Eq (2.4), the Gaussian curvature $K_T^N$ and the mean curvature $H_T^N$ of the ruled surface $\varphi_T^N$ are given by

$$\begin{aligned}K_T^N &= \varepsilon\frac{\kappa^2\tau^2}{\left(\varepsilon_2\tau^2 + \varepsilon_3 v^2\kappa^2\right)\left|\varepsilon_2\tau^2 + \varepsilon_3 v^2\kappa^2\right|},\\ H_T^N &= \varepsilon\frac{2\varepsilon_2\kappa^2\tau - \tau^3 - \varepsilon_1\left(\left(-v^2 + \varepsilon_3\right)\kappa^2\tau + v\tau\kappa' - v\kappa\tau'\right)}{2\left(\varepsilon_2\tau^2 + \varepsilon_3 v^2\kappa^2\right)\sqrt{\left|\varepsilon_2\tau^2 + \varepsilon_3 v^2\kappa^2\right|}}.\end{aligned} \tag{3.11}$$

Therefore, based on Eqs (3.6) and (3.11), we can conclude that the $TN$-partner-ruled surfaces satisfy the conditions stated in the hypothesis and they are simultaneously developable and minimal (maximal) surfaces.

**Theorem 3.2.** *Let $\varphi_N^T$ and $\varphi_T^N$ be a pair of the $TN$-partner-ruled surfaces in $\mathbb{R}_1^3$, then the $s$-parameter curves of the $TN$-partner-ruled surfaces are simultaneously*

*(i) not geodesics,*
*(ii) asymptotics if $\tau = 0$ and $\kappa \neq 0$.*

*Proof.* Let $\varphi_N^T$ and $\varphi_T^N$ be a pair of the $TN$-partner-ruled surfaces in $\mathbb{R}_1^3$.

(i) The cross products of second partial derivatives of $\varphi_N^T$ and $\varphi_T^N$ with the normal vector fields of the $TN$-partner-ruled surfaces are found as:

$$\left(\varphi_N^T\right)_{ss} \times U_N^T = \left(\frac{\kappa\left(v\left(-\varepsilon_2\kappa^2 + \tau^2\right) + \varepsilon_1\kappa'\right)}{\sqrt{\left|\varepsilon_1\tau^2 + \varepsilon_3\kappa^2\right|}}T + \frac{\varepsilon_3\left(\kappa\left(\kappa + \tau\right) + v\left(\kappa' + \tau'\right)\right)}{\sqrt{\left|\varepsilon_1\tau^2 + \varepsilon_3\kappa^2\right|}}N + \frac{\tau\left(v\left(\varepsilon_1\kappa^2 + \varepsilon_3\tau^2\right) - \varepsilon_2\kappa'\right)}{\sqrt{\left|\varepsilon_1\tau^2 + \varepsilon_3\kappa^2\right|}}B\right)$$

and

$$\left(\varphi_T^N\right)_{ss} \times U_T^N = \left( \frac{v\varepsilon_2\kappa\left(\varepsilon_3\kappa^2 + \varepsilon_1\tau^2 + v\kappa'\right) + \varepsilon_3\tau\left(v\kappa\tau + \tau'\right)}{\sqrt{\left|\varepsilon_2\tau^2 + \varepsilon_3 v^2\kappa^2\right|}} T + \frac{\varepsilon_2 v\kappa\left(v\kappa^2 + \kappa'\right)}{\sqrt{\left|\varepsilon_2\tau^2 + \varepsilon_3 v^2\kappa^2\right|}} N + \frac{\varepsilon_2\tau\left(v\kappa^2 + \kappa'\right)}{\sqrt{\left|\varepsilon_2\tau^2 + \varepsilon_3 v^2\kappa^2\right|}} B \right).$$

Since $\left(\varphi_N^T\right)_{ss} \times U_N^T \neq 0$ and $\left(\varphi_T^N\right)_{ss} \times U_T^N \neq 0$, $s$-parameter curves of the $TN$-partner-ruled surfaces simultaneously are not geodesic.

(ii) The scalar products of second partial derivatives of $\varphi_N^T$ and $\varphi_T^N$ with the normal vector fields of the $TN$-partner-ruled surfaces are given by

$$\left\langle \left(\varphi_N^T\right)_{ss}, U_N^T \right\rangle = \frac{\varepsilon_3 v\left(\tau\kappa' - \kappa\tau'\right)}{\sqrt{\left|\varepsilon_1\tau^2 + \varepsilon_3\kappa^2\right|}}$$

and

$$\left\langle \left(\varphi_T^N\right)_{ss}, U_T^N \right\rangle = \frac{-\tau\left(\varepsilon_3\kappa^2 + \varepsilon_1\tau^2 + v\kappa'\right) + v\kappa\left(v\kappa\tau + \tau'\right)}{\sqrt{\left|\varepsilon_2\tau^2 + \varepsilon_3 v^2\kappa^2\right|}}.$$

From here, if $\tau = 0$ and $\kappa \neq 0$, then $\left\langle \left(\varphi_N^T\right)_{ss}, U_N^T \right\rangle = 0$ and $\left\langle \left(\varphi_T^N\right)_{ss}, U_T^N \right\rangle = 0$. So, we can say that $s$-parameter curves of the $TN$-partner-ruled surfaces are simultaneously asymptotic if $\tau = 0$ and $\kappa \neq 0$.

