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Research article

Study on the oscillation of solution to second-order impulsive systems

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Abstract: In the present article, we set the if and only if conditions for the solutions of the class of neutral impulsive delay second-order differential equations. We consider two cases when it is non-increasing and non-decreasing for quotient of two positive odd integers. Our main tool is the Lebesgue's dominated convergence theorem. Examples illustrating the applicability of the results are also given, and state an open problem.

Keywords: nonlinear; nonoscillation; delay argument; second-order differential equation; Lebesgue's dominated convergence theorem
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1. Introduction

In modern era, delay differential equations ($\mathbb{DE}$s) have become almost the center of interest. Many things in the world are directed by differential systems. We assume that the systems are independent of past state and future state depends on present state neutral delays. $\mathbb{DE}$s are natural extensions of the delay $\mathbb{DE}$ which involve derivatives of the unknown at the delayed argument. Mathematical modeling with delay $\mathbb{DE}$s is widely used for analysis and predictions in various areas of life science, for example, population dynamics, epidemiology, immunology, neutral networks, chemistry, physics, engineering, etc. The literature connected to impulsive delay differential system is vast.

Below, we are going to provide some background of oscillation theory of impulsive $\mathbb{DE}$s. The authors in [1] are concerned with the asymptotic behavior of a class of higher-order sublinear Emden-Fowler delay differential equations

$$\left(\mathrm{q}_2(\iota)\varrho^{(n-1)}(\iota)\right)' + \mathrm{q}_1(\iota)\varrho^{\nu}(\tau(\iota)) = 0, \qquad \text{for } \iota \geq \iota_0, \tag{1.1}$$

where $0 < \nu < 1$ is a ratio of odd natural numbers, $\mathrm{q}_2 \in C^1[\iota_0, \infty)$, $\mathrm{q}_2 > 0$, $\mathrm{q}_2' \geq 0$, $\mathrm{q}_1, \tau \in C[\iota_0, \infty)$, $\tau(\iota) < \iota$, $\lim_{\iota\to\infty} \tau(\iota) = \infty$, $\mathrm{q}_1(\iota) \geq 0$ and $\mathrm{q}_2(\iota)$ is not identically zero for large ι (for instanse, consider [2–5]). Shen et al. have taken the impulsive system ($\mathbb{IS}$)

$$\begin{cases} \varrho'(\iota) + \mathrm{q}(\iota)\varrho(\iota - \mu_1) = 0, & \iota \neq \iota_k, \\ \varrho(\iota_k^+) - \varrho(\iota_k^-) = I_k(\varrho(\iota_k)), & k \in \mathbb{N}, \end{cases} \tag{1.2}$$

where q, $I_k \in C(\mathbb{R}, \mathbb{R})$, and they established the sufficient conditions for the oscillatory and asymptotic behavior of (1.2) [6]. Graef et al. in [7] considered the $\mathbb{IS}$

$$\begin{cases} \left(\varrho(\iota) - \mathrm{q}_2(\iota)\varrho(\iota - \mu_2)\right)' + \mathrm{q}_1(\iota)|\varrho(\iota - \mu_1)|^{\iota} \operatorname{sgn} \varrho(\iota - \mu_1) = 0, & \iota \geq \iota_0, \\ \varrho(\iota_k^+) = \eta_k \varrho(\iota_k), & k \in \mathbb{N}, \end{cases} \tag{1.3}$$

and considering $\mathrm{q}_2(\iota) \in PC([\iota_0, \infty), \mathbb{R}_+)$ ($\mathrm{q}_2(\iota)$ piece wisely continuous in $[\iota_0, \infty)$) set up results on sufficient conditions for oscillation (1.3). Shen et al. have established new sufficient conditions for oscillation of the $\mathbb{IS}$

$$\begin{cases} (\varrho(\iota) - \mathrm{q}_3(\iota)\varrho(\iota - \mu_3))' + \mathrm{q}_2(\iota)\varrho(\iota - \mu_2) - \mathrm{q}_1(\iota)\varrho(\iota - \mu_1) = 0, & \mu_2 \geq \mu_1 > 0, \\ \varrho(\iota_k^+) = \eta_k(\varrho(\iota_k)), & k \in \mathbb{N}, \end{cases} \tag{1.4}$$

and established some new conditions for the oscillation of (1.4) when $\mathrm{q}_3(\iota) \in PC([\iota_0, \infty), \mathbb{R}_+)$ and $\eta_i \leq \frac{J_i(\varrho)}{\varrho} \leq 1$ [8]. Karpuz et al. [9] studied on advanced case, that is, taking a non homogeneous system and established the results for sufficient conditions for oscillation of (1.4). Tripathy et al. [10] have taken the following equations to establish the oscillatory and non-oscillatory character of a second order neutral impulsive differential system ($\mathbb{IDS}$)

$$\begin{cases} (\varrho(\iota) - \eta_2\varrho(\iota - \mu_2))'' + \eta_1\varrho(\iota - \mu_1) = 0, & \iota \neq \iota_k, \\ \Delta(\varrho(\iota_k) - \eta_2\varrho(\iota_k - \mu_2))' + \eta_1\varrho(\iota_k - \mu_1) = 0, \end{cases} \tag{1.5}$$

here $k \in \mathbb{N}$, all coefficients and delays are constants. In [11] new result established for second-order neutral delay $\mathbb{DS}$

$$\begin{cases} \left(\mathrm{q}_3(\iota)(\varrho(\iota) + \mathrm{q}_2(\iota)\varrho(\iota - \mu_2))'\right)' + \mathrm{q}_1(\iota)\wp(\varrho(\iota - \mu_1)) = 0, & \iota \neq \iota_k, \\ \Delta\left(\mathrm{q}_3(\iota_k)(\varrho(\iota_k) + \mathrm{q}_2(\iota_k)\varrho(\iota_k - \mu_2))'\right) + \mathrm{q}_4(\iota_k)\wp(\varrho(\iota_k - \mu_1)) = 0, \end{cases} \tag{1.6}$$

where $k \in \mathbb{N}$. Santra et al. [12] observed the characteristic of solutions for first-order neutral delay $\mathbb{IS}$ of the form

$$\begin{cases} (\varrho(\iota) - \mathrm{q}_2(\iota)\varrho(\iota-\mu_2))' + \mathrm{q}_1(\iota)\,\wp(\varrho(\iota-\mu_1)) = 0, \\ \varrho(\iota_k^+) = J_k(\varrho(\iota_k)), \\ \varrho(\iota_k^+ - \mu_3) = J_k(\varrho(\iota_k^+ - \mu_3)), \end{cases} \tag{1.7}$$

here $k \in \mathbb{N}$, taking varying values of the neutral coefficient q_2. Also, Santra et al. in [13] established the necessary and sufficient results for oscillation of the solutions of the below systems with impulses applying Lebesgue's Dominated convergent theorem,

$$\begin{cases} \left(\acute{\mathrm{q}}(\iota)(w'(\iota))^{\alpha}\right)' + \sum_{i=1}^{m} \mathrm{q}_i(\iota)\,\wp_i(\varrho(\sigma(\iota))) = 0, \\ \Delta\left(\acute{\mathrm{q}}(\iota_k)(w'(\iota))^{\alpha}\right) + \sum_{i=1}^{m} \mathrm{q}_i(\iota_k)\,\wp_i(\varrho(\sigma(\iota_k))) = 0, \end{cases} \tag{1.8}$$

where $w(\iota) = \varrho(\iota) + \acute{\mathrm{q}}(\iota)\varrho(\acute{\sigma}(\iota))$, with

$$\Delta\varrho(\eta) = \lim_{s\to\eta^+} \varrho(s) - \lim_{s\to\eta^-} \varrho(s),$$

and $-1 \le \acute{\mathrm{q}}(\iota) \le 0$. In 2020, Li et al. studied the dynamic behavior of a computer worm system under a discontinuous control strategy and some conditions for globally asymptotically stable solutions of the discontinuous system were obtained by using the Bendixson–Dulac theorem, Green's formula and the Lyapunov function [14]. Also, they investigated the global dynamics of a controlled discontinuous diffusive SIR epidemic system under Neumann boundary conditions [15].

