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Research article

Orbital stability of periodic traveling waves to some coupled BBM equations

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Abstract: In this work, we show some results concerning the orbital stability of dnoidal wave solutions to some Benjamin-Bona-Mahony equations (BBM equations henceforth). First, by the standard argument, we prove the existence of a smooth curve of positive traveling wave solutions of dnoidal type. Then, we show that this type of solutions are orbitally stable by perturbations with the same period L. The major tools to obtain these results are the Grillaks, Shatah and Strauss' general theory in the periodic case. The results in the present paper extend some previous stability results for the BBM equations.

Keywords: coupled BBM equations; periodic traveling waves; orbital stability
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1. Introduction

The study of existence and stability or instability of periodic traveling waves of dispersive type partial differential equations has increased significantly in the last decade. Additionally, many researchers are interested in solving a rich variety of new mathematical problems due to physical importance related to them. Because properties of traveling waves are of fundamental importance in the development of a broad range of disturbances, the study of orbital stability for periodic traveling waves is of interest for many researchers in a nonlinear analysis. We recall that the first work about the existence and orbital stability of periodic traveling waves was determined by Benjamin [1], who studied the periodic waves of *cnoidal* type for the KdV equation. A complete study was carried out by Pava et al. [2] in 2006. The main idea is the classical theory form Grillaks, Shatah and Strauss [3] to the periodic context. Particularly, the spectral information about the related linear operator is fundamental for the study of the stability and is one of the main difficulties in applying the classical theory. By using the Fourier techniques associated to positive linear operators, Angulo and Natali [4] introduced a new approach for analysing the spectral properties for the general linear operator. This approach can be possibly used to some non-local linear operators. In addition, based on tools from ODES and Evans function methods, and under some restriction on the manifold of initial data of perturbations, some other criterium for obtaining the required spectral information was given by Johnson [5]. There have been many research works on periodic traveling wave solutions [6–15], since study of the existent nonlinear stability/instability for periodic traveling waves is valuable. On the other hand, some numerical solutions of the solitary wave are also investigated by many authors [16–18]. With respect to soliton resolution conjecture, some interesting work in deriving the solutions of Wadati-Konno-Ichikawa equation and complex short pulse equation has been done, and the long-time asymptotic behavior of the solutions of these equations has been solved. The soliton resolution conjecture and the asymptotic stability of solutions of these equations have been proved in [19–21].

In the present paper, we are interested in a special case of the following coupled BBM equations

$$\begin{cases} \eta_t + u_x + (u^{p+1}\eta^p)_x - \eta_{xxt} = 0, \\ u_t + \eta_x + (u^p\eta^{p+1})_x - u_{xxt} = 0, \end{cases} \tag{1.1}$$

where p is a positive integer and η, u are real-valued functions. It is well known that the BBM equation has been derived as a model to describe water waves in the long-wave regime. The modified BBM equation describes wave propagation in a one-dimensional nonlinear lattice. Thus, the generalization considered here is not only of mathematical interest. Among (1.1), *sech* type solitary waves have been studied by Cui [22]. Furthermore, the orbital stability/instability of the waves are obtained. Now we focus on periodic waves for $p = 1$ in (1.1). Namely, the equations are

$$\begin{cases} \eta_t + u_x + (u^2\eta)_x - \eta_{xxt} = 0, \\ u_t + \eta_x + (u\eta^2)_x - u_{xxt} = 0. \end{cases} \tag{1.2}$$

The solutions considered here are the general form $(\eta(x,t) = \varphi(x-ct), u(x,t) = \psi(x-ct))$. $\varphi, \psi : R \to R$ are smooth periodic functions with the same arbitrary fundamental period $L > 0$ and the wave speed $c > 1$. Substituting this form of solutions in (1.2) and integrating with respect to $\xi(\xi = x - ct)$ once, and assuming the integration constant equals to zero, we obtain that φ, ψ have to satisfy the following ordinary differential system

$$\begin{cases} -c\varphi + \psi + \varphi\psi^2 + c\varphi'' = 0, \\ -c\psi + \varphi + \varphi^2\psi + c\psi'' = 0. \end{cases} \tag{1.3}$$

We consider the solutions (φ, ψ) as the critical points for the functional $\mathcal{E}(\vec{u}) + cQ(\vec{u})$, where $\mathcal{E}(\vec{u})$ and $Q(\vec{u})$ represent conserved quantities with respect to (1.2), namely

$$\begin{aligned} \mathcal{E}(\vec{u}) &= \mathcal{E}(\eta, u) = -\int_0^L (u\eta + \tfrac{1}{2}u^2\eta^2)dx, \\ Q(\vec{u}) &= Q(\eta, u) = \tfrac{1}{2}\int_0^L (\eta^2 + u^2 + \eta_x^2 + u_x^2)dx, \end{aligned} \tag{1.4}$$

where $\vec{u} = (\eta, u)$.

