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Nonhomogeneous nonlinear integral equations on bounded domains

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Abstract: This paper investigates the existence of positive solutions for a nonhomogeneous nonlinear integral equation of the form

$$u^{p-1}(x) = \int_\Omega \frac{u(y)}{|x-y|^{n-\alpha}}dy + \int_\Omega \frac{f(y)}{|x-y|^{n-\alpha}}dy,\ x \in \bar{\Omega}$$

where $\frac{2n}{n+\alpha} \le p < 2$, $1 < \alpha < n$, $n > 2$, Ω is a bounded domain in $\mathbb{R}^n$. We show that under suitable assumptions on f, the integral equation admits a positive solution in $L^{\frac{2n}{n+\alpha}}(\Omega)$. Our method combines the Ekeland variational principle, a blow-up argument and a rescaling argument which allows us to overcome the difficulties arising from the lack of Brezis-Lieb lemma in $L^{\frac{2n}{n+\alpha}}(\Omega)$.

Keywords: integral equation; Hardy-Littlewood-Sobolev inequality; blowing-up and rescaling argument; Ekeland variational principle
Mathematics Subject Classification: 45G05, 35A01, 35B44

1. Introduction

This paper concerns the existence of positive solutions for the following integral equation:

$$u^{p-1}(x) = \int_\Omega \frac{u(y)}{|x-y|^{n-\alpha}}dy + \int_\Omega \frac{f(y)}{|x-y|^{n-\alpha}}dy,\ x \in \bar{\Omega} \tag{1.1}$$

where $u \in L^p(\Omega)$, $f \in L^p(\Omega)$, $n > 2$, $\frac{2n}{n+\alpha} = p_\alpha \le p < 2$, $1 < \alpha < n$ and Ω is a bounded domain in $\mathbb{R}^n$.

When we set $f(x) = 0$, Eq (1.1) simplifies to the subsequent integral equation:

$$u^{p-1}(x) = \int_\Omega \frac{u(y)}{|x-y|^{n-\alpha}}dy,\ x \in \bar{\Omega}. \tag{1.2}$$

Indeed, the existence of solutions for problem (1.2) is connected to the classic sharp Hardy-Littlewood-Sobolev (HLS) inequality:

Theorem A. Let $\alpha \in (0, n)$. The classical sharp HLS inequality ([15, 16, 19–21]) states that

$$|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)|x-y|^{-(n-\alpha)}g(y)dxdy| \leq N(p,\alpha,n)\|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^t(\mathbb{R}^n)} \tag{1.3}$$

for all $f \in L^p(\mathbb{R}^n), g \in L^t(\mathbb{R}^n), 1 < p, t < \infty, 0 < \alpha < n$ and $1/p + 1/t + (n-\alpha)/n = 2$. If $p = t = 2n/(n+\alpha)$, then

$$|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)|x-y|^{-(n-\alpha)}g(y)dxdy| \leq N_\alpha\|f\|_{L^{\frac{2n}{n+\alpha}}(\mathbb{R}^n)}\|g\|_{L^{\frac{2n}{n+\alpha}}(\mathbb{R}^n)}$$

holds for all $f, g \in L^{\frac{2n}{n+\alpha}}(\mathbb{R}^n)$ where

$$N_\alpha := N\left(\frac{2n}{n+\alpha}, \alpha, n\right) = \pi^{(n-\alpha)/2}\frac{\Gamma(\alpha/2)}{\Gamma(n/2+\alpha/2)}\left\{\frac{\Gamma(n/2)}{\Gamma(n)}\right\}^{-\alpha/n}.$$

And the equality holds if and only if

$$f(x) = c_1 g(x) = c_2\left(\frac{1}{c_3 + |x-x_0|^2}\right)^{\frac{n+\alpha}{2}}$$

where c_2 is any constant, c_1, c_3 are positive constants and $x_0 \in \mathbb{R}^n$. Clearly, inequality (1.3) is applicable to bounded domains as well. Motivated by this, Dou and Zhu in [11] recently explored the Euler-Lagrange equation for inequality (1.3) in a bounded domain, as per the following equation:

$$u^{p-1} = \int_\Omega \frac{u(y)}{|x-y|^{n-\alpha}}dy, x \in \bar{\Omega} \tag{1.4}$$

where Ω is a bounded domain in $\mathbb{R}^n$. Additionally, Dou and Zhu examined the subsequent general equation:

$$u^{p-1}(x) = \int_\Omega \frac{u(y)}{|x-y|^{n-\alpha}}dy + \lambda\int \frac{u(y)}{|x-y|^{n-\alpha-1}}dy, \quad u \geq 0, \quad x \in \bar{\Omega}. \tag{1.5}$$

Using the compact embedding theorem along with a blowing-up and rescaling argument (as mentioned in Lemma 4.3 of [11]), they established the following theorem.

Theorem B. Assume $\alpha \in (1, n)$ and Ω is a smooth bounded domain.
(1) For $\frac{2n}{n+\alpha} < p < 2$ (subcritical case), there is a positive solution $u \in C^1(\bar{\Omega})$ to Eq (1.5) for any given $\lambda \in \mathbb{R}$;
(2) For $p = \frac{2n}{n+\alpha}$ (critical case) and $\lambda > 0$, there is a positive solution $u \in C(\bar{\Omega})$ to Eq (1.5).

Dou and Zhu in [11] established the existence results for weak solutions to (1.5) when $\lambda > 0$ and $p = p_\alpha$. They considered the functional

$$Q_\lambda(\Omega) := \sup_{u \in L^{p_\alpha}(\Omega)\setminus\{0\}} \frac{\int_\Omega\int_\Omega u(x)\left(|x-y|^{-(n-\alpha)} + \lambda|x-y|^{-(n-\alpha-1)}\right)u(y)dxdy}{\|u\|^2_{L^{p_\alpha}(\Omega)}}.$$

Due to homogeneity, we know that the corresponding Euler-Lagrange equation for nonnegative extremal functions up to a constant multiplier is the integral equation (1.5) for $p = p_\alpha$. It should be noted that Eq (1.5) differs from Eq (1.1) due to the nonhomogeneous nature of Eq (1.1). Therefore, we cannot directly obtain the existence results for weak solutions to (1.1) using the approach of setting up extremal problems as done in [11]. Integral equations or systems of integral equations on the whole space, bounded domains or upper half space have been extensively studied previously as shown in [6–10, 12–14, 17, 18, 23–25] and the references therein.

In relation to the nonhomogeneous critical semilinear elliptic equation associated with Eq (1.1),

$$\begin{cases} -\Delta u = |u|^{2^*-2}u + f(x) & x \in \Omega \\ u \in H_0^1(\Omega), \end{cases} \tag{1.6}$$

where $2^* = \frac{2n}{n-2}$ is the critical Sobolev exponent, $n > 2$, Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary. Tarantello [22] demonstrated that problem (1.6) possesses at least two solutions. The fundamental idea is to partition the Nehari manifold $\Lambda = \left\{u \in W_0^{1,2}(\Omega); \langle I'(u), u\rangle = 0\right\}$ into three disjoint subsets, namely Λ^+, Λ^- and Λ_0 and to employ the Ekeland variational principle to obtain one solution in Λ^+ and another solution in Λ^-. The existence results for an elliptic problem of (p, q)-Laplacian type, involving a critical term, a power-type nonlinearity at the critical level with a subcritical term, nonnegative weights and a positive parameter λ have been discussed in the literature, specifically in references [2, 3], for the entire space $\mathbb{R}^N$.

There exists a notable distinction between integral and differential equations. For instance, consider

$$u(x) = \frac{1}{c(n,\alpha)} \int_\Omega \frac{f(u(y))}{|x-y|^{n-\alpha}} dy, \ \ x \in \bar{\Omega}$$

where Ω is a bounded domain and $c(n, \alpha)$ is a constant, dependent only on n, α. Given $f = u^{\frac{n-2}{n+2}}$ and $\alpha = 2$, it can be observed that u must fulfill

$$\begin{cases} -\Delta u = c(n)u^{\frac{n+2}{n-2}}, u > 0 & \text{in } \Omega, \\ u(x) = \int_\Omega \frac{u^{\frac{n+2}{n-2}}(y)}{|x-y|^{n-2}} dy & \text{on } \partial\Omega, \end{cases}$$

but not conversely, as seen in [11]. Furthermore, the difference between $W^{1,2}(\Omega)$ and $L^p(\Omega)(1 < p < 2)$ arises challenges when attempting to treat integral equations in the same manner as differential equations. For instance, the Brezis-Lieb lemma [4] cannot be applied in $L^p(\Omega)(1 < p < 2)$ because almost-everywhere convergence of sequences cannot be inferred from weak convergence of sequences in $L^p(\Omega)(1 < p < 2)$. This fact complicates our attempts to prove the existence of the solution to Eq (1.1) using the variational method to handle Eq (1.6).