The authors observed oscillatory and non-oscillatory both conditions for the solutions of the non linear neutral $\mathbb{DE}$ of the form

$$\begin{cases} \left(\mathrm{q}_3(\iota)(\varrho(\iota) + \mathrm{q}_2(\iota)\varrho(\iota-\mu_2))'\right)' + \mathrm{q}_1(\iota)\,\wp(\varrho(\iota-\mu_1)) = \hbar(\iota), \\ \Delta\left(\mathrm{q}_3(\iota_k)(\varrho(\iota_k) + \mathrm{q}_2(\iota_k)\varrho(\iota_k-\mu_2))'\right) + \mathrm{q}_1(\iota_k)\,\wp(\varrho(\iota_k-\mu_1)) = \hbar(\iota_k), \quad k \in \mathbb{N}. \end{cases} \tag{1.9}$$

At last we observed some modern results in [16] where Tripathy and Santra improved oscillatory results of non-linear neutral $\mathbb{IS}$ of the form

$$\begin{cases} \left(\acute{\mathrm{q}}(\iota)(w'(\iota))^{\alpha}\right)' + \sum_{i=1}^{m} \mathrm{q}_i(\iota)\,\varrho^{\beta_i}(\sigma_i(\iota)) = 0, & \iota \ge \iota_0,\ \iota \ne \iota_k, \\ \Delta\left(\acute{\mathrm{q}}(\iota_k)(w'(\iota_k))^{\alpha}\right) + \sum_{i=1}^{m} \mathrm{q}_i(\iota)\,\varrho^{\beta_i}(\sigma_i(\iota_k)) = 0, & k \in \mathbb{N}, \end{cases}$$

where $w(\iota) = \varrho(\iota) + \bar{\mathrm{q}}(\iota)\varrho(\acute{\sigma}(\iota))$ with $-1 \le \bar{\mathrm{q}}(\iota) \le 0$ [17].

Motivated by the above works, in this paper, we consider the $\mathbb{IS}$

$$\begin{cases} \left(\acute{\mathrm{q}}(\iota)(\varrho'(\iota))^{\alpha}\right)' + \sum_{i=1}^{m} \mathrm{q}_i(\iota)\,\wp_i(\varrho(\sigma_i(\iota))) = 0, & \iota \ge \iota_0, \\ \Delta\left(\acute{\mathrm{q}}(\iota_k)(\varrho'(\iota_k))^{\alpha}\right) + \sum_{i=1}^{m} \mathrm{q}_i(\iota_k)\,\wp_i(\varrho(\sigma_i(\iota_k))) = 0, & \iota \ne \iota_k \end{cases} \tag{1.10}$$

where $\wp_i, \mathrm{q}_i, \acute{\mathrm{q}}, \sigma_i$ are continuous and α be the quotient of two positive odd integers which satisfy the given following postulate as

(B1) $\sigma_i \in C([0,\infty),\mathbb{R})$, $\sigma_i(\iota) < \iota$, $\lim_{\iota\to\infty} \sigma_i(\iota) = \infty$;
(B2) $\acute{q} \in C^1([0,\infty),\mathbb{R})$, $q_i \in C([0,\infty),\mathbb{R})$; $\acute{q}(\iota) > 0$, $q_i(\iota) \geq 0$, for each $\iota \geq 0$ & $i = 1,2,\ldots,m$, $\sum q_i(\iota) \neq 0$ in any $[\tau,\infty)$;
(B3) $\varrho_i \in C(\mathbb{R},\mathbb{R})$ is non-decreasing and $\wp_i(\varrho)\varrho > 0$ for $\varrho \neq 0$ $(i = 1,2,\ldots,m)$;
(B4) $\lim_{\iota\to\infty} \mathring{R}(\iota) = \infty$ where

$$\mathring{R}(\iota) = \int_0^\iota (\acute{q}(\eta))^{-1/\alpha}\, d\eta; \tag{1.11}$$

(B5) α be the quotient of two positive odd integers and the sequence ι_k satisfies $\iota_1 < \iota_2 < \cdots < \iota_k \to \infty$, as $k \to \infty$.

The main objective of this paper is to find out both necessary and sufficient conditions for the oscillation of all solutions to $\mathbb{IS}$ (1.10). In this direction, we refer [18–32] to the readers for more details on this study. All functional inequalities assumed here should be held eventually i.e., for all large ι that also satisfy whereas the domain is not clearly given.

In Section 2, we recall some essential definition and necessary lemmas. Section 3 contains our main results in this work, while an example is presented to support the validity of our obtained results in Section 4. In Section 5, conclusion are presented.

2. Primary consequences

First we start and prove the following key lemma.

Lemma 2.1. *Consider postulates* (B1)–(B4)*, and that ϱ is converges to zero for the Eq* (1.10)*. So there exist $\iota_1 \geq \iota_0$ and $\delta > 0$, so we have*

$$0 < \varrho(\iota) \leq \delta \mathring{R}(\iota), \tag{2.1}$$

$$\begin{aligned}\left(\mathring{R}(\iota) - \mathring{R}(\iota_1)\right)\left[\int_\iota^\infty \sum_{i=1}^m q_i(\iota)\wp_i(\varrho(\sigma_i(\iota)))\, d\iota + \sum_{\iota_k\geq\iota}\sum_{i=1}^m q_i(\iota_k)\wp_i(\varrho(\sigma_i(\iota_k)))\right]^{1/\alpha}&\\ \leq \varrho(\iota), \qquad \forall\, \iota \geq \iota_1.&\end{aligned} \tag{2.2}$$

Proof. Suppose ϱ is converge to zero. From (B1) There exists ι^* so that $\varrho(\iota) > 0$ and $\varrho(\sigma_i(\iota)) > 0$, for all $\iota \geq \iota^*$ and $i = 1,2,\ldots,m$. Then by (1.10) we get

$$\begin{cases} \left(\acute{q}(\iota)\left(\varrho'(\iota)\right)^\alpha\right)' = -\sum_{i=1}^m q_i(\iota)\wp_i(\varrho(\sigma_i(\iota))) \leq 0, \\ \Delta\left(\acute{q}(\iota_k)\left(\varrho'(\iota_k)\right)^\alpha\right) = -\sum_{i=1}^m q_i(\iota_k)\wp_i(\varrho(\sigma_i(\iota_k))) \leq 0. \end{cases} \tag{2.3}$$

So, $\acute{q}(\iota)(\varrho'(\iota))^\alpha$ is non-increasing for $\iota \geq \iota^*$. Then $\acute{q}(\iota)(\varrho'(\iota))^\alpha > 0$. For contradiction let us consider

$$\acute{q}(\iota)\left(\varrho'(\iota)\right)^\alpha \leq 0,$$

at a certain time $\iota \geq \iota^*$. Applying $\sum q_i \neq 0$ in $[\tau,\infty)$, and that $\wp(\varrho) > 0$ for $\varrho > 0$, by (2.3), there exist $\iota_2 \geq \iota^*$ we get

$$\acute{q}(\iota)\left(\varrho'(\iota)\right)^\alpha \leq \acute{q}(\iota_2)\left(\varrho'(\iota_2)\right)^\alpha < 0, \qquad \forall\, \iota \geq \iota_2.$$