For (1.3), the author in [22] obtained the solitary waves

$$\begin{aligned} &\varphi_c(x-ct) = \psi_c(x-ct) = \pm\sqrt{2(c-1)}\,sech\sqrt{\tfrac{c-1}{c}}(x-ct), \quad \text{or} \\ &-\varphi_c(x-ct) = \psi_c(x-ct) = \pm\sqrt{-2(c+1)}\,sech\sqrt{\tfrac{c+1}{c}}(x-ct), \end{aligned} \tag{1.5}$$

and proved the stability for $c > 1$ and $c < -1$, respectively. In this work, we establish the existence of a smooth curve of *dnoidal* type solutions of (1.3) with minimal periodic $L > \sqrt{2}\pi$ and $c > \frac{L^2}{L^2-2\pi^2}$, which enrich and extend the results obtained in [22]. As far as we know, most of the results on orbital stability of periodic waves for the BBM equations are about single equations. For more general physical situations, such as the waves not assumed to be uni-directional, we extend the results for single equation to systems of equations. The main motivation of the present work is to extend the existence/stability results from *sech* type solitary waves to periodic *dnoidal* type solutions.

This paper is organized as follows. In Section 2, we study the existence of a smooth curve of *dnoidal* wave solutions for system (1.2). Section 3 is devoted to the spectral analysis of some certain self-adjoint operators necessary to obtain our stability result. In Section 4, we show stability result of the *dnoidal* waves solutions for (1.2).

2. Existence of dnoidal wave solutions for Eq (1.2)

In this section, we first show the existence of positive and periodic solutions with a fixed period $L > 0$ for some parameter interval I. Motivated by [22], we suppose $\varphi = \psi$, hence (1.3) reduces to the following equation

$$c\varphi'' + \varphi^3 - (c-1)\varphi = 0. \tag{2.1}$$

Multiplying (2.1) by φ' and integrating once, we get that φ must satisfy

$$(\varphi')^2 = \frac{1}{2c}[-\varphi^4 + 2(c-1)\varphi^2 + 4A_\varphi], \tag{2.2}$$

where A_φ is a needed nonzero integration constant. Let $F(t) = -t^4 + 2(c-1)t^2 + 4A_\varphi$. Assume that $c > 1$, and for $A_\varphi < 0$, we have

$$\begin{aligned} F(t) &= -[t^2 - (c-1)]^2 + (c-1)^2 + 4A_\varphi \\ &= \{\sqrt{(c-1)^2 + 4A_\varphi} - [t^2 - (c-1)]\}\{\sqrt{(c-1)^2 + 4A_\varphi} + [t^2 - (c-1)]\}. \end{aligned}$$

Therefore, $F(t)$ has the real and symmetric roots $\pm\eta_1$ and $\pm\eta_2$. Without loss of generality, we can suppose that $\eta_1 > \eta_2 > 0$. Thus we have

$$(\varphi')^2 = \frac{1}{2c}(\varphi^2 - \eta_2^2)(\eta_1^2 - \varphi^2). \tag{2.3}$$

Therefore, we should have $\eta_1 \geq \varphi \geq \eta_2$, $\eta_1 \in (\sqrt{c-1}, \sqrt{2(c-1)})$ and roots $\eta_i's (i = 1, 2)$ must satisfy

$$\begin{cases} \eta_1^2 + \eta_2^2 = 2(c-1), \\ \eta_1^2 \cdot \eta_2^2 = -4A_\varphi > 0. \end{cases} \tag{2.4}$$

The positive and periodic solution for (2.3) can be obtained by the standard direct integration method in [4], namely,

$$\varphi(\xi) = \eta_1 dn(\frac{\eta_1}{\sqrt{2c}}\xi; k). \tag{2.5}$$

Since dn has fundamental period $2K$, where $K = K(k)$ represents the complete elliptic integral of first kind (see [24]), we obtain that φ has fundamental period

$$T_\varphi = \frac{2\sqrt{2c}}{\eta_1}K(k). \tag{2.6}$$

Fix $c > 1$ and from (2.4), for $k \in (0, 1)$ we have

$$0 < \eta_2 < \sqrt{c-1} < \eta_1 < \sqrt{2(c-1)}$$

and the fundamental period T_φ can be seen as a function of variable η_1 only, that is

$$T_\varphi(\eta_1) = \frac{2\sqrt{2c}}{\eta_1}K(k(\eta_1)), \text{ with } k^2(\eta_1, c) = 2 - \frac{2(c-1)}{\eta_1^2}. \tag{2.7}$$

From the asymptotic properties of K, it follows that $T_\varphi > \sqrt{\frac{2c}{c-1}}\pi$.

