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Research article

On the variational principle and applications for a class of damped vibration systems with a small forcing term

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Abstract: This paper is dedicated to studying the existence of periodic solutions to a new class of forced damped vibration systems by the variational method. The advantage of this kind of system is that the coefficient of its second order term is a symmetric $N\times N$ matrix valued function rather than the identity matrix previously studied. The variational principle of this problem is obtained by using two methods: the direct method of the calculus of variations and the semi-inverse method. New existence conditions of periodic solutions are created through several auxiliary functions so that two existence theorems of periodic solutions of the forced damped vibration systems are obtained by using the least action principle and the saddle point theorem in the critical point theory. Our results improve and extend many previously known results.

Keywords: forced damped vibration system; the variational principle; periodic solutions; the least action principle; the saddle point theorem
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1. Introduction

Consider the following forced damped vibration systems:

$$\begin{cases}(A(t)\dot{u}(t))'+A(t)q(t)\dot{u}(t)=\nabla F(t,u(t))+f(t),\\ u(0)-u(T)=\dot{u}(0)-e^{Q(T)}\dot{u}(T)=0, \qquad a.e.\,t\in[0,T].\end{cases} \tag{1.1}$$

where $T>0$, $q\in L^1(0,T;R)$, $Q(t)=\int_0^t q(s)ds$, $f\in L^1(0,T;R^N)$, $A(t)=[a_{ij}(t)]$ is an invertible symmetric $N\times N$ matrix-valued function defined in $[0,T]$ with $a_{ij}\in C([0,T])$ for all $i,j=1,2,\cdots N$ and there exists a positive constant λ such that $\lambda|\xi|^2\le (A(t)\xi,\xi)$, for all $\xi\in R^N$, $a.e.\ t\in[0,T]$ and $F:[0,T]\times R^N\to R$ satisfies the following assumption:

(A) $F(t,x)$ is measurable in t for every $x\in R^N$ and continuously differentiable in x for $a.e.\ t\in[0,T]$, and there exist $a\in C(R^+,R^+)$, $b\in L^1(0,T;R^+)$ such that

$$|F(t,x)|\le a(|x|)b(t),\quad |\nabla F(t,x)|\le a(|x|)b(t),\ \text{for all } x\in R^N \text{ and } a.e.\ t\in[0,T].$$

When $q(t)\equiv 0$, $A(t)=I_{N\times N}$ and $f(t)\equiv 0$, the forced damped vibration systems [i.e., (1.1)] reduce to the following classical second order non-autonomous Hamiltonian systems:

$$\begin{cases}\ddot{u}(t)=\nabla F(t,u(t)),\\ u(0)-u(T)=\dot{u}(0)-\dot{u}(T)=0.\end{cases} \tag{1.2}$$

By the variational method, many existence results have been obtained under some suitable conditions in the last two decades. The readers may refer to [1–12] for more relevant results. In particular, Wang and Zhang [4] gave the following two existence theorems of periodic solutions of problem (1.2).

Theorem A. *Suppose that* F *satisfies assumption* (A) *and the following conditions:*

(H_1^*) *There exist constants* $C_0>0$, $K_1\ge 0$, $K_2\ge 0$, $\alpha\in[0,1)$ *and a non-negative function* $h\in C([0,+\infty);[0,+\infty))$ *with the properties:*

(i) $h(s)\le h(t)$ $\quad\forall s\le t, s,t\in[0,+\infty)$,

(ii) $h(s+t)\le C_0(h(s)+h(t))$ $\quad\forall s,t\in[0,+\infty)$,

(iii) $0\le h(t)\le K_1t^\alpha+K_2$ $\quad\forall t\in[0,+\infty)$,

(iv) $h(t)\to+\infty$ *as* $t\to+\infty$.

Moreover, there exist $f\in L^1(0,T;R^+)$ *and* $g\in L^1(0,T;R^+)$ *such that*

$$|\nabla F(t,x)|\le f(t)h(|x|)+g(t),\ \textit{for}\ x\in R^N,\ a.e.\ t\in[0,T].$$

(H_2^*) *There exists a non-negative function* $h\in C([0,+\infty);[0,+\infty))$ *which satisfies the conditions* (i)–(iv) *and* $\dfrac{1}{h^2(|x|)}\int_0^T F(t,x)dt\to+\infty$, *as* $|x|\to+\infty$.

Then problem (1.2) *has at least one solution which minimizes the functional* φ *on* H_T^1.