From (B5), we get

$$\varrho'(\iota) \le \left(\frac{\acute{q}(\iota_2)}{\acute{q}(\iota)}\right)^{1/\alpha} \varrho'(\iota_2), \qquad \forall \iota \ge \iota_2 .$$

Taking integration from ι_2 to ι, we get

$$\varrho(\iota) \le \varrho(\iota_2) + (\acute{q}(\iota_2))^{1/\alpha}\, \varrho'(\iota_2)\left(\mathring{R}(\iota) - \mathring{R}(\iota_2)\right) . \tag{2.4}$$

Applying (B4), in the right part goes to $-\infty$; so $\lim_{\iota\to\infty}\varrho(\iota) = -\infty$. That contradict to $\varrho(\iota) > 0$. Consequently

$$\acute{q}(\iota)\left(\varrho'(\iota)\right)^{\alpha} > 0, \qquad \forall\, \iota \ge \iota^{*}.$$

By $\acute{q}(\iota)(\varrho'(\iota))^{\alpha}$ is non-increasing, so we get

$$\varrho'(\iota) \le \left(\frac{\acute{q}(\iota_1)}{\acute{q}(\iota)}\right)^{1/\alpha} \varrho'(\iota_1), \qquad \forall\, \iota \ge \iota_1 .$$

Now we integrate the above inequality ι_1 to ι and applying ϱ is continuous we have

$$\varrho(\iota) \le \varrho(\iota_1) + (\acute{q}(\iota_1))^{1/\alpha}\, \varrho'(\iota_1)\left(\mathring{R}(\iota) - \mathring{R}(\iota_1)\right) .$$

As $\lim_{\iota\to\infty}\mathring{R}(\iota) = \infty$, then $\exists\, \delta > 0$ so that (2.1) satisfies. As

$$\acute{q}(\iota)\left(\varrho'(\iota)\right)^{\alpha} > 0,$$

and non-increasing, so the limit of

$$\lim_{\iota\to\infty}\acute{q}(\iota)(\varrho'(\iota))^{\alpha},$$

non negatively exists. Taking integration (1.10) from ι to τ, we have

$$\acute{q}(\tau)\left(\varrho'(\tau)\right)^{\alpha} - \acute{q}(\iota)\left(\varrho'(\iota)\right)^{\alpha} + \int_{\iota}^{\infty}\sum_{i=1}^{m} q_i(\eta)\,\wp_i\left(\varrho(\sigma_i(\eta))\right) d\eta + \sum_{\iota_k\ge\iota}\sum_{i=1}^{m} q_i(\iota_k)\,\wp_i(\varrho(\sigma_i(\iota_k))) = 0.$$

Calculating the limit when $\tau \to \infty$,

$$\acute{q}(\tau)\left(\varrho'(\tau)\right)^{\alpha} \ge \int_{\iota}^{\infty}\sum_{i=1}^{m} q_i(\zeta)\,\wp_i\left(\varrho(\sigma_i(\eta))\right) d\eta + \sum_{\iota\le\iota_k}\sum_{i=1}^{m} q_i(\iota_k)\,\wp_i(\varrho(\sigma_i(\iota_k))). \tag{2.5}$$

Therefore

$$\varrho'(\iota) \ge \left[\frac{1}{\acute{q}(\iota)}\left[\int_{\iota}^{\infty}\sum_{i=1}^{m} q_i(\zeta)\,\wp_i\left(\varrho(\sigma_i(\eta))\right) d\eta + \sum_{\iota\le\iota_k}\sum_{i=1}^{m} q_i(\iota_k)\,\wp_i(\varrho(\sigma_i(\iota_k)))\right]\right]^{1/\alpha}.$$

As $\varrho(\iota_1) > 0$, integrating this inequality derives

$$\varrho(\iota) \ge \int_{\iota_1}^{\zeta}\left[\frac{1}{\acute{q}(\zeta)}\int_{\eta}^{\infty}\sum_{i=1}^{m} q_i(\eta)\,\wp_i\big(\varrho(\sigma_i(\eta))\big)\, d\eta + \sum_{\iota\le\iota_k}\sum_{i=1}^{m} q_i(\iota_k)\,\wp_i(\varrho(\sigma_i(\iota_k)))\right]^{1/\alpha} d\zeta.$$

As the integrand is positive, increasing the lower limit from η to ι, and after that using the definition of $\mathring{R}(\iota)$, we have

$$\varrho(\iota) \geq \left(\mathring{R}(\iota) - \mathring{R}(\iota_1)\right)\left[\int_{\iota}^{\infty} \sum_{i=1}^{m} \mathrm{q}_i(\zeta)\, \wp_i\left(\varrho(\sigma_i(\zeta))\right)\, \mathrm{d}\zeta + \sum_{\iota \leq \iota_k} \sum_{i=1}^{m} \mathrm{q}_i(\iota_k)\, \wp_i(\varrho(\sigma_i(\iota_k)))\right]^{1/\alpha},$$

this yields (2.2). □

3. Main results

Now for next result we consider the a constant γ, which satisfy (B5) with $\gamma < \alpha$, so that

$$\frac{\wp_i(z)}{z^{\gamma}}, \tag{3.1}$$

is non-increasing for $z > 0\,(i = 1, 2, \ldots, m)$.

Example 3.1. *An instant* $\wp_i(z) = |z|^{\beta}\,\mathrm{sgn}(z)$*, with* $0 < \beta < \gamma$ *holds this condition.*

Theorem 3.2. *Letting* (B1)–(B5) *and* (3.1)*, each solution of* (1.10) *is oscillatory iff*

$$\left[\int_{0}^{\infty} \sum_{i=1}^{m} \mathrm{q}_i(\eta)\, \wp_i(\delta \mathring{R}(\sigma_i(\eta)))\, \mathrm{d}\eta + \sum_{i=1}^{\infty} \sum_{i=1}^{m} \mathrm{q}_i(\iota_k)\, \wp_i(\delta \mathring{R}(\sigma_i(\iota_k)))\right] = \infty, \qquad \forall\, \delta > 0\,.$$

Proof. We prove the sufficient part by contradiction. For the purpose of sufficient part prove, at the beginning let ϱ is eventually positive solution. As, Lemma 2.1 satisfies, and so that there exists $\iota_1 \geq \iota_0$, we have

$$\varrho(\iota) \geq \left(\mathring{R}(\iota) - \mathring{R}(\iota_1)\right) w^{1/\alpha}(\iota) \geq 0, \qquad \forall\, \iota \geq \iota_1\,, \tag{3.2}$$

where

$$w(\iota) = \int_{\iota}^{\infty} \sum_{i=1}^{m} \mathrm{q}_i(\eta)\, \wp_i\left(\varrho(\sigma_i(\eta))\right)\, \mathrm{d}\eta + \sum_{\iota_k \geq \iota} \sum_{i=1}^{m} \mathrm{q}_i(\iota_k)\, \wp_i(\varrho(\sigma_i(\iota_k))) \geq 0.$$