□

**Theorem 3.3.** *Let $\varphi_N^T$ and $\varphi_T^N$ be a pair of the $TN$-partner-ruled surfaces in $\mathbb{R}_1^3$, then the $v$-parameter curves of the $TN$-partner-ruled surfaces are simultaneously*

*(i) geodesics,*
*(ii) asymptotic curves.*

*Proof.* Let $\varphi_N^T$ and $\varphi_T^N$ be a pair of the $TN$-partner-ruled surfaces in $\mathbb{R}_1^3$.

(i) Since $\left(\varphi_N^T\right)_{vv} \times U_N^T = 0$ and $\left(\varphi_T^N\right)_{vv} \times U_T^N = 0$, the $v$-parameter curves of the $TN$-partner-ruled surfaces simultaneously are geodesics.
(ii) Since $\left\langle \left(\varphi_N^T\right)_{vv}, U_N^T \right\rangle = 0$ and $\left\langle \left(\varphi_T^N\right)_{vv}, U_T^N \right\rangle = 0$, the $v$-parameter curves of the $TN$-partner-ruled surfaces simultaneously asymptotic curves.

□

**Theorem 3.4.** *Let $\varphi_N^T$ and $\varphi_T^N$ be a pair of the $TN$-partner-ruled surfaces in $\mathbb{R}_1^3$, then the $s$ and $v$-parameter curves of the $TN$-partner-ruled surfaces are simultaneously lines of curvature if and only if $\kappa = 0$ and $\tau \neq 0$.*

*Proof.* Let $\varphi_N^T$ and $\varphi_T^N$ be a pair of the $TN$-partner-ruled surfaces in $\mathbb{R}_1^3$, From Eqs (3.4), (3.5), (3.9) and (3.10), we have

$$F_N^T = f_N^T = F_T^N = f_T^N = 0,$$

for $\kappa = 0$ and $\tau \neq 0$, thus, the proof is completed. □

**Definition 3.2.** *Let $\alpha : I \to \mathbb{R}$ be a differentiable unit speed non-null space curve parametrized by arc-length $s$ in $\mathbb{R}^3_1$ with Frenet elements $\{T, N, B, \kappa, \tau\}$ such that $\kappa(s) \neq -\varepsilon_1 v\tau(s)$ and $\tau(s) \neq -\varepsilon_1 v$ for all $s \in I$. The two ruled surfaces represented by*

$$\begin{aligned} \varphi^T_B(s, v) &= T(s) + vB(s), \\ \varphi^B_T(s, v) &= B(s) + vT(s) \end{aligned} \tag{3.12}$$

*are called $TB$-partner-ruled surfaces with respect to the Frenet frame of the curve $\alpha$ in $\mathbb{R}^3_1$.*

**Theorem 3.5.** *Let the surfaces $\varphi^T_B$ and $\varphi^B_T$ be a $TB$-partner-ruled surfaces in $\mathbb{R}^3_1$, then the $TB$-partner-ruled surfaces are simultaneously*

*(i) developable surfaces,*
*(ii) not minimal (maximal) surfaces.*

*Proof.* By differentiating the first equation of Eq (3.12) with respect to $s$ and $v$, respectively, and using Eq (2.1), one can obtain

$$\left(\varphi^T_B\right)_s = (\kappa + \varepsilon_1 v\tau) N, \ \left(\varphi^T_B\right)_v = B. \tag{3.13}$$

Then, by considering the cross product of the partial derivatives of the surface $\varphi^T_B$ given by Eq (3.13), the normal vector field of the surface $\varphi^T_B$ is found as follows:

$$U^T_B = \frac{\left(\varphi^T_B\right)_s \times \left(\varphi^T_B\right)_v}{\left\|\left(\varphi^T_B\right)_s \times \left(\varphi^T_B\right)_v\right\|} = \frac{(-\varepsilon_1\kappa - v\tau)}{|\varepsilon_1\kappa + v\tau|} T = \pm T. \tag{3.14}$$

Here $\kappa \neq -\varepsilon_1 v\tau$ satisfies $\varepsilon_1\kappa + v\tau \neq 0$. By applying the scalar product for both vectors in Eq (3.13), we have the components of the first fundamental form of the ruled surface $\varphi^T_B$ as follows:

$$E^T_B = \varepsilon_2(\kappa + \varepsilon_1 v\tau)^2, \ F^T_B = 0, \ G^T_B = \varepsilon_3. \tag{3.15}$$