Inspired by the work described above, our study differs from previous works on integral equations which primarily focused on the homogeneous case in that we instead handle the nonhomogeneous case. Therefore, we consider the existence of positive solutions for Eq (1.1) for $p_\alpha \le p < 2$. A function $u \in L^p(\Omega)$ is said to be a solution of (1.1) if u satisfies

$$\int_\Omega |u|^{p-1}w - \int_\Omega \int_\Omega \frac{u(x)w(y)}{|x-y|^{n-\alpha}} dxdy - \int_\Omega \int_\Omega \frac{w(x)f(y)}{|x-y|^{n-\alpha}} dxdy = 0 \text{ for all } w \in L^p(\Omega).$$

Consider functionals $\mathrm{I} : L^p(\Omega) \to \mathbb{R}$:

$$\mathrm{I}(u) = \frac{1}{p} \int_\Omega |u|^p - \frac{1}{2} \int_\Omega \int_\Omega \frac{u(x)u(y)}{|x-y|^{n-\alpha}} dxdy - \int_\Omega \int_\Omega \frac{u(x)f(y)}{|x-y|^{n-\alpha}} dxdy.$$

Let

$$\tilde{u}(x) = \begin{cases} u(x), & x \in \Omega, \\ 0, & x \in \mathbb{R}^n \backslash \Omega, \end{cases} \quad \tilde{w}(x) = \begin{cases} w(x), & x \in \Omega, \\ 0, & x \in \mathbb{R}^n \backslash \Omega. \end{cases}$$

For $u, w \in L^p(\Omega)$, due to HLS inequality and Hölder inequality, we have

$$\begin{aligned} &\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \tilde{u}(x)|x-y|^{-(n-\alpha)} \tilde{w}(y) dx dy \\ &\leq N_\alpha \|\tilde{u}\|_{L^{p_\alpha}(\mathbb{R}^n)} \|\tilde{w}\|_{L^{p_\alpha}(\mathbb{R}^n)} = N_\alpha \|u\|_{L^{p_\alpha}(\Omega)} \|w\|_{L^{p_\alpha}(\Omega)} \\ &\leq C(n, p, \alpha, \Omega) \|u\|_{L^p(\Omega)} \|w\|_{L^p(\Omega)}. \end{aligned}$$

This implies that $I \in C^1(L^p(\Omega), \mathbb{R})$.

We first investigate the critical problem, leading to the following existence result, which is the principal outcome of this paper.

Theorem 1.1. *Assume that $f(x)$ is a non-negative function satisfying the following conditions:*
(A_1) For small enough ϵ, $\|f\|_{p_\alpha} < \min\{C(n, p_\alpha, \alpha, \Omega) N_\alpha^{\frac{1}{p_\alpha - 2}}, \epsilon^{\frac{n+\alpha}{2}}\}$;
(A_2) $f(x) \in C^0(B_\delta(x_*)) \bigcap L^{p_\alpha + \delta}(\Omega), f(x_*) > 0$ where $B_\delta(x_*) \subseteq \Omega$ for some $x_* \in \Omega$ and $\delta > 0$ is small enough.
Then, problem (1.1) has at least one positive solution $u \in L^{p_\alpha}(\Omega)$, $1 < \alpha < (\sqrt{2} - 1)n, n > 2$ and Ω is a bounded domain in $\mathbb{R}^n$.

Next, we examine the existence result for Eq (1.1) in the subcritical case.

Theorem 1.2. *Let $f(x) \in L^p(\Omega), f(x) \neq 0$, $\|f\|_p < C(n, p, \alpha, \Omega) N_\alpha^{\frac{1}{p-2}}$. Then problem (1.1) has at least two positive solutions $u_0, u_1 \in L^p(\Omega)$, $\frac{2n}{n+\alpha} < p < 2$, $1 < \alpha < n$, $n > 2$ and Ω is a bounded domain in $\mathbb{R}^n$.*

Remark 1.1. In what follows, we proceed with the proof of these theorems. For the critical case, we employ the Ekeland variational principle (see [22]) and a blow-up argument and a rescaling argument to find a weak solution of (1.1). In the process of proving the main theorem (Theorem 1.1), we encounter difficulties similar to those in [22]. In [22], the following core lemma is required to be proved:

Lemma 1.1. *For $f \neq 0$, $n > 2$, $p = \frac{2n}{n-2}$,*

$$\mu_0 =: \inf_{\|u\|_p = 1} \left(c_n \|\nabla u\|^{(n+2)/2} - \int_\Omega f u \right)$$

is achieved, where c_n is a constant that only depends on n.

Similarly, we aim to show that for $p = p_\alpha$,

$$Q_p(\Omega) =: \inf_{\|u\|_p = 1} \left(c_{n,\alpha} (\int_\Omega \int_\Omega \frac{u(x)u(y)}{|x-y|^{n-\alpha}} dx dy)^{\frac{(p-1)}{p-2}} - \int_\Omega \int_\Omega \frac{u(x) f(y)}{|x-y|^{n-\alpha}} dx dy \right)$$

is achieved in this paper. It's important to note that the Brezis-Lieb lemma [4] plays a crucial role in proving Lemma 1.1 through the variational method. However, since $1 < p_\alpha < 2$, the Brezis-Lieb lemma [4] does not hold in $L^{p_\alpha}(\Omega)$. Consequently, the proof method of Lemma 1.1 fails to prove that

$Q_{p_\alpha}(\Omega)$ is achieved. To solve the problem, we use a blow-up argument and a rescaling argument in this paper. First, for $\frac{2n}{n+\alpha} < p < 2$, we can show $Q_p(\Omega)$ is achieved at a point u_p. For $p = p_\alpha$, we will show $\lim_{p\to p_\alpha} \left\|u_p\right\|_{L^\infty(\Omega)} \le C$, by a blow-up argument and a rescaling argument. Thus, $u_p \to u_*$ as $p \to p_\alpha$ in $C(\bar{\Omega})$. Once $Q_p(\Omega)$ is achieved, we can prove that problem (1.1) has at least one positive solution by Ekeland variational principle.

The structure of this paper is as follows: In Section 2, we provide preliminary results. In Section 3, we prove Theorems 1.1 and 1.2.

Throughout this paper, we utilize the symbols c and C to represent various positive constants, the value of which may change from one line to another.

2. Preliminaries

To obtain the proof of the main theorems, several preliminary are needed. Let

$$\begin{aligned}
\Lambda &= \left\{u \in L^p(\Omega) : \|u\|_p^p - \int_\Omega\int_\Omega \frac{u(x)u(y)}{|x-y|^{n-\alpha}}dxdy - \int_\Omega\int_\Omega \frac{u(x)f(y)}{|x-y|^{n-\alpha}}dxdy = 0\right\},\\
\Lambda^+ &= \left\{u \in \Lambda : (p-1)\|u\|_p^p - \int_\Omega\int_\Omega \frac{u(x)u(y)}{|x-y|^{n-\alpha}}dxdy > 0\right\},\\
\Lambda_0 &= \left\{u \in \Lambda : (p-1)\|u\|_p^p - \int_\Omega\int_\Omega \frac{u(x)u(y)}{|x-y|^{n-\alpha}}dxdy = 0\right\},\\
\Lambda^- &= \left\{u \in \Lambda : (p-1)\|u\|_p^p - \int_\Omega\int_\Omega \frac{u(x)u(y)}{|x-y|^{n-\alpha}}dxdy < 0\right\}.
\end{aligned}$$

Lemma 2.1. *Let $f \neq 0$ satisfy (A_1). For every $u \in L^p(\Omega), p_\alpha \le p < 2, u \neq 0$, there exists unique $t^+ = t^+(u) > 0$ such that $t^+u \in \Lambda^-$. In particular:*

$$t^+ > \left[\frac{\int_\Omega\int_\Omega \frac{u(x)u(y)}{|x-y|^{n-\alpha}}dxdy}{(p-1)\|u\|_p^p}\right]^{1/(p-2)} := t_{\max}$$

and $I(t^+u) = \max_{t\ge t_{\max}} I(tu)$. Moreover, if $\int_\Omega\int_\Omega \frac{u(x)f(y)}{|x-y|^{n-\alpha}}dxdy > 0$, then there exists a unique $t^- = t^-(u) > 0$ such that $t^-(u) \in \Lambda^+$. In particular,

$$t^- < \left[\frac{\int_\Omega\int_\Omega \frac{u(x)u(y)}{|x-y|^{n-\alpha}}dxdy}{(p-1)\|u\|_p^p}\right]^{1/(p-2)},$$

$I(t^-u) \le I(tu), \forall t \in [0, t^+]$.

