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Research article

Sharp Adams type inequalities in Lorentz-Sobole space

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Abstract: This article addresses several sharp weighted Adams type inequalities in Lorentz-Sobolev spaces by using symmetry, rearrangement and the Riesz representation formula. In particular, the sharpness of these inequalities were also obtained by constructing a proper test sequence.

Keywords: Adams type inequalities; Lorentz-Sobolev space; Moser-Trudinger type inequalities; Hardy-Littlewood inequality; Riesz representation
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1. Introduction

Sharp Moser-Trudinger inequality and its high-order form (which is called Adams inequality) have received a lot of attention due to their wide applications to problems in geometric analysis, partial differential equations, spectral theory and stability of matter [2, 3, 5, 8–12, 24–27]. This paper is concerned with the problem of finding optimal Adams type inequalities in Lorentz-Sobolev space.

The Trudinger inequality, which can be seen as the critical case of the Sobolev imbedding, was first obtained by Trudinger [30]. More precisely, Trudinger employed the power series expansion to prove that there exists $\beta > 0$, such that

$$\sup_{\|\nabla u\|_n^n \le 1, u \in W_0^{1,n}(\Omega)} \int_\Omega \exp(\beta |u|^{\frac{n}{n-1}}) dx < \infty, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain and $W_0^{1,p}(\Omega)$ denotes the usual Sobolev space on Ω, i.e., the completion of $C_0^\infty(\Omega)$(the space of all functions being infinity-times continuously differential in Ω with compact support) with the norm

$$\|u\|_{W_0^{1,p}(\Omega)} = \int_\Omega (|\nabla u(x)|^p + |u(x)|^p)\, dx.$$

Let $\Omega \subset \mathbb{R}^n$ be an open domain with finite measure. It is well known that for a positive integer $k < n$ and $1 \le p < \frac{n}{k}$, the Sobolev space $W_0^{k,p}(\Omega)$ embeds continuously into $L^{\frac{np}{n-kp}}(\Omega)$, but in the borderline case $p = \frac{n}{k}$, $W_0^{k,\frac{n}{k}}(\Omega) \subsetneq L^\infty(\Omega)$, unless $k = n$. For the case $k = 1$, Yudovich [31] and Trudinger [30] have shown that

$$W_0^{1,n}(\Omega) \subset \{u \in L^1(\Omega) : E_\beta := \int_\Omega e^{\beta|u|^{\frac{n}{n-1}}} dx < \infty\}, for\ any\ \beta < \infty$$

and the function E_β is continuous on $W_0^{1,n}(\Omega)$. In 1971, Moser sharped the Trudinger inequality and gave the sharp constant $\beta = nw_{n-1}^{\frac{1}{n-1}}$ of (1.1) by using the technique of the symmetry and rearrangement in [20].

Theorem A. *[20]* Let $\Omega \subset \mathbb{R}^n$ be an open domain with finite measure. Then, there exists a sharp constant $\beta_n = n\left(\frac{n\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})+1}\right)^{\frac{1}{n-1}}$, such that

$$\frac{1}{|\Omega|}\int_\Omega \exp(\beta|f|^{\frac{n}{n-1}})dx \le C_0 < \infty$$

for any $\beta \le \beta_n$ and any $f \in C_0^\infty(\Omega)$ with $\int_\Omega |\triangledown f|^n dx \le 1$. The constant β_n is sharp in the sense that the above inequality can no longer hold with some C_0 independent of f if $\beta > \beta_n$.

Theorem A has been extended in many directions, one of which states that

$$\sup_{u\in W_0^{1,n}(\Omega),\ \|\nabla u\|_n \le 1} \frac{1}{|\Omega|}\int_\Omega \exp(\beta\,|u|^{\frac{n}{n-1}})dx < \infty$$

for any $\beta \le \beta_n = n\omega_{n-1}^{\frac{1}{n-1}}$, plays an important role in analysis, where ω_{n-1} is the surface measure of the unit ball in $\mathbb{R}^n$. In fact, the constant β_n is sharp in the sense that if $\beta > \beta_n$, the supremum is infinity.

Since the Polyá-Szegö inequality, on which the technique of the symmetry and rearrangement depends, is not valid on the high-order Sobolev space, many challenges arise in the research of high-order Trudinger-Moser inequalities. In 1988, Adams [1] utilized the method of representative formulas and potential theory to establish the sharp Adams inequalities on bounded domains.

Theorem B. *[1]* Let Ω be an open and bounded set in $\mathbb{R}^n$. If m is a positive integer less than n, then there exists a constant $C_0 = C(n,m) > 0$ such that for any $u \in W_0^{m,\frac{n}{m}}(\Omega)$ with $\|\nabla^m u\|_{L^{\frac{n}{m}}(\Omega)} \le 1$,

$$\frac{1}{|\Omega|}\int_\Omega \exp(\beta|u(x)|^{\frac{n}{n-m}})dx \le C_0\ for\ all\ \beta \le \beta(n,m), \tag{1.2}$$

where

$$\beta(n,m) = \begin{cases} \frac{n}{\omega_{n-1}}\left[\frac{\pi^{\frac{n}{2}}2^m\Gamma(\frac{m+1}{2})}{\Gamma(\frac{n-m+1}{2})}\right]^{\frac{n}{n-m}}, & m \text{ is odd}, \\ \frac{n}{\omega_{n-1}}\left[\frac{\pi^{\frac{n}{2}}2^m\Gamma(\frac{m}{2})}{\Gamma(\frac{n-m}{2})}\right]^{\frac{n}{n-m}}, & m \text{ is even}. \end{cases}$$

Furthermore, the constant $\beta(n,m)$ is best possible in the sense that for any $\beta > \beta(n,m)$, the integral can be made as large as possible. In the case of Sobolev space with homogeneous Navier boundary conditions $W_N^{m,\frac{n}{m}}(\Omega)$, the Adams inequality was extended by Cassani and Tarsi in [6]. It is easy to check that $W_N^{m,\frac{n}{m}}(\Omega)$ contains $W_0^{m,\frac{n}{m}}(\Omega)$ as a closed subspace.