As $\lim_{\iota\to\infty} \mathring{R}(\iota) = \infty$, then there exists $\iota_2 \geq \iota_1$, so that $\mathring{R}(\iota) - \mathring{R}(\iota_1) \geq \frac{1}{2}\mathring{R}(\iota)$, for $\iota \geq \iota_2$. Then

$$\varrho(\iota) \geq \frac{1}{2}\mathring{R}(\iota)\, w^{1/\alpha}(\iota)\,. \tag{3.3}$$

Therefore

$$w'(\iota) = -\sum_{i=1}^{m} \mathrm{q}_i(\iota)\, \wp_i\left(\varrho(\sigma_i(\iota))\right),$$

$$\Delta w(\iota_k) = -\sum_{i=1}^{m} \mathrm{q}_i(\iota_k\, \wp_i\left(\varrho(\sigma_i(\iota_k))\right) \leq 0.$$

Therefore from the above we see that $w \geq 0$ and decreasing. As ϱ is positive, from (B3), $\wp_i\big(\varrho(\sigma_i(\iota))\big)$ is also positive, and by (B2), it gives us

$$\sum_{i=1}^{m} \mathrm{q}_i(\iota)\, \wp_i\big(\varrho(\sigma_i(\iota))\big) \neq 0,$$

in any $[\tau, \infty)$; thus $w' \neq 0$ and w never be a constant in any interval $[\tau, \infty)$. Thus $w(\iota)$ be also positive for $\iota \geq \iota_1$. Calculating derivative,

$$\left(w^{1-\gamma/\alpha}(\iota)\right)' = \left(1 - \frac{\gamma}{\alpha}\right) w^{-\gamma/\alpha}(\iota) w'(\iota)\,. \tag{3.4}$$

Integrating (3.4) from ι_2 to ι, and applying $w > 0$, we get

$$\begin{aligned} w^{1-\gamma/\alpha}(\iota_2) &\geq \left(1 - \frac{\gamma}{\alpha}\right)\left[- \int_{\iota_2}^{\iota} w^{-\gamma/\alpha}(\eta) w'(\eta)\,\mathrm{d}\eta - \sum_{\iota_2 \leq \iota_k} w^{-\gamma/\alpha}(\iota_k)\,\Delta w(\iota_k)\right] \\ &= \left(1 - \frac{\gamma}{\alpha}\right)\left[\int_{\iota_2}^{\iota} w^{-\gamma/\alpha}(\eta) \left(\sum_{i=1}^{m} \mathrm{q}_i(\eta)\, \wp_i\big(\varrho(\sigma_i(\eta))\big)\right) \mathrm{d}\eta \right. \\ &\qquad \left. + \sum_{\iota_k \leq \iota} w^{-\gamma/\alpha}(\iota_k) \sum_{i=1}^{m} \mathrm{q}_i(\iota_k)\, \wp_i(\varrho(\sigma_i(\iota_k)))\right]. \end{aligned} \tag{3.5}$$

Next we search a lower bound for the right part of (3.5), which is not dependent of the solution ϱ. From (B3), (2.1), (3.1) and (3.3), we get

$$\begin{aligned} \wp_i(\varrho(\iota)) &= \wp_i\big(\varrho(\iota)\big)\frac{\varrho^{\gamma}(\iota)}{\varrho^{\alpha}(\iota)} \\ &\geq \frac{\wp_i\left(\delta \mathring{R}(\iota)\right)}{\left(\delta \mathring{R}(\iota)\right)^{\gamma}} \varrho^{\gamma}(\iota) \\ &\geq \frac{\wp_i\left(\delta \mathring{R}(\iota)\right)}{\left(\delta \mathring{R}(\iota)\right)^{\gamma}} \left(\frac{\mathring{R}(\iota) w^{1/\alpha}(\iota)}{2}\right)^{\gamma} \\ &= \frac{\wp_i\left(\delta \mathring{R}(\iota)\right)}{(2\delta)^{\gamma}} w^{\gamma/\alpha}(\iota), \qquad \forall\, \iota \geq \iota_2\,. \end{aligned}$$

As w is non-increasing, $\frac{\gamma}{\alpha} > 0$, and $\sigma_i(\eta) < \eta$, it ensure us that

$$\begin{aligned} \wp_i\left(\varrho(\sigma_i(\eta))\right) &\geq \frac{\wp_i\left(\delta \mathring{R}(\sigma_i(\eta))\right)}{(2\delta)^{\gamma}} w^{\gamma/\alpha}(\sigma_i(\eta)) \\ &\geq \frac{\wp_i\left(\delta \mathring{R}(\sigma_i(\eta))\right)}{(2\delta)^{\gamma}} w^{\gamma/\alpha}(\eta)\,. \end{aligned} \tag{3.6}$$

Returning to (3.5), we get

$$w^{1-\gamma/\alpha}(\iota_2) \geq \frac{1 - \frac{\gamma}{\alpha}}{(2\delta)^{\gamma}}\left[\int_{\iota_2}^{\iota} \sum_{i=1}^{m} \mathrm{q}_i(\eta)\, \wp_i\left(\delta \mathring{R}(\sigma_i(\eta))\right) \mathrm{d}\eta\right.$$

$$+\sum_{\iota_k\leq\iota}\sum_{i=1}^{m}\mathrm{q}_i(\iota_k)\,\wp_i\left(\delta\mathring{R}(\sigma_i(\iota_k))\right)\Bigg]. \tag{3.7}$$

As $1-\frac{\Upsilon}{\alpha}$ is positive, from (3.2) the right part goes to infinity as $\iota\to\infty$. It is a contradiction (3.7) and completes the sufficient part of the eventually positive solutions. Now we find solution for negative ϱ, for that we set the variables $\acute{\varrho}=-\varrho$ and

$$\acute{\wp}_i(\acute{\varrho})=-\wp_i(\acute{\varrho}).$$

Thus (1.10) converted to positive solution of $\acute{\varrho}$ and $\acute{\wp}_i$ in exchange with $\wp_i$. Write after $\acute{\wp}_i$ satisfies (B3) and (3.1) then using the method for the solution $\acute{\varrho}$ from the above. In the subsequent part we prove the necessary condition by contrapositive thought. Whenever (3.2) does not satisfy we search an eventually positive solution which diverge to zero. Then for positive δ and for each positive ϵ there exists $\iota_1\geq\iota_0$ if (3.2) does not satisfy so that

$$\int_{\eta}^{\infty}\sum_{i=1}^{m}\mathrm{q}_i(\eta)\,\wp_i\left(\delta\mathring{R}(\sigma_i(\eta))\right)\,\mathrm{d}\eta+\sum_{\iota_k\geq S}\sum_{i=1}^{m}\mathrm{q}_i(\iota_k)\wp_i\left(\delta\mathring{R}(\sigma_i(\iota_k))\right)\leq\epsilon, \tag{3.8}$$

for all $\eta\geq\iota_1$. Here ι_1 rely on δ. Now here we are assume there exist set of continuous function

$$\Upsilon=\left\{\varrho\in C([0,\infty))\;:\;\left(\frac{\epsilon}{2}\right)^{1/\alpha}\left(\mathring{R}(\iota)-\mathring{R}(\iota_1)\right)\leq\varrho(\iota)\leq\epsilon^{1/\alpha}\left(\mathring{R}(\iota)-\mathring{R}(\iota_1)\right),\;\iota\geq\iota_1\right\}.$$