Proof. Let $\varphi(t) = t^{p-1}\|u\|_p^p - \mathrm{t}\int_\Omega\int_\Omega \frac{u(x)u(y)}{|x-y|^{n-\alpha}}dxdy$. Easy computations show that φ is concave and achieves its maximum at

$$t_{\max} = \left[\frac{\int_\Omega\int_\Omega \frac{u(x)u(y)}{|x-y|^{n-\alpha}}dxdy}{(p-1)\|u\|_p^p}\right]^{1/(p-2)}.$$

Also

$$\varphi(t_{\max}) = \left[\frac{1}{p-1}\right]^{(p-1)/(p-2)}(2-p)\left[\frac{(\int_\Omega\int_\Omega \frac{u(x)u(y)}{|x-y|^{n-\alpha}}dxdy)^{(p-1)}}{\|u\|_p^p}\right]^{1/(p-2)},$$

that is

$$\varphi\left(t_{\max}\right)=c_{n,\alpha}\left[\frac{\left(\int_\Omega\int_\Omega \frac{u(x)u(y)}{|x-y|^{n-\alpha}}dxdy\right)^{(p-1)}}{\|u\|_p^p}\right]^{1/(p-2)}.$$

Thus, if $\int_\Omega\int_\Omega \frac{u(x)f(y)}{|x-y|^{n-\alpha}}dxdy\leq 0$, then there exists a unique $t^+>t_{\max}$ such that: $\varphi\left(t^+\right)=\int_\Omega\int_\Omega \frac{u(x)f(y)}{|x-y|^{n-\alpha}}dxdy$ and $\varphi'\left(t^+\right)<0$. Equivalently $t^+u\in\Lambda^-$ and $\mathrm{I}\left(t^+u\right)\geq \mathrm{I}(tu)$, $\forall t\geq t_{\max}$. In case $\int_\Omega\int_\Omega \frac{u(x)f(y)}{|x-y|^{n-\alpha}}dxdy>0$, by assumption (A_1) we have that necessarily,

$$\int_\Omega\int_\Omega \frac{u(x)f(y)}{|x-y|^{n-\alpha}}dxdy<c_{n,\alpha}\left[\frac{\left(\int_\Omega\int_\Omega \frac{u(x)u(y)}{|x-y|^{n-\alpha}}dxdy\right)^{(p-1)}}{\|u\|_p^p}\right]^{1/(p-2)}=\varphi\left(t_{\max}\right).$$

Therefore, in this case, we have unique $0<t^-<t_{\max}<t^+$ such that

$$\varphi\left(t^+\right)=\int_\Omega\int_\Omega \frac{u(x)f(y)}{|x-y|^{n-\alpha}}dxdy=\varphi\left(t^-\right)$$

and

$$\varphi'\left(t^-\right)>0>\varphi'\left(t^+\right).$$

Equivalently $t^+u\in\Lambda^-$ and $t^-u\in\Lambda^+$. □

Let

$$Q_p(\Omega)=\inf_{u\in L^p(\Omega)\backslash\{0\}}\frac{\|u\|_{L^p(\Omega)}^{2\frac{p-1}{2-p}}}{c_{n,\alpha}\left(\int_\Omega\int_\Omega u(x)\left(|x-y|^{-(n-\alpha)}u(y)\right)dydx\right)^{\frac{p-1}{2-p}}}-\|u\|_{L^p(\Omega)}^{-1}\int_\Omega\int_\Omega u(x)\left(|x-y|^{-(n-\alpha)}f(y)\right)dydx,$$

we show

Lemma 2.2. *Assume that $f(x)$ is a non-negative function satisfying (A_2). Then, $Q_p(\Omega)<\frac{1}{c_{n,\alpha}(N_\alpha)^{\frac{p-1}{2-p}}}$, where $p=p_\alpha$.*

Proof. Similar to the proof of Lemma 4.1 of [11], let $x_*\in\Omega$. For small positive ϵ and a fixed $R>0$ so that $B_R\left(x_*\right)\subset\Omega$, we define

$$\tilde{u}_\epsilon(x)=\begin{cases}u_\epsilon(x) & x\in B_R\left(x_*\right)\subset\Omega,\\ 0 & x\in\mathbb{R}^n\backslash B_R\left(x_*\right),\end{cases}$$

where

$$u_\epsilon(x)=e^{-\frac{n+\alpha}{2}}u\left(\frac{|x-x_*|}{\epsilon}\right)=\left(\frac{\epsilon}{\epsilon^2+|x-x_*|^2}\right)^{\frac{n+\alpha}{2}}.$$

Obviously, $\tilde{u}_\epsilon\in L^{p_\alpha}\left(\mathbb{R}^n\right)$. Thus, similar to the proof of Proposition 2.1 of [11] we have

$$\begin{aligned}&\int_\Omega\int_\Omega \frac{1}{|x-y|^{n-\alpha}}\tilde{u}_\epsilon(x)\tilde{u}_\epsilon(y)dxdy\\&=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{1}{|x-y|^{n-\alpha}}u_\epsilon(x)u_\epsilon(y)dxdy\\&-2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n\backslash B_R(x_*)}\frac{u_\epsilon(x)u_\epsilon(y)}{|x-y|^{n-\alpha}}dxdy+\int_{\mathbb{R}^n\backslash B_R(x_*)}\int_{\mathbb{R}^n\backslash B_R(x_*)}\frac{u_\epsilon(x)u_\epsilon(y)}{|x-y|^{n-\alpha}}dxdy\\&=N_\alpha\|u_\epsilon\|_{L^{p_\alpha}(\mathbb{R}^n)}^2-I_1+I_2\end{aligned}\tag{2.1}$$

where

$$I_1 = 2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n\setminus B_R(x_*)} \frac{u_\epsilon(x)u_\epsilon(y)}{|x-y|^{n-\alpha}}dxdy = C\int_{\mathbb{R}^n\setminus B_R(x_0)} u_\epsilon^{\frac{2n}{n+\alpha}}(x)dx = O\left(\frac{R}{\epsilon}\right)^{-n} as \quad \epsilon \to 0,$$

$$I_2 = \int_{\mathbb{R}^n\setminus B_R(x_*)}\int_{\mathbb{R}^n\setminus B_R(x_*)} \frac{u_\epsilon(x)u_\epsilon(y)}{|x-y|^{n-\alpha}}dxdy \le N_\alpha \, \|u_\epsilon\|^2_{L^{p_\alpha}(\mathbb{R}^n \setminus B_R(x_0))} = O\left(\frac{R}{\epsilon}\right)^{-n-\alpha} \quad as \quad \epsilon \to 0.$$

By (A_2), we have $f(x) \in C^0(B_\delta(x_*)), B_\delta(x_*) \subseteq \Omega$ for some point x_* within Ω and a positive real number δ. Subsequently, we can select δ_1 such that $0 < \delta_1 < \delta$ thereby ensuring $f(x) > C$ for every x in the ball $B_{\delta_1}(x_*)$ where C is a constant independent of x. Choose $\epsilon < R$ so that $|\epsilon\eta| < \delta_1$ if $\eta \in B_1(0)$. Set

$$I_3 := \int_{B_R(x_*)}\int_\Omega \frac{u_\epsilon(x)f(y)}{|x-y|^{n-\alpha}}dxdy.$$