Theorem B. *Suppose that* (H_1^*) *and assumption* (A) *hold. Assume that*

(H_3^*) $\dfrac{1}{h^2(|x|)}\int_0^T F(t,x)dt\to-\infty$ *as* $|x|\to+\infty$. *Then problem* (1.2) *has at least one solution in* H_T^1.

With the increase in research, scholars began to study a more general form of Hamiltonian systems: damped vibration systems. In damped vibration, due to the need of the system overcoming resistance, the displacement and energy of vibration continuously reduced and their decreasing trend is correlated to factors such as the natural frequency and damping coefficient of the system. Therefore, the damped vibration system is greatly based on physics and can be one of the important mathematical models.

Wu [13] studied the existence of periodic solutions of the following damped vibration systems:

$$\begin{cases} \ddot{u}(t)+q(t)\dot{u}(t)=A(t)u(t)+\nabla F(t,u(t)), \\ u(0)-u(T)=\dot{u}(0)-e^{Q(T)}\dot{u}(T)=0. \end{cases} \tag{1.3}$$

where $q(t)\dot{u}(t)$ is called damping term. The systems [i.e., (1.3)] are called damped vibration systems in physics. Wu put forward the variational principle of problem (1.3) for the first time and studied further the existence of periodic solution of problem (1.3). Subsequently, in case $A(t)=0$, Wang [14] studied the existence of periodic solution of the corresponding system.

In addition, the vibration of a nonlinear vibration system under the action of a periodic dynamic force $f(t)$ is called forced vibration. Take the spring oscillator model as an example. Suppose that the spring oscillator is subjected to both resistance $-\gamma\dfrac{\mathrm{d}x}{\mathrm{d}t}$ and dynamic force $F_0\cos\omega t$, and then Dynamic equation of the spring oscillator is

$$\mathrm{m}\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=-kx-\gamma\frac{\mathrm{d}x}{\mathrm{d}t}+F_0\cos\omega t.$$

This equation is a special forced damped vibration system. Another example is the famous Duffing oscillator model: Assume that it is subjected to both resistance $c\dot{x}$ and dynamic force $f\cos\omega t$, and then the dynamic equation of the forced damped vibration of the Duffing oscillator is

$$\mathrm{m}\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+\mathrm{c}\dot{x}+k(x+\beta x^3)=f\cos\omega t.$$

The forced damped vibration systems [i.e., (1.1)] we have studied are more general than the two equations above. Therefore, the systems [i.e., (1.1)] are proved to not only have a very strong physical background but also be a more general class of new systems.

Generally, a nonlinear vibration system is complex and it is difficult to get a strong solution to a differential equation. In recent years, the variational method has been used by many scholars to study the existence of solutions of differential equations, such as the classical second order non-autonomous Hamiltonian systems [i.e., (1.2)] (see [1–12]), the damped vibration systems [i.e., (1.3)] (See [13,14]) and the damped random impulsive differential equations under Dirichlet boundary value conditions (See [15–17]). The variational principle, including the Hamilton principle, is widely used in the nonlinear vibration theory. For the case of Hamiltonian-based frequency formulation for nonlinear oscillators (See [18]), its Hamilton principle is established by the semi-inverse method.

Inspired by [4,13], we obtain a new class of forced damped vibration systems [i.e., (1.1)] and decide to study the existence of periodic solutions of this problem by the variational method. We explore, in depth, the existence of variational construction for problem (1.1) and study further the existence of periodic solutions of it under some solvability conditions by following the least action principle and the Saddle Point Theorem 4.7 in [3], and obtain two new existence theorems.

2. The variational principle

Let us suppose $H_T^1=\{u:[0,T]\to R^N \mid u$ is absolutely continuous, $u(0)=u(T)$, $\dot{u}\in L^2(0,T;R^N)\}$ with the inner product

$$\langle u,v\rangle = \int_0^T (u(t),v(t))dt + \int_0^T (\dot{u}(t),\dot{v}(t))dt \text{, for any } u,v\in H_T^1,$$

where $(\cdot\,,\cdot)$ and $|\cdot|$ are the usual inner product and norm of R^N. The corresponding norm is defined by

$$\|u\| = \left(\int_0^T |u(t)|^2\, dt + \int_0^T |\dot{u}(t)|^2 dt\right)^{\frac{1}{2}} \text{, for } u\in H_T^1.$$

Then, H_T^1 is obviously a Hilbert space.

Set

$$\|u\|_0 = \left(\int_0^T e^{Q(t)}(A(t)\dot{u}(t),\dot{u}(t))dt + \int_0^T e^{Q(t)}(u(t),u(t))dt\right)^{\frac{1}{2}} \text{, for } u\in H_T^1.$$

Obviously, the norm $\|\cdot\|_0$ is equivalent to the usual one $\|\cdot\|$ on H_T^1. The proof is similar to the corresponding parts in [19].

