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Research article

A note on the boundedness of Hardy operators in grand Herz spaces with variable exponent

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Abstract: The fractional Hardy-type operators of variable order is shown to be bounded from the grand Herz spaces $\dot{K}^{a(\cdot),u),\theta}_{p(\cdot)}(\mathbb{R}^n)$ with variable exponent into the weighted space $\dot{K}^{a(\cdot),u),\theta}_{\rho,q(\cdot)}(\mathbb{R}^n)$, where $\rho = (1 + |z_1|)^{-\lambda}$ and

$$\frac{1}{q(z)} = \frac{1}{p(z)} - \frac{\zeta(z)}{n}$$

when $p(z)$ is not necessarily constant at infinity.

Keywords: Lebesgue spaces; weighted estimates; Hardy operators; grand Herz spaces
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1. Introduction

Let g is a non-negative integrable function in $\mathbb{R}^+$. The classical Hardy operator can be defined as

$$Hg(z) := \frac{1}{z}\int_0^z g(x)dx, \quad H^*g(z) := \int_x^\infty \frac{g(x)}{x}dx, \quad z > 0.$$

The Hardy operators H and H^* are mutually adjoint.

$$\int_0^\infty g(z)Hf(z)dz = \int_0^\infty f(z)H^*g(z)dz,$$

where $f \in L^p(\mathbb{R}^+)$, $g \in L^q(\mathbb{R}^+)$ with $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Nowadays, there is a vast boom of research related to the study of variable exponent function spaces and the development of operator theory in these spaces. This is due to the influence of some of their possible applications in modeling with non-standard local growth conditions (in differential equations, fluid mechanics, elasticity theory, see for example [1–5]). The boundedness of various operators in these spaces has intensively been studied in the last few years. In [6], remarkable results were proven in which authors discussed the boundedness of sublinear operators on homogenous Herz spaces $K_{v(\cdot)}^{\zeta(\cdot),u}(\mathbb{R}^n)$ and non-homogenous Herz spaces $\dot{K}_{v(\cdot)}^{\zeta(\cdot),u}(\mathbb{R}^n)$ with variable exponents.

Besides, fractional integral operators are of frequent interest among the researchers, for reference see [7–9]. In [10] authors considered the Herz-Morrey spaces $M\dot{K}_{u,v(\cdot)}^{\zeta(\cdot),\lambda}(\mathbb{R}^n)$ with variable exponent and investigated mapping properties for the fractional Hardy type operators $\mathcal{H}_{\beta(\cdot)}$ and $\mathcal{H}^*_{\beta(\cdot)}$ in these spaces. In [11] authors proved the boundedness for variable potential operator $I^{\zeta(x)}$ from the Lebesgue space $L^{p(\cdot)}$ into the weighted Lebesgue space $L_w^{q(\cdot)}$ in $\mathbb{R}^n$, under the conditions that $p(x)$ is satisfying the logarithmic condition locally and at infinity. It was not only supposed that $p(x)$ is constant at infinity but also assumed that $p(x)$ took its minimal value at infinity.

Variable exponent grand Herz spaces was introduced in [12] and proved boundedness for sublinear operators in these spaces. Boundedness of other operators on grand variable Herz spaces can be seen in [13]. Sultan et al. [14] introduced the idea of grand variable Herz-Morrey spaces and proved boundedness for Riesz potential operator in these spaces. Grand weighted Herz spaces and grand weighted Herz-Morrey spaces was introduced by Sultan et al. in [15, 16] respectively. Inspired by the concept, in this article we will define the idea of fractional Hardy-type operators of variable order and obtain its boundedness from grand Herz spaces to weighted space under some proper assumptions on weight.

We divided this article into different sections. Apart from introduction, a section is dedicated to basic lemmas and definitions. One section is for Sobolev type theorem for fractional Hardy-type operators of variable order in grand Herz spaces.

2. Preliminaries

For this section we refer to [17–20].

2.1. Lebesgue space with variable exponent

Definition 2.1. If H is a measurable set in $\mathbb{R}^n$ and $p(\cdot)$: $H \to [1,\infty)$ is a measurable function.

(a) Lebesgue space with variable exponent $L^{p(\cdot)}(H)$ can be defined as

$$L^{p(\cdot)}(H) = \left\{ f \textit{ measurable} : \int_H \left(\frac{|f(y)|}{\gamma} \right)^{p(y)} dy < \infty \textit{ where } \gamma \textit{ is a constant} \right\}.$$

Norm in $L^{p(\cdot)}(H)$ can be defined as

$$\|f\|_{L^{p(\cdot)}(H)} = \inf \left\{ \gamma > 0 : \int_H \left(\frac{|f(y)|}{\gamma} \right)^{p(y)} dy \le 1 \right\}.$$

(b) The space $L^{p(\cdot)}_{\text{loc}}(H)$ can be defined as

$$L^{p(\cdot)}_{\text{loc}}(H) := \left\{ f : f \in L^{p(\cdot)}(K) \textit{ for all compact subsets } K \subset H \right\}.$$

We use these notations in this article:

(i) Let $f \in L^1_{\text{loc}}(H)$ be a locally integrable function. Then, the Hardy-Littlewood maximal operator M is defined as

$$Mf(y) := \sup_{s>0} s^{-n} \int_{B(y,s)} |f(y)| dy \quad (y \in H),$$

where

$$B(y, s) := \{x \in H : |y - x| < s\}.$$

(ii) The set $\mathfrak{P}(H)$ consists of all measuable functions $p(\cdot)$ satisfying

$$p_- := \operatorname*{ess\,inf}_{h\in H} p(h) > 1, \quad p_+ := \operatorname*{ess\,sup}_{h\in H} p(h) < \infty. \tag{2.1}$$

(iii) $\mathfrak{P}^{\log} = \mathfrak{P}^{\log}(H)$ consists of all functions $q \in \mathfrak{P}(H)$ satisfying (2.1) and log condition defined as

$$|q(x) - q(y)| \leq \frac{C(q)}{-\ln|x - y|}, \quad |x - y| \leq \frac{1}{2}, \quad x, y \in H. \tag{2.2}$$

(iv) Let H be unbounded. $\mathfrak{P}_\infty(H)$ and $\mathfrak{P}_{0,\infty}(H)$ are the subsets of $\mathfrak{P}(H)$ and values of these subsets lies in $[1, \infty)$. $\mathfrak{P}_\infty(H)$ and $\mathfrak{P}_{0,\infty}(H)$ satisfy following conditions:

$$|q(h) - q_\infty| \leq \frac{C}{\ln(e + |h|)}, \tag{2.3}$$

where $q_\infty \in (1, \infty)$.

$$|q(h) - q_0| \leq \frac{C}{\ln|h|}, |h| \leq \frac{1}{2}, \tag{2.4}$$

respectively.