By differentiating Eq (3.13) in terms of $s$ and $v$, we have

$$\begin{aligned} &\left(\varphi^T_B\right)_{ss} = \left(\varepsilon_3\kappa^2 - \varepsilon_2 v\kappa\tau\right) T + (\kappa' + \varepsilon_1 v\tau') N + \left(\kappa\tau + \varepsilon_1 v\tau^2\right) B, \\ &\left(\varphi^T_B\right)_{sv} = \varepsilon_1\tau N, \ \left(\varphi^T_B\right)_{vv} = 0, \end{aligned}$$

and taking the scalar product of the last equation with the normal vector field found as Eq (3.14), we have the component of the second fundamental form of the ruled surface $\varphi^T_B$ as follows:

$$e^T_B = \frac{-\varepsilon_1\kappa (v\tau + \varepsilon_1\kappa)(\varepsilon_3\kappa - v\varepsilon_2\tau)}{|\varepsilon_1\kappa + v\tau|}, \ f^T_B = 0, \ g^T_B = 0. \tag{3.16}$$

Thus, by substituting Eqs (3.15) and (3.16) into Eq (2.4), the Gaussian curvature and the mean curvature of the ruled surface $\varphi^T_B$ are given by

$$K^T_B = 0, \ H^T_B = \varepsilon . \frac{\kappa (\varepsilon_3\kappa - \varepsilon_2 v\tau)}{2\varepsilon_2 |\varepsilon_1\kappa + v\tau| (\kappa + \varepsilon_1 v\tau)}. \tag{3.17}$$

On the other hand, by differentiating the second equation of Eq (3.12) with respect to $s$ and $v$, respectively, and using the Frenet frame derivative formulae, we obtain:

$$\left(\varphi_T^B\right)_s = (v\kappa + \varepsilon_1\tau)\, N, \ \left(\varphi_T^B\right)_v = T. \tag{3.18}$$

Then, by considering the cross product of the partial derivatives of the surface $\varphi_T^B$ given by Eq (3.18), the normal vector field of the surface $\varphi_T^B$ is found as:

$$U_T^B = \frac{\left(\varphi_T^B\right)_s \times \left(\varphi_T^B\right)_v}{\left\|\left(\varphi_T^B\right)_s \times \left(\varphi_T^B\right)_v\right\|} = \frac{\varepsilon_3\,(v\kappa + \varepsilon_1\tau)}{|v\kappa + \varepsilon_1\tau|} B = \mp\varepsilon_3 B. \tag{3.19}$$

Here $\tau \neq -\varepsilon_1 v\kappa$ guarantees $v\kappa + \varepsilon_1\tau \neq 0$. By applying the scalar product for both vectors in Eq (3.18), we have the components of the first fundamental form of the ruled surface $\varphi_T^B$ as follows:

$$E_T^B = \varepsilon_2(v\kappa + \varepsilon_1\tau)^2, F_T^B = 0, \ G_T^B = \varepsilon_1. \tag{3.20}$$

By differentiating Eq (3.18) with respect to $s$ and $v$, we get

$$\begin{aligned}
&\left(\varphi_T^B\right)_{ss} = \left(\varepsilon_3 v\kappa^2 - \varepsilon_2\kappa\tau\right) T + (v\kappa' + \varepsilon_1\tau')\, N + \left(v\kappa\tau + \varepsilon_1\tau^2\right) B, \\
&\left(\varphi_T^B\right)_{sv} = \kappa N, \ \left(\varphi_T^B\right)_{vv} = 0,
\end{aligned}$$

and from the scalar product of the last equations with the normal vector field given by Eq (3.19), we have the component of the second fundamental form of the ruled surface $\varphi_T^B$ as follows:

$$e_T^B = \frac{\tau(v\kappa + \varepsilon_1\tau)^2}{|v\kappa + \varepsilon_1\tau|}, \ f_T^B = 0, \ g_T^B = 0. \tag{3.21}$$

So, by substituting Eqs (3.20) and (3.21) into Eq (2.4), the Gaussian curvature $K_T^B$ and the mean curvature $H_T^B$ of the ruled surface $\varphi_T^B$ are given by

$$K_T^B = 0, \ H_T^B = -\varepsilon\frac{\tau}{2\varepsilon_2\,|v\kappa + \varepsilon_1\tau|}. \tag{3.22}$$

Consequently, from Eqs (3.17) and (3.22), it can easily be said that the $TB$-partner-ruled surfaces simultaneously can be developable but not minimal (maximal) surfaces. □

In the same way, as for $TN$-partner ruled surfaces, we can prove the following three theorems:

**Theorem 3.6.** *Let $\varphi_B^T$ and $\varphi_T^B$ be a pair of the $TB$-partner-ruled surfaces in $\mathbb{R}_1^3$, then $s$-parameter curves of the $TB$-partner-ruled surfaces are simultaneously*

*(i) not geodesics,*
*(ii) not asymptotic curves.*

**Theorem 3.7.** *Let $\varphi_B^T$ and $\varphi_T^B$ be a pair of the $TB$-partner-ruled surfaces in $\mathbb{R}_1^3$, then the $v$-parameter curves of the $BT$-partner-ruled surfaces are simultaneously*

*(i) geodesics,*

*(ii) asymptotic curves.*

**Theorem 3.8.** *Let $\varphi_B^T$ and $\varphi_T^B$ be a pair of the $TB$-partner-ruled surfaces in $\mathbb{R}_1^3$, then the $s$ and $v$-parameter curves of $TB$-partner-ruled surfaces are simultaneously lines of curvature.*