Next, we define an operator O on Υ by

$$(O\varrho)(\iota)=\begin{cases}0, & \iota\leq\iota_1,\\ \displaystyle\int_{\iota_1}^{\iota}\left[\frac{1}{\acute{\mathrm{q}}(\zeta)}\left[\frac{\epsilon}{2}+\int_{\zeta}^{\infty}\sum_{i=1}^{m}\mathrm{q}_i(\eta)\,\wp_i(\varrho(\sigma_i(\eta)))\,\mathrm{d}\eta\right.\right. & \\ \displaystyle\qquad\left.\left.+\sum_{\iota_k\geq S}\sum_{i=1}^{m}\mathrm{q}_i(\iota_k)\,\wp_i\left(\varrho\left(\delta\mathring{R}(\sigma_i(\iota_k))\right)\right)\right]\right]^{1/\alpha}\mathrm{d}\zeta, & \iota>\iota_1.\end{cases}$$

Here we see that when ϱ is continuous, $O\varrho$ is also continuous on $[0,\infty)$. If $O\varrho=\varrho$, i.e., ϱ is a fixed point of $O(\varrho)$ is a solution of (1.10). Initially we calculate $(O\varrho)(\iota)$ from below. Since $\varrho\in\Upsilon$, we get

$$0\leq\epsilon^{1/\alpha}\left(\mathring{R}(\iota)-\mathring{R}(\iota_1)\right)\leq\varrho(\iota).$$

By (B3), we get $0\leq\wp_i(\varrho(\sigma_i(\eta)))$ and by (B2) we get

$$\begin{aligned}(O\varrho)(\iota)&\geq 0+\int_{\iota_1}^{\iota}\left[\frac{1}{\acute{\mathrm{q}}(\zeta)}\left[\frac{\epsilon}{2}+0+0\right]\right]^{1/\alpha}\mathrm{d}\zeta\\&=\left(\frac{\epsilon}{2}\right)^{1/\alpha}\left(\mathring{R}(\iota)-\mathring{R}(\iota_1)\right).\end{aligned}$$

Then we calculate $(O\varrho)(\iota)$ from above. For ϱ in Υ, from (B2) and (B3), we get

$$\wp_i(\varrho(\sigma_i(\eta)))\leq\wp_i\left(\delta\mathring{R}(\sigma_i(\eta))\right).$$

From (3.8),

$$(O\varrho)(\iota) \le \int_{\iota_1}^{\iota} \left[\frac{1}{\acute{q}(\eta)} \left[\frac{\epsilon}{2} + \int_{\eta}^{\infty} \sum_{i=1}^{m} q_i(\zeta)\, \wp_i\left(\delta \mathring{R}(\sigma_i(\zeta))\right) d\zeta + \sum_{\iota_k \ge S} \sum_{i=1}^{m} q_i(\iota_k)\, \wp_i\left(\delta \mathring{R}(\sigma_i(\iota_k))\right) \right]\right]^{1/\alpha} d\eta$$
$$\le \epsilon^{1/\alpha} \left(\mathring{R}(\iota) - \mathring{R}(\iota_1) \right).$$

Thus, O maps Υ to Υ. Later on we will look for O in Υ. We are going to explain a sequence of function Υ by the iterative formula

$$\begin{aligned} & z_0(\iota) = \bar{0}, && \iota \ge \iota_0, \\ & z_1(\iota) = (Oz_0)(\iota) = \begin{cases} \bar{0}, & \iota < \iota_1, \\ \epsilon^{1/\alpha}\left(\mathring{R}(\iota) - \mathring{R}(\iota_1)\right), & \iota \ge \iota_1, \end{cases} \\ & z_{n+1}(\iota) = (Oz_n)(\iota), && n \ge 1,\ \iota \ge \iota_1 . \end{aligned}$$

Now when we fixed ι, we can get $z_1(\iota) \ge z_0(\iota)$. Applying that $\wp$ is non-decreasing and also using induction formula of mathematics, we can formulate that $z_{n+1}(\iota) \ge z_n(\iota)$. Thus, $\{z_n\}$ convergent sequence which converges to z^* pointwise. Here we find the fixed point z^* for the operator O in Υ applying dominated convergence theorem of Lebesgue. From consideration (3.8) shows that the solution is eventually positive i.e., does not converge to zero. Hence the proof of the theorem is complete. □

For subsequent theorem, let us consider there exists a continuously differentiable function σ_0 satisfying

$$0 < \sigma_0(\iota) \le \sigma_i(\iota), \qquad \exists\, \gamma > 0 : \gamma \le \sigma_0'(\iota)\ (\iota \ge \iota_0,\ i = 1, 2, \ldots, m). \tag{3.9}$$

Also, we suppose a constant γ, satisfy first part of (B5), and $\alpha < \gamma$, such that

$$\frac{\wp_i(z)}{z^{\gamma}}, \tag{3.10}$$

is non-decreasing for $z > 0\ (i = 1, 2, \ldots, m)$. The Example 3.1, $\wp_i(z) = |z|^{\beta}\,\mathrm{sgn}(z)$ with $\gamma < \beta$ holds this condition.

Theorem 3.3. *Under assumptions* (B1)–(B4)*,* (3.9)*,* (3.10)*, and* $\acute{q}(\iota)$ *is non-decreasing, every solution of* (1.10) *is converges to zero iff*

$$\int_{\iota_1}^{\infty} \left[\frac{1}{\acute{q}(\zeta)} \left[\int_{\zeta}^{\infty} \sum_{i=1}^{m} q_i(\eta)\, d\eta + \sum_{\iota_k \ge \iota} \sum_{i=1}^{m} q_i(\iota_k) \right]\right]^{1/\alpha} d\zeta = \infty . \tag{3.11}$$

Proof. Our aim to prove sufficient part by contradiction method. First we consider that the solution ϱ does not converges to zero. Applying similar logic same as in Lemma 2.1, we get $\iota_1 \ge \iota_0$ and $\varrho(\sigma_i(\iota))$ is positive and

$$\acute{q}(\iota)\left(\varrho'(\iota)\right)^{\alpha} > 0,$$

and non-increasing. Since $\acute{q}(\iota) > 0$ so $\varrho(\iota)$ is increasing for $\iota \geq \iota_1$. From (B3), $\varrho(\iota) \geq \varrho(\iota_1)$ and (3.10), we have

$$\wp_i(\varrho(\iota)) \geq \frac{\wp_i(\varrho(\iota))}{\varrho^\gamma(\iota)}\varrho^\gamma(\iota) \geq \frac{\wp_i(\varrho(\iota_1))}{\varrho^\gamma(\iota_1)}\varrho^\gamma(\iota). \tag{3.12}$$

From (B1) we can find $\iota_2 \geq \iota_1$ and also $\sigma_i(\iota) \geq \iota_1$ when $\iota \geq \iota_2$. Therefore

$$\wp_i(\varrho(\sigma_i(\iota))) \geq \frac{\wp_i(\varrho(\iota_1))}{\varrho^\gamma(\iota_1)}\varrho^\gamma(\sigma_i(\iota)), \qquad \forall\, \iota \geq \iota_2. \tag{3.13}$$

Using this inequality, (2.5), we have $\sigma_i(\iota) \geq \sigma_0(\iota)$ which shows that σ is increasing, and ϱ is also so, thus

$$\acute{q}(\iota)\left(\varrho'(\iota)\right)^\alpha \geq \frac{\varrho^\gamma(\sigma_0(\iota))}{\varrho^\gamma(\iota_1)}\left[\int_\iota^\infty \sum_{i=1}^m q_i(\eta)\,\wp_i(\varrho(\iota_1))\,d\eta + \sum_{\iota_k \geq \iota}\sum_{i=1}^m q_i(\iota_k)\,\wp_i(\varrho(\iota_1))\right],$$

for $\iota \geq \iota_2$. From $\acute{q}(\iota)\left(\varrho'(\iota)\right)^\alpha$ being non-increasing and $\sigma_0(\iota) \leq \iota$, we get

$$\acute{q}(\sigma_0(\iota))\left(\varrho'(\sigma_0(\iota))\right)^\alpha \geq \acute{q}(\iota)\left(\varrho'(\iota)\right)^\alpha.$$