(v) $\chi_l = \chi_{F_l}$, $F_l = D_l \setminus D_{l-1}$, $D_l = D(0, 2^l) = \{x \in \mathbb{R}^n : |x| < 2^l\}$ for all $l \in \mathbb{Z}$.

C is a constant, its value varies from line to line and independent of main parameters involved.

We are assuming that order of Riesz potential operator $\zeta(x)$ is not continuous rather we are assuming that it is a measurable function in $\mathbb{R}^n$ satisfying the following conditions:

(1) $\zeta_0 := \operatorname{ess\,inf}_{x\in\mathbb{R}^n} \zeta(x) > 0$,

(2) $\operatorname{ess\,sup}_{x\in\mathbb{R}^n} p(x)\zeta(x) < n$,

(3) $\operatorname{ess\,sup}_{x\in\mathbb{R}^n} p(\infty)\zeta(x) < n$,

where $p(\infty) = \lim_{|x|\to\infty} p(x)$.

The following proposition is one of the main requirement to prove our main results. This proposition was proved in [11] and commonly known as Sobolev theorem for Riesz potential operator in Lebesgue spaces under the some necessary assumptions on exponent.

We consider the Riesz potential operator

$$I^{\zeta(z)} f(z) = \int_{\mathbb{R}^n} \frac{f(x)}{|z - x|^{n-\zeta(z)}} dx, \qquad 0 < \zeta(z) < n. \tag{2.5}$$

Note that the $\zeta(z)$ is the order of the Riesz potential operator which is variable.

Proposition 2.2. *Suppose that*

$$p(\cdot) \in \mathfrak{B}^{\log}(\mathbb{R}^n) \cap \mathfrak{B}_{0,\infty}(\mathbb{R}^n) \cap \mathfrak{B}(\mathbb{R}^n)$$

and assume

$$1 < p(\infty) \leq p(x) \leq p_+ < \infty.$$

Let $\zeta(x)$ satisfy the above conditions (1)–(3). Then, we have following weighted Sobolev-type estimate for the fractional operator $I^{\zeta(z)}$

$$\|(1+|z|)^{-\lambda(z)} I^{\zeta(z)}(f)\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

where

$$\frac{1}{q(z)} = \frac{1}{p(z)} - \frac{\zeta(z)}{n}$$

is the Sobolev exponent.

$$\lambda(z) = C\zeta(z)\left(1 - \frac{\zeta(z)}{n}\right) \leq \frac{n}{4}C,$$

with C is being the Dini-Lipschitz constant from the inequality (2.3) *in which $a(\cdot)$ replaced by $p(\cdot)$.*

Remark 2.3. (i) If $\zeta(z)$ is satisfying the condition (2.3):

$$|\zeta(z) - \zeta_\infty| \leq \frac{C_\infty}{\ln(e + \||z|)}$$

for $x \in \mathbb{R}^n$. Then, $(1+|z|)^{-\lambda(z)}$ is equivalent to the weight $(1+|z|)^{-\lambda_\infty}$.
(ii) One can replace the variable order of Riesz potential operator $\zeta(x)$ by $\zeta(y)$ in the case of potentials over bounded domain, such potentials differ unessential if the function $\zeta(x)$ is satisfying the smoothness logarithmic condition as (2.2) since

$$C_1|z_1 - z_2|^{n-\zeta(z_2)}| \leq |z_1 - z_2|^{n-\zeta(z_1)} \leq C_2|z_1 - z_2|^{n-\zeta(z_2)}.$$

Lemma 2.4. *[21] Let $D > 1$ and $p \in \mathfrak{P}_{0,\infty}(\mathbb{R}^n)$. Then,*

$$\frac{1}{t_0} s^{\frac{n}{p(0)}} \leq \|\chi_{R_{s,D_s}}\|_{p(\cdot)} \leq t_0 s^{\frac{n}{p(0)}} \tag{2.6}$$

for $0 < s \leq 1$ and

$$\frac{1}{t_\infty} s^{\frac{n}{p_\infty}} \leq \|\chi_{R_{s,D_s}}\|_{p(\cdot)} \leq t_\infty s^{\frac{n}{p_\infty}} \tag{2.7}$$

for $s \geq 1$, respectively, where $t_0 \geq 1$ and $t_\infty \geq 1$ is depending on D but not depending on s.

Lemma 2.5. *[17] (Generalized Hölder's inequality) Consider a measurable subset H such that $H \subseteq \mathbb{R}^n$ and $1 \leq p_-(H) \leq p_+(H) \leq \infty$. Then,*

$$\|fg\|_{L^{r(\cdot)}(H)} \leq \|f\|_{L^{p(\cdot)}(H)}\|g\|_{L^{q(\cdot)}(H)}$$

holds, where $f \in L^{p(\cdot)}(H)$, $g \in L^{q(\cdot)}(H)$ and

$$\frac{1}{r(t)} = \frac{1}{p(t)} + \frac{1}{q(t)}$$

for every $t \in H$.

2.2. Variable exponent Herz spaces and Herz-Morrey spaces

In this section we will define variable exponent Herz spaces.

Definition 2.6. Let $u, v \in [1,\infty)$, $\zeta \in \mathbb{R}$, the classical versions of Herz spaces commonly known as homogenous and non-homogenous are defined by their norms such as,

$$\|g\|_{K^{\zeta}_{u,v}(\mathbb{R}^n)} = \|g\|_{L^u(D(0,1))} + \left\{ \sum_{\ell\in\mathbb{N}} 2^{\ell\zeta v} \left(\int\limits_{F_{2^{\ell-1},2^{\ell}}} |g(y)|^u dy \right)^{\frac{v}{u}} \right\}^{\frac{1}{v}}, \tag{2.8}$$

$$\|g\|_{\dot{K}^{\zeta}_{u,v}(\mathbb{R}^n)} = \left\{ \sum_{\ell\in\mathbb{Z}} 2^{\ell\zeta v} \left(\int\limits_{F_{2^{\ell-1},2^{\ell}}} |g(y)|^u dy \right)^{\frac{v}{u}} \right\}^{\frac{1}{v}}, \tag{2.9}$$

respectively, such that $F_{t,\tau}$ denotes the annulus $F_{t,\tau} := D(0,\tau)\backslash D(0,t)$.