Let $\tilde{H}_T^1 = \{u\in H_T^1 \mid \int_0^T u dt = 0\}$, it is easy to know that $\tilde{H}_T^1$ is a subset of H_T^1, and $H_T^1 = R^N \oplus \tilde{H}_T^1$. It follows from Proposition 1.3 in [3] that

$$\int_0^T |u(t)|^2 dt \le \frac{T^2}{4\pi^2}\int_0^T |\dot{u}(t)|^2 dt \text{, for every } u\in \tilde{H}_T^1 \text{ (Wirtinger's inequality),}$$

and

$$\|u\|_\infty^2 \le \frac{T}{12}\int_0^T |\dot{u}(t)|^2 dt \text{, for every } u\in \tilde{H}_T^1 \text{ (Sobolev's inequality).}$$

Hence,

$$\|u\|^2 \le (1+\frac{T^2}{4\pi^2})\int_0^T |\dot{u}(t)|^2 dt, \text{ for every } u\in \tilde{H}_T^1. \tag{2.1}$$

Define the functional $\varphi(u)$ on H_T^1 by

$$\varphi(\mathrm{u}) = \frac{1}{2}\int_0^T e^{Q(t)}(A(t)\dot{u}(t),\dot{u}(t))dt + \int_0^T e^{Q(t)}F(t,u(t))dt + \int_0^T e^{Q(t)}(f(t),u(t))dt. \tag{2.2}$$

We have the following facts.

Theorem 2.1. *The functional* $\varphi(u)$ *is continuously differentiable and weak lower semi-continuous on* H_T^1.

Proof. Set $L(t,x,y) = e^{Q(t)}[\frac{1}{2}(A(t)y,y) + F(t,x) + (f(t),x)]$ for all $x,y\in R^N$ and $t\in[0,T]$. Then $L(t,x,y)$ satisfies all assumptions of Theorem 1.4 in [3]. Hence, by Theorem 1.4 in [3], we know that the functional $\varphi(u)$ is continuously differentiable on H_T^1 and

$$(\varphi'(u),v) = \int_0^T e^{Q(t)}(A(t)\dot{u}(t),\dot{v}(t))dt + \int_0^T e^{Q(t)}(\nabla F(t,u(t)),v(t))dt + \int_0^T e^{Q(t)}(f(t),v(t))dt,$$

for all $u,v \in H_T^1$. Moreover, the proof for the weak lower semi-continuity of $\varphi(u)$ is similar to the corresponding parts in [3, P12-13].

Theorem 2.2. *If* $u \in H_T^1$ *is a solution of the Euler equation* $\varphi'(u)=0$, *then* u *is a solution of problem* (1.1).

Proof. Since $\varphi'(u)=0$, then

$$0=(\varphi'(u),v)=\int_0^T e^{Q(t)}(A(t)\dot{u}(t),\dot{v}(t))dt+\int_0^T e^{Q(t)}(\nabla F(t,u(t)),v(t))dt+\int_0^T e^{Q(t)}(f(t),v(t))dt,$$

for all $u,v \in H_T^1$. i.e.,

$$\int_0^T e^{Q(t)}(A(t)\dot{u}(t),\dot{v}(t))dt=-\int_0^T e^{Q(t)}(\nabla F(t,u(t))+f(t),v(t))dt, \text{ for all } v\in H_T^1.$$

By the Fundamental Lemma and Remarks in [3, P6-7], we know that $e^{Q(t)}A(t)\dot{u}(t)$ has a weak-derivative, and

$$(e^{Q(t)}A(t)\dot{u}(t))'=e^{Q(t)}[\nabla F(t,u(t))+f(t)], \quad a.e.\, t\in[0,T]. \tag{2.3}$$

$$e^{Q(t)}A(t)\dot{u}(t)=\int_0^t e^{Q(s)}[\nabla F(s,u(s))+f(s)]ds+C, \quad a.e.\, t\in[0,T]. \tag{2.4}$$

$$\int_0^T e^{Q(t)}[\nabla F(t,u(t))+f(t)]dt=0, \tag{2.5}$$

where C is a constant. We identify the equivalence class $e^{Q(t)}A(t)\dot{u}(t)$ and its continuous representation

$$\int_0^t e^{Q(s)}[\nabla F(s,u(s))+f(s)]ds+C.$$

Then, by (2.4), we have

$$e^{Q(0)}A(0)\dot{u}(0)=0.$$

i.e., $\dot{u}(0)=0$.