Adimurthi and Sandeep proved a singular Moser-Trudinger inequality with the sharp constant in [2]. Since then, Moser's results for the first order derivatives and Adams' result for the high order derivatives were extended to the unbounded domain case. Earlier research of the Moser-Trudinger inequalities on the whole space goes back to Cao's work in [7]. Later, Li and Ruf [19, 23] improved Cao's result and established the following result

$$\sup_{\|u\|_{W^{1,n}(\mathbb{R}^n)}\le 1}\int_{\mathbb{R}^n}\Phi(\beta_n|u|^{\frac{n}{n-1}})dx \le C_n, \tag{1.3}$$

where proof relies on the rearrangement argument and the Polyá-Szegö inequality. For more on the rearrangement argument, see [21, 29]. In 2013, Lam and Lu [17] used a symmetrization-free approach to give a simple proof for the sharp Moser-Trudinger inequalities in $W^{1,n}(\mathbb{R}^n)$. It should be pointed out that this approach is surprisingly simple and can be easily applied to other settings where symmetrization argument does not work. Furthermore, they also developed a new tool to establish the Moser-Trudinger inequalities on the Heisenberg group and the Fractional Adams inequalities in $W^{s,\frac{n}{s}}(\mathbb{R}^n)$ $(0 < s < n)$ ([16]). For more applications of the symmetrization-free method, see also [18, 32]. The Adams type inequality on $W_0^{m,\frac{n}{m}}(\Omega)$ when Ω has infinite volume and m is an even integer was studied recently by Ruf and Sani in [22].

In [22], Ruf and Sani used the norm $\|u\|_{m,n} = \left\|(-\triangle + I)^{\frac{m}{2}}u\right\|_{\frac{n}{m}}$, which is equivalent to the standard Sobolev norm

$$\|u\|_{W^{m,\frac{n}{m}}} = (\|u\|_{\frac{n}{m}}^{\frac{n}{m}} + \sum_{j=1}^{m}\left\|\nabla^j u\right\|_{\frac{n}{m}}^{\frac{n}{m}})^{\frac{m}{n}}.$$

In particular, if $u \in W_0^{m,\frac{n}{m}}(\Omega)$ or $u \in W^{m,\frac{n}{m}}(\mathbb{R}^n)$, then $\|u\|_{W^{m,\frac{n}{m}}} \le \|u\|_{m,n}$. Since Ruf and Sani only considered the case when m is even, it leaves an open question if Ruf and Sani's result is still right when m is odd. Recently, the authors of [17] solved the problem and proved the results of Adams type inequalities on unbounded domains when m is odd.

We notice that when Ω has infinite volume, the usual Moser-Truding inequality become meaningless. In the case $|\Omega| = +\infty$, a modified Moser-Truding type inequality was established in [13].

Theorem C. *[13]* Assume $n \ge 2, \beta > 0, -\infty < s \le \alpha < n$ and $u \in L^n(\mathbb{R}^n; |x|^{-s}dx) \cap W^{1,n}(\mathbb{R}^n)$, there esists a positive constant $C = C(n, s, \alpha, \beta)$ such that the inequality

$$\int_{\mathbb{R}^n}\frac{\phi(\beta\,|u|^{\frac{n}{n-1}})}{|x|^\alpha}dx \le C\|u\|_{L^n(\mathbb{R}^n;|x|^{-s}dx)}^{\frac{n(n-\alpha)}{n-s}}.$$

Furthermore, for all $\beta \le (1-\frac{\alpha}{n})\beta_n$, there holds

$$\int_{\mathbb{R}^n}\frac{\phi(\beta\,|u|^{\frac{n}{n-1}})}{|x|^\alpha}dx \le C\|u\|_{L^n(\mathbb{R}^n;|x|^{-s}dx)}^{\frac{n(n-\alpha)}{n-s}},$$

where $\phi(t) = e^t - \sum_{j=0}^{n-2}\frac{t^j}{j!}$ and $L^n(\mathbb{R}^n; |x|^{-s}dx)$ denotes the weighted Lebesgue space endowed with the norm

$$\|u\|_{L^n(\mathbb{R}^n;|x|^{-s}dx)} := \left(\int_{\mathbb{R}^n}|u(x)|^n|x|^{-s}dx\right)^{\frac{1}{n}}.$$

Moreover the constant $(1-\frac{\alpha}{n})\beta_n$ is sharp in the sense that if $\beta > (1-\frac{\alpha}{n})\beta_n$, the supremum is infinity.

When $\alpha = 0$, Ruf in [23] and Li-Ruf in [19] proved the above modified Moser-Truding type inequality in $\mathbb{R}^2$. Such type of inequality on unbounded domains in the subcritical case ($\beta < \beta_n,\ \alpha = 0$) was first established by Cao in [7] for $n = 2$ and Adachi Tanaka in [4] for $n \geq 3$ in high dimension.

In this paper, we will consider some sharp Adams type inequalities in Lorentz-Sobolev space $W^{\alpha}_{\frac{n}{m},q}(\Omega \subseteq \mathbb{R}^n)$ with $q \neq n$ (If $q = n$, the Lorentz norm becomes the $L^n(\mathbb{R}^n)$ domain norm). Let $1 < p < +\infty$ and $1 \leq q < +\infty$. Then we recall the Lorentz space $L_{p,q}(\mathbb{R}^n)$ as: $\psi \in L_{p,q}(\mathbb{R}^n)$ if

$$\|\psi\|^*_{p,q} = \begin{cases} (\int_0^{+\infty}[\psi^*(t)t^{\frac{1}{p}}]^q \frac{dt}{t})^{\frac{1}{q}} < \infty, & 1 \leq q < \infty, \\ \sup_{t>0} \psi^*(t)t^{\frac{1}{p}} < \infty, & q = \infty. \end{cases} \tag{1.4}$$