Definition 2.7. Let $u \in [1,\infty)$, $v(\cdot) \in \mathfrak{P}(\mathbb{R}^n)$ and $\zeta \in \mathbb{R}$. $\dot{K}^{\zeta,u}_{v(\cdot)}(\mathbb{R}^n)$ is the homogenous version of Herz space and its norm is given as

$$\dot{K}^{\zeta,u}_{v(\cdot)}(\mathbb{R}^n) = \left\{ g \in L^{v(\cdot)}_{\rm loc}(\mathbb{R}^n \setminus \{0\}) : \|g\|_{\dot{K}^{\zeta,u}_{v(\cdot)}(\mathbb{R}^n)} < \infty \right\}, \tag{2.10}$$

where

$$\|g\|_{\dot{K}^{\zeta,u}_{v(\cdot)}(\mathbb{R}^n)} = \left(\sum_{\ell=-\infty}^{\ell=\infty} \|2^{\ell\zeta} g\chi_\ell\|^u_{L^{v(\cdot)}} \right)^{\frac{1}{u}}.$$

Definition 2.8. Let $u \in [1,\infty)$, $\zeta \in \mathbb{R}$ and $v(\cdot) \in \mathfrak{P}(\mathbb{R}^n)$. The non-homogenous Herz space $K^{\zeta,u}_{v(\cdot)}(\mathbb{R}^n)$ can be defined as

$$K^{\zeta,u}_{v(\cdot)}(\mathbb{R}^n) = \left\{ g \in L^{v(\cdot)}_{\rm loc}(\mathbb{R}^n \setminus \{0\}) : \|g\|_{K^{\zeta,u}_{v(\cdot)}(\mathbb{R}^n)} < \infty \right\}, \tag{2.11}$$

where

$$\|g\|_{K^{\zeta,u}_{v(\cdot)}(\mathbb{R}^n)} = \|g\|_{L^{v(\cdot)}(D(0,1))} + \left(\sum_{\ell=-\infty}^{\ell=\infty} \|2^{\ell\zeta} g\chi_\ell\|^u_{L^{v(\cdot)}} \right)^{\frac{1}{u}}.$$

Now, we will define variable Herz-Morrey spaces.

Definition 2.9. For $a(\cdot)\colon \mathbb{R}^n \to \mathbb{R}$, $0 < u < \infty$, $v(\cdot) \in \mathfrak{P}(\mathbb{R}^n)$ and $0 \le \beta < \infty$. A variable Herz-Morrey spaces $M\dot{K}^{a(\cdot),\beta}_{u,v(\cdot)}(\mathbb{R}^n)$ is defined by

$$M\dot{K}^{a(\cdot),\beta}_{u,v(\cdot)}(\mathbb{R}^n) = \left\{ g \in L^{v(\cdot)}_{\rm loc}(\mathbb{R}^n \setminus \{0\}) : \|g\|_{M\dot{K}^{a(\cdot),\beta}_{u,v(\cdot)}(\mathbb{R}^n)} < \infty \right\},$$

where

$$\|g\|_{M\dot{K}^{a(\cdot),\beta}_{u,v(\cdot)}(\mathbb{R}^n)} = \sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta} \left(\sum_{\ell=-\infty}^{k_0} 2^{\ell a(\cdot)u} \|g\chi_\ell\|^u_{L^{v(\cdot)}(\mathbb{R}^n)} \right)^{\frac{1}{u}}.$$

2.3. Variable exponent grand Herz spaces

Next we define variable exponent grand Herz spaces.

Definition 2.10. Let $a(\cdot) \in L^{\infty}(\mathbb{R}^n)$, $u \in [1,\infty)$, v: $\mathbb{R}^n \to [1,\infty)$, $\theta > 0$. A grand Herz spaces with variable exponent $\dot{K}_{v(\cdot)}^{a(\cdot),u),\theta}$ is defined by

$$\dot{K}_{v(\cdot)}^{a(\cdot),u),\theta} = \left\{ g \in L_{\text{loc}}^{v(\cdot)}(\mathbb{R}^n \setminus \{0\}) : \|g\|_{\dot{K}_{v(\cdot)}^{a(\cdot),u),\theta}} < \infty \right\},$$

where

$$\begin{aligned}\|g\|_{\dot{K}_{v(\cdot)}^{a(\cdot),u),\theta}} &= \sup_{\psi>0} \left(\psi^{\theta} \sum_{\ell\in\mathbb{Z}} 2^{\ell a(\cdot)u(1+\psi)} \|g\chi_{\ell}\|_{L^{v(\cdot)}}^{u(1+\psi)} \right)^{\frac{1}{u(1+\psi)}} \\ &= \sup_{\psi>0} \psi^{\frac{\theta}{u(1+\psi)}} \|g\|_{\dot{K}_{v(\cdot)}^{a(\cdot),u(1+\psi)}}.\end{aligned}$$

Definition 2.11. Let g is a locally integrable function on $\mathbb{R}^n$, $0 \leq \zeta(z) < n$. The n-dimensional fractional Hardy operators of variable order $\zeta(z)$ can be defined as

$$\mathcal{H}g(z) := \frac{1}{|z|^{n-\zeta(z)}} \int_{|x|<|z|} g(x)dx, \quad \mathcal{H}^{*}g(z) := \int_{|x|\geq|z|} \frac{g(x)}{|x|^{n-\zeta(z)}}dx, \quad z \in \mathbb{R}^n \setminus 0.$$

3. Sobolev type theorem for grand Herz spaces

Now we will prove main results of our paper:

Theorem 3.1. *Let* $1 < u < \infty$*,*

$$1/q_1(z_1) - 1/q_2(z_1) = \zeta(\cdot)/n,$$

$0 < \zeta(\cdot) < n$ *and* $a, q_2 \in \mathfrak{P}_{0,\infty}(\mathbb{R}^n)$*, such that*

$$\frac{-n}{q_{1\infty}} < a_{\infty} < \frac{n}{q'_{1\infty}}, \quad \frac{-n}{q_1(0)} < a(0) < \frac{n}{q'_1(0)}.$$

Then,

$$\|(1+|z_1|)^{-\lambda(z_1)}\mathcal{H}(f)\|_{\dot{K}_{q_2(\cdot)}^{a(\cdot),u),\theta}(\mathbb{R}^n)} \leq C\|f\|_{\dot{K}_{q_1(\cdot)}^{a(\cdot),u),\theta}(\mathbb{R}^n)}.$$

Proof. Let $f \in \dot{K}_{q_2(\cdot)}^{a(\cdot),u),\theta}(\mathbb{R}^n)$ and

$$f(z_1) = \sum_{j=-\infty}^{\infty} f(z_1)\chi_j(z_1) = \sum_{j=-\infty}^{\infty} f_j(z_1),$$

we have

$$\begin{aligned}|\mathcal{H}(f)(z_1).\chi_{\ell}(z_1)| &\leq \frac{1}{|z_1|^{n-\zeta(z_1)}} \int_{D_{\ell}} |f(x)|dx.\chi_{\ell}(z_1) \\ &\leq C2^{-\ell n} \sum_{j=-\infty}^{\ell} \|f_j\|_{q_1(\cdot)} \|\chi_j\|_{q'_1(\cdot)}.|z|^{\zeta(z_1)}\chi_{\ell}(z_1).\end{aligned}$$