For I_3, we have

$$\begin{aligned}
I_3 &:= \int_{B_R(x_*)}\int_\Omega |x-y|^{-(n-\alpha)}\left(\frac{\epsilon}{\epsilon^2+|x-x_*|^2}\right)^{\frac{n+\alpha}{2}} f(y)dxdy \\
&\ge \int_{B_R(x_*)}\int_{B_R(x_*)} |x-y|^{-(n-\alpha)}\left(\frac{\epsilon}{\epsilon^2+|x-x_*|^2}\right)^{\frac{n+\alpha}{2}} f(y)dxdy \\
&= \epsilon^{-\frac{n+\alpha}{2}+\alpha-n+2n}\int_{B_{\frac{R}{\epsilon}}(0)}\int_{B_{\frac{R}{\epsilon}}(0)} |\xi-\eta|^{-(n-\alpha)}\left(1+|\xi|^2\right)^{-\frac{n+\alpha}{2}} f(\epsilon\eta+x_*)d\xi d\eta \\
&\ge \epsilon^{-\frac{n+\alpha}{2}+\alpha-n+2n}\int_{B_1(0)}\int_{B_1(0)} |\xi-\eta|^{-(n-\alpha)}\left(1+|\xi|^2\right)^{-\frac{n+\alpha}{2}} Cd\xi d\eta \\
&\ge C_0\epsilon^{\frac{n+\alpha}{2}}.
\end{aligned}$$

So, for $1 < \alpha < (\sqrt{2}-1)n$ and small enough ϵ, we get

$$\begin{aligned}
Q_p(\Omega) &\le \frac{\|u_\epsilon\|^{2\frac{p-1}{2-p}}_{L^p(\Omega)}}{c_{n,\alpha}(\int_\Omega\int_\Omega u_\epsilon(x)\left(|x-y|^{-(n-\alpha)}u_\epsilon(y)\right)dydx))^{\frac{p-1}{2-p}}} \\
&\quad - \|u_\epsilon\|^{-1}_{L^p(\Omega)}\int_\Omega\int_\Omega u_\epsilon(x)\left(|x-y|^{-(n-\alpha)}f(y)\right)dydx \\
&= \frac{\|u_\epsilon\|^{2\frac{p-1}{2-p}}_{L^p(\Omega)}}{(N_\alpha\,\|u_\epsilon\|^2_{L^p(\mathbb{R}^n)} - I_1 + I_2)^{\frac{p-1}{2-p}}} - \|u_\epsilon\|^{-1}_{L^p(\Omega)}I_3 \\
&\le \frac{1}{(N_\alpha - I_1\|u_\epsilon\|^{-2}_{L^p(\Omega)})^{\frac{p-1}{2-p}}} - \|u_\epsilon\|^{-1}_{L^p(\Omega)}I_3 \\
&\le \frac{1}{(N_\alpha)^{\frac{p-1}{2-p}}} + C(I_1)^{\frac{p-1}{2-p}} - \|u_\epsilon\|^{-1}_{L^p(\Omega)}I_3 \\
&\le \frac{1}{(N_\alpha)^{\frac{p-1}{2-p}}} + C_1\left(\frac{R}{\epsilon}\right)^{-n(\frac{n-\alpha}{2\alpha})} - C_0\epsilon^{\frac{n+\alpha}{2}} \\
&\le \frac{1}{(N_\alpha)^{\frac{p-1}{2-p}}} - C_0\epsilon^{\frac{n+\alpha}{2}}.
\end{aligned}$$

□

Notation: For any function $u(x)$ defined on Ω, we always use

$$I_{\alpha,\Omega}u(x) = \int_\Omega \frac{u(y)}{|x-y|^{n-\alpha}}\, dy.$$

Lemma 2.3. *(Lemma 3.1 of [11]) Let $p > p_\alpha$ and $p^{'} = \frac{p}{p-1}$ be its conjugate. There exists a positive constant $C(n,\alpha,\Omega) > 0$ such that*

$$\|I_{\alpha,\Omega}u\|_{L^{p'}(\Omega)} \leq C(n,p,\alpha,\Omega)\|u\|_{L^p(\Omega)} \tag{2.2}$$

holds for any $u \in L^p(\Omega)$. Moreover, for $\alpha > 1$ operator $I_{\alpha,\Omega} : L^p(\Omega) \hookrightarrow L^{p^{'}}(\Omega)$ is a compact embedding.

Lemma 2.4. *Assume $f(x)$ is a non-negative function satisfying (A_1), (A_2). Then,*

$$\inf_{\|u\|_p=1}\left(c_{n,\alpha}(\int_\Omega\int_\Omega \frac{u(x)u(y)}{|x-y|^{n-\alpha}}dxdy)^{\frac{(p-1)}{p-2}} - \int_\Omega\int_\Omega \frac{u(x)f(y)}{|x-y|^{n-\alpha}}dxdy\right) := Q_p(\Omega)$$

is achieved and $Q_p(\Omega) > 0$, where $p = p_\alpha$.

Proof. In order to establish the conclusion, we need to prove that

$$\begin{aligned} Q_p(\Omega) = \inf_{u\in L^p(\Omega), \|u\|_{L^p(\Omega)}^{-p\frac{p-1}{p-2}}=1} & c_{n,\alpha}(\int_\Omega\int_\Omega u(x)\left(|x-y|^{-(n-\alpha)}u(y)\right)dydx))^{\frac{p-1}{p-2}} \\ & - \int_\Omega\int_\Omega u(x)\left(|x-y|^{-(n-\alpha)}f(y)\right)dydx \end{aligned}$$

is achieved, where $p = p_\alpha$. For this purpose, for $2 > p > p_\alpha$, we wil show that the infinum is attained by a positive function u_p. To do this, all we have to do is show

$$\begin{aligned} Q_p(\Omega) = \inf_{u\in L^p(\Omega)\setminus\{0\}} & c_{n,\alpha}\|u\|_{L^p(\Omega)}^{-2\frac{p-1}{p-2}}(\int_\Omega\int_\Omega u(x)\left(|x-y|^{-(n-\alpha)}u(y)\right)dydx))^{\frac{p-1}{p-2}} \\ & - \|u\|_{L^p(\Omega)\setminus\{0\}}^{-1}\int_\Omega\int_\Omega u(x)\left(|x-y|^{-(n-\alpha)}f(y)\right)dydx \end{aligned}$$

is achieved. By Lemma 2.3, we have

$$\left\|I_{\alpha,\Omega}u\right\|_{L^{p'}(\Omega)} \leq C(N,p,\alpha,\Omega)\|u\|_{L^p(\Omega)}$$

where $p^{'} = \frac{p}{p-1}$. Together with the HLS inequality this implies:

$$\begin{aligned} & c_{n,\alpha}\|u\|_{L^p(\Omega)}^{-2\frac{p-1}{p-2}}(\int_\Omega\int_\Omega u(x)\left(|x-y|^{-(n-\alpha)}u(y)\right)dydx))^{\frac{p-1}{p-2}} \\ & - \|u\|_{L^p(\Omega)}^{-1}\int_\Omega\int_\Omega u(x)\left(|x-y|^{-(n-\alpha)}f(y)\right)dydx \\ & \geq c_{n,\alpha}(\frac{\|u\|_{L^p(\Omega)}}{\|I_{\alpha,\Omega}u\|_{L^{p'}(\Omega)}})^{\frac{p-1}{2-p}} - \frac{\|I_{\alpha,\Omega}u\|_{L^{p'}(\Omega)}\|f\|_{L^p(\Omega)}}{\|u\|_{L^p(\Omega)}} \\ & \geq c_{n,\alpha}(\frac{1}{C(n,p,\alpha,\Omega)})^{\frac{p-1}{2-p}} - C(n,p,\alpha,\Omega)\|f\|_{L^p(\Omega)}. \end{aligned}$$

Select a minimizing positive sequence $\left\{u_j\right\}_{j=1}^{\infty}$ such that $\left\|I_{\alpha,\Omega}u_j\right\|_{L^{p'}(\Omega)} = 1$. Thus, $\left\{u_j\right\}$ is bounded in $L^p(\Omega)$. It follows that there exists a subsequence $\{u_j\}$(still denoted as $\{u_j\}$) and $u_* \in L^p(\Omega)$ such that

$$u_j \rightharpoonup u_* \text{ in } L^p(\Omega), \text{ so } \|u_*\|_{L^p(\Omega)} \leq \liminf_{j\to\infty} \left\|u_j\right\|_{L^p(\Omega)}.$$