**Definition 3.3.** *Let $\alpha : I \to \mathbb{R}$ be a differentiable unit speed non-null space curve parametrized by arc-length $s$ in $\mathbb{R}_1^3$ with Frenet elements $\{T, N, B, \kappa, \tau\}$ such that $\kappa(s) \neq \mp v\tau(s)$ and $\kappa(s) \neq \mp\tau(s)$ for all $s \in I$. The two ruled surfaces defined by*

$$\begin{cases} \varphi_B^N(s,v) = N(s) + vB(s), \\ \varphi_N^B(s,v) = B(s) + vN(s) \end{cases} \tag{3.23}$$

*are called $NB$-partner-ruled surfaces with respect to the Frenet frame of the curve $\alpha$ in $\mathbb{R}_1^3$.*

**Theorem 3.9.** *Let $\varphi_B^N$ and $\varphi_N^B$ be a pair of the $NB$-partner-ruled surfaces in $\mathbb{R}_1^3$, then the $NB$-partner-ruled surfaces are simultaneously*

*(i) developable surfaces if and only if $\kappa = 0$ or $\tau = 0$,*
*(ii) minimal (maximal) surfaces if and only if $\kappa = 0$.*

*Proof.* By differentiating the first equation of Eq (3.23) with respect to $s$ and $v$, respectively, and using Frenet frame derivative formulae, one can obtain

$$\left(\varphi_B^N\right)_s = \varepsilon_3\kappa T + \varepsilon_1 v\tau N + \tau B,\ \left(\varphi_B^N\right)_v = B. \tag{3.24}$$

Then, by considering the partial derivatives of the surface $\varphi_B^N$ given by Eq (3.24) and the cross product of both vectors, the normal vector field of the surface $\varphi_B^N$ is found as:

$$U_B^N = \frac{\left(\varphi_B^N\right)_s \times \left(\varphi_B^N\right)_v}{\left\|\left(\varphi_B^N\right)_s \times \left(\varphi_B^N\right)_v\right\|} = \frac{-v\tau T - \varepsilon_1\kappa N}{\sqrt{\left|\varepsilon_1 v^2\tau^2 + \varepsilon_2\kappa^2\right|}}. \tag{3.25}$$

Here $\kappa \neq \mp v\tau$ satisfies $\varepsilon_1 v^2\tau^2 + \varepsilon_2\kappa^2 \neq 0$. By applying the scalar product for both vectors in Eq (3.24), we have the components of the first fundamental form of the ruled surface $\varphi_B^N$ as follows:

$$E_B^N = \varepsilon_1\kappa^2 + \left(\varepsilon_3 + \varepsilon_2 v^2\right)\tau^2,\ F_B^N = \varepsilon_3\tau,\ G_B^N = \varepsilon_3. \tag{3.26}$$

By differentiating Eq (3.24) in terms of $s$ and $v$, we get

$$\begin{aligned} &\left(\varphi_B^N\right)_{ss} = \left(-\varepsilon_2 v\kappa\tau + \varepsilon_3\kappa'\right)T + \left(\varepsilon_3\kappa^2 + \varepsilon_1\tau^2 + \varepsilon_1 v\tau'\right)N + \left(\varepsilon_1 v\tau^2 + \tau'\right)B, \\ &\left(\varphi_B^N\right)_{sv} = \varepsilon_1\tau N,\ \left(\varphi_B^N\right)_{vv} = 0, \end{aligned}$$

and from the scalar product of the last equations with the normal vector field given by Eq (3.25), we have the component of the second fundamental form of the ruled surface $\varphi_B^N$ as follows:

$$e_B^N = \frac{v\tau\left(-\varepsilon_3 v\kappa\tau + \varepsilon_2\kappa'\right) + \kappa\left(\kappa^2 - \varepsilon_2\left(\tau^2 + v\tau'\right)\right)}{\sqrt{\left|\varepsilon_1 v^2\tau^2 + \varepsilon_2\kappa^2\right|}},\ f_B^N = \frac{-\varepsilon_2\kappa\tau}{\sqrt{\left|\varepsilon_1 v^2\tau^2 + \varepsilon_2\kappa^2\right|}},\ g_B^N = 0. \tag{3.27}$$

Thus, by substituting Eqs (3.26) and (3.27) into Eq (2.4), the Gaussian curvature $K_B^N$ and the mean curvature $H_B^N$ of the ruled surface $\varphi_B^N$ are given by

$$\begin{aligned} K_B^N &= \varepsilon \frac{-\kappa^2\tau^2}{\varepsilon_3 \left|\varepsilon_1 v^2\tau^2 + \varepsilon_2\kappa^2\right| \left(\varepsilon_1\kappa^2 + \varepsilon_2 v^2\tau^2\right)}, \\ H_B^N &= \varepsilon \frac{\varepsilon_2 \left(v\left(\kappa\tau' - \tau\kappa'\right) + \kappa\tau^2\right) - \varepsilon_2\kappa\tau^2\left(\varepsilon_1 v^2 - 2\right) - \kappa^3}{2\sqrt{\left|\varepsilon_1 v^2\tau^2 + \varepsilon_2\kappa^2\right|}\left(\varepsilon_1\kappa^2 + \varepsilon_2 v^2\tau^2\right)}. \end{aligned} \tag{3.28}$$