Now we obtain a family of dnoidal waves solutions with period $L > \sqrt{2}\pi$. Consider $c_0 > \frac{L^2}{L^2-2\pi^2}$ fixed, there is a unique $\eta_{1,0} \in (\sqrt{c_0-1}, \sqrt{2(c_0-1)})$ such that $T_\varphi(\eta_{1,0}) = L$.

Remark 2.1. If

$$\eta_1 \to \sqrt{2(c-1)}, \ k(\eta_1) \to 1^-.$$

On the basis of limitation $dn(\xi, 1) = sech(\xi)$, we obtain that

$$\varphi(\xi; \sqrt{2(c-1)}, 0) \to \sqrt{2(c-1)}sech\sqrt{\frac{c-1}{c}}\xi.$$

These solitary wave solutions are the same as those obtained by Cui [22].

Theorem 2.1. Let $L > \sqrt{2}\pi$ fixed. Consider $c_0 > \frac{L^2}{L^2-2\pi^2}$ and the unique $\eta_{1,0} = \eta_1(c_0)$ such that $T_{\varphi_{c_0}}(\eta_{1,0}) = L$, then

(i) there exist intervals $I(c_0)$ and $J(\eta_{1,0})$ around c_0 and $\eta_{1,0}$ respectively, and a unique smooth function $\Pi : I(c_0) \to J(\eta_{1,0})$ such that $\Pi(c_0) = \eta_{1,0}$ and

$$\frac{2\sqrt{2c}}{\eta_1}K(k) = L,$$

where $c \in I(c_0)$, $\eta_1 \in \Pi(c)$ and $k = k(c)$ is given by $k^2 = 2 - \frac{2(c-1)}{\eta_1^2} \in (0, 1)$;

(ii) the dnoidal wave solution in (2.5), $\varphi_c(\cdot; \eta_1, \eta_2)$, determined by $\eta_i = \eta_i(c), (i = 1, 2)$, has fundamental period L and satisfies (2.1). Moreover, the mapping $c \in I(c_0) \mapsto \varphi_c \in H^n_{per}$ is smooth for any integer $n \geq 1$.

(iii) $I(c_0)$ can be chosen as $\mathcal{I} = (\frac{L^2}{L^2-2\pi^2}, \infty)$.

Proof. The proof is just an application of the Implicit Function Theorem and the arguments are standard, so we sketch the outline of the proof. Consider the open set $\Omega = \{(\eta, c) \in R^2 : c > \frac{L^2}{L^2-2\pi^2}, \eta \in (\sqrt{c-1}, \sqrt{2(c-1)})\} \subseteq R^2$, and define $\Lambda : \Omega \to R$ by

$$\Lambda(\eta, c) = \frac{2\sqrt{2c}}{\eta}K(k) - L,$$

where $k^2 = 2 - \frac{2(c-1)}{\eta^2}$. From the hypotheses, we get $\Lambda(\eta_{1,0}, c_0) = 0$. Using the relation $\frac{dK}{dk} = \frac{E-k'^2K}{kk'^2}$, and $\frac{\partial k}{\partial \eta} = \frac{2(c-1)}{k\eta^3} > 0$, we obtain

$$\frac{\partial \Lambda}{\partial \eta} = \frac{2\sqrt{2c}}{k^2k'^2\eta^2}[(1+k'^2)E - 2k'^2K] > 0, \tag{2.8}$$

since $f(k') \equiv (1+k'^2)E - 2k'^2K > 0$ for $k' \in (0,1)$. Here $E = E(k)$ is the complete elliptic integral of the second type and $k'^2 = 1 - k^2$ is the complementary modulus. Therefore, by the Implicit Function Theorem we can obtain (i) of Theorem 2.1. Since c_0 is chosen arbitrarily in the interval $\mathcal{I} = (\frac{L^2}{L^2-2\pi^2}, \infty)$, from the uniqueness of the function Λ, it follows that we can extend $I(c_0)$ to $\mathcal{I}$. Finally, using the smoothness of the function involved, we can immediately obtain part (ii).

Corollary 2.1. The map $\Pi : I(c_0) \to J(\eta_{1,0})$ is a strictly increasing function. Moreover, the modulus function $c \mapsto k(c)$ strictly increases with respect to c, that is $\frac{dk}{dc} > 0$.