It is well known that $\|\cdot\|^*_{p,q}$ is not a norm, and

$$\|\psi\|_{p,q} = \left(\int_0^{+\infty}[\psi^{**}(t)t^{\frac{1}{p}}]^q \frac{dt}{t}\right)^{\frac{1}{q}}$$

is a norm for any p and q. However, they are equivalent in the sense that

$$\|\psi\|_{p,q} \leq \|\psi\|^*_{p,q} \leq C(p,q)\|\psi\|_{p,q}.$$

The Sobolev-Lorentz space ([15])

$$W^{\alpha}_{\frac{n}{m},q}(\mathbb{R}^n) := (I - \Delta)^{-\frac{\alpha}{2}} L_{\frac{n}{m},q}(\mathbb{R}^n)$$

equipped with the norm

$$\|u\|_{W^{\alpha}_{\frac{n}{m},q}} = \|(I - \Delta)^{\frac{\alpha}{2}} u\|_{\frac{n}{m},q}$$

for $0 < \alpha < n, m < n, 1 < q < \infty$. For simplicity of notation, we write

$$\overline{W^{m}_{\frac{n}{m},q}(\Omega)} = \left\{u \in W^{m}_{\frac{n}{m},q}(\Omega),\ \left\|(I - \triangle)^{\frac{m}{2}} u\right\|_{\frac{n}{m},q} \leq 1\right\}$$

for any $\Omega \subseteq \mathbb{R}^n$. Then we can formulate our main results as follows.

Theorem 1. *Let $m \leq n$ be an integer, $0 \leq \alpha < n$, $1 < q < +\infty$ and A be a positive real number. Then for any bounded domain $\Omega \subset \mathbb{R}^n$ with $|\Omega| \geq A > 0$, we have*

(1) $\sup\limits_{u \in \overline{W^{m}_{\frac{n}{m},q}(\Omega)}} \frac{1}{|\Omega|} \int_\Omega \exp(\beta_{n,m,q} |u|^{\frac{q}{q-1}}) dx \leq C_{m,n,q}.$

Additionally, the constant $\beta_{n,m,q} = \left(\frac{n}{\omega_{n-1}}\right)^{q' \frac{n-m}{n}} K^{-q'}_{m,n}$ is sharp in the sense that the supremum is infinity if $\beta > \beta_{n,m,q}$, where $K_{m,n} = \frac{\Gamma(\frac{n-m}{2})}{\pi^{\frac{n}{2}} 2^m \Gamma(\frac{m}{2})}$.

(2) $\sup\limits_{u \in \overline{W^{m}_{\frac{n}{m},q}(\Omega)}} \int_\Omega \frac{\exp[\beta_{n,m,q}(1-\frac{\alpha}{n})|u|^{\frac{q}{q-1}}]}{|x|^\alpha} \leq C_{m,n,q,\alpha}.$

Additionally, the constant $\beta_{n,m,q}$ is sharp in the sense that the supremum is infinity if $\beta > \beta_{n,m,q}$.

For the unbounded domain, we take $\mathbb{R}^n$ for example to have the following inequalities.

Theorem 2. *Let m, q, α be the same as in Theorem 1. Then we have*

$$\sup_{u \in W^m_{\frac{n}{m},q}(\mathbb{R}^n)} \int_{\mathbb{R}^n} \Phi(\beta_{n,m,q}\,|u|^{\frac{q}{q-1}})dx \le C_{m,n,q},$$

and

$$\sup_{u \in W^m_{\frac{n}{m},q}(\mathbb{R}^n)} \int_{\mathbb{R}^n} \frac{\Phi[\beta_{n,m,q}(1-\frac{\alpha}{n})|u|^{\frac{q}{q-1}}]}{|x|^\alpha} dx \le \tilde{C}_{m,n,q,\alpha},$$

where $\Phi(x) = e^x - \sum_{j=0}^{k_0} \frac{x^j}{j!}, k_0 = [\frac{q-1}{q}\frac{n}{m}]$ and $\beta_{n,m,q}$ is sharp in the sense that the supremum is infinity if $\beta > \beta_{n,m,q}$.

2. Proofs of the main results

We begin this section with some preparations which are necessary for the proofs of our main results. Let $f : \mathbb{R}^n \to \mathbb{R}$ such that

$$|\{x \in \mathbb{R}^n : |f(x)| > t\}| = \int_{\{x \in \mathbb{R}^n : |f(x)| > t\}} dx < +\infty$$

for every $t > 0$. Its distribution function $d_f(t)$ and its decreasing rearrangement f^* are defined by

$$d_f(t) = |\{x : |f(x)| > t\}|,$$

and

$$f^*(s) = \sup\{t > 0, \mu_f(t) > s\},$$

respectively. Now, define $f^\sharp : \mathbb{R}^n \to \mathbb{R}$ by

$$f^\sharp(x) = f^*(v_n|x|^n),$$

where v_n is the volume of the unit ball in $\mathbb{R}^n$. Then for every continuous increasing function $\Psi : [0, +\infty) \to [0, +\infty)$, it follows from [14] that

$$\int_{\mathbb{R}^n} \Psi(f)dx = \int_{\mathbb{R}^n} \Psi(f^\sharp)dx.$$

Since f^* is nonincreasing, the maximal function of f^*, which is defined by

$$f^{**} := \frac{1}{s}\int_0^s f^* dt \, for \; s \ge 0$$

is also nonincreasing and $f^* \le f^{**}$. For more properties of the rearrangement, we refer the reader to [14, 28].

Lemma 2.1. *Let $0 < \alpha \le 1, 1 < p < \infty$ and $a(s,t)$ be a non-negative measurable function on $(-\infty, \infty) \times [0, \infty]$ such that*

$$a(s,t) \le 1, when \;\; 0 < s < t,$$

$$\sup_{t>0}\left(\int_{-\infty}^{0} a(s,t)^{p'}ds + \int_{t}^{\infty} a(s,t)^{p'}ds\right)^{1/p'} = b < \infty.$$

Then there is a constant $c_0 = c_0(p, b, \alpha)$ *such that if*

$$\int_{-\infty}^{\infty} \phi(s)^p ds \le 1, \ \textit{for } \phi \ge 0,$$

then

$$\int_0^{\infty} e^{-F_\alpha(t)}dt \le c_0, \ \textit{where } F_\alpha(t) = \alpha t - \alpha\left(\int_{-\infty}^{\infty} a(s,t)\phi(s)ds\right)^{p'}. \tag{2.1}$$

Proof. The integral in (2.1) can be written as

$$\int_{-\infty}^{\infty} |E_{\alpha\lambda}|e^{-\lambda}d\lambda = \int_0^{\infty} e^{-F_\alpha(t)}dt,$$

where $F_\alpha(t) \le \lambda$ and $E_{\alpha\lambda} = \int_\Omega e^{\alpha\lambda|u|^{\frac{n}{n-1}}}dx$.