It is known, see e.g. [10] that

$$I^{\zeta(\cdot)}(\chi_{D_\ell})(z_1) \geq I^{\zeta(\cdot)}(\chi_{D_\ell})(z_1).(\chi_{D_\ell})(z_1) = \int_{D_\ell} \frac{1}{|z_1 - z_2|^{\zeta(z_1)-n}} dy.\chi_{D_\ell}(z_1)$$
$$\geq C|z_1|^{\zeta(z_1)}.\chi_{D_\ell}(z_1)$$
$$\geq C|z_1|^{\zeta(z_1)}.\chi_\ell(z_1).$$

Consequently, we have

$$\begin{aligned}
\|\chi_k(1+|z_1|)^{-\lambda(z_1)}\mathcal{H}(f)\|_{q_2(\cdot)} &\leq C2^{-\ell n}\|f_j\|_{L^{q_1(\cdot)}}\|\chi_j\|_{q_1'(\cdot)}\|(1+|z_1|)^{-\lambda(z_1)}\mathcal{H}(\chi_{D_\ell})\|_{q_2(\cdot)} \\
&\leq C2^{-\ell n}\sum_{j=-\infty}^{\ell}\|f_j\|_{q_1(\cdot)}\|\chi_j\|_{q_1'(\cdot)}\|(1+|z_1|)^{-\lambda(z_1)}\mathcal{H}(\chi_{D_\ell})\|_{q_2(\cdot)} \\
&\leq C2^{-\ell n}\sum_{j=-\infty}^{\ell}\|f_j\|_{q_1(\cdot)}\|\chi_j\|_{q_1'(\cdot)}\|\chi_{D_\ell}\|_{q_1(\cdot)} \\
&\leq C\sum_{j=-\infty}^{\ell}2^{-\ell n}\|f_j\|_{q_1(\cdot)}\|\chi_j\|_{q_1'(\cdot)}\|\chi_{D_\ell}\|_{q_1(\cdot)}.
\end{aligned}$$

$$\begin{aligned}
\|(1+|z_1|)^{-\lambda(z_1)}\mathcal{H}(f)\|_{\dot{K}^{a(\cdot),u),\theta}_{q_2(\cdot)}(\mathbb{R}^n)} &= \sup_{\psi>0}\left(\psi^\theta\sum_{\ell\in\mathbb{Z}}2^{\ell a(\cdot)u(1+\psi)}\|\chi_\ell(1+|z_1|)^{-\lambda(z_1)}\mathcal{H}(f)\|_{q_2(\cdot)}^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}} \\
&\leq \sup_{\psi>0}\left(\psi^\theta\sum_{\ell\in\mathbb{Z}}2^{\ell a(\cdot)u(1+\psi)}\left(\sum_{j=-\infty}^{\ell}2^{-\ell n}\|f_j\|_{q_1(\cdot)}\|\chi_j\|_{q_1'(\cdot)}\|\chi_{D_\ell}\|_{q_1(\cdot)}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}} \\
&\leq \sup_{\psi>0}\left(\psi^\theta\sum_{\ell=-\infty}^{-1}2^{\ell a(\cdot)u(1+\psi)}\left(\sum_{j=-\infty}^{\ell}2^{-\ell n}\|f_j\|_{q_1(\cdot)}\|\chi_j\|_{q_1'(\cdot)}\|\chi_{D_\ell}\|_{q_1(\cdot)}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}} \\
&+ \sup_{\psi>0}\left(\psi^\theta\sum_{\ell=0}^{\infty}2^{\ell a(\cdot)u(1+\psi)}\left(\sum_{j=-\infty}^{\ell}2^{-\ell n}\|f_j\|_{q_1(\cdot)}\|\chi_j\|_{q_1'(\cdot)}\|\chi_{D_\ell}\|_{q_1(\cdot)}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}} \\
&=: E_1 + E_2.
\end{aligned}$$

Now, we will find the estimate for E_1. By the Lemma (2.4)

$$2^{-\ell n}\|\chi_j\|_{q_1'(\cdot)}\|\chi_{D_\ell}\|_{q_1(\cdot)} \leq C2^{-\ell n}2^{\frac{kn}{q_1(0)}}2^{\frac{jn}{q_1'(0)}} \leq C2^{\frac{(j-\ell)n}{q_1'(0)}}. \tag{3.1}$$

Applying above results to E_1 to get

$$\begin{aligned}
E_1 &\leq \sup_{\psi>0}\left(\psi^\theta\sum_{\ell=-\infty}^{-1}2^{\ell a(\cdot)u(1+\psi)}\left(\sum_{j=-\infty}^{\ell}2^{-\ell n}\|f_j\|_{q_1(\cdot)}\|\chi_j\|_{q_1'(\cdot)}\|\chi_{D_\ell}\|_{q_1(\cdot)}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}} \\
&\leq C\sup_{\psi>0}\left(\psi^\theta\sum_{\ell=-\infty}^{-1}2^{\ell a(0)u(1+\psi)}\left(\sum_{j=-\infty}^{\ell}2^{\frac{(j-\ell)n}{q_1'(0)}}\|f_j\|_{q_1(\cdot)}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}.
\end{aligned}$$

Let $b := \frac{n}{q_1'(0)} - a(0)$. Applying the fact $2^{-u(1+\psi)} < 2^{-u}$, the Hölder's inequality and Fubini's theorem to get,