Proof. Note that $\Lambda(\Pi(c), c) = 0$ for all $c \in I(c_0)$ in Theorem 3.1, for $\frac{d\Pi(c)}{dc} > 0$, we have to prove $\frac{\partial \Lambda}{\partial c} < 0$ in $I(c_0)$. Since $\frac{\partial k}{\partial c} = -\frac{1}{k\eta^2} < 0$ and $\eta^2 = \frac{2(c-1)}{2-k^2}$, we have

$$\frac{\partial \Lambda}{\partial c} = \frac{\sqrt{2}}{\sqrt{c}(2-k^2)k^2k'^2\eta^3}\{2c[2k'^2K - (1+k'^2)E] - 2k^2k'^2K\} < 0. \tag{2.9}$$

Hence, from the relation $\frac{\partial \Lambda}{\partial \eta}\frac{d\Pi(c)}{dc} + \frac{\partial \Lambda}{\partial c} = 0$, we obtain $\frac{d\Pi(c)}{dc} > 0$. Next, differentiating the modulus function $k(c)^2 = 2 - \frac{2(c-1)}{\eta(c)^2}$ with respect to c, we have

$$\frac{dk}{dc} = \frac{1}{k\eta^3}[2(c-1)\eta' - \eta].$$

Note that,

$$\frac{d\Pi(c)}{dc} = -\frac{\frac{\partial \Lambda}{\partial c}}{\frac{\partial \Lambda}{\partial \eta}} = \frac{\eta}{2(c-1)} + \Delta_+,$$

where

$$\Delta_+ = \frac{k^2k'^2K}{c(2-k^2)\eta[(1+k'^2)E - 2k'^2K]} > 0.$$

Thus $2(c-1)\eta' - \eta = 2(c-1)\Delta_+ > 0$, which implies $\frac{dk}{dc} > 0$.

3. Spectral analysis

In this section we will present the spectral analysis to the linear operator $H_c = \mathcal{E}''(\vec{\phi}) + c\mathcal{Q}''(\vec{\phi})$, where $\vec{\phi} = (\varphi_c, \varphi_c)$ and φ_c is the dnoidal wave solution (2.5) with the fundamental period L and $c \in (\frac{L^2}{L^2-2\pi^2}, \infty)$. So we have the matrix operator

$$H_c = \begin{pmatrix} -c\frac{\partial^2}{\partial x^2} - \varphi_c^2 + c & -1 - 2\varphi_c^2 \\ -1 - 2\varphi_c^2 & -c\frac{\partial^2}{\partial x^2} - \varphi_c^2 + c \end{pmatrix}. \tag{3.1}$$

We shall prove that the spectrum of (3.1) is discrete and has its first two eigenvalues simple, with the eigenvalue zero is the second one with eigenfunction (φ_c', φ_c'). Initially, we consider the operator $\mathcal{L}_{dn} = -c\frac{\partial^2}{\partial x^2} - 3\varphi_c^2 + c - 1$. In fact, we have the following result.

Theorem 3.1. Let the dnoidal wave solution φ_c given by Theorem 2.1. Then, the operator $\mathcal{L}_{dn}$ defined on $H^2_{per}([0, L])$ with domain $L^2_{per}([0, L])$ has exactly its first three simple eigenvalues where zero eigenvalue is the second one with associated eigenfunction φ'_c. Moreover, the remainder of the spectrum is constituted by a discrete set of eigenvalues which are double.

Theorem 3.1 is a consequence of the Floquet theory in [23] together with some particular facts about the Lam $\acute{e}$ equation. By convenience of the readers we quickly outline the basic results that are needed in the proof of Theorem 3.1. Firstly, it follows from the Wely's essential spectral theorem that $\sigma_{ess}\mathcal{L}_{dn} = \sigma_{ess}(-c\frac{\partial^2}{\partial x^2} + c - 1)$. The spectral problem in question is

$$\begin{cases} \mathcal{L}_{dn}\zeta = \lambda\zeta, \\ \zeta(0) = \zeta(L), \zeta'(0) = \zeta'(L). \end{cases} \tag{3.2}$$

Introduce the so called semi-periodic eigenvalue problem considered on $[0, L]$

$$\begin{cases} \mathcal{L}_{dn}\xi = \mu\xi, \\ \xi(0) = -\xi(L), \xi'(0) = -\xi'(L). \end{cases} \tag{3.3}$$

From the spectral theory of compact symmetric operators, (3.2) and (3.3) determine a countable infinite set of eigenvalue respectively, $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4 \cdots$, and $\mu_0 \leq \mu_1 \leq \mu_2 \leq \mu_3 \leq \mu_4 \cdots$, that accumulate only at $+\infty$. Here double eigenvalue is counted twice. Let ζ_n and ξ_n be the eigenfunctions associated to the eigenvalue λ_n and μ_n, respectively, $n = 0, 1, 2, \cdots$. By the boundary conditions in (3.2), ζ_n can be extended to the whole of $(-\infty, +\infty)$ as a continuously differentiable function with period L. Similarly, the boundary conditions on ξ_n in (3.3) allow it to be extended as a periodic solution of periodic $2L$ of