We first show that there is a constant $C = C(p, b, \alpha) > 0$ such that $F_\alpha(t) \ge -C$ for all $t \ge 0$. To do so, we claim that if $E_{\alpha\lambda} \ne \emptyset$, then $\lambda \ge -C$, and furthermore that if $t \in E_{\alpha\lambda}$, then there are $A_1 > 0$ and $B_1 > 0$ such that

$$(b^{p'} + t)^{\frac{1}{p}}\left(\int_t^{\infty} \phi(s)^p ds\right)^{\frac{1}{p}} \le A_1 + B_1|\lambda|^{\frac{1}{p}}.$$

In fact, if $E_{\alpha\lambda} \ne \emptyset$, and $t \in E_{\alpha\lambda}$, then

$$t - \frac{\lambda}{\alpha} \le t - \frac{F_\alpha(t)}{\alpha} \le \left(\int_{-\infty}^{\infty} a(s,t)\phi(s)ds\right)^{p'}.$$

Hence the desired result can be obtained by repeating the argument as in the proof of [1, Lemma 1].

The second is to prove that $|E_{\alpha\lambda}| \le A|\lambda| + B$ for constants A and B depending only on p, b and α, which is straightforward via modifying the argument of [1, Lemma 1]. Thus, we complete the proof of Lemma 2.1.

Lemma 2.2. *[15] There exists a constant* $K_{n,m}$ *depending only on m and n such that*

$$u^*(t) \le K_{n,m} \min\left\{(\log(e + \frac{1}{t}))^{\frac{1}{q'}}, t^{-\frac{m}{n}}\right\}\|u\|_{W^{\alpha}_{\frac{n}{m},q}(\mathbb{R}^n)}$$

for all $u \in W^{\alpha}_{\frac{n}{m},q}(\mathbb{R}^n)$ *and* $1 < q \le +\infty$.

Having disposed of the above lemmas, we can now turn to the proofs of Theorems 1 and 2.

2.1. Proof of Theorem 1

Since $u \in W^{m}_{\frac{n}{m},q}(\mathbb{R}^n)$, there exists a function $f \in L_{\frac{n}{m},q}(\mathbb{R}^n)$ with $u = (I - \Delta)^{-\frac{m}{2}}f$ and $\|f\|_{\frac{n}{m},q} \le 1$. Then $u = G_m * f$, where

$$G_m(x) = \frac{1}{(4\pi)^{m/2}\Gamma(m/2)}\int_0^{+\infty} e^{-\pi\frac{|x|^2}{t} - \frac{t}{4\pi}} t^{\frac{m-n}{2}}\frac{dt}{t}.$$

It follows from O'Neil's lemma [21] that for all $t \geq 0$,

$$u^*(t) \leq u^{**}(t) \leq tG_m^{**}(t)f^{**}(t) + \int_t^{+\infty} f^*(r)G_r^*(r)dr \;\; = \frac{1}{t}\int_0^t f^*(r)dr\int_0^t G_m^*(r)dr + \int_t^{+\infty} f^*(r)G_m^*(r)dr.$$

Since G_m is radial and decreasing, $G_m^*(r) = G_m(v_n^{\frac{1}{n}} r^{\frac{1}{n}})$. Therefore, by taking

$$\begin{cases} \phi(t) = |\Omega|^{\frac{m}{n}} e^{-\frac{m}{n}t} f^*(|\Omega|e^{-t}), \\ \psi(t) = (\beta_{n,m,q})^{\frac{q-1}{q}} |\Omega|^{1-\frac{m}{n}} e^{-(1-\frac{m}{n})t} G_m^*(|\Omega|e^{-t}), \end{cases}$$

and using the Hardy-Littlewood inequality, we find

$$\begin{aligned}
&\frac{1}{|\Omega|}\int_\Omega \exp\left[\beta_{n,m,q}|u|^{\frac{q}{q-1}}\right]dx \leq \frac{1}{|\Omega|}\int_\Omega \exp\left[\beta_{n,m,q}(u^*(t))^{\frac{q}{q-1}}\right]dx \\
&\leq \frac{1}{|\Omega|}\int_0^{+\infty} \exp\left[\beta_{n,m,q}|u^*(e^{-s}|\Omega|)|^{\frac{q}{q-1}}\right] e^{-s}|\Omega|ds \\
&\leq \int_0^{+\infty} \exp[\beta_{n,m,q}|u^*(e^{-s}|\Omega|)|^{\frac{q}{q-1}}]e^{-s}ds \\
&\leq \int_0^{+\infty} \exp\Big\{\beta_{n,m,q}[\frac{e^s}{|\Omega|}\int_0^{|\Omega|e^{-s}} f^*(r)dr\int_0^{|\Omega|e^{-s}} G_m^*(r)dr + \int_{\frac{|\Omega|}{e^s}}^{+\infty} f^*(r)G_m^*(r)dr]^{\frac{q}{q-1}}\Big\}e^{-s}ds \\
&\leq \int_0^{+\infty} \exp\Big\{\beta_{n,m,q}[|\Omega|e^s\int_s^{+\infty} f^*(|\Omega|e^{-t})e^{-t}dt\int_s^{+\infty} G_m^*(|\Omega|e^{-t})e^{-t}dt \\
&+ |\Omega|\int_{-\infty}^s f^*(|\Omega|e^{-t})G_m^*(|\Omega|e^{-t})e^{-t}dt]^{\frac{q}{q-1}}\Big\}e^{-s}ds \\
&= \int_0^{+\infty} \exp\Big\{\Big[e^s\int_s^{+\infty} \phi(t)e^{(\frac{m}{n}-1)t}dt\int_s^{+\infty} \psi(t)e^{-\frac{m}{n}t}dt + \int_{-\infty}^s \phi(t)\psi(t)dt\Big]^{\frac{q}{q-1}}\Big\}e^{-s}ds \\
&\leq \int_0^{+\infty} \exp(-F(s))ds,
\end{aligned}$$

where

$$F(s) = s - \left[e^s\int_s^{+\infty} \phi(t)e^{(\frac{m}{n}-1)t}dt\int_s^{+\infty} \psi(t)e^{-\frac{m}{n}t}dt + \int_{-\infty}^s \phi(t)\psi(t)dt\right]^{\frac{q}{q-1}}.$$