On the other hand, by differentiating the second equation of Eq (3.23) with respect to $s$ and $v$, respectively, and using the Frenet frame derivative formulae, we find

$$\left(\varphi_N^B\right)_s = v\varepsilon_3\kappa T + \varepsilon_1\tau N + v\tau B, \ \left(\varphi_N^B\right)_v = N. \tag{3.29}$$

Then, by considering the partial derivatives of the surface $\varphi_N^B$ given by Eq (3.29) and the cross product of both vectors, the normal vector field of the surface $\varphi_N^B$ is found as follows:

$$U_N^B = \frac{\left(\varphi_N^B\right)_s \times \left(\varphi_N^B\right)_v}{\left\|\left(\varphi_N^B\right)_s \times \left(\varphi_N^B\right)_v\right\|} = \frac{\varepsilon_1\tau T - \kappa B}{\sqrt{\left|\varepsilon_1\tau^2 + \varepsilon_3\kappa^2\right|}}. \tag{3.30}$$

Here $\kappa \neq \mp\tau$ satisfies $\varepsilon_1\tau^2 + \varepsilon_3\kappa^2 \neq 0$. By applying the scalar product for both vectors in Eq (3.30), we have the components of the first fundamental form of the ruled surface $\varphi_N^B$ as follows:

$$E_N^B = \varepsilon_1 v^2\kappa^2 + \left(\varepsilon_3 v^2 + \varepsilon_2\right)\tau^2, \ F_N^B = -\varepsilon_3\tau, \ G_N^B = \varepsilon_2. \tag{3.31}$$

By differentiating Eq (3.29) in terms of $s$ and $v$, we have

$$\begin{aligned} \left(\varphi_N^B\right)_{ss} &= \left(-\varepsilon_2\kappa\tau + \varepsilon_3 v\kappa'\right) T + \left(\varepsilon_3 v\kappa^2 + \varepsilon_1 v\tau^2 + \varepsilon_1\tau'\right) N + \left(\varepsilon_1\tau^2 + v\tau'\right) B, \\ \left(\varphi_N^B\right)_{sv} &= \varepsilon_3\kappa T + \tau B, \ \left(\varphi_N^B\right)_{vv} = 0, \end{aligned}$$

and taking the scalar product of the last equations with the normal vector field Eq (3.30), we have the component of the second fundamental form of the ruled surface $\varphi_N^B$ as follows:

$$e_N^B = \frac{\tau\left(-\varepsilon_2\kappa\tau + \varepsilon_3\kappa'\right) - \varepsilon_3\kappa\left(\varepsilon_1\tau^2 + v\tau'\right)}{\sqrt{\left|\varepsilon_1\tau^2 + \varepsilon_3\kappa^2\right|}}, \ f_N^B = 0, \ g_N^B = 0. \tag{3.32}$$

Thus, by substituting Eqs (3.31) and (3.32) into Eq (2.4), the Gaussian curvature $K_N^B$ and the mean curvature $H_N^B$ of the ruled surface $\varphi_N^B$ is given by

$$K = 0, \ H_N^B = \varepsilon \frac{\varepsilon_1\left(\kappa\tau' - \tau\kappa'\right)}{2v\left(\varepsilon_1\tau^2 + \varepsilon_3\kappa^2\right)\sqrt{\left|\varepsilon_1\tau^2 + \varepsilon_3\kappa^2\right|}}. \tag{3.33}$$

Consequently, from Eqs (3.28) and (3.33), it can easily be implied that the $NB$-partner-ruled surfaces are simultaneously developable and minimal (maximal) surfaces under the conditions stated in the hypothesis. □

In the same way, as for $TN$-partner ruled surfaces, we can prove the following three theorems:

**Theorem 3.10.** *Let $\varphi_B^N$ and $\varphi_N^B$ be a pair of the $NB$-partner-ruled surfaces in $\mathbb{R}_1^3$, then the s-parameter curves of the $NB$-partner-ruled surfaces are simultaneously*

*(i) not geodesics,*
*(ii) asymptotics if and only if $\kappa = 0$ and $\tau \neq 0$.*

**Theorem 3.11.** *Let $\varphi_B^N$ and $\varphi_N^B$ be a pair of the $NB$-partner-ruled surfaces in $\mathbb{R}_1^3$, then the v-parameter curves of the $NB$-partner-ruled surfaces are simultaneously*

*(i) geodesics,*
*(ii) asymptotics curve.*

**Theorem 3.12.** *Let $\varphi_B^N$ and $\varphi_N^B$ be a pair of the $NB$-partner-ruled surfaces in $\mathbb{R}_1^3$, then the s and v-parameter curves of the $NB$-partner-ruled surfaces are simultaneously are a line of curvature if and only if $\tau = 0$ and $\kappa \neq 0$.*

## 4. Applications with the partner-ruled surfaces

In this section, three examples are given according to cases of the curve being timelike or spacelike, and graphs of these examples are drawn.