$$
\begin{aligned}
E_1 &\le C \sup_{\psi>0}\left(\psi^\theta \sum_{\ell=-\infty}^{-1}\left(\sum_{j=-\infty}^{\ell} 2^{a(0)j}\|f_j\|_{q_1(\cdot)} 2^{b(j-\ell)}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}\\
&\le C \sup_{\psi>0}\left(\psi^\theta \sum_{\ell=-\infty}^{-1}\left(\sum_{j=-\infty}^{\ell} 2^{a(0)j}\|f_j\|_{q_1(\cdot)} 2^{b(j-\ell)}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}\\
&\le C \sup_{\psi>0}\left[\psi^\theta \sum_{\ell=-\infty}^{-1}\left(\sum_{j=-\infty}^{\ell} 2^{a(0)u(1+\psi)j}\|f_j\|_{q_1(\cdot)}^{u(1+\psi)} 2^{bu(1+\psi)(j-\ell)/2}\right)\times\left(\sum_{j=-\infty}^{\ell} 2^{b(u(1+\psi))'(j-\ell)/2}\right)^{\frac{u(1+\psi)}{(u(1+\psi))'}}\right]^{\frac{1}{u(1+\psi)}}\\
&\le C \sup_{\psi>0}\left[\psi^\theta \sum_{\ell=-\infty}^{-1}\sum_{j=-\infty}^{\ell} 2^{a(0)u(1+\psi)j}\|f_j\|_{q_1(\cdot)}^{u(1+\psi)} 2^{bu(1+\psi)(j-\ell)/2}\right]^{\frac{1}{u(1+\psi)}}\\
&\le C \sup_{\psi>0}\left[\psi^\theta \sum_{j=-\infty}^{-1} 2^{a(\cdot)u(1+\psi)j}\|f_j\|_{q_1(\cdot)}^{u(1+\psi)} \sum_{\ell=j}^{-1} 2^{bu(1+\psi)(j-\ell)/2}\right]^{\frac{1}{u(1+\psi)}}\\
&\le C \sup_{\psi>0}\left(\psi^\theta \sum_{j=-\infty}^{-1} 2^{a(0)u(1+\psi)j}\|f_j\|_{q_1(\cdot)}^{u(1+\psi)} \sum_{\ell=j}^{-1} 2^{bu(1+\psi)(j-\ell)/2}\right)^{\frac{1}{u(1+\psi)}}\\
&\le C \sup_{\psi>0}\left(\psi^\theta \sum_{l=-\infty}^{-1} 2^{a(0)u(1+\psi)j}\|f_j\|_{q_1(\cdot)}^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}\\
&= C \sup_{\psi>0}\left(\psi^\theta \sum_{j\in\mathbb{Z}} 2^{a(\cdot)u(1+\psi)j}\|f_j\|_{q_1(\cdot)}^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}\\
&\le C\|f\|_{\dot{K}_{q_1(\cdot)}^{a(\cdot),u),\theta}(\mathbb{R}^n)}.
\end{aligned}
$$

Now for E_2, using Minkowski's inequality we have

$$
\begin{aligned}
E_2 &\le \sup_{\psi>0}\left(\psi^\theta \sum_{\ell=0}^{\infty} 2^{\ell a(\cdot)u(1+\psi)}\left(\sum_{j=-\infty}^{\ell} \|\chi_\ell(1+|z_1|)^{-\lambda(z_1)}\mathcal{H}(f_j)\|_{q_2(\cdot)}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}\\
&\le \sup_{\psi>0}\left(\psi^\theta \sum_{\ell=0}^{\infty} 2^{\ell a(\cdot)u(1+\psi)}\left(\sum_{j=-\infty}^{-1} \|\chi_\ell(1+|z_1|)^{-\lambda(z_1)}\mathcal{H}(f_j)\|_{q_2(\cdot)}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}\\
&+ \sup_{\psi>0}\left(\psi^\theta \sum_{\ell=0}^{\infty} 2^{\ell a(\cdot)u(1+\psi)}\left(\sum_{j=0}^{\ell} \|\chi_\ell(1+|z_1|)^{-\lambda(z_1)}\mathcal{H}(f_j)\|_{q_2(\cdot)}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}\\
&:= A_1 + A_2.
\end{aligned}
$$

As we can easily find the estimate for A_2 in a similar manner to E_1. We will replace $q_1'(0)$ with $q_{1\infty}'$ and by virtue of the fact $b := \frac{n}{q'_{1\infty}} - a_\infty > 0$ to get our desired results. For A_1, we have

$$2^{-\ell n}\|\chi_{D_\ell}\|_{q_1(\cdot)}\|\chi_j\|_{q'_1(\cdot)} \le C2^{-\ell n}2^{\frac{\ell n}{q_{1\infty}}}2^{\frac{jn}{q'_1(0)}} \le C2^{\frac{-\ell n}{q_{1\infty}}}2^{\frac{jn}{q'_1(0)}}. \tag{3.2}$$

As $a_\infty - \frac{n}{q'_{1\infty}} < 0$ we have

$$\begin{aligned}
A_1 &\le C\sup_{\psi>0}\left(\psi^\theta\sum_{\ell=0}^{\infty}2^{\ell a_\infty u(1+\psi)}\left(\sum_{j=-\infty}^{-1}\|\chi_\ell(1+|z_1|)^{-\lambda(z_1)}\mathcal{H}(f_j)\|_{q_2(\cdot)}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}\\
&\le C\sup_{\psi>0}\left[\psi^\theta\sum_{\ell=0}^{\infty}2^{\ell a_\infty u(1+\psi)}\left(\sum_{j=-\infty}^{-1}2^{-\ell n}2^{\frac{\ell n}{q_{1\infty}}}2^{\frac{jn}{q'_1(0)}}\|f_j\|_{q_1(\cdot)}\right)^{u(1+\psi)}\right]^{\frac{1}{u(1+\psi)}}\\
&\le C\sup_{\psi>0}\left[\psi^\theta\sum_{\ell=0}^{\infty}2^{\frac{\ell a_\infty-\ell n}{q'_{1\infty}}u(1+\psi)}\left(\sum_{j=-\infty}^{-1}2^{\frac{jn}{q'_1(0)}}\|f_j\|_{q_1(\cdot)}\right)^{u(1+\psi)}\right]^{\frac{1}{u(1+\psi)}}\\
&\le C\sup_{\psi>0}\left(\psi^\theta\left(\sum_{j=-\infty}^{-1}2^{\frac{jn}{q'_1(0)}}\|f_j\|_{q_1(\cdot)}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}\\
&\le C\sup_{\psi>0}\left(\psi^\theta\left(\sum_{j=-\infty}^{-1}2^{a(0)j}\|f_j\|_{q_1(\cdot)}2^{\frac{jn}{q'_1(0)}-a(0)j}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}.
\end{aligned}$$

Now, by applying Hölder's inequality and using the fact that $\frac{n}{q'_1(0)} - a(0) > 0$ we have

$$\begin{aligned}
A_1 &\le C\sup_{\psi>0}\left[\psi^\theta\left(\sum_{j=-\infty}^{-1}2^{a(0)ju(1+\psi)}\|f_j\|_{q_1(\cdot)}^{u(1+\psi)}\right)\times\left(\sum_{j=-\infty}^{-1}2^{(\frac{jn}{q'_1(0)}-a(0)j)(u(1+\psi))'}\right)^{\frac{u(1+\psi)}{(u(1+\psi))'}}\right]^{\frac{1}{u(1+\psi)}}\\
&\le C\sup_{\psi>0}\left(\psi^\theta\left(\sum_{j\in\mathbb{Z}}2^{a(0)ju(1+\psi)}\|f_j\|_{q_1(\cdot)}^{u(1+\psi)}\right)\right)^{\frac{1}{u(1+\psi)}}\\
&\le C\|f\|_{\dot{K}^{a(\cdot),u),\theta}_{q_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Combining these estimates we get

$$\|(1+|z_1|)^{-\lambda(z_1)}\mathcal{H}(f)\|_{\dot{K}^{a(\cdot),u),\theta}_{q_2(\cdot)}(\mathbb{R}^n)} \le C\|f\|_{\dot{K}^{a(\cdot),u),\theta}_{q_1(\cdot)}(\mathbb{R}^n)}.$$