By Lemma 2.3, we get

$$I_{\alpha,\Omega}u_j \to I_{\alpha,\Omega}u_* \quad \text{ in } L^{p'}(\Omega).$$

Then,

$$\begin{aligned}
Q_p(\Omega) = \lim_{j\to\infty} c_{n,\alpha}\|u_j\|_{L^p(\Omega)}^{-2\frac{p-1}{p-2}}(\int_\Omega\int_\Omega u_j(x)\left(|x-y|^{-(n-\alpha)}u_j(y)\right)dydx))^{\frac{p-1}{p-2}}\\
- \|u_j\|_{L^p(\Omega)\setminus\{0\}}^{-1}\int_\Omega\int_\Omega u_j(x)\left(|x-y|^{-(n-\alpha)}f(y)\right)dydx\\
\geq c_{n,\alpha}\|u_*\|_{L^p(\Omega)}^{-2\frac{p-1}{p-2}}(\int_\Omega\int_\Omega u_*(x)\left(|x-y|^{-(n-\alpha)}u_*(y)\right)dydx))^{\frac{p-1}{p-2}}\\
- \|u_*\|_{L^p(\Omega)\setminus\{0\}}^{-1}\int_\Omega\int_\Omega u_*(x)\left(|x-y|^{-(n-\alpha)}f(y)\right)dydx.
\end{aligned}$$

Therefore, u_* is a minimizer. Thus, we have

$$\begin{aligned}
Q_p(\Omega) = \inf_{u\in L^p(\Omega)\setminus\{0\}} c_{n,\alpha}\|u\|_{L^p(\Omega)}^{-2\frac{p-1}{p-2}}(\int_\Omega\int_\Omega u(x)\left(|x-y|^{-(n-\alpha)}u(y)\right)dydx))^{\frac{p-1}{p-2}}\\
- \|u\|_{L^p(\Omega)\setminus\{0\}}^{-1}\int_\Omega\int_\Omega u(x)\left(|x-y|^{-(n-\alpha)}f(y)\right)dydx.
\end{aligned}$$

Also, by considering $\frac{u}{\|u\|_p}$, we have

$$\begin{aligned}
Q_p(\Omega) = \inf_{u\in L^p(\Omega), \|u\|_{L^p(\Omega)}^{-p\frac{p-1}{p-2}}=1} c_{n,\alpha}(\int_\Omega\int_\Omega u(x)\left(|x-y|^{-(n-\alpha)}u(y)\right)dydx))^{\frac{p-1}{p-2}}\\
- \int_\Omega\int_\Omega u(x)\left(|x-y|^{-(n-\alpha)}f(y)\right)dydx
\end{aligned}$$

is achieved, where $2 > p > p_\alpha$. Thus, for $2 > p > p_\alpha$, the infinum is attained by a positive function u_p, which satisfies the following equation with subcritical exponent

$$\begin{aligned}
(Q_p(\Omega) + \int_\Omega\int_\Omega \frac{p}{2(p-1)}\frac{u_p(x)f(y)}{|x-y|^{n-\alpha}}dxdy)\frac{u_p^{p-1}(x)}{\|u_p\|_{L^{q_\alpha}(\Omega)}^{p\frac{p-1}{2-p}+p}}\\
= c_{n,\alpha}(\int_\Omega\int_\Omega \frac{u_p(x)u_p(y)}{|x-y|^{n-\alpha}}dxdy)^{\frac{1}{p-2}}\int_\Omega \frac{u_p(y)}{|x-y|^{n-\alpha}}dy - \frac{p-2}{2(p-1)}\int_\Omega \frac{f(y)}{|x-y|^{n-\alpha}}dy, \quad x \in \bar{\Omega},
\end{aligned} \tag{2.3}$$

where $\left\|u_p\right\|_p = 1$. We claim that $u_p \in C(\bar{\Omega})$ and $Q_p \to Q_{p_\alpha}$ for $p \to p_\alpha$. First, we prove that $u_p \in C(\bar{\Omega})$. According to Eq (2.3), by writing $g(x) = u^{p-1}(x)$, we can obtain a weak positive solution $g(x) \in L^{p'}(\Omega)$ to

$$g(x) = C(n,p,\alpha,\Omega)\int_\Omega \frac{g^{p'-1}(y)}{|x-y|^{n-\alpha}}dy + C(n,p,\alpha,\Omega)\int_\Omega \frac{f(y)}{|x-y|^{n-\alpha}}dy, \quad x \in \bar{\Omega}, \tag{2.4}$$

for $p' < \frac{2n}{n-\alpha} = q_\alpha$. By (2.4) and HLS inequality, we have

$$\|g\|_{L^s(\Omega)} = \left\|I_{\alpha,\Omega} g^{p'-1}\right\|_{L^s(\Omega)} \le C(n,p,\alpha,\Omega)\left\|u^{p'-1}\right\|_{L^t(\Omega)} + C(n,p,\alpha,\Omega)\|f\|_p$$

for $1/s = 1/t - \alpha/n$. By employing a similar method as in Lemma 3.3 of [11], we can use the above inequality in an iterative process to obtain $g \in C(\bar{\Omega})$. Therefore, we can conclude that $u_p \in C(\bar{\Omega})$. Using a similar method as in Lemma 2.3 of [5], we apply Proposition 2.1 in [11] and the Hölder inequality to find a minimizing sequence of Q_{p_α} from the minimizer u_p. Consequently, we can establish that $Q_p \to Q_{p_\alpha}$ as $p \to p_\alpha$.

Next, we need to show $\lim_{p\to p_\alpha} \left\|u_p\right\|_{L^\infty(\Omega)} \le C$. We prove this by contradiction. Suppose not. Let $u_p\left(x_p\right) = \max_{\bar{\Omega}} u_p(x)$. Then $u_p\left(x_p\right) \to \infty$ as $p \to p_\alpha$. Let $\mu_p = u_p^{p-2+\frac{p}{2-p}}\left(x_p\right)$ and $\Omega_\mu = \frac{\Omega - x_p}{\mu_p} := \left\{z | z = \frac{x-x_p}{\mu_p} \text{ for } x \in \Omega\right\}$. We define $g_p(z) = \mu_p^{\frac{-p^2+4+p}{p-2}} u_p\left(\mu_p z + x_p\right)$ for $z \in \Omega_\mu$. Then, g_p satisfies

$$\begin{aligned}
&(Q_p(\Omega)\frac{g_p^{p-1}(z)}{\|g_p\|_{L^p(\Omega)}^{p\frac{p-1}{2-p}+p}} + \int_\Omega\int_\Omega \frac{p}{2(p-1)}\frac{u_p(x)f(y)}{|x-y|^{n-\alpha}}dxdy\frac{g_p^{p-1}(z)}{\|g_p\|_{L^p(\Omega)}^{p\frac{p-1}{2-p}+p}}\\
&= c_{n,\alpha}(\int_{\Omega_\mu}\int_{\Omega_\mu}\frac{g_p(x)g_p(y)}{|x-y|^{n-\alpha}}dxdy)^{\frac{1}{p-2}}\int_{\Omega_\mu}\frac{g_p(y)}{|z-y|^{n-\alpha}}dy\\
&-\frac{p-2}{2(p-1)}\int_\Omega \frac{u_p^{(1-p)((p-2+\frac{p}{p-1})n)(\frac{p-1}{2-p}+1)}(x_p)f(y)}{|x-y|^{n-\alpha}}dy,
\end{aligned}$$

and $g_p(0) = 1, g_p(z) \in (0,1]$.