We apply this in the left part of the above inequality. Additionally, dividing by $\acute{q}(\sigma_0(\iota)) > 0$, uplift right and left part to $\frac{1}{\alpha}$ index, and divided by $\varrho^{\beta/\gamma}(\sigma_0(\iota)) > 0$, we get

$$\frac{\varrho'(\sigma_0(\iota))}{\varrho^{\gamma/\alpha}(\sigma_0(\iota))} \geq \left[\frac{1}{\acute{q}(\sigma_0(\iota))\varrho^\gamma(\iota_1)}\left[\int_\iota^\infty \sum_{i=1}^m q_i(\eta)\,\wp_i(\varrho(\iota_1))\,d\eta + \sum_{\iota_k \geq \iota}\sum_{i=1}^m q_i(\iota_k)\,\wp_i(\varrho(\iota_1))\right]\right]^{1/\alpha},$$

for $\iota \geq \iota_2$. Multiply by $\sigma_0'(\iota)/\beta \geq 1$ left part, and taking integration from ι_1 to ι,

$$\frac{1}{\beta}\int_{\iota_1}^\iota \frac{\varrho'(\sigma_0(\eta))\sigma_0'(\eta)}{\varrho^{\gamma/\alpha}(\sigma_0(\eta))}\,d\eta \geq \frac{1}{\varrho^{\gamma/\alpha}(\iota_1)}\left[\int_{\iota_1}^\iota\left[\frac{1}{\acute{q}(\sigma_0(\eta))}\int_\eta^\infty \sum_{i=1}^m q_i(\zeta)\,\wp_i(\varrho(\iota_1))\,d\zeta + \sum_{s\leq\iota_k}\sum_{i=1}^m q_i(\iota_k)\,\wp_i(\varrho(\iota_1))\right]\right]^{1/\alpha} d\eta. \tag{3.14}$$

As $\alpha < \gamma$, taking integration left part of above inequality, we finally reach

$$\frac{1}{\beta(1-\gamma/\alpha)}\left[\varrho^{1-\gamma/\alpha}(\sigma_0(\eta))\right]_{s=\iota_2}^\iota \leq \frac{1}{\gamma(\gamma/\alpha-1)}\varrho^{1-\gamma/\alpha}(\sigma_0(\iota_2)).$$

Our main task is to show that (3.11) right part going to infinity as ι tends to infinity for that here apply

$$\min_{1\leq i\leq m}\wp_i(\varrho(\iota_1)) > 0,$$

and $\acute{q}(\sigma_0(s)) \leq \acute{q}(s)$, (3.14) right part. For eventually negative solutions, we use the same change of variables as in Theorem 3.2, and proceed as above. To prove the necessary part we assume that (3.11) does not hold, and obtain an eventually positive solution that does not converge to zero. If (3.11) does not hold, then for each $\epsilon > 0$ there exists $\iota_1 \geq \iota_0$ such that

$$\int_{\iota_1}^{\infty}\left[\frac{1}{\acute{q}(\eta)}\int_{\eta}^{\infty}\sum_{i=1}^{m} q_i(\zeta)\,d\zeta+\sum_{\iota_k\geq\iota}\sum_{i=1}^{m} q_i(\iota_k)\right]^{1/\alpha} d\eta < \frac{\epsilon}{2}\left(\wp_i(\epsilon)\right)^{1/\alpha}, \qquad \forall\,\iota \geq \iota_1\,. \tag{3.15}$$

Construct the continuous functions

$$\Upsilon=\left\{\varrho\in C([0,\infty))\,:\,\frac{\epsilon}{2}\leq\varrho(\iota)\leq\epsilon \text{ when } \iota\geq\iota_1\right\}. \tag{3.16}$$

Now we define the operator O,

$$(O\varrho)(\iota)=\begin{cases} 0, & \iota\leq\iota_1,\\ \dfrac{\epsilon}{2}+\Bigg[\displaystyle\int_{\iota_1}^{\iota}\frac{1}{\acute{q}(\zeta)}\Bigg[\int_{\zeta}^{\infty}\sum_{i=1}^{m} q_i(\eta)\,\wp_i(\varrho(\sigma_i(\eta)))\,d\eta & \\ \qquad +\displaystyle\sum_{\iota_k\geq S}\sum_{i=1}^{m} q_i(\iota_k)\,\wp_i(\varrho(\sigma_i(\iota_k)))\Bigg]\Bigg]^{1/\alpha} d\zeta, & \iota>\iota_1\,. \end{cases}$$

Note that if ϱ is continuous, for $\iota = \iota_1$, $O(\varrho)$ is a continuous function. Also as ϱ is a fixed point i.e., $O\varrho = \varrho$ it give us that ϱ is a solution of (1.10). Our main criteria to calculate $(O\varrho)(\iota)$ from both equations for first part let $\varrho \in \Upsilon$. By $0 < \frac{\epsilon}{2} \leq \varrho$, we have

$$(O\varrho)(\iota)\geq\frac{\epsilon}{2}+0+0,$$

on $[\iota_1, \infty)$. For the next part let $\varrho \in \Upsilon$. Then $\varrho \leq \epsilon$ and from (3.15), we have

$$\begin{aligned}(O\varrho)(\iota) &\leq \frac{\epsilon}{2}+\left(\wp_i(\epsilon)\right)^{1/\alpha}\int_{\iota_1}^{\iota}\left[\frac{1}{\acute{q}(\eta)}\int_{\eta}^{\infty}\sum_{i=1}^{m} q_i(\zeta)\,d\zeta+\sum_{\iota_k\geq S}\sum_{i=1}^{m} q_i(\iota_k)\right]^{1/\alpha} d\eta\\ &\leq\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.\end{aligned}$$

Hence O is a rules from Υ to Υ. For finding a fixed point of O we can construct a sequence of function by recursive rules

$$\begin{aligned} z_0(\iota) &= \bar{0}, & \iota &\geq \iota_0,\\ z_1(\iota) &= (Oz_0)(\iota) = 1, & \iota &\geq \iota_1,\\ z_{n+1}(\iota) &= (Oz_n)(\iota), & \iota &\geq \iota_1,\ n\geq 1\,. \end{aligned}$$

Now when we fixed ι, thus $z_1(\iota) \geq z_0(\iota)$. Applying $\wp$ is non-decreasing and also induction formula of mathematics, we can establish $z_{n+1}(\iota) \geq z_n(\iota)$ so that $\{z_n\}$ convergent sequence which converges to z in Υ pointwise. Hence, z be a positive solution of (1.10). This completes the proof. □

4. Conclusions and future scope

In this section, we are going to conclude the paper by providing two examples to show the effectiveness and feasibility of the main results.