By (2.4) and (2.5), one has

$$e^{Q(T)}A(T)\dot{u}(T)=0.$$

i.e., $e^{Q(T)}\dot{u}(T)=0$.

By the existence of $\dot{u}(t)$, we draw a conclusion similar to (2.5), that is

$$\int_0^T \dot{u}(t)dt=0.$$

i.e., $u(0)-u(T)=0$.

Therefore, u satisfies the following periodic boundary condition

$$\dot{u}(0)-e^{Q(T)}\dot{u}(T)=u(0)-u(T)=0.$$

Moreover, by (2.3), u satisfies the following forced damped vibration equation

$$(A(t)\dot{u}(t))'+A(t)q(t)\dot{u}(t)=\nabla F(t,u(t))+f(t),\quad a.e.\ t\in[0,T].$$

Hence, u is a solution of problem (1.1). This completes the proof.

From the proof of Theorems 2.1 and 2.2, it can be seen that the variational principle of problem (1.1) is indeed the $\varphi(u)$ [i.e., (2.2)] we defined above.

In fact, we can also directly derive the variational principle of problem (1.1) by using the semi-inverse method [18]. The derivation process is as follows.

In case $q(t)\equiv 0$, we can easily obtain the following variational principle:

$$\varphi_1(u)=\int_0^T \frac{1}{2}(A(t)\dot{u}(t),\dot{u}(t))+F(t,u(t))+(f(t),u(t))dt.$$

In order to obtain the variational principle of problem (1.1), we introduce an integrating factor $g(t)$ which is an unknown function of time, and consider the following integral:

$$\varphi_2(u)=\int_0^T \{g(t)[\frac{1}{2}(A(t)\dot{u}(t),\dot{u}(t))+F(t,u(t))+(f(t),u(t))]+G(u,u_t,u_{tt},\cdots)\}dt, \tag{2.6}$$

where G is an unknown function of u and/or its derivatives. The semi-inverse method is to identify such g and G that the stationary condition of Eq (2.6) satisfies problem (1.1). The Euler–Lagrange equation of Eq (2.6) reads

$$g(t)(\nabla F(t,u(t))+f(t))-(g(t)[(A(t)\dot{u}(t)])'+\frac{\delta G}{\delta u}=0, \tag{2.7}$$

where $\frac{\delta G}{\delta u}$ is called variational derivative [20–22] defined as

$$\frac{\delta G}{\delta u}=\frac{\partial G}{\partial u}-\frac{\partial}{\partial t}\frac{\partial G}{\partial u_t}+\frac{\partial^2}{\partial t^2}\frac{\partial G}{\partial u_{tt}}-\cdots.$$

We re-write Eq (2.7) in the form

$$\frac{g'}{g}(A(t)\dot{u}(t))+(A(t)\dot{u}(t))'=(\nabla F(t,u(t))+f(t))+\frac{1}{g}\frac{\partial G}{\partial u}. \tag{2.8}$$

Comparing Eq (2.8) with problem (1.1), we set

$$\frac{g'}{g}=q(t),\ \frac{\partial G}{\partial u}=0.$$

Therefore, we have

$$g = \exp \int_0^t q(s)ds = e^{Q(t)}, G = 0.$$

Consequently, we obtain the needed variational principle for problem (1.1), which reads

$$\varphi_2(u) = \int_0^T e^{Q(t)}[\frac{1}{2}(A(t)\dot{u}(t),\dot{u}(t)) + F(t,u(t)) + (f(t),u(t))]dt.$$

Obviously, $\varphi_2(u) = \varphi(u)$.

3. Existence of solutions for the forced damped vibration systems

In convenience, we set

$$d_1 = \max_{t\in[0,T]} e^{Q(t)}, \quad d_2 = \min_{t\in[0,T]} e^{Q(t)}, \quad a = \max_{\substack{i,j=1,\cdots,N\\ t\in[0,T]}} \{|a_{ij}|\}.$$

Theorem 3.1. *Let* $F(t,x) = F_1(t,x) + F_2(x)$, *suppose that* $F_1(t,x)$ *and* $F_2(x)$ *satisfy assumption* (A) *and the following conditions*:

(H_1) *There exist constants* $C_1 > 0$, $K_1 \geq 0$, $\alpha \in (\frac{1}{2},1)$ *and a non-negative function* $h_1 \in C([0,+\infty);[0,+\infty))$ *with the properties:*

(i) $h_1(s) \leq h_1(t) \qquad \forall s \leq t, \quad s,t \in [0,+\infty)$,

(ii) $h_1(s+t) \leq C_1(h_1(s) + h_1(t)) \qquad \forall s,t \in [0,+\infty)$,

(iii) $\lim_{s\to+\infty} \frac{h_1(s)}{s^\alpha} \leq K_1$.