□

Theorem 3.2. *Let* $1 < u < \infty$,

$$1/q_1(z_1) - 1/q_2(z_1) = \zeta(\cdot)/n,$$

$0 < \zeta(\cdot) < n$, *and* $a, q_2 \in \mathfrak{P}_{0,\infty}(\mathbb{R}^n)$ *such that*

$$\frac{-n}{q_{2\infty}} < a_\infty < \frac{n}{q'_{2\infty}}, \quad \frac{-n}{q_2(0)} < a(0) < \frac{n}{q'_2(0)}.$$

Then,

$$\|(1+|z_1|)^{-\lambda(z_1)}\mathcal{H}^*(f)\|_{\dot{K}^{a(\cdot),u),\theta}_{q_2(\cdot)}(\mathbb{R}^n)} \le C\|f\|_{\dot{K}^{a(\cdot),u),\theta}_{q_1(\cdot)}(\mathbb{R}^n)}.$$

Proof. Let $f \in \dot{K}_{q_2(\cdot)}^{a(\cdot),u),\theta}(\mathbb{R}^n)$ and

$$f(z_1) = \sum_{j=-\infty}^{\infty} f(z_1)\chi_j(z_1) = \sum_{j=-\infty}^{\infty} f_j(z_1).$$

We have

$$\begin{aligned}|(1+|z_1|)^{-\lambda(z_1)}\mathcal{H}^*(f)(z_1).\chi_\ell(z_1)| &\leq \int_{\mathbb{R}^n\setminus\ell} \frac{1}{|z_1|^{n-\zeta(z_1)}}|f(x)|dx.(1+|z_1|)^{-\lambda(z_1)}\chi_\ell(z_1)\\ &\leq C\sum_{j=\ell+1}^{\infty}\|f_j\|_{q_1(\cdot)}\|(1+|z_1|)^{-\lambda(z_1)}|\cdot|^{\zeta(z_1)-n}\chi_j\|_{q_1'(\cdot)}\chi_\ell(z_1).\end{aligned}$$

It is known, see e.g. [10] that

$$\begin{aligned}I^{\zeta(\cdot)}(\chi_{D_j})(z_1) &\geq I^{\zeta(\cdot)}(\chi_{D_j})(z_1).(\chi_{D_j})(z_1)\\ &= \int_{D_j}\frac{1}{|z_1-z_2|^{\zeta(z_1)-n}}dy.\chi_{D_j}(z_1)\\ &\geq C|z_1|^{\zeta(z_1)}.\chi_{D_j}(z_1)\\ &\geq C|z_1|^{\zeta(z_1)}.\chi_j(z_1).\end{aligned}$$

Consequently, we have

$$\begin{aligned}\|\chi_\ell(1+|z_1|)^{-\lambda(z_1)}\mathcal{H}^*(f)\|_{q_2(\cdot)} &\leq C\sum_{j=\ell+1}^{\infty}\|f_j\|_{q_1(\cdot)}\|(1+|z_1|)^{-\lambda(z_1)}|\cdot|^{\zeta(z_1)-n}\chi_j\|_{L^{q_1'(\cdot)}(\tau)}\|\chi_\ell\|_{q_2(\cdot)}\\ &\leq C\sum_{j=\ell+1}^{\infty}2^{-jn}\|f_j\|_{q_1(\cdot)}\|(1+|z_1|)^{-\lambda(z_1)}|z_1|^{\zeta(z_1)}\chi_j\|_{q_1'(\cdot)}\|\chi_\ell\|_{q_2(\cdot)}\\ &\leq C2^{-jn}\sum_{j=\ell+1}^{\infty}\|f_j\|_{L^{q_1(\cdot)}}\|\chi_j\|_{q_2'(\cdot)}\|\chi_\ell\|_{q_2(\cdot)}.\end{aligned}$$

$$\begin{aligned}\|(1+|z_1|)^{-\lambda(z_1)}\mathcal{H}^*(f)\|_{\dot{K}_{q_2(\cdot)}^{a(\cdot),u),\theta}(\mathbb{R}^n)} &= \sup_{\psi>0}\left(\psi^\theta\sum_{\ell\in\mathbb{Z}}2^{\ell a(\cdot)u(1+\psi)}\|\chi_\ell(1+|z_1|)^{-\lambda(z_1)}\mathcal{H}^*(f)\|_{q_2(\cdot)}^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}\\ &\leq \sup_{\psi>0}\left(\psi^\theta\sum_{\ell\in\mathbb{Z}}2^{\ell a(\cdot)u(1+\psi)}\left(\sum_{j=\ell+1}^{\infty}2^{-jn}\|f_j\|_{q_1(\cdot)}\|\chi_j\|_{q_2'(\cdot)}\|\chi_{D_\ell}\|_{q_2(\cdot)}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}\\ &\leq \sup_{\psi>0}\left(\psi^\theta\sum_{\ell=-\infty}^{-1}2^{\ell a(\cdot)u(1+\psi)}\left(\sum_{j=\ell+1}^{\infty}2^{-jn}\|f_j\|_{q_1(\cdot)}\|\chi_j\|_{q_2'(\cdot)}\|\chi_{D_\ell}\|_{q_2(\cdot)}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}\\ &+ \sup_{\psi>0}\left(\psi^\theta\sum_{\ell=0}^{\infty}2^{\ell a(\cdot)u(1+\psi)}\left(\sum_{j=\ell+1}^{\infty}2^{-jn}\|f_j\|_{q_1(\cdot)}\|\chi_j\|_{q_2'(\cdot)}\|\chi_{D_\ell}\|_{q_2(\cdot)}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}\\ &=: E_1 + E_2.\end{aligned}$$