For p close to p_α with $1 < \alpha < n$, we have $(1-p)((p-2+\frac{p}{p-1})n)(\frac{p-1}{2-p}+1) < 0$.

$$\begin{aligned}
\int_\Omega \frac{u_p^{(1-p)((p-2+\frac{p}{p-1})n)(\frac{p-1}{2-p}+1)}(x_p)f(y)}{|x-y|^{n-\alpha}}dy &\le C_{n,\alpha}u_p^{(1-p)((p-2+\frac{p}{p-1})n)(\frac{p-1}{2-p}+1)}(x_p)\|f\|_p\\
&\to 0, \quad \text{as} \quad p \to p_\alpha.
\end{aligned} \tag{2.5}$$

Additionally, let $\Omega_R^c = \Omega \backslash \bar{B}_{R\mu_p}\left(x_p\right)$. For p close to p_α, we know $\alpha < n/p$. We can observe that for any fixed $\left|x - x_p\right| < C\mu_p$, as R being chosen large enough

$$\begin{aligned}
\int_{\Omega_R^c}\frac{u_p(y)}{|x-y|^{n-\alpha}}dy &\le \left\|u_p\right\|_p \cdot \left\{\int_{\Omega_R^e}\left[\frac{1}{|x-y|^{n-\alpha}}\right]^{\frac{p}{p-1}}dy\right\}^{\frac{p-1}{p}}\\
&\le C\left(R\mu_p\right)^{\alpha-\frac{n}{p}}.
\end{aligned}$$

Thus,

$$\int_{\Omega_R^c}\frac{u_p^{(1-p)((p-2+\frac{p}{p-1})n)(\frac{p-1}{2-p}+1)}(x_p)\cdot u_p(y)}{|x-y|^{n-\alpha}}dy \le CR^{\alpha-\frac{n}{p}}\cdot u_p^{(1-p)((p-2+\frac{p}{p-1})n)(\frac{p-1}{2-p}+1)}(x_p) \to 0 \tag{2.6}$$

as $p \to p_\alpha$ and $R \to \infty$. As $p \to p_\alpha$, there are two cases:

Case 1. $\Omega_\mu \to \mathbb{R}^n$, and $u_p(z) \to g(z)$ point-wise in $\mathbb{R}^n$ where $g(z)$ satisfies from estimates (2.5) and (2.6) :

$$\begin{aligned}&(Q_{p_\alpha}(\Omega)\frac{g_{p_\alpha}^{p_\alpha-1}(z)}{\|g_{p_\alpha}\|_{L^{p_\alpha}(\mathbb{R}^n)}^{p_\alpha\frac{p_\alpha-1}{2-p_\alpha}+p_\alpha}}+\lim_{p\to p_\alpha}\int_\Omega\int_\Omega\frac{p}{2(p-1)}\frac{u_p(x)f(y)}{|x-y|^{n-\alpha}}dxdy\frac{g_{p_\alpha}^{p_\alpha-1}(z)}{\|g_{p_\alpha}\|_{L^{p_\alpha}(\mathbb{R}^n)}^{p_\alpha\frac{p_\alpha-1}{2-p_\alpha}+p_\alpha}}\\&=c_{n,\alpha}(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{g_{p_\alpha}(x)g_{p_\alpha}(y)}{|x-y|^{n-\alpha}}dxdy)^{\frac{1}{p_\alpha-2}}\int_{\mathbb{R}^n}\frac{g_{p_\alpha}(y)}{|z-y|^{n-\alpha}}dy,\quad z\in\bar{\Omega}.\end{aligned}\tag{2.7}$$

Also, direct computation yields

$$1=\int_\Omega u_p^p(y)dy=u_p^{(p-2+\frac{p}{p-1})n+p}(x_p)g_p^pdz\geq\int_{\Omega_\mu}g_p^pdz.$$

Thus $\int_{\mathbb{R}^n}g^{p_\alpha}dz\leq 1$. Combining this with (2.7) and Lemma 2.2, we have

$$\begin{aligned}\frac{1}{c_{n,\alpha}(N_\alpha)^{\frac{p_\alpha-1}{2-p_\alpha}}}&\leq\frac{\|g\|_{L^{p_\alpha}(\mathbb{R}^n)}^{2\frac{p_\alpha-1}{2-p_\alpha}}}{c_{n,\alpha}(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}g(x)\left(|x-y|^{-(n-\alpha)}g(y)\right)dydx))^{\frac{p_\alpha-1}{2-p_\alpha}}}\\&\leq\frac{\|g\|_{L^{p_\alpha}(\mathbb{R}^n)}^{p_\alpha\frac{p_\alpha-1}{2-p_\alpha}}}{c_{n,\alpha}(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}g(x)\left(|x-y|^{-(n-\alpha)}g(y)\right)dydx))^{\frac{p_\alpha-1}{2-p_\alpha}}}\\&=Q_{p_\alpha}(\Omega)+\lim_{p\to p_\alpha}\int_\Omega\int_\Omega\frac{p}{2(p-1)}\frac{u_p(x)f(y)}{|x-y|^{n-\alpha}}dxdy\\&<\frac{1}{c_{n,\alpha}(N_\alpha)^{\frac{p_\alpha-1}{2-p_\alpha}}}-C_0\epsilon^{\frac{n+\alpha}{2}}+C\|f\|_{L_{p_\alpha}(\Omega)}\\&<\frac{1}{c_{n,\alpha}(N_\alpha)^{\frac{p_\alpha-1}{2-p_\alpha}}}.\end{aligned}$$

This is a contradiction.

Case 2. $\Omega_\mu \to \mathbb{R}^n_T := \{(z_1,z_2,\cdots,z_n)\,|z_n>-T\}$ for some $T>0, g_q(z)\to g(z)$ pointwise in $\mathbb{R}^n_T$, where $g(z)$ satisfies from estimates (2.5) and (2.6) :

$$Q_{p_\alpha}(\Omega)g^{q_\alpha-1}=\int_{\mathbb{R}^n_T}\frac{g(y)}{|z-y|^{n-\alpha}}dy,\quad g(0)=1.$$

Similarly, we know $\int_{\mathbb{R}^n}g^{p_\alpha}dz\leq 1$. Combining this with (2.7), A_2 and Lemma 2.2, we have

$$\begin{aligned}\frac{1}{c_{n,\alpha}(N_\alpha)^{\frac{p_\alpha-1}{2-p_\alpha}}} &\le \frac{\|g\|_{L^{p_\alpha}(\mathbb{R}_{\mathbb{T}}{}^n)}^{2\frac{p_\alpha-1}{2-p_\alpha}}}{c_{n,\alpha}(\int_{\mathbb{R}_{\mathbb{T}}{}^n}\int_{\mathbb{R}_{\mathbb{T}}{}^n} g(x)\,(|x-y|^{-(n-\alpha)}g(y))\,dydx))^{\frac{p_\alpha-1}{2-p_\alpha}}}\\ &\le \frac{\|g\|_{L^{p_\alpha}(\mathbb{R}^n)}^{p_\alpha\frac{p_\alpha-1}{2-p_\alpha}}}{c_{n,\alpha}(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} g(x)\,(|x-y|^{-(n-\alpha)}g(y))\,dydx))^{\frac{p_\alpha-1}{2-p_\alpha}}}\\ &= Q_{p_\alpha}(\Omega) + \lim_{p\to p_\alpha}\int_\Omega\int_\Omega \frac{p}{2(p-1)}\frac{u_p(x)f(y)}{|x-y|^{n-\alpha}}dxdy\\ &< \frac{1}{c_{n,\alpha}(N_\alpha)^{\frac{p_\alpha-1}{2-p_\alpha}}} - C_0\epsilon^{\frac{n+\alpha}{2}} + C\|f\|_{L^{p_\alpha}(\Omega)}\\ &< \frac{1}{c_{n,\alpha}(N_\alpha)^{\frac{p_\alpha-1}{2-p_\alpha}}}.\end{aligned}$$

This is a contradiction.

Let $u_p > 0$ be solutions to (2.3) for $p \in (p_\alpha, 2)$ which are also the minimizers of the energy $Q_p(\Omega)$. Then, $\|u_p\|_{L^\infty(\bar{\Omega})} \le C$, which yields that u_p is uniformly bounded and equi-continuous due to Eq (2.3). Thus, $u_p \to u_*$ as $p \to p_\alpha$ in $C(\bar{\Omega})$, and u_* is the energy minimizer for $Q_{p_\alpha}(\Omega)$. □

Lemma 2.5. *Let f be a non-negative function satisfying (A_1), (A_2). For every $u \in \Lambda, u \neq 0$, $p = p_\alpha$ we have*

$$(p-1)\|u\|_p^p - \int_\Omega\int_\Omega \frac{u(x)u(y)}{|x-y|^{n-\alpha}}dxdy \neq 0,$$

(i.e., $\Lambda_0 = \{0\}$).