$$\mathcal{L}_{dn}g = \sigma g, \tag{3.4}$$

with $\sigma = \mu_n$ (define $\xi_n(L + x) = \xi_n(L - x)$ for $0 \leq x \leq L$). Indeed, (3.4) has a solution of period L if and only if $\sigma = \lambda_n, n = 0, 1, 2 \cdots$. As well as, it has a solution of period $2L$ if and only if $\sigma = \mu_n, n = 0, 1, 2 \cdots$. For a given value σ, if all solutions of (3.4) are bounded, then σ is called a stable value; otherwise σ is called unstable. Sturm's oscillation theory implies that

$$\lambda_0 < \mu_0 \leq \mu_1 < \lambda_1 \leq \lambda_2 < \mu_2 \leq \mu_3 < \lambda_3 \leq \lambda_4 \cdots . \tag{3.5}$$

The open intervals $(\lambda_0, \mu_0), (\mu_1, \lambda_1), (\lambda_2, \mu_2), (\mu_3, \lambda_3), \cdots$, are called intervals of stability. At the endpoints of these intervals the solutions of (3.4) are generally unstable. This is always so for $\sigma = \lambda_0$ as λ_0 is always simple. The intervals, $(-\infty, \lambda_0), (\mu_0, \mu_1)$, $(\lambda_1, \lambda_2), (\mu_2, \mu_3), \cdots$, are called intervals of instability. However, at a double eigenvalue, the intervals is empty and omitted from the discussion. The interval of instability $(-\infty, \lambda_0)$ will always be present. Absence of an instability interval means there is a value of σ for which all solution of (3.4) are either periodic of period L or periodic with basic period $2L$.

We remind the reader that the number of zeros of ζ_n and ξ_n is determined in the following form:

(a) ζ_0 has no zeros in $[0, \mathrm{L}]$.
(b) $\zeta_{2\mathrm{n}+1}$ and $\zeta_{2\mathrm{n}+2}$ have exactly $2\mathrm{n} + 2$ zeros in $[0, \mathrm{L})$. (3.6)
(c) $\xi_{2\mathrm{n}}$ and $\xi_{2\mathrm{n}+1}$ have exactly $2\mathrm{n} + 1$ zeros in $[0, \mathrm{L})$.

Proof of Theorem 3.1. From (3.5), we need to show that $0 = \lambda_1 < \lambda_2$. Firstly, differentiating Eq (2.1) with respect to ξ, we have $\mathcal{L}_{dn}\varphi_c' = 0$. Also, φ_c' has two zeros in $[0, L)$, then the eigenvalue 0 is either λ_1 or λ_2. We claim that $0 = \lambda_1$. Let $\Psi(x) = \zeta(\eta x)$ with $\eta^2 = \frac{2c}{\eta_1^2}$, we have that (3.2) turn to the eigenvalue problem

$$\begin{cases} \Psi'' + [\rho - 6k^2 sn^2(x;k)]\Psi = 0, \\ \Psi(0) = \Psi(2K), \Psi'(0) = \Psi'(2K), \end{cases} \tag{3.7}$$

which is called the Jacobian form of Lamé's equation. Here, $\rho = \frac{2c}{\eta_1^2}(\lambda + \frac{3\eta_1^2}{c} - \frac{c-1}{c})$. By Floquet theory, (3.7) has exactly three intervals of instability which are $(-\infty, \rho_0), (\mu_0', \mu_1'), (\rho_1, \rho_2)$ (where $\mu_i', i \geq 0$ are the eigenvalues associated to the semi-periodic problem determined by Lamé's equation (3.7)). Therefore, the first three eigenvalues ρ_0, ρ_1, ρ_2 associated to (3.7) are simple and the rest of the eigenvalues for (3.7) are double, so, $\rho_3 = \rho_4, \rho_5 = \rho_6, \cdots$, etc.

We note that $\rho_1 = 4 + k^2$ is an eigenvalue to (3.7) with eigenfunction $\Psi_1(x) = sn(x)cn(x)$, so $\lambda = 0$ is a simple eigenvalue to (3.2). From Ince [25] we have that the *Lam é polynomials*,

$$\Psi_0 = 1 - (1 + k^2 - \sqrt{1 - k^2 + k^4})sn^2(x)$$

and

$$\Psi_2 = 1 - (1 + k^2 + \sqrt{1 - k^2 + k^4})sn^2(x),$$

with periodic $2K$, are the eigenfunctions associated to the others two eigenvalues ρ_0, ρ_2 given by

$$\rho_0 = 2(1 + k^2 - 2\sqrt{1 - k^2 + k^4}), \;\; \rho_2 = 2(1 + k^2 + 2\sqrt{1 - k^2 + k^4}).$$