We will find estimate of E_2,

$$2^{-jn}\|\chi_\ell\|_{q_2(\cdot)}\|\chi_j\|_{q_2'(\cdot)} \leq C2^{-jn}2^{\frac{\ell n}{q_{2\infty}}}2^{\frac{jn}{q_{2\infty}'}} \leq C2^{\frac{(\ell-j)n}{q_{2\infty}}}. \tag{3.3}$$

$$E_2 \leq C \sup_{\psi>0}\left(\psi^\theta \sum_{\ell=0}^{\infty} 2^{\ell a(\cdot)u(1+\psi)}\left(\sum_{j=\ell+1}^{\infty} \|\chi_\ell(1+|z_1|)^{-\lambda(z_1)}\mathcal{H}^*(f_j)\|_{q_2(\cdot)}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}$$
$$\leq C \sup_{\psi>0}\left(\psi^\theta \sum_{\ell=0}^{\infty}\left(\sum_{l=\ell+1}^{\infty} 2^{a_\infty j}\|f_j\|_{q_1(\cdot)}2^{d(\ell-j)}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}},$$

where
$$d = \frac{n}{q_{2\infty}} + a_\infty > 0.$$

Then, by virtue of the well known Hölder's theorem for series and $2^{-u(1+\psi)} < 2^{-u}$ yields

$$E_2 \leq C \sup_{\psi>0}\left[\psi^\theta \sum_{\ell=0}^{\infty}\left(\sum_{j=\ell+1}^{\infty} 2^{a_\infty u(1+\psi)j}\|f_j\|_{q_1(\cdot)}^{u(1+\psi)}2^{du(1+\psi)(\ell-j)/2}\right)\times\left(\sum_{j=\ell+1}^{\infty} 2^{d(u(1+\psi))'(\ell-j)/2}\right)^{\frac{u(1+\psi)}{(u(1+\psi))'}}\right]^{\frac{1}{u(1+\psi)}}$$
$$\leq C \sup_{\psi>0}\left[\psi^\theta \sum_{\ell=0}^{\infty}\sum_{j=\ell+1}^{\infty} 2^{a_\infty u(1+\psi)j}\|f_j\|_{q_1(\cdot)}^{u(1+\psi)}2^{du(1+\psi)(\ell-j)/2}\right]^{\frac{1}{u(1+\psi)}}$$
$$\leq C \sup_{\psi>0}\left(\psi^\theta \sum_{j=0}^{\infty} 2^{a_\infty u(1+\psi)j}\|f_j\|_{q_1(\cdot)}^{u(1+\psi)}\sum_{\ell=0}^{j-1} 2^{du(1+\psi)(\ell-j)/2}\right)^{\frac{1}{u(1+\psi)}}$$
$$< C \sup_{\psi>0}\left(\psi^\theta \sum_{j\in\mathbb{Z}} 2^{a_\infty u(1+\psi)j}\|f_j\|_{q_1(\cdot)}^{u(1+\psi)}\sum_{\ell=-\infty}^{j-1} 2^{du(1+\psi)(\ell-j)/2}\right)^{\frac{1}{u(1+\psi)}}$$
$$= C \sup_{\psi>0}\left(\psi^\theta \sum_{j\in\mathbb{Z}} 2^{a(\cdot)u(1+\psi)j}\|f_j\|_{q_1(\cdot)}^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}$$
$$\leq C\|f\|_{\dot{K}^{a(\cdot),u),\theta}_{q_1(\cdot)}(\mathbb{R}^n)}.$$

For E_1, by using Minkowski's inequality

$$E_1 \leq \sup_{\psi>0}\left(\psi^\theta \sum_{\ell=-\infty}^{-1} 2^{\ell a(\cdot)u(1+\psi)}\left(\sum_{j=\ell+1}^{\infty} \|\chi_\ell(1+|z_1|)^{-\lambda(z_1)}\mathcal{H}^*(f_j)\|_{q_2(\cdot)}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}$$
$$\leq \sup_{\psi>0}\left(\psi^\theta \sum_{\ell=-\infty}^{-1} 2^{\ell a(\cdot)u(1+\psi)}\left(\sum_{j=\ell+1}^{-1} \|\chi_\ell(1+|z_1|)^{-\lambda(z_1)}\mathcal{H}^*(f_j)\|_{q_2(\cdot)}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}$$
$$+ \sup_{\psi>0}\left(\psi^\theta \sum_{\ell=-\infty}^{-1} 2^{\ell a(\cdot)u(1+\psi)}\left(\sum_{j=0}^{\infty} \|\chi_\ell(1+|z_1|)^{-\lambda(z_1)}\mathcal{H}^*(f_j)\|_{q_2(\cdot)}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}$$
$$:= D_1 + D_2.$$

The estimate of D_1 can be obtained in a similar way to E_2 by replacing $q_{2\infty}$ with $q_2(0)$ and using the inequality
$$\frac{n}{q_2(0)} + a(0) > 0$$

and

$$\frac{n}{q_{2\infty}}+a_\infty>0.$$

For D_2 we have

$$2^{-jn}\|\chi_{D_\ell}\|_{q_2(\cdot)}\|\chi_j\|_{q_2'(\cdot)} \le C2^{-jn}2^{\frac{\ell n}{q_2(0)}}2^{\frac{jn}{q'2\infty}} \le C2^{\frac{\ell n}{q_2(0)}}2^{\frac{-jn}{q_{2\infty}}}, \tag{3.4}$$

$$\begin{aligned}
D_2 &\le C\sup_{\psi>0}\left(\psi^\theta\sum_{\ell=-\infty}^{-1}2^{\ell a(0)u(1+\psi)}\left(\sum_{j=0}^{\infty}\|\chi_\ell(1+|z_1|)^{-\lambda(z_1)}\mathcal{H}^*(f_j)\|_{q_2(\cdot)}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}\\
&\le C\sup_{\psi>0}\left(\psi^\theta\sum_{\ell=-\infty}^{-1}2^{\ell a(0)u(1+\psi)}\left(\sum_{j=0}^{\infty}2^{-jn}2^{\frac{\ell n}{q_2(0)}}2^{\frac{jn}{q'_{2\infty}}}\|f_j\|_{q_1(\cdot)}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}\\
&\le C\sup_{\psi>0}\left(\psi^\theta\sum_{\ell=-\infty}^{-1}2^{\ell a(0)u(1+\psi)}\left(\sum_{j=0}^{\infty}2^{\frac{\ell n}{q_2(0)}}2^{\frac{-jn}{q_{2\infty}}}\|f_j\|_{q_1(\cdot)}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}\\
&\le C\sup_{\psi>0}\left(\psi^\theta\sum_{\ell=-\infty}^{-1}2^{\ell(a(0)+n)/q_2(0)u(1+\psi)}\left(\sum_{j=0}^{\infty}2^{\frac{-jn}{q_{2\infty}}}\|f_j\|_{q_1(\cdot)}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}\\
&\le C\sup_{\psi>0}\left(\psi^\theta\left(\sum_{j=0}^{\infty}2^{a_\infty j}\|f_j\|_{q_1(\cdot)}2^{j(nq_{2\infty}+a_\infty)}\right)^{u(1+\psi)}\right)^{\frac{1}{u(1+\psi)}}\\
&\le C\sup_{\psi>0}\left(\psi^\theta\left(\sum_{j=0}^{\infty}2^{a_\infty ju(1+\psi)}\|f_j\|_{q_1(\cdot)}^{u(1+\psi)}\right)^{u(1+\psi)}\left(\sum_{j=0}^{\infty}2^{j(nq_{2\infty}+a_\infty)u(1+\psi)}\right)^{\frac{u(1+\psi)}{(u(1+\psi))'}}\right)^{\frac{1}{u(1+\psi)}}\\
&\le C\sup_{\psi>0}\left(\psi^\theta\left(\sum_{j\in\mathbb{Z}}2^{a(\cdot)ju(1+\psi)}\|f_j\|_{q_1(\cdot)}^{u(1+\psi)}\right)\right)^{\frac{1}{u(1+\psi)}}\\
&\le C\|f\|_{\dot{K}^{a(\cdot),u),\theta}_{q_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Combining the estimates for E_1 and E_2 yields