Hence,

$$\int_{-\infty}^{+\infty} \Phi^q(t)dt = \int_{-\infty}^{+\infty} (|\Omega|^{\frac{m}{n}} e^{-\frac{m}{n}t} f^*(|\Omega|e^{-t}))^q dr = \int_0^{+\infty} (f^*(s)\frac{1}{s^{\frac{n}{m}}})^q\frac{ds}{s} = \|(I-\triangle)^{\frac{m}{2}}u\|_{\frac{n}{m},q}^q \leq 1.$$

Set

$$a(t,s) = \begin{cases} \psi(t), & \text{if } t \leq s, \\ e^{(\frac{m}{n}-1)t}(\int_s^{+\infty} \psi(r)e^{-\frac{m}{n}r}dr)e^s, & \text{if } s < t. \end{cases}$$

Since

$$G_m(x) \approx \begin{cases} |x|^{-n+m}, & \text{if } |x| \leq 2, \\ e^{-|x|}, & \text{if } |x| > 2, \end{cases}$$

and $|\Omega| > A > 0$, we get

$$\begin{aligned}
&\int_{-\infty}^{0} a(t,s)^{q'}dt = \int_{-\infty}^{0} \psi(t)^{q'}dt = C_n \int_{-\infty}^{0} (|\Omega|^{1-\frac{m}{n}} e^{-(1-\frac{m}{n})t} G_m^*(|\Omega| e^{-t}))^{q'}dt \\
&= C_n \int_{|\Omega|}^{\infty} (s^{1-\frac{m}{n}} G_m(v_n^{-1/n} s^{1/n}))^{q'} \frac{ds}{s} \\
&= C_n \int_{v_n^{-\frac{1}{n}}|\Omega|^{\frac{1}{n}}}^{\infty} ((t^n v_n)^{1-\frac{m}{n}} G_m(t))^{q'} t^n v_n^{-1} v_n^{\frac{1}{n}} n (t^n v_n)^{1-\frac{1}{n}} dt \\
&= C_n \int_{v_n^{-\frac{1}{n}}|\Omega|^{\frac{1}{n}}}^{\infty} \frac{n}{t} (t^{n-m} v_n^{\frac{n-m}{n}} G_m(t))^{q'} dt \\
&= C_n\Big(\int_{v_n^{-\frac{1}{n}}|\Omega|^{\frac{1}{n}}}^{2} \frac{n}{t} (t^{n-m} v_n^{\frac{n-m}{n}} t^{m-n})^{q'} dt + \int_{2}^{+\infty} \frac{n}{t} (t^{n-m} v_n^{\frac{n-m}{n}} e^{-t})^{q'} dt\Big) \\
&\leq C_{n,m,q,A} < +\infty,
\end{aligned}$$

and

$$\begin{aligned}
&\int_{s}^{+\infty} a(t,s)^{q'}dt = e^{sq'} \int_{s}^{+\infty} e^{(\frac{m}{n}-1)tq'} dt (\int_{s}^{+\infty} \psi(t) e^{-\frac{m}{n}t} dt)^{q'} \\
&= C_{n,m,q} e^{sq'(\frac{m}{n})} (\int_{s}^{\infty} |\Omega|^{1-\frac{m}{n}} e^{-t} G_m^*(|\Omega| e^{-t}) dt)^{q'} \\
&\leq C_{n,m,q} e^{sq'(\frac{m}{n})} e^{-sq'(\frac{m}{n})} = C_{n,m,q} < \infty.
\end{aligned}$$

It's easy to check that when $0 < s < t$, $a(s,t) \leq 1$. This, along with Lemma 2.1 gives $\int_0^{+\infty} \exp[-F(s)]ds \leq C_0$. Therefore, we have obtained

$$\frac{1}{|\Omega|} \int_{\Omega} \exp[\beta_{n,m,q} |u|^{\frac{q}{q-1}}] dx \leq C.$$

Next, we show the sharpness of $\beta_{n,m,q}$ according to Adams method in [1]. The equivalent form of Theorem 1(1) is

$$\frac{1}{|\Omega|} \int_{\Omega} \exp(\beta \left| \frac{G_m * f(x)}{\|f\|_{\frac{n}{m},q}} \right|^{q'}) dx \leq C_{m,n,q}.$$

We need to prove that $\left(\frac{n}{\omega_{n-1}}\right)^{q'\frac{(n-m)}{n}}$ is the best one for $\Omega = B$ (the unit ball centered at the origin). Choose $f \geq 0$ such that $G_m * f \geq 1$ for $x \in B_r := \{x \in \mathbb{R} : |x| \leq r\}$ with $0 < r < 1$. The equivalent form gives

$$\frac{|B_r|}{|B|} \times e^{\alpha \|f\|^{-q'}_{L^{\frac{n}{m},q}(B)}} \leq C,$$

and hence

$$\alpha \leq \|f\|^{q'}_{\frac{n}{m},q} \left(\log \frac{|B|}{|B_r|} + \log C \right),$$

thereby finding

$$\alpha \leq n \lim_{r\to 0} \log \frac{1}{r} [Cap_{W^m L^{\frac{n}{m},q}}(B_r, B)]^{q'},$$

with $Cap_{W^m L^{\frac{n}{m},q}}(B_r, B) = \inf \|f\|^{q'}_{L^{\frac{n}{m},q}}(B)$. Here the infimum is taken over all $f > 0$ vanishing on the complement of B, and $G_m * f(x) \geq 1$ on E. It follows from the proof of [1, Theorem 2] that for any $\varepsilon > 0$, one can find $0 < r < 1$ small enough such that

$$G_m * f_r(y) \geq 1, \quad on \ \ B_r,$$

with

$$f_r(y) = \begin{cases} \frac{1}{\omega_{n-1}(1-\varepsilon)}(\log \frac{1}{r})^{-1}|y|^{-m}, & r < |y| < 1, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$h(y) = \begin{cases} |y|^{-m}, & r < |y| < 1, \\ 0, & \text{otherwise}. \end{cases}$$