Example 4.1. *Consider the* $\mathbb{IS}$

$$\begin{cases} \left(e^{-\iota}\left(\varrho'(\iota)\right)^{11/3}\right)' + \frac{1}{\iota+1}(\varrho(\iota-2))^{1/3} + \frac{1}{\iota+2}(\varrho(\iota-1))^{5/3} = 0, \\ \left(e^{-k}\left(\varrho'(k)\right)^{11/3}\right)' + \frac{1}{\iota+4}(\varrho(k-2))^{1/3} + \frac{1}{\iota+5}(\varrho(k-1))^{5/3} = 0\,. \end{cases} \tag{4.1}$$

Comparing with said systems we get $\alpha = \frac{11}{3}$, $\acute{\mathrm{q}}(\iota) = e^{-\iota}$, $\sigma_1(\iota) = \iota - 2$, $\sigma_2(\iota) = \iota - 1$, *from* (1.11)

$$\mathring{R}(\iota) = \int_0^{\iota} \left(\acute{\mathrm{q}}(\eta)\right)^{-1/\alpha}\, \mathrm{d}\eta = \int_0^{\iota} e^{-3\eta/11}\, \mathrm{d}\eta = \frac{-11}{3}\left(e^{-3\iota/11} - 1\right), \tag{4.2}$$

$\wp_1(\varrho) = \varrho^{1/3}$ *and* $\wp_2(\varrho) = \varrho^{5/3}$. *For* $\beta = \frac{7}{3}$, *we have*

$$0 < \max\left\{\gamma_1, \gamma_2\right\} = \max\left\{\frac{1}{3}, \frac{5}{3}\right\} = \frac{5}{3} < \frac{7}{3} = \beta < \frac{11}{3} = \alpha,$$

and

$$\frac{\wp_1(\varrho)}{\varrho^{\beta}} = \frac{\varrho^{1/3}}{\varrho^{7/3}} = \varrho^{-2}, \qquad \frac{\wp_2(\varrho)}{\varrho^{\beta}} = \frac{\varrho^{5/3}}{\varrho^{7/3}} = \varrho^{-2/3},$$

which both are non increasing. To verify (3.2), *by employing* (4.2), *we have*

$$\begin{aligned} \Bigg[\int_0^{\infty} \sum_{i=1}^{m} \mathrm{q}_i(\eta)\, \wp_i\left(\delta \mathring{R}(\sigma_i(\eta))\right)\, \mathrm{d}\eta & \\ + \sum_{k=1}^{\infty}\sum_{i=1}^{m} \mathrm{q}_i(\iota_k)\wp_i\left(\delta \mathring{R}(\sigma_i(\iota_k))\right) \Bigg] & \\ \geq \int_0^{\infty} \sum_{i=1}^{m} \mathrm{q}_i(\eta)\wp_i\left(\delta \mathring{R}(\sigma_i(\eta))\right)\, \mathrm{d}\eta & \\ \geq \int_0^{\infty} \mathrm{q}_1(\eta)\, \wp_1\left(\delta \mathring{R}(\sigma_1(\eta))\right)\, \mathrm{d}\eta & \\ = \int_0^{\infty} \frac{1}{\eta+1}\left(\delta\, \frac{11}{3}\left(1 - e^{-3(\eta-2)/11}\right)\right)^{1/3} \mathrm{d}\eta = \infty, \qquad \forall\, \delta > 0, & \end{aligned} \tag{4.3}$$

as integrand goes to $+\infty$ *since* η *to positive infinity.*

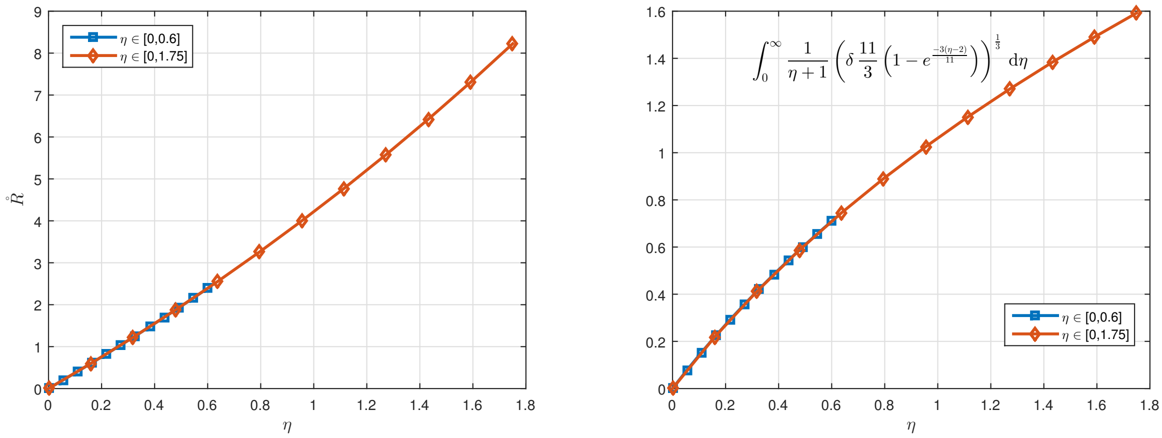
One can see these results in Tables 1 and 2. We can see graphical representation of the inequality (4.3) *for* $\eta \in [0,\, 0.6]$ *and* $\eta \in [0,\, 1.75]$ *in Figure 1 (a) and (b), respectively, for* $\eta = 1.5$ *and* $\delta \in \{0.5, 2.5\}$ *in Figure 2. Algorithmes 1 and 2 can be used for this purpose.*

Table 1. Numerical results of the integral inequality (4.3) of $\mathbb{IS}$ for $\eta \in \{0.6, 1.75\}$ in Example 4.1.

	$\eta \in [0, 0.6]$			$\eta \in [0, 1.75]$		
n	η	$\mathring{R}$	$\mathbb{IS}$ (4.3)	η	$\mathring{R}$	$\mathbb{IS}$ (4.3)
1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
2	0.0545	0.2015	0.0784	0.1591	0.5962	0.2188
3	0.1091	0.4060	0.1531	0.3182	1.2188	0.4122
4	0.1636	0.6136	0.2247	0.4773	1.8690	0.5861
5	0.2182	0.8243	0.2933	0.6364	2.5481	0.7444
6	0.2727	1.0381	0.3592	0.7955	3.2572	0.8901
7	0.3273	1.2552	0.4226	0.9545	3.9978	1.0254
8	0.3818	1.4755	0.4839	1.1136	4.7713	1.1519
9	0.4364	1.6991	0.5430	1.2727	5.5790	1.2709
10	0.4909	1.9261	0.6002	1.4318	6.4226	1.3835
11	0.5455	2.1564	0.6556	1.5909	7.3036	1.4905
12	0.6000	2.3902	0.7094	1.7500	8.2236	1.5926

Table 2. Numerical results of the integral inequality (4.3) of $\mathbb{IS}$ for $\eta \in [0, 1.5]$ and $\delta \in \{0.5, 2.5\}$ in Example 4.1.

	$\delta = 0.5$		$\delta = 2.5$	
η	$\mathring{R}$	$\mathbb{IS}$ (4.3)	$\mathring{R}$	$\mathbb{IS}$ (4.3)
0.0000	0.0000	0.0000	0.0000	0.0000
0.1364	0.5094	0.1312	0.5094	0.2244
0.2727	1.0381	0.2490	1.0381	0.4259
0.4091	1.5869	0.3562	1.5869	0.6090
0.5455	2.1564	0.4546	2.1564	0.7773
0.6818	2.7475	0.5458	2.7475	0.9333
0.8182	3.3611	0.6310	3.3611	1.0790
0.9545	3.9978	0.7110	3.9978	1.2157
1.0909	4.6587	0.7865	4.6587	1.3449
1.2273	5.3447	0.8581	5.3447	1.4674
1.3636	6.0566	0.9263	6.0566	1.5840
1.5000	6.7955	0.9915	6.7955	1.6955



(a) $\eta \in [0, 0.6]$ **(b)** $\eta \in [0, 1.75]$

Figure 1. 2D plot numerical results of $\mathring{R}$ and the integral inequality of $\mathbb{IS}$ (4.3) for $\eta \in \{0.6, 1.75\}$ in Example 4.1.