Since Ψ_0 has no zeros in $[0, 2K)$ and Ψ_2 has exactly two zeros in $[0, 2K)$, and $\rho_0 < \rho_1 < \rho_2$ for every $k^2 \in (0, 1)$, then ρ_0 is the first eigenvalue, ρ_1 is the second eigenvalue and ρ_2 is the third eigenvalue. By the relation

$$\lambda = \frac{\eta_1^2}{2c}(\rho - \frac{3\eta_1^2}{c} + \frac{c-1}{c}) = \frac{\eta_1^2}{2c}(\rho - 6) + \frac{c-1}{c},$$

we can see that λ is an increasing function with respect to ρ, then $\lambda_0 < \lambda_1 < \lambda_2$. Combining $k^2 = 2 - \frac{2(c-1)}{\eta_1^2}$ with $\lambda_0 < \lambda_1 < \lambda_2$, we conclude that $\lambda(\rho_1) = 0 = \lambda_1$ and $\lambda_0 < 0$.

Next, by using the Lamé's equation in (3.7) with the conditions $\Psi(0) = -\Psi(2K), \Psi'(0) = -\Psi'(2K)$. We can see that the eigenvalues μ_i' associated to semi-periodic problem are related to the μ_i via the relation

$$\mu_i' = \frac{2c}{\eta_1^2}(\mu + \frac{3\eta_1^2}{c} - \frac{c-1}{c}).$$

It is straightforward to ascertain that the first two eigenvalues of Lamé's equation in the semi-periodic case are $\mu_0' = 1 + k^2$ and $\mu_1' = 1 + 4k^2$. The associated eigenfunctions are $\Psi_{0,sm}(x) = cn(x)dn(x)$ and $\Psi_{1,sm}(x) = sn(x)dn(x)$, respectively. It is concluded from the relation $\mu_i = \frac{\eta_1^2}{2c}(\mu_i' - 6) + \frac{c-1}{c}$ and $\Psi(x) = \zeta(\eta x)$, that the first two eigenvalues are $\mu_0 = -\frac{3}{(2-k^2)} \cdot \frac{c-1}{c}$ and $\mu_1 = -\frac{3(1-k^2)}{(2-k^2)} \cdot \frac{c-1}{c}$. Hence, ρ_0, ρ_1 and ρ_2 are simple and the rest of the eigenvalues are double. This concludes the proof.

Theorem 3.2. $L > \sqrt{2}\pi$, $c > \frac{L^2}{L^2 - 2\pi^2}$. For any y satisfying $< y, \Psi_0(\frac{1}{\sqrt{2c}}\eta_1 x) >=< y, \varphi' >= 0$, there exists $\delta > 0$, such that

$$< \mathcal{L}_{dn} y, y >\geq \delta ||y||^2, \tag{3.8}$$

where Ψ_0 is the eigenfunction associated to eigenvalue $\lambda_0 < 0$ of $\mathcal{L}_{dn}$ given in Theorem 3.1.

Proof. $\Psi_0(\frac{1}{\sqrt{2c}}\eta_1 x) = 1 - (1 + k^2 - \sqrt{1-k^2+k^4})sn^2(\frac{1}{\sqrt{2c}}\eta_1 x)$. Combining (2.5), we have $<\varphi', \Psi_0(\frac{1}{\sqrt{2c}}\eta_1 x)>= 0$, which implies

$$\mathcal{L}_{dn}y = \alpha y + \lambda\varphi_x + \theta\Psi_0.$$

By

$$<\mathcal{L}_{dn}y, \varphi'>=<y, \varphi'>=<\varphi', \Psi_0(\frac{1}{\sqrt{2c}}\eta_1 x)>= 0,$$

we immediately have $\lambda = 0$. Then by $\mathcal{L}_{dn}\Psi_0(\frac{1}{\sqrt{2c}}\eta_1 x) = \lambda_0\Psi_0(\frac{1}{\sqrt{2c}}\eta_1 x)$ and $<y, \Psi_0(\frac{1}{\sqrt{2c}}\eta_1 x)>= 0$, we have $\theta < \Psi_0, \Psi_0 >= 0$, which implies $\theta = 0$. Finally, $<\mathcal{L}_{dn}y, y>= \alpha < y, y>$ and $\alpha > 0$. This completes the proof.