Then the domain of $h^*(t)$ is $(r^n \frac{\omega_{n-1}}{n},\ \infty)$, where

$$h^*(t) = \begin{cases} (\frac{tn}{\omega_{n-1}})^{-\frac{m}{n}}, & r^n \frac{\omega_{n-1}}{n} < t < \frac{\omega_{n-1}}{n}, \\ 0, & \text{otherwise}. \end{cases}$$

Consequently,

$$\begin{aligned} &\|f_r\|_{L^{\frac{n}{m},q}(B)} = \|t^{\frac{m}{n}-\frac{1}{q}} f_r^*(t)\|_{L^q(0,|B|)} \\ &\leq \frac{1}{\omega_{n-1}(1-\varepsilon)} \left(\log \frac{1}{r}\right)^{-1} \left(\int_{r^n \frac{\omega_{n-1}}{n}}^{\frac{\omega_{n-1}}{n}} \left[(\frac{tn}{\omega_{n-1}})^{-\frac{m}{n}} t^{\frac{m}{n}-\frac{1}{q}} \right]^q dt \right)^{\frac{1}{q}} \\ &= \frac{n^{\frac{1}{q}}}{\omega_{n-1}(1-\varepsilon)} \left(\frac{\omega_{n-1}}{n}\right)^{\frac{m}{n}} \left(\log \frac{1}{r}\right)^{\frac{1-q}{q}}. \end{aligned}$$

This gives

$$Cap_{W^m L^{\frac{n}{m},q}}(B_r; B) \leq \|f_r\|_{L^{\frac{n}{m},q}(B)} = \frac{n^{\frac{1}{q}}}{\omega_{n-1}(1-\varepsilon)} \left(\frac{\omega_{n-1}}{n}\right)^{\frac{s}{n}} \left(\log \frac{1}{r}\right)^{\frac{1-q}{q}}.$$

Finally, a simple computation yields

$$\alpha \leq n \lim_{r\to 0} \log \frac{1}{r} \left(\frac{n^{\frac{1}{q}}}{\omega_{n-1}\,(1-\varepsilon)} (\frac{\omega_{n-1}}{n})^{\frac{m}{n}} (\log \frac{1}{r})^{\frac{1-q}{q}} \right)^{q'} = \left(\frac{n}{\omega_{n-1}}\right)^{q' \frac{n-m}{n}},$$

which complete the proof of (1).

The statement (2) can be proved similarly as that of (1), we only pay attention to the difference arguments as follows. The Hardy-Littlewood inequality shows that

$$\begin{aligned} &\frac{1}{|\Omega|^{1-\frac{\alpha}{n}}} \int_\Omega \frac{\exp[(1-\frac{\alpha}{n})\beta_{n,m,q}|u|^{\frac{q}{q-1}}]}{|x|^\alpha} dx \\ &\leq \frac{1}{|\Omega|^{1-\frac{\alpha}{n}}} \int_0^{|\Omega|} \exp\left[(1-\frac{\alpha}{n})\beta_{n,m,q}(u^*(t))^{\frac{q}{q-1}})\right] \left(\frac{t}{v_n}\right)^{-\frac{\alpha}{n}} dt \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{|\Omega|^{1-\frac{\alpha}{n}}} \int_0^{+\infty} \exp\left[(1-\frac{\alpha}{n})\beta_{n,m,q}|u^*(e^{-s}|\Omega|)|^{\frac{q}{q-1}}\right]\left(\frac{e^{-s}|\Omega|}{v_n}\right)^{-\frac{\alpha}{n}} e^{-s}|\Omega|ds \\
&= v_n^{\frac{\alpha}{n}} \int_0^{+\infty} \exp\left[(1-\frac{\alpha}{n})\beta_{n,m,q}|u^*(e^{-s}|\Omega|)|^{\frac{q}{q-1}}\right] e^{-s(1-\frac{\alpha}{n})}ds \\
&\le v_n^{\frac{\alpha}{n}} \int_0^{+\infty} \exp\{(1-\frac{\alpha}{n})\beta_{n,m,q}[\frac{e^s}{|\Omega|}\int_0^{|\Omega|e^{-s}} f^*(r)dr \int_0^{|\Omega|e^{-s}} G_m^*(r)dr \\
&+ \int_{\frac{|\Omega|}{e^s}}^{+\infty} f^*(r)G_m^*(r)dr]^{\frac{q}{q-1}}\}e^{-(1-\frac{\alpha}{n})s}ds \\
&= v_n^{\frac{\alpha}{n}} \int_0^{+\infty} \exp\{(1-\frac{\alpha}{n})\beta_{n,m,q}[|\Omega|e^s \int_s^{+\infty} f^*(|\Omega|e^{-t})e^{-t}dt \int_s^{+\infty} G_m^*(|\Omega|e^{-t})e^{-t}dt \\
&+ |\Omega| \int_{-\infty}^{s} f^*(|\Omega|e^{-t})G_m^*(|\Omega|e^{-t})e^{-t}dt]^{\frac{q}{q-1}}\}e^{-(1-\frac{\alpha}{n})s}ds \\
&= v_n^{\frac{\alpha}{n}} \int_0^{+\infty} \exp\left\{(1-\frac{\alpha}{n})\left[e^s \int_s^{+\infty} \phi(t)e^{(\frac{m}{n}-1)t}dt \int_s^{+\infty} \psi(t)e^{-\frac{m}{n}t}dt + \int_{-\infty}^{r} \phi(t)\psi(t)dt\right]^{\frac{q}{q-1}}\right\}\times \\
&e^{(1-\frac{\alpha}{n})s}ds \\
&\le v_n^{\frac{\alpha}{n}} \int_0^{+\infty} \exp\left[-F_{1-\frac{\alpha}{n}}(s)\right]ds,
\end{aligned}
$$

where

$$
F_{1-\frac{\alpha}{n}}(s) = (1-\frac{\alpha}{n})s - (1-\frac{\alpha}{n})\left[e^s \int_s^{+\infty} \phi(t)e^{(\frac{m}{n}-1)t}dt \int_s^{+\infty} \psi(t)e^{-\frac{m}{n}t}dt + \int_{-\infty}^{s} \phi(t)\psi(t)dt\right]^{\frac{q}{q-1}}.
$$