$$\|(1+|z_1|)^{-\lambda(z_1)}\mathcal{H}^*(f)\|_{\dot{K}^{a(\cdot),u),\theta}_{q_2(\cdot)}(\mathbb{R}^n)} \le C\|f\|_{\dot{K}^{a(\cdot),u),\theta}_{q_1(\cdot)}(\mathbb{R}^n)}.$$

□

4. Conclusions

In this paper we proved the boundedness of fractional type Hardy operator of variable order on grand Herz spaces with variable exponent under some proper assumptions on exponent.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

The authors declare no conflicts of interest.

References

1. M. Růžička, *Electroreological fluids: modeling and mathematical theory*, Springer, 2000. https://doi.org/10.1007/BFb0104029
2. L. Diening, M. Růžička, Calderon-Zygmund operators on generalized Lebesgue spaces $L^{p(\cdot)}$ and problems related to fluid dynamics, *J. Reine Angew. Math.*, **563** (2003), 197–220. https://doi.org/10.1515/crll.2003.081
3. O. A. Omer, K. Saibi, M. Z. Abidin, M. Osman, Parametric Marcinkiewicz integral and its higher-order commutators on variable exponents Morrey-Herz spaces, *J. Funct. Spaces*, **2022** (2022), 7209977. https://doi.org/10.1155/2022/7209977
4. M. A. Ragusa, Regularity of solutions of divergence form elliptic equations, *Proc. Amer. Math. Soc.*, **128** (2000), 533–540. https://doi.org/10.1090/s0002-9939-99-05165-5
5. A. Scapellato, Homogeneous Herz spaces with variable exponents and regularity results, *Electron. J. Qual. Theory Differ. Equ.*, **2018** (2018), 82. https://doi.org/10.14232/ejqtde.2018.1.82
6. A. Almeida, D. Drihem, Maximal, potential and singular type operators on Herz spaces with variable exponents, *J. Math. Anal. Appl.*, **394** (2012), 781–795. https://doi.org/10.1016/j.jmaa.2012.04.043
7. A. Hussain, M. Asim, F. Jarad, Variable lambda-central morrey space estimates for the fractional Hardy operators and commutators, *J. Math.*, **2022** (2022), 5855068. https://doi.org/10.1155/2022/5855068
8. M. Asim, A. Hussain, N. Sarfraz, Weighted variable Morrey-Herz estimates for fractional Hardy operators, *J. Inequal. Appl.*, **2022** (2022), 2. https://doi.org/10.1186/s13660-021-02739-z
9. A. Hussain, M. Asim, M. Aslam, F. Jarad, Commutators of the fractional Hardy operator on weighted variable Herz-Morrey spaces, *J. Funct. Spaces*, **2021** (2021), 9705250. https://doi.org/10.1155/2021/9705250
10. J. L. Wu, W. J. Zhao, Boundedness for fractional Hardy-type operator on variable-exponent Herz-Morrey spaces, *Kyoto J. Math.*, **56** (2016), 831–845. https://doi.org/10.1215/21562261-3664932
11. V. Kokilashvili, S. Samko, On Sobolev theorem for Riesz-type potentials in the Lebesgue spaces with variable exponent, *Z. Anal. Anwend.*, **22** (2003), 899–910. https://doi.org/10.4171/ZAA/1178

12. H. Nafis, H. Rafeiro, M. A. Zaighum, A note on the boundedness of sublinear operators on grand variable Herz spaces, *J. Inequal. Appl.*, **2020** (2020), 1. https://doi.org/10.1186/s13660-019-2265-6
13. M. Sultan, B. Sultan, A. Aloqaily, N. Mlaiki, Boundedness of some operators on grand Herz spaces with variable exponent, *AIMS Math.*, **8** (2023), 12964–12985. https://doi.org/10.3934/math.2023653
14. B. Sultan, F. Azmi, M. Sultan, M. Mehmood, N. Mlaiki, Boundedness of Riesz potential operator on grand Herz-Morrey spaces, *Axioms*, **11** (2022), 583. https://doi.org/10.3390/axioms11110583
15. B. Sultan, M. Sultan, M. Mehmood, F. Azmi, M. A. Alghafli, N. Mlaiki, Boundedness of fractional integrals on grand weighted Herz spaces with variable exponent, *AIMS Math.*, **8** (2023), 752–764. https://doi.org/10.3934/math.2023036
16. B. Sultan, F. M. Azmi, M. Sultan, T. Mahmood, N. Mlaiki, N. Souayah, Boundedness of fractional integrals on grand weighted Herz-Morrey spaces with variable exponent, *Fractal Fract.*, **6** (2022), 660. https://doi.org/10.3390/fractalfract6110660
17. D. Cruz-Uribe, A. Fiorenza, *Variable Lebesgue spaces*, Springer, 2013.
18. V. Kokilashvili, A. Meskhi, H. Rafeiro, S. Samko, *Integral operators in non-standard function spaces, volume 1: variable exponent Lebesgue and amalgam spaces*, Springer, 2016. https://doi.org/10.1007/978-3-319-21015-5
19. V. Kokilashvili, A. Meskhi, H. Rafeiro, S. Samko, *Integral operators in non-standard function spaces, volume 2: variable exponent Hölder, Morrey Campanato and grand spaces*, Springer, 2016. https://doi.org/10.1007/978-3-319-21018-6
20. O. Kováčik, J. Rákosník, On spaces $L^{p(x)}$ and $W^{k,p(x)}$, *Czechoslov. Math. J.*, **41** (1991), 592–618. https://doi.org/10.21136/CMJ.1991.102493
21. S. Samko, Variable exponent Herz spaces, *Mediterr. J. Math.*, **10** (2013), 2007–2025. https://doi.org/10.1007/s00009-013-0285-x

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