Moreover, there exist $r_1, r_2 \in L^1(0,T;R^+)$ *such that*

$$|\nabla F_1(t,x)| \leq r_1(t)h_1(|x|) + r_2(t), \text{ for } x \in R^N, \ a.e.\, t \in [0,T].$$

(H_2) *There exist a constant* $K_2 > 0$ *and two functions* $k \in L^1(0,T;R^+)$ *with* $\int_0^T k(t)dt < \frac{3\lambda d_2}{d_1 K_2 T}$ *and* $h_2 \in C([0,+\infty);[0,+\infty))$ *which is non-decreasing, such that*

$$(\nabla F_2(x) - \nabla F_2(y), x-y) \leq k(t)h_2(|x-y|), \text{ for } x,y \in R^N, \ a.e.\, t \in [0,T],$$

and

$$\limsup_{s\to+\infty} \frac{h_2(s)}{s^2} \leq K_2.$$

(H_3) $\lim_{|x|\to+\infty} \frac{1}{|x|^{2\alpha}} \int_0^T e^{Q(t)}F(t,x)dt > \frac{C_1^2 d_1^2 T K_1^2}{3\lambda d_2} (\int_0^T r_1(t)dt)^2.$

Then problem (1.1) *has at least one solution which minimizes the functional* $\varphi(u)$ *on* H_T^1.

Proof. It is clear that

$$\frac{1}{2}\int_0^T e^{Q(t)}(A(t)\dot{u}(t),\dot{u}(t))dt \geq \frac{1}{2}\int_0^T \lambda d_2 \,|\,\dot{u}(t)\,|^2\, dt \quad = \frac{1}{2}\lambda d_2 \|\dot{u}\|_2^2 . \tag{3.1}$$

It follows from condition (H_1) and Sobolev's inequality that

$$\begin{aligned}
&\left|\int_0^T e^{Q(t)}[F_1(t,u(t)) - F_1(t,\bar{u})]dt\right| \\
&= \left|\int_0^T e^{Q(t)} \int_0^1 \big(\nabla F_1(t,\bar{u} + s\tilde{u}(t)),\tilde{u}(t)\big) ds dt\right| \\
&\leq \int_0^T \int_0^1 e^{Q(t)} r_1(t) h_1(|\,\bar{u} + s\tilde{u}(t)\,|)\cdot|\,\tilde{u}(t)\,|\, ds dt + \int_0^T \int_0^1 e^{Q(t)} r_2(t)\,|\,\tilde{u}(t)\,|\, ds dt \\
&\leq \int_0^T \int_0^1 e^{Q(t)} r_1(t) C_1 (h_1(|\,\bar{u}\,|) + h_1(s\,|\,\tilde{u}(t)\,|))\,|\,\tilde{u}(t)\,|\, ds dt + \int_0^T e^{Q(t)} r_2(t)\,|\,\tilde{u}(t)\,|\, dt \\
&\leq \int_0^T e^{Q(t)} r_1(t) C_1 (h_1(|\,\bar{u}\,|) + h_1(|\,\tilde{u}(t)\,|))\,|\,\tilde{u}(t)\,|\, dt + \|\tilde{u}\|_\infty \int_0^T e^{Q(t)} r_2(t) dt \\
&\leq C_1 d_1 (h_1(|\,\bar{u}\,|) + h_1(\|\tilde{u}\|_\infty))\|\tilde{u}\|_\infty \int_0^T r_1(t) dt + C_2 \Big(\int_0^T |\,\dot{u}(t)\,|^2\, dt\Big)^{\frac{1}{2}} \\
&= C_1 d_1 h_1(|\,\bar{u}\,|)\|\tilde{u}\|_\infty \int_0^T r_1(t) dt + C_1 d_1 h_1(\|\tilde{u}\|_\infty)\|\tilde{u}\|_\infty \int_0^T r_1(t) dt + C_2 \Big(\int_0^T |\,\dot{u}(t)\,|^2\, dt\Big)^{\frac{1}{2}} \\
&\leq C_1 d_1 \Big(\frac{3\lambda d_2}{C_1 d_1 T}\|\tilde{u}\|_\infty^2 + \frac{C_1 d_1 T}{3\lambda d_2} h_1^2(|\,\bar{u}\,|)\Big(\int_0^T r_1(t) dt\Big)^2\Big) + C_1 d_1 (K_1 \|\tilde{u}\|_\infty^\alpha + C_3)\|