Choosing

$$\Psi^- = (\Psi_0(\frac{1}{\sqrt{2c}}\eta_1 x, \Psi_0(\frac{1}{\sqrt{2c}}\eta_1 x)),$$

then

$$<H_c\Psi^-, \Psi^->= 2\lambda_0 < \Psi_0, \Psi_0 >< 0.$$

Denoting $\tau'(0)\Phi = (\varphi', \varphi')$. Let

$$P = \{\vec{p} \in X|\vec{p} = (p_1, p_2), <p_i, \Psi_0(\tfrac{1}{\sqrt{2c}}\eta_1 x)>=<p_i, \varphi_{cx}>= 0, i = 1, 2\},$$
$$Z = \{k_1\tau'(0)\Phi|k_1 \in R\}, \quad N = \{k_2\Psi^-|k_2 \in R\}.$$

By (3.1), we have

$$<H_c\vec{p}, \vec{p}>= \frac{1}{2} < \mathcal{L}_{dn}(p_1 + p_2), (p_1 + p_2) > +\frac{1}{2} < \widetilde{L}(p_1 - p_2), (p_1 - p_2) >,$$

where $\widetilde{L} = -c\frac{\partial^2}{\partial x^2} + \varphi_c^2 + c + 1$, which is a positive operator. By (3.8), we have

$$<H_c\vec{p}, \vec{p}>\geq \frac{1}{2}\delta||p_1 + p_2||^2 + \frac{1}{2}\delta_1||p_1 - p_2||^2 \geq \delta_2||\vec{p}||^2,$$

where $\delta_2 = \min(\frac{1}{2}\delta, \frac{1}{2}\delta_1)$.

Thus X can be decomposed into the direct sum $N \oplus Z \oplus P$. This implies that the spectral structure required for H_c.

Theorem 3.3. Let $c > \frac{L^2}{L^2-2\pi^2}$ and φ_c given by Theorem 2.1. Then the linear operator H_c in (3.1) and defined on $H^2_{per}([0, L]) \times H^2_{per}([0, L])$ has exactly its first two eigenvalues simple. The eigenvalue zero is the second one with associated eigenfunction (φ'_c, φ'_c). Moreover, the remainder of the spectrum is constituted by a discrete set of eigenvalues.

4. Stability of dnoidal waves

We start establishing the definition of stability we are considering. Let $\tau_s f(x) = f(x + s)$ for $x \in R$. For $\vec{\phi} = (\varphi_c, \psi_c)$ we define $\vec{\phi}$–orbit, as $\Omega_{\vec{\phi}} = \vec{\phi}(\cdot + s) : s \in R$. For $\epsilon > 0$ the neighbourhood U_ϵ around of $\Omega_{\vec{\phi}}$ in $X = H^1_{per}([0, L]) \times H^1_{per}([0, L])$ is defined as $U_\epsilon = \{\vec{u} \in X : inf_{s\in R}||\vec{u} - \tau_s\vec{\phi}||_X\} \leq \epsilon$.

Definition 4.1. Let $\vec{\phi} \in X$ be a periodic traveling wave solution of (1.3) with period L. We say that the orbit $\Omega_{\vec{\phi}}$ is orbitally stable in X by the flow of Eq (1.2) if for each $\epsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$ such that if $\vec{u_0} \in X$ and $inf_{r\in R}||\vec{u_0} - \tau_r\vec{\phi}||_X < \delta$ then the solution $\vec{u}(t)$ of (1.2) with $\vec{u}(0) = \vec{u_0}$ exists globally and satisfies

$$sup_{t\in R}inf_{r\in R}||\vec{u}(t) - \tau_r\vec{\phi}||_X < \epsilon.$$

Otherwise, we say that $\Omega_{\vec{\phi}}$ is X-unstable.

Because the stability refers to perturbations of the periodic-wave profile itself, we first introduce the initial-value problem for Eq (1.2). In fact, by applying the classical theory of semigroups, we have the following general lemma.

Lemma 4.1. For any fixed initial data $(\eta_0, u_0) \in X$, there exists $T \geq 0$ and a unique solution $(\eta, u) \in C([0, T); X)$.

Proof. Rewriting the Eq (1.2) as

$$\begin{pmatrix} \eta \\ u \end{pmatrix}_t = M\begin{pmatrix} \eta \\ u \end{pmatrix} + \mathcal{G}\begin{pmatrix} \eta \\ u \end{pmatrix},$$

where

$$M = \begin{pmatrix} (1-\partial_x^2)^{-1}\partial_x & 0 \\ 0 & (1-\partial_x^2)^{-1}\partial_x \end{pmatrix},$$

$$\mathcal{G}\begin{pmatrix} \eta \\ u \end{pmatrix} = (1-\partial_x^2)^{-1}\partial_x\begin{pmatrix} u^2\eta \\ u\eta^2 \end{pmatrix}.$$