Let

$$
a(t,s) = \begin{cases} \psi(t), & \text{if } t \le s, \\ e^{(\frac{m}{n}-1)t}(\int_s^{+\infty} \psi(r)e^{-\frac{m}{n}r}dr)e^s, & \text{if } s < t. \end{cases}
$$

Then

$$
\begin{aligned}
&\int_{-\infty}^{0} a(t,s)^{q'}dt = \int_{-\infty}^{0} \psi(t)^{q'}dt \\
&= C_n \int_{-\infty}^{0} (|\Omega|^{1-\frac{m}{n}} e^{-(1-\frac{m}{n})t} G_m^*(|\Omega|e^{-t}))^{q'}dt \\
&= C_n \int_{|\Omega|}^{\infty} (s^{1-\frac{m}{n}} G_m(v_n^{-1/n}s^{1/n}))^{q'} \frac{ds}{s} \\
&\le C_{n,m,q} < +\infty,
\end{aligned}
$$

and

$$
\int_s^{+\infty} a(t,s)^{q'}dt = e^{sq'} \int_s^{+\infty} e^{(\frac{m}{n}-1)tq'}dt(\int_s^{+\infty} \psi(t)e^{-\frac{m}{n}t}dt)q' \le C_{n,m,q} < \infty.
$$

Since $a(s,t) \le 1$ for $0 < s < t$, we have $\int_0^{+\infty} \exp[-F_{1-\frac{\beta}{n}}(s)]ds$ by Lemma 2.1. Hence

$$
\frac{1}{|\Omega|^{1-\frac{\alpha}{n}}} \int_\Omega \frac{\exp[(1-\frac{\alpha}{n})\beta_{n,m,q}|u|^{\frac{q}{q-1}}]}{|x|^\alpha}dx \le C.
$$

What is left is to show the sharpness of $(1-\frac{\alpha}{n})\beta_{n,m,q}$, which also inspired by [1]. Since the equivalent form of (2) is

$$\int_\Omega \frac{\exp\left[(1-\frac{\alpha}{n})\beta\left|\frac{I_m * f(x)}{\|f\|_{L^{\frac{n}{m}},\, q(\Omega)}}\right|^{q'}\right]}{|x|^\alpha}dx \le C_{n,p}|\Omega|^{1-\frac{\alpha}{n}},\ \ \beta \le \left(\frac{n}{\omega_{n-1}}\right)^{q'\frac{n-m}{n}}, \tag{2.2}$$

we only need to prove that $\left(\frac{n}{\omega_{n-1}}\right)^{q'\frac{n-m}{n}}$ is the best one for $\Omega = B$. Similarly analysis as that of (1), we choose $f \ge 0$ such that $G_m * f \ge 1$ for $x \in B_r$ with $0 < r < 1$, it follows from (1) that

$$\begin{aligned}
\left|\frac{B_r}{B}\right|^{1-\frac{\alpha}{n}} |B_r|^{\frac{\alpha}{n}} \frac{1}{r^\alpha} e^{\frac{(1-\frac{\alpha}{n})\beta}{\|f\|^{q'}_{L^{\frac{n}{s},q}}}} &\le \left|\frac{B_r}{B}\right|^{1-\frac{\alpha}{n}} \frac{1}{|B_r|^{1-\frac{\alpha}{n}}} \int_{B_r} \frac{e^{\frac{(1-\frac{\alpha}{n})\beta}{\|f\|^{q'}_{L^{\frac{n}{m},q}}}}}{|x|^\alpha}dx \\
&\le \left|\frac{B_r}{B}\right|^{1-\frac{\alpha}{n}} \frac{1}{|B_r|^{1-\frac{\alpha}{n}}} \int_{B_r} \frac{e^{\frac{(1-\frac{\alpha}{n})\beta G_m * f(x)}{\|f\|^{q'}_{L^{\frac{n}{m},q}}}}}{|x|^\alpha}dx \\
&\le \frac{1}{|B_r|^{1-\frac{\alpha}{n}}} \int_{B} \frac{e^{\frac{(1-\frac{\alpha}{n})\beta G_m * f(x)}{\|f\|^{q'}_{L^{\frac{n}{m},q}}}}}{|x|^\alpha}dx \\
&\le C,
\end{aligned}$$

and

$$\begin{aligned}
(1-\frac{\alpha}{n})\beta &\le \|f\|^{q'}_{L^{\frac{n}{s},q}(B)}\left((1-\frac{\alpha}{n})\log\left|\frac{B}{B_r}\right| + \log(r^\alpha|B_r|^{-\frac{\alpha}{n}}) + \log C\right) \\
&\le \|f\|^{q'}_{L^{\frac{n}{s},q}(B)}\left((1-\frac{\alpha}{n})\log\left|\frac{B}{B_r}\right| + \log|B|^{\frac{\alpha}{n}} + \log C\right).
\end{aligned}$$