**Example 4.1.** *Let us consider a timelike curve parameterized as*

$$\alpha(s) = \left(\frac{-5}{9}\sinh(3s), \frac{-5}{9}\cosh(3s), \frac{4}{3}s\right).$$

*Then, the Frenet vectors of $\alpha$ are given by*

$$\begin{aligned} T(s) &= \left(-\frac{5}{3}\cosh(3s), -\frac{5}{3}\sinh(3s), \frac{4}{3}\right), \\ N(s) &= (-\sinh(3s), -\cosh(3s), 0), \\ B(s) &= \left(\frac{-4}{3}\cosh(3s), \frac{-4}{3}\sinh(3s), \frac{5}{3}\right). \end{aligned}$$

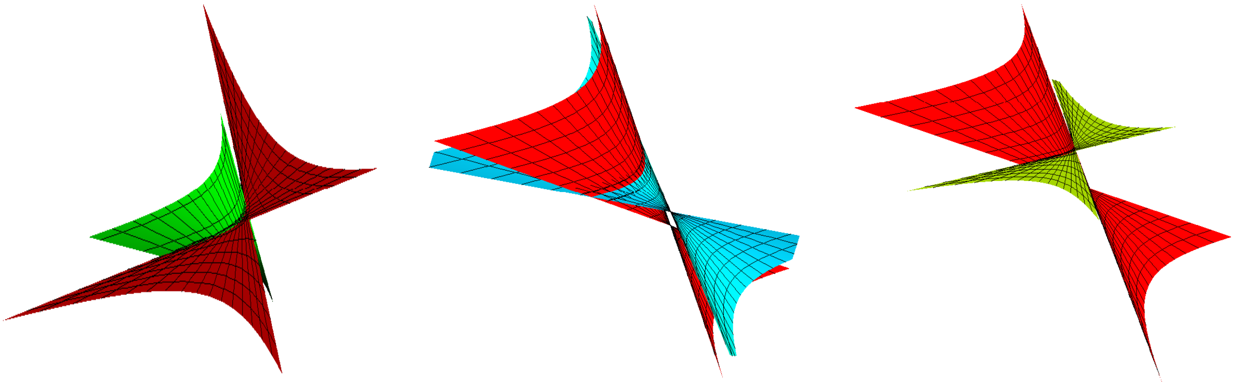
*Thus, the partner-ruled surfaces with the parametric forms*

$$\begin{cases} \varphi_N^T = \left(-\frac{5}{3}\cosh(3s) - v\sinh(3s), -v\cosh(3s) - \frac{5}{3}\sinh(3s), \frac{4}{3}\right), \\ \varphi_T^N = \left(-\frac{5}{3}v\cosh(3s) - \sinh(3s), -\cosh(3s) - \frac{5}{3}v\sinh(3s), \frac{4v}{3}\right), \end{cases}$$

$$\begin{cases} \varphi_B^T = \left(-\frac{1}{3}(5+4v)\cosh(3s), -\frac{1}{3}(5+4v)\sinh(3s), \frac{1}{3}(4+5v)\right), \\ \varphi_T^B = \left(-\frac{1}{3}(4+5v)\cosh(3s), -\frac{1}{3}(4+5v)\sinh(3s), \frac{1}{3}(5+4v)\right), \end{cases}$$

$$\begin{cases} \varphi_B^N = \left(-\frac{4}{3}v\cosh(3s) - \sinh(3s), -\cosh(3s) - \frac{4}{3}v\sinh(3s), \frac{5v}{3}\right), \\ \varphi_N^B = \left(-\frac{4}{3}\cosh(3s) - v\sinh(3s), -v\cosh(3s) - \frac{4}{3}\sinh(3s), \frac{5}{3}\right) \end{cases}$$

*are drawn in Figure 1, respectively.*



**(a)** $TN$-partner-ruled surfaces ( $\varphi_N^T$ (red) and $\varphi_T^N$ (green)).

**(b)** $TB$-partner-ruled surfaces ( $\varphi_B^T$ (red) and $\varphi_T^B$ (cyan)).

**(c)** $NB$-partner-ruled surfaces ( $\varphi_B^N$ (red) and $\varphi_N^B$ (yellow)).

**Figure 1.** The partner-ruled surfaces generated by the timelike curve $\alpha$ for $s = [-\pi/8,\ \pi/8]$ and $v = [-5, 5]$.