We have that $M \in \mathcal{L}(H^2_{per}([0, L]) \times H^2_{per}([0, L]), X)$. Moreover, M generates a bounded C_0-group $S(t)$ on X. On the other hand, since $(1-\partial_x^2)^{-1}$ is a bounded linear operator from $L^2_{per}([0, L])$ to $H^2_{per}([0, L])$, and $H^1_{per}([0, L]) \hookrightarrow L^\infty(R)$, thus we have

$$||(1-\partial_x^2)^{-1}\begin{pmatrix} (u_1^2\eta_1)_x \\ (u_1\eta_1^2)_x \end{pmatrix} - (1-\partial_x^2)^{-1}\begin{pmatrix} (u_2^2\eta_2)_x \\ (u_2\eta_2^2)_x \end{pmatrix}||_{H^2_{per}\times H^2_{per}} \leq c||(\eta_1, u_1) - (\eta_2, u_2)||_X$$

implies that $\mathcal{G}$ is locally Lipschitz from X to $H^2_{per} \times H^2_{per}$. By classical semigroup theory, we can show that a local mild L-periodic solution $(\eta, u) \in X$ exists with initial data $(\eta_0, u_0) \in X$.

Stability theorem. Let $L > \sqrt{2}\pi$. Consider $c > \frac{L^2}{L^2-2\pi^2}$ and φ_c given by Theorem 2.1. Then for $\vec{\phi} = (\varphi_c, \varphi_c)$ the orbit $\Omega_{\vec{\phi}}$ is orbitally stable in X by the periodic flow of the Eq (1.2).

Proof. First, by (2.5) we have

$$\int_0^L \varphi_c^2 d\xi = \frac{8c}{L}KE. \tag{4.1}$$

On the other hand, multiplying (2.1) by φ_c and integrating from 0 to L we get,

$$\int_0^L \varphi_c^4 d\xi = c\int_0^L (\varphi_c')^2 d\xi + (c-1)\int_0^L \varphi_c^2 d\xi. \tag{4.2}$$

Integrating (2.2) from 0 to L yields,

$$\int_0^L (\varphi_c')^2 d\xi = -\frac{1}{2c}\int_0^L \varphi_c^4 d\xi + \frac{c-1}{c}\int_0^L \varphi_c^2 d\xi + \frac{2}{c}A_{\varphi_c}L. \tag{4.3}$$

Combining (4.2) and (4.3), it is easy to see that,

$$\int_0^L (\varphi_c')^2 d\xi = \frac{c-1}{3c}\int_0^L \varphi_c^2 d\xi + \frac{4}{3c}A_{\varphi_c}L.$$

From the last identity, (2.4) and (4.1)

$$\int_0^L \varphi_c^2 + (\varphi_c')^2 d\xi = \frac{8}{3L}(4c-1)KE + \frac{64}{3L^3}cK^4 - \frac{16}{3L}(c-1)K^2.$$

From (2.7), we have $c - 1 = \frac{\eta_1^2}{2}(2-k^2)$. Thus

$$\int_0^L \varphi_c^2 + (\varphi_c')^2 d\xi = \frac{8L}{3[L^2 - 4(2-k^2)K^2]}\{3KE + \frac{4}{L^2}K^3[(2-k^2)E - 2(1-k^2)K]\} := J_L(k).$$

By detailed calculation, we have

$$\begin{aligned}\frac{dJ_L(k)}{dk} = \frac{64LK[(2-k^2)E-2(1-k^2)K]}{3kk'^2[L^2-4(2-k^2)K^2]^2} & \{3KE + \tfrac{4}{L^2}K^3[(2-k^2)E - 2(1-k^2)K]\} \\ +\frac{8L}{kk'^2[L^2-4(2-k^2)K^2]} & \{(E^2 - k'^2K^2) + \tfrac{4}{L^2}K^3k^2k'^2(K-E) \\ & + \tfrac{4}{L^2}[(2-k^2)E - 2(1-k^2)K](E - k'^2K)K^2\}.\end{aligned}$$

It follows from $K > E$, $(2-k^2)E > 2(1-k^2)K$, $E > k'^2K$, $E^2 > k'^2K^2$, we conclude

$$\frac{d}{dc}\int_0^L \varphi_c^2 + (\varphi_c')^2 d\xi = \frac{dJ_L(k)}{dk}\frac{dk}{dc} > 0.$$

This finishes the proof.

5. Conclusions

In this article, we are interested in studying the stability of periodic traveling wave solutions for the coupled BBM equations (1.2). First, we obtain a family of dnoidal waves solutions by the standard direct integration method. Second, by Floquet theory and detailed spectral analysis, we prove the spectral properties to some matrix operator required for stability result. Finally, we prove the theorem on orbital stability of periodic traveling wave solutions for the system (1.2). Our results on existence and stability extend the results obtained in [22]. Furthermore, for the other cases $p = 2, 3, ...,$ in (1.1), we believe that there are chances of getting periodic waves and we will discuss these problems in the future study.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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