Hence, $\beta \le n\lim_{r\to 0}(\log\frac{1}{r})[Cap_{\dot{w}L^{\frac{n}{m},q}}(B_r; B)]^{q'}$, with $Cap_{\dot{w}L^{\frac{n}{m},q}}(E; B) = \inf\|f\|_{L^{\frac{n}{s},q}(B)}$, and E is a compact subset of B, where the infimum is taken over all $f \ge 0$ vanishing on the complement of B, and $G_m * f(x) \ge 1$ on E. Analysis similar as that of (1), for any $\varepsilon > 0$, we can choose $0 < r < 1$ small enough such that

$$G_m * f_r(y) \ge 1, \qquad \text{on} \quad B_r,$$

with

$$f_r(y) = \begin{cases} \frac{1}{\omega_{n-1}(1-\varepsilon)}(\log\frac{1}{r})^{-1}|y|^{-m}, & r < |y| < 1, \\ 0, & \text{otherwise}. \end{cases} \ \& \ h(y) = \begin{cases} |y|^{-m}, & r < |y| < 1, \\ 0, & \text{otherwise}. \end{cases}$$

Consequently, we get

$$\|f_r\|_{L^{\frac{n}{m},q}(B)} = \|t^{\frac{m}{n}-\frac{1}{q}} f_r^*(t)\|_{L^q(0,|B|)} \le \frac{n^{\frac{1}{q}}}{\omega_{n-1}(1-\varepsilon)}(\frac{\omega_{n-1}}{n})^{\frac{s}{n}}(\log\frac{1}{r})^{\frac{1-q}{q}}.$$

This shows

$$Cap_{\dot{w}L^{\frac{n}{m},q}}(B_r; B) \le \|f_r\|_{L^{\frac{n}{m},q}(B)} = \frac{n^{\frac{1}{q}}}{\omega_{n-1}(1-\varepsilon)}\left(\frac{\omega_{n-1}}{n}\right)^{\frac{s}{n}}\left(\log\frac{1}{r}\right)^{\frac{1-q}{q}},$$

which gives

$$\beta \le n \lim_{r\to 0} \log \frac{1}{r}\left(\frac{n^{\frac{1}{q}}}{\omega_{n-1}(1-\varepsilon)}(\frac{\omega_{n-1}}{n})^{\frac{m}{n}}(\log\frac{1}{r})^{\frac{1-q}{q}}\right)^{q'} = \left(\frac{n}{\omega_{n-1}}\right)^{q'\frac{n-m}{n}}$$

as desired.

2.2. Proof of Theorem 2

For any $u \in W^m_{\frac{n}{m},q}(\mathbb{R}^n)$ with $\|(I-\Delta)^{\frac{m}{2}}u\|_{\frac{n}{m},q} \le 1$, set $A(u) = \|u\|_{w_{\frac{n}{m},q}}$ and $\Omega = \{x \in \mathbb{R}^n : |u| > A(u)\}$. Then it is clear that $A(u) \le 1$. By the property of the rearrangement, we know that for any $t \in [0, |\Omega|)$,

$$u^*(t) > \|u\|_{w_{\frac{n}{m},q}}. \tag{2.3}$$

At the same time, Lemma 2.2 shows

$$u^*(t) \le K_{n,m} t^{-\frac{m}{n}} \|u\|_{w_{\frac{n}{m},q}}. \tag{2.4}$$

Combining (2.3) with (2.4), we have $t \le K^{\frac{n}{m}}_{n,m}$ for any $t \in [0, |\Omega|)$. Therefore $|\Omega| \le K^{\frac{n}{m}}_{n,m}$. Write

$$\int_{\mathbb{R}^n} \Phi[\beta_{n,m,q}|u|^{\frac{q}{q-1}}]dx = I_1 + I_2,$$

where

$$I_1 = \int_{\Omega} \Phi[\beta_{n,m,q}|u|^{\frac{q}{q-1}}]dx, \quad I_2 = \int_{\mathbb{R}^n\setminus\Omega} \Phi[\beta_{n,m,q}|u|^{\frac{q}{q-1}}]dx.$$

Choose Ω' such that $\Omega \subset \Omega'$ and $|\Omega'| = K^{\frac{n}{m}}_{n,m}$. Then by Theorem B, we have

$$\int_{\Omega'} \exp(\beta_{n,m,q}|u|^{\frac{q}{q-1}}) \le C_{n,m,q}|\Omega'| \le C_{n,m,q},$$

thereby finding

$$I_1 = \int_{\Omega} \Phi(\beta_{n,m,q}|u|^{\frac{q}{q-1}})dx \le C_{n,m,q}.$$

For the term I_2, since $\mathbb{R}^n \setminus \Omega \subset \{|u(x)| < 1\}$ and $(k_0+1)\frac{q}{q-1} = ([\frac{q}{q-1}\frac{n}{m}]+1)\frac{q}{q-1} > \frac{n}{m}$, the Hardy-Littlewood inequality and Lemma 2.2 shows that

$$\begin{aligned}
I_2 &\le \int_{\{|u|\le 1\}} \sum_{j=k_0+1}^{\infty} \frac{\beta^j_{n,m,q}}{j!}|u|^{j\frac{q}{q-1}}dx \le \sum_{j=k_0+1}^{\infty} \frac{\beta^j_{n,m,q}}{j!}\int_{\{|u|\le 1\}} |u|^{(k_0+1)\frac{q}{q-1}}dx \\
&\le C_{n,m,q}\int_0^{+\infty} [u'(t)]^{(k_0+1)\frac{q}{q-1}}dt = C_{n,m,q}\left(\int_0^1 [u'(t)]^{(k_0+1)\frac{q}{q-1}}dt + \int_1^{+\infty} [u'(t)]^{(k_0+1)\frac{q}{q-1}}dt\right) \\
&\le C_{n,m,q}(\int_0^1 [\ln(e+\frac{1}{t})]^{(k_0+1)}\|u\|^m_{W_{\frac{n}{m},q}}dt + \int_1^{+\infty} t^{-\frac{n}{m}(k_0+1)\frac{q}{q-1}}\|u\|^m_{W_{\frac{n}{m},q}}dt) \\
&\le C_{n,m,q}.
\end{aligned}$$

This is the first desired result.

The second inequality of Theorem 2 can be proved similarly via Theorem 1 and the above arguments, we omit its proof here.

3. Conclusions

We deal mainly with several sharp weighted Adams type inequalities in Lorentz-Sobolev spaces. In particular, the sharpness of these inequalities were also obtained by constructing a proper test sequence. Moreover, we discuss the boundedness of partial fractional integral operators.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest in this paper.

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