Proof. By contradiction, assume that for some $u \in \Lambda$ with $u \neq 0$, we have

$$(p-1)\|u\|_p^p - \int_\Omega\int_\Omega \frac{u(x)u(y)}{|x-y|^{n-\alpha}}dxdy = 0. \tag{2.8}$$

Thus, we have

$$0 = \|u\|_p^p - \int_\Omega\int_\Omega \frac{u(x)u(y)}{|x-y|^{n-\alpha}}dxdy - \int_\Omega\int_\Omega \frac{u(x)f(y)}{|x-y|^{n-\alpha}}dxdy = (2-p)\|u\|_p^p - \int_\Omega\int_\Omega \frac{u(x)f(y)}{|x-y|^{n-\alpha}}dxdy. \tag{2.9}$$

Using the HLS inequality and the condition (2.8), we have

$$\|u\|_p \geqq \left(\frac{p-1}{N_\alpha}\right)^{1/(2-p)} := \gamma,$$

and from (2.9) we obtain:

$$0 < Q_p(\Omega)\gamma \leqq \psi(u) =$$

$$\left[\frac{1}{p-1}\right]^{(p-1)/(p-2)}(2-p)\left[\frac{\int_\Omega\int_\Omega \frac{u(x)u(y)}{|x-y|^{n-\alpha}}dxdy^{(p-1)}}{\|u\|_p^p}\right]^{1/(p-2)} - \int_\Omega\int_\Omega \frac{u(x)f(y)}{|x-y|^{n-\alpha}}dxdy$$

$$= (2-p)\left(\left[\frac{1}{p-1}\right]^{(p-1)/(p-2)}\left[\frac{\int_\Omega\int_\Omega \frac{u(x)u(y)}{|x-y|^{n-\alpha}}dxdy^{(p-1)}}{\|u\|_p^p}\right]^{1/(p-2)} - \|u\|_p^p\right)$$

$$= (2-p)\|u\|_p^p\left(\left[\frac{\int_\Omega\int_\Omega \frac{u(x)u(y)}{|x-y|^{n-\alpha}}dxdy}{(p-1)\|u\|_p^p}\right]^{(p-1)/(p-2)} - 1\right) = 0,$$

which leads to a contradiction. □

As a consequence of Lemma 2.5 we have:

Lemma 2.6. *Let $f(x)$ be a non-negative function satisfying (A_1), (A_2). Given $u \in \Lambda, u \neq 0$, $p = p_\alpha$, there exist $\varepsilon > 0$ and a differentiable function $t = t(w) > 0$, $w \in L^p(\Omega)$, $\|w\| < \varepsilon$ satisfying the following properties:*

$$t(0) = 1, \quad t(w)(u-w) \in \Lambda, \quad \textit{for } \|w\| < \varepsilon$$

and

$$\langle t'(0), w\rangle = \frac{p\int_\Omega |u|^{p-2}uw - 2\int_\Omega\int_\Omega \frac{u(x)w(y)}{|x-y|^{n-\alpha}}dxdy - \int_\Omega\int_\Omega \frac{f(x)w(y)}{|x-y|^{n-\alpha}}dxdy}{(p-1)\|u\|_p^p - \int_\Omega\int_\Omega \frac{u(x)u(y)}{|x-y|^{n-\alpha}}dxdy}. \tag{2.10}$$

Proof. Define $F : \mathbb{R} \times L^p(\Omega) \to \mathbb{R}$ as follows:

$$\begin{aligned} F(t,w) &= t^{p-1}\|u-w\|_p^p - t\int_\Omega\int_\Omega \frac{(u(x)-w(x))(u(y)-w(y))}{|x-y|^{n-\alpha}}dxdy \\ &\quad - \int_\Omega\int_\Omega \frac{f(x)(u(y)-w(y))}{|x-y|^{n-\alpha}}dxdy. \end{aligned}$$

Since $F(1,0) = 0$ and $\mathrm{F}_t(1,0) = (p-1)\|u\|_p^p - \int_\Omega\int_\Omega \frac{u(x)u(y)}{|x-y|^{n-\alpha}}dxdy \neq 0$ (by Lemma 2.5), we can apply the implicit function theorem at the point $(1,0)$ and obtain the desired result. □

3. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. Let us denote

$$p = p_\alpha, \ \inf_\Lambda I = c_0. \tag{3.1}$$

We will first show that I is bounded from below in Λ. For $u \in \Lambda$, we have:

$$\int_\Omega |u|^p - \int_\Omega\int_\Omega \frac{u(x)u(y)}{|x-y|^{n-\alpha}}dxdy - \int_\Omega\int_\Omega \frac{u(x)f(y)}{|x-y|^{n-\alpha}}dxdy = 0.$$

Thus,

$$\begin{aligned} \mathrm{I}(u) &= \frac{1}{p}\int_\Omega |u|^p - \frac{1}{2}\int_\Omega\int_\Omega \frac{u(x)u(y)}{|x-y|^{n-\alpha}}dxdy - \int_\Omega\int_\Omega \frac{u(x)f(y)}{|x-y|^{n-\alpha}}dxdy \\ &\geq \left(\frac{1}{p} - \frac{1}{2}\right)\|u\|_p^p - \frac{N_\alpha}{2}\|u\|_p\|f\|_p \geq C\|f\|_p^{\frac{p}{p-1}}. \end{aligned}$$

In particular, we have

$$c_0 \geq C\|f\|_p^{\frac{p}{p-1}}.$$

To obtain an upper bound for c_0, let $v \in L^p(\Omega)$ be a positive solutions for $u^{p-1} = \int_\Omega \frac{f(y)}{|x-y|^{n-\alpha}} dy$. So, for $f \neq 0$

$$\int_\Omega \int_\Omega \frac{f(x)v(y)}{|x-y|^{n-\alpha}} dxdy = \|v\|_p^p > 0.$$

Set $t_0 = t^-(v) > 0$ as defined by Lemma 2.1. It follows that $t_0 v \in \Lambda^+$ and

$$\begin{aligned} \mathrm{I}(t_0 v) &= \frac{t_0^p}{p} \int_\Omega |v|^p - \frac{t_0^2}{2} \int_\Omega \int_\Omega \frac{v(x)v(y)}{|x-y|^{n-\alpha}} dxdy - t_0 \int_\Omega \int_\Omega \frac{v(x)f(y)}{|x-y|^{n-\alpha}} dxdy \\ &= t_0^p (\frac{1}{p} - 1) \int_\Omega |v|^p + \frac{t_0^2}{2} \int_\Omega \int_\Omega \frac{v(x)v(y)}{|x-y|^{n-\alpha}} dxdy < \frac{-2n\alpha + 2\alpha^2}{n+\alpha} t_0^p \int_\Omega |v|^p. \end{aligned}$$

This implies

$$c_0 < \frac{-2n\alpha + 2\alpha^2}{n+\alpha} t_0^p \int_\Omega |v|^p < 0. \tag{3.2}$$

It is clear that Ekeland's variational principle (see [1], Corollary 5.3.2) holds for the minimization problem (3.1). This principle provides a minimizing sequence $\{u_m\} \subset \Lambda$ with the following properties: (i) $\mathrm{I}(u_m) < c_0 + \frac{1}{m}$, (ii) $\mathrm{I}(w) \geq \mathrm{I}(u_m) - \frac{1}{m} \|(w - u_m)\|_p$, $\forall w \in \Lambda$. By taking m large, from (3.2) we have

$$\begin{aligned} \mathrm{I}(u_m) \quad &= (\tfrac{1}{p} - \tfrac{1}{2}) \textstyle\int_\Omega |u_m|^p - \tfrac{1}{2} \int_\Omega \int_\Omega \frac{u_m(x)f(y)}{|x-y|^{n-\alpha}} dxdy \\ &< c_0 + \tfrac{1}{m} < \tfrac{-2n\alpha+2\alpha^2}{n+\alpha} t_0^p \textstyle\int_\Omega |v|^p. \end{aligned} \tag{3.3}$$

Thus, it follows that

$$\frac{1}{2} \int_\Omega \int_\Omega \frac{u_m(x)f(y)}{|x-y|^{n-\alpha}} dxdy \geqq \frac{2n\alpha - 2\alpha^2}{n+\alpha} t_0^p \int_\Omega |v|^p > 0. \tag{3.4}$$