**Example 4.2.** *Let us consider a spacelike curve with timelike normal parameterized as*

$$\alpha(s) = \frac{1}{\sqrt{2}}\left(\cosh(s), \sinh(s), s\right).$$

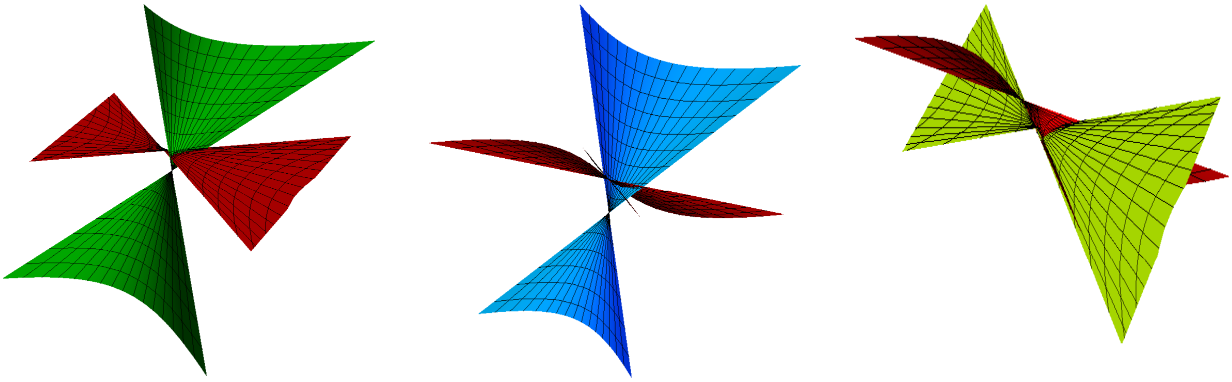
*Then, the Frenet vectors of the spacelike curve with timelike normal $\alpha$ are given by*

$$\begin{aligned} T(s) &= \frac{1}{\sqrt{2}}\left(\sinh(s), \cosh(s), 1\right), \\ N(s) &= \left(\cosh(s), \sinh(s), 0\right), \\ B(s) &= \frac{1}{\sqrt{2}}\left(\sinh(s), \cosh(s), -1\right). \end{aligned}$$

*Thus, the graphs of the partner-ruled surfaces with the parametric forms*

$$\begin{cases} \varphi_N^T = \left(v\cosh(s) + \frac{\sinh(s)}{\sqrt{2}}, \frac{\cosh(s)}{\sqrt{2}} + v\sinh(s), \frac{1}{\sqrt{2}}\right), \\ \varphi_T^N = \left(\cosh(s) + \frac{v\sinh(s)}{\sqrt{2}}, \frac{v\cosh(s)}{\sqrt{2}} + \sinh(s), \frac{v}{\sqrt{2}}\right), \end{cases}$$
$$\begin{cases} \varphi_B^T = \left(\frac{(1+v)\sinh(s)}{\sqrt{2}}, \frac{(1+v)\cosh(s)}{\sqrt{2}}, -\frac{-1+v}{\sqrt{2}}\right), \\ \varphi_T^B = \left(\frac{(1+v)\sinh(s)}{\sqrt{2}}, \frac{(1+v)\cosh(s)}{\sqrt{2}}, \frac{-1+v}{\sqrt{2}}\right), \end{cases}$$
$$\begin{cases} \varphi_B^N = \left(\cosh(s) + \frac{v\sinh(s)}{\sqrt{2}}, \frac{v\cosh(s)}{\sqrt{2}} + \sinh(s), -\frac{v}{\sqrt{2}}\right), \\ \varphi_N^B = \left(v\cosh(s) + \frac{\sinh(s)}{\sqrt{2}}, \frac{\cosh(s)}{\sqrt{2}} + v\sinh(s), -\frac{1}{\sqrt{2}}\right) \end{cases}$$

*are given in Figure 2, respectively.*



**(a)** $TN$-partner-ruled surfaces ( $\varphi_N^T$ (red) and $\varphi_T^N$ (green)).

**(b)** $TB$-partner-ruled surfaces ( $\varphi_B^T$ (red) and $\varphi_T^B$ (cyan)).

**(c)** $NB$-partner-ruled surfaces ( $\varphi_B^N$ (red) and $\varphi_N^B$ (yellow)).

**Figure 2.** The partner-ruled surfaces generated by the spacelike with timeline normal curve with $s = [-1, 1]$ and $v = [-10, 10]$.

**Example 4.3.** *Let us consider a spacelike curve with timelike binormal parameterized as*

$$\alpha(s) = (s,\ s\sin(\ln(s)),\ s\cos(\ln(s))).$$

*Then, the Frenet vectors of the spacelike curve with timelike binormal $\alpha$ are given by*

$$\begin{aligned}
T(s) &= (1, \cos(\ln(s)) + \sin(\ln(s)), \cos(\ln(s)) - \sin(\ln(s))),\\
N(s) &= \frac{1}{\sqrt{2}}(0, \cos(\ln(s)) - \sin(\ln(s)), -\cos(\ln(s)) - \sin(\ln(s))),\\
B(s) &= \frac{1}{\sqrt{2}}\left(2\sqrt{2}, \cos(\ln(s)) + \sin(\ln(s)), \cos(\ln(s)) - \sin(\ln(s))\right).
\end{aligned}$$