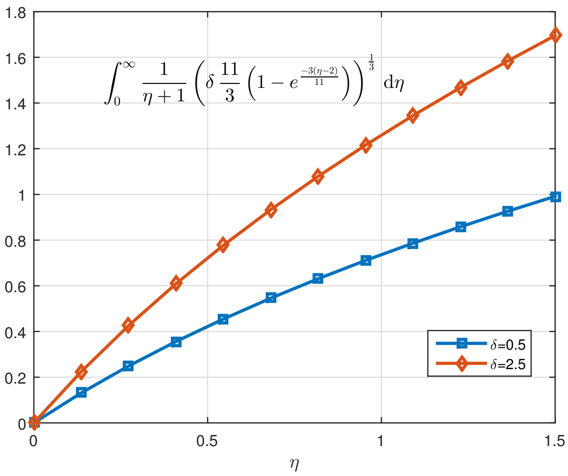


Figure 2. Graphical representation of of the integral inequality of $\mathbb{IS}$ (4.3) for $\eta = 1.5$ and $\delta \in \{0.5,\ 2.5\}$ in Example 4.1.

Therefore, all the postulates of Theorem 3.2 hold true. Hence, by Theorem 3.2 all solution of (4.1) *is oscillatory.*

Example 4.2. *Let us assume nonlinear* $\mathbb{IS}$

$$\begin{cases} \left(\left(\varrho'(\iota)\right)^{1/3}\right)' + \iota(\varrho(\iota-2))^{7/3} + (\iota+1)(\varrho(\iota-1))^{11/3} = 0 \\ \left(\left(\varrho'(3^k)\right)^{1/3}\right)' + (\iota+3)\varrho(3^k-2)^{7/3} + (\iota+4)\left(\varrho(3^k-1)\right)^{11/3} = 0\,. \end{cases} \tag{4.4}$$

Now comparing with given system we have $\alpha = \frac{1}{3}$, $\acute{q}(\iota) = 1$, $\sigma_1(\iota) = \iota-2$, $\sigma_2(\iota) = \iota-1$, *from* (1.11)

$$\mathring{R}(\iota) = \int_0^\iota \left(\acute{q}(\eta)\right)^{-1/\alpha}\,\mathrm{d}\eta = \int_0^\iota \mathrm{d}\eta = \iota, \tag{4.5}$$

$\wp_1(\varrho) = \varrho^{7/3}$ *and* $\wp_2(\varrho) = \varrho^{11/3}$. *For* $\gamma = \frac{5}{3}$, *thus*

$$\min\left\{\gamma_1, \gamma_2\right\} = \left\{\frac{7}{3}, \frac{11}{3}\right\} = \frac{7}{3} > \frac{5}{3} = \gamma > \frac{1}{3} = \alpha,$$

also

$$\frac{\wp_1(\varrho)}{\varrho^{\gamma}} = \frac{\varrho^{7/3}}{\varrho^{5/3}} = \varrho^{2/3}, \qquad \frac{\wp_2(\varrho)}{\varrho^{\gamma}} = \frac{\varrho^{11/3}}{\varrho^{5/3}} = \varrho^{2},$$

two functions are increasing functions. To verify (3.11) *we get*

$$\begin{aligned}
\int_{\iota_0}^{\infty} \left[\frac{1}{\acute{q}(\zeta)} \int_{S}^{\infty} \sum_{i=1}^{m} q_i(\eta)\, d\eta + \sum_{\iota_k \geq S} \sum_{i=1}^{m} q_i(\iota_k)\right]^{1/\alpha} d\zeta
&\geq \int_{\iota_0}^{\infty} \left[\frac{1}{\acute{q}(\zeta)} \int_{\zeta}^{\infty} \sum_{i=1}^{m} q_i(\eta)\, d\eta\right]^{1/\alpha} d\zeta \\
&\geq \int_{\iota_0}^{\infty} \left[\frac{1}{\acute{q}(\zeta)} \int_{\zeta}^{\infty} q_1(\eta)\, d\eta\right]^{1/\alpha} d\zeta \\
&\geq \int_{2}^{\infty} \left[\int_{\zeta}^{\infty} \eta\, d\eta\right]^{3} d\zeta = \infty.
\end{aligned}$$

Therefore, all postulate of Theorem 3.3 hold true. Hence, by Theorem 3.3, all solution of (4.4) *is oscillatory or converges to zero.*

5. Conclusions

After concluding the paper and introducing [16, 17, 22, 23, 25, 29, 30, 33–35], we have an open question that “Can we find the necessary and sufficient conditions for the oscillatory solution of the second order neutral impulsive delay differential system with several delays and arguments”?

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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Appendix

Algorithm 1: MATLAB lines for calculation all variables in Example 4.1 when η changes and the δ is constant.

```
clear;
format long;
syms v e;
q=[0.6 1.75];
[xq yq]=size(q);
upalpha=11/3; Δ=3/2;
acutemathrmq=exp(-v);
mathrmq_1=1/(v+1); mathrmq_2=1/(v+2);
wp_1=v^(1/3); wp_2=v^(5/3);
sigma_1=v-2; sigma_2=v-1;
column=1;
for s=1:yq
    eta=q(s);
    h=eta/11;
    t=0;
    n=1;
    while t≤eta+0.05
        paramsmatrix(n, column)=n;
        paramsmatrix(n, column+1)=t;
        I1=int((acutemathrmq)^(-1/upalpha), v);
        paramsmatrix(n, column+2)=int(subs(I1,{v},{e}), 0,t);
        I2=Δ*subs(I1,{v},sigma_1);
        I3=mathrmq_1 * subs(wp_1,{v},I2);
        I4=int(subs(I3,{v},{e}), e, 0, t);
        paramsmatrix(n, column+3)=I4;
        t=t+h;
        n=n+1;
    end;
    column=column+4;
end;
```

Algorithm 2: MATLAB lines for calculation all variables in Example 4.1 whenever η is constant and when the δ changes.

```
clear;
format long;
syms v e;
Δ=[0.5 2.5];
[xΔ yΔ]=size(Δ);
upalpha=11/3;
eta=3/2;
acutemathrmq=exp(-v);
mathrmq_1=1/(v+1); mathrmq_2=1/(v+2);
wp_1=v^(1/3); wp_2=v^(5/3);
sigma_1=v-2; sigma_2=v-1;
column=1;
for s=1:yΔ
    h=eta/11;
    t=0;
    n=1;
    while t≤eta+0.05
        paramsmatrix(n, column)=n;
        paramsmatrix(n, column+1)=t;
        I1=int((acutemathrmq)^(-1/upalpha), v);
        paramsmatrix(n, column+2)=int(subs(I1,{v},{e}), 0,t);
        I2=Δ(s)*subs(I1,{v},sigma_1);
        I3=mathrmq_1 * subs(wp_1,{v},I2);
        I4=int(subs(I3,{v},{e}), e, 0, t);
        paramsmatrix(n, column+3)=I4;
        t=t+h;
        n=n+1;
    end;
    column=column+4;
end;
```