Therefore, we have $u_m \neq 0$. By applying HLS inequality, $u_m \neq 0$ and (3.3), we obtain

$$\|u_m\|_p \leq C_{n,\alpha} \|f\|_p^{\frac{1}{p-1}}. \tag{3.5}$$

Using HLS inequality and (3.4), we have

$$C_{n,\alpha} \frac{\|v\|_p^p}{\|f\|_p} \leq \|u_m\|_p. \tag{3.6}$$

Applying (3.5) and (3.6), we obtain

$$C_{n,\alpha} \frac{\|v\|_p^p}{\|f\|_p} \leq \|u_m\|_p \leq C_{n,\alpha} \|f\|_p^{\frac{1}{p-1}}. \tag{3.7}$$

Our goal is to show that $\|\mathrm{I}'(u_m)\|_p \to 0$ as $m \to +\infty$. Hence, let us assume $\|I'(u_m)\|_p > 0$ for m large (otherwise we are done). Applying Lemma 2.6 with $u = u_m$ and $w = \delta \frac{I'(u_m)}{\|I'(u_m)\|_p}$, $\delta > 0$ small, we find $t_m(\delta) := t\left[\delta \frac{\mathrm{I}'(u_m)}{\|\mathrm{I}'(u_m)\|_p}\right]$ such that

$$w_\delta = t_m(\delta)\left[u_m - \delta \frac{I'(u_m)}{\|I'(u_m)\|_p}\right] \in \Lambda.$$

Using condition (ii) we have

$$\frac{1}{m}\|(w_\delta - u_m)\|_p \geq \mathrm{I}(u_m) - \mathrm{I}(w_\delta) = (1 - t_m(\delta))\langle \mathrm{I}'(w_\delta), u_m\rangle + \delta t_m(\delta)\left\langle \mathrm{I}'(w_\delta), \frac{\mathrm{I}'(u_m)}{\|\mathrm{I}'(u_m)\|_p}\right\rangle + o(\delta).$$

Dividing by $\delta > 0$ and passing to the limit as $\delta \to 0$ we derive

$$\frac{1}{m}\left(1 + \left|t_m'(0)\right| \|u_m\|_p\right) \geq -t_m'(0)\langle \mathrm{I}'(u_m), u_m\rangle + \|\mathrm{I}'(u_m)\|_p = \|\mathrm{I}'(u_m)\|_p$$

where we set $t_m'(0) = \left\langle t'(0), \frac{\mathrm{I}'(u_m)}{\|\mathrm{I}'(u_m)\|_p}\right\rangle$. Thus, from (3.7) we conclude that

$$\|\mathrm{I}'(u_m)\|_p \leq \frac{\mathrm{C}}{m}\left(1 + \left|t_m'(0)\right|\right)$$

for a suitable positive constant C. We do this once we show that $\left|t_m'(0)\right|$ is bounded uniformly on m. From (2.10) and the estimate (3.7) we get

$$|t_m'(0)| \leq \frac{C_1}{|\ (p-1)\|u_m\|_p^p - \int_\Omega \int_\Omega \frac{u_m(x)u_m(y)}{|x-y|^{n-\alpha}}dxdy\ |},$$

$\mathrm{C}_1 > 0$ suitable constant. Hence, we need to show that $|\ (p-1)\|u_m\|_p^p - \int_\Omega \int_\Omega \frac{u_m(x)u_m(y)}{|x-y|^{n-\alpha}}dxdy\ |$ is bounded away from zero.

On the contrary, suppose that for a subsequence which we still call u_m we have

$$(p-1)\|u_m\|_p^p - \int_\Omega \int_\Omega \frac{u_m(x)u_m(y)}{|x-y|^{n-\alpha}}dxdy = o(1). \tag{3.8}$$

Using the estimates (3.7) and (3.8), we obtain

$$\|u_m\|_p \geq \gamma \quad (\gamma > 0 \text{ suitable constant}) \tag{3.9}$$

and

$$\left[\frac{\int_\Omega \int_\Omega \frac{u_m(x)u_m(y)}{|x-y|^{n-\alpha}}dxdy}{p-1}\right]^{(p-1)/(p-2)} - \left[\|u_m\|_p^p\right]^{(p-1)/(p-2)} = o(1).$$

Furthermore, combining (3.8) with the fact that $u_m \in \Lambda$ we also have

$$\int_\Omega \int_\Omega \frac{u_m(x)f(y)}{|x-y|^{n-\alpha}}dxdy = (2-p)\|u_m\|_p^p + o(1).$$

This, together with (3.9) and Lemma 2.4 implies

$$\begin{aligned} 0 < Q_p(\Omega)\gamma^{2/2-p} &\leq \|u_m\|_p^{p/(2-p)}\psi(u_m) \\ &= (2-p)\left[\left[\frac{\int_\Omega \int_\Omega \frac{u_m(x)u_m(y)}{|x-y|^{n-\alpha}}dxdy}{p-1}\right]^{(p-1)/(p-2)} - \left[\|u_m\|_p^{(p-1)/(p-2)}\right] = o(1), \right. \end{aligned}$$

which is clearly impossible. Therefore,

$$\|I'(u_m)\|_p \to 0 \quad \text{as } n \to +\infty. \tag{3.10}$$

Let $u_0 \in L^p(\Omega)$ be the weak limit in $L^p(\Omega)$ of (a subsequence of) u_m. From (3.7) we derive that

$$\int_\Omega \int_\Omega \frac{u_0(x) f(y)}{|x-y|^{n-\alpha}} dxdy > 0$$

and from (3.10) we have

$$\langle I'(u_m), w\rangle = 0, \quad \forall w \in L^p(\Omega),$$

i.e., u_0 is a weak solution for (1.1). In particular, $u_0 \in \Lambda$. Therefore,

$$c_0 \le \mathrm{I}(u_0) = (\frac{1}{p} - \frac{1}{2}) \int_\Omega |u_0|^p - \frac{1}{2} \int_\Omega \int_\Omega \frac{u_0(x) f(y)}{|x-y|^{n-\alpha}} dxdy \le \lim_{n \to +\infty} \mathrm{I}(u_m) = c_0.$$

Consequently $u_m \to u_0$ strongly in $L^p(\Omega)$ and $I(u_0) = c_0 = \inf_\Lambda I$. Also, from Lemma 2.1 and (3.10), we can conclude that $u_0 \in \Lambda^+$. Finally, since $f \ge 0$ we can easily deduce that $u_0 \ge 0$ from [22]. Therefore, for $p = p_\alpha$, the problem (1.1) has a positive solution $u_0 \in \Lambda^+$.

Proof of Theorem 1.2. Let $\frac{2n}{n+\alpha} < p < 2$(subcritical case) and

$$\inf_{\Lambda^-} \mathrm{I} = c_1.$$

Similar to the proof of Theorem 1.1, we can show that there is a solution $u_0 \in \Lambda^+$ to Eq (1.1) using compactness imbedding theorem (see Lemma 2.3). Analogously to the proof of the first solution, one can show that the Ekeland's variational principle gives a sequence $\{u_m\} \subset \Lambda^-$ satisfying:

$$\mathrm{I}(u_m) \to c_1,$$
$$\|\mathrm{I}'(u_m)\|_p \to 0.$$

Furthermore, by the compactness imbedding theorem (Lemma 2.3) it can be proved that the functional I satisfies the usual $(PS)_{c_1}$ condition for the subcritical equation. For $\frac{2n}{n+\alpha} < p < 2$, there is another solution $u_1 \in \Lambda^-$ to Eq (1.1). We can also deduce that $u_1 \ge 0$ from $f \ge 0$ (see [22]).

4. Conclusions

In this paper, we demonstrate that under suitable assumptions on f, the integral equation admits a positive solution in $L^{\frac{2n}{n+\alpha}}(\Omega)$. Our approach combines the Ekeland variational principle, a blow-up argument, and a rescaling argument. Additionally, we establish the existence of multiple solutions for this equation in the subcritical case. In the next section, we will investigate the existence of multiple solutions in the critical case.

Use of AI tools declaration

The author declares that he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares there is no conflicts of interest.

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