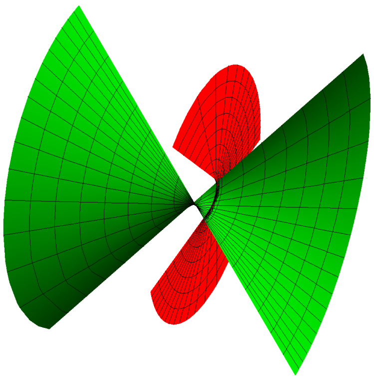
*Thus, the parametric forms of the partner-ruled surfaces are given as follows:*

$$\begin{cases}
\varphi_N^T = \begin{pmatrix} 1, \cos(\ln(s)) + \sin(\ln(s)) + \dfrac{v(\cos(\ln(s)) - \sin(\ln(s)))}{\sqrt{2}} \\ \cos(\ln(s)) - \sin(\ln(s)) - \dfrac{v(\cos(\ln(s)) + \sin(\ln(s)))}{\sqrt{2}} \end{pmatrix}, \\
\varphi_T^N = \begin{pmatrix} u, u(\cos(\ln(s)) + \sin(\ln(s))) + \dfrac{\cos(\ln(s)) - \sin(\ln(s))}{\sqrt{2}} + \\ u(\cos(\ln(s)) - \sin(\ln(s))) - \dfrac{\cos(\ln(s)) + s\sin(\ln(s))}{\sqrt{2}} \end{pmatrix},
\end{cases}$$

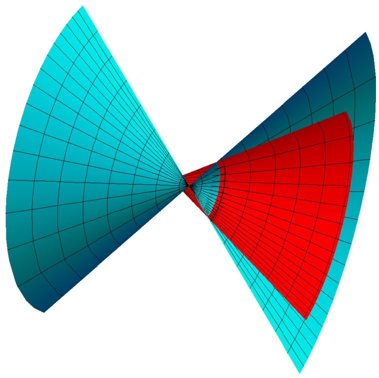
$$\begin{cases}
\varphi_B^T = \begin{pmatrix} 1 + \sqrt{2}v, \dfrac{1}{2}\left(2 + \sqrt{2}v\right)(\cos(\ln(s)) + \sin(\ln(s))) \\ \dfrac{1}{2}\left(2 + \sqrt{2}v\right)(\cos(\ln(s)) - \sin(\ln(s))) \end{pmatrix}, \\
\varphi_T^B = \begin{pmatrix} \sqrt{2} + v, \dfrac{1}{2}\left(\sqrt{2} + 2v\right)(\cos(\ln(s)) + \sin(\ln(s))) \\ \dfrac{1}{2}\left(\sqrt{2} + 2v\right)(\cos(\ln(s)) - \sin(\ln(s))) \end{pmatrix},
\end{cases}$$

$$\begin{cases} \varphi_B^N = \left( \sqrt{2}v, \dfrac{(1+v)\cos(\ln(s)) + (-1+v)\sin(\ln(s))}{\sqrt{2}}, \dfrac{(-1+v)\cos(\ln(s)) - (1+v)\sin(\ln(s))}{\sqrt{2}} \right), \\ \varphi_N^B = \left( \sqrt{2}, \dfrac{(1+v)\cos(\ln(s)) + (1-v)\sin(\ln(s))}{\sqrt{2}}, \dfrac{(1-v)\cos(\ln(s)) - (1+v)\sin(\ln(s))}{\sqrt{2}} \right) \end{cases}$$

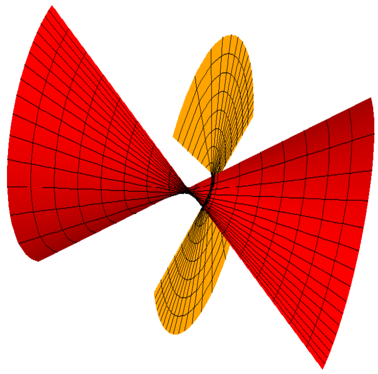
*and their graphics are drawn in Figure 3, respectively.*



**(a)** $TN$-partner-ruled surfaces ( $\varphi_N^T$ (red) and $\varphi_T^N$ (green)).



**(b)** $TB$-partner-ruled surfaces ( $\varphi_B^T$ (red) and $\varphi_T^B$ (cyan)).



**(c)** $NB$-partner-ruled surfaces ( $\varphi_B^N$ (red) and $\varphi_N^B$ (yellow)).

**Figure 3.** The partner-ruled surfaces generated by the spacelike with timelike binormal curve with $s = [1, 10]$ and $v = [-10, 10]$.

## 5. Conclusions

In this paper, the invariants of the partner-ruled surfaces formed by tangent, normal and binormal vector fields of non-null space curves simultaneously have been presented in Minkowski 3-space. As it is recalled, two ruling lines generate the partner-ruled surfaces if they simultaneously move along their respective curves. The simultaneous characterizations of such couples of surfaces can provide insights into the surface theory in Minkowski space. This comprehensive knowledge may lead to the development of surfaces of the dynamics of cosmic objects. With this motivation, some characterizations of the parameter curves have been examined. Examples of these surfaces have been given, and their graphics have been drawn. In future research, we will delve into the practical applications of our main discoveries by integrating concepts from singularity theory, submanifold theory, and other relevant results in [20–37]. These integrations offer promising avenues for future investigation within this article.

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare no conflict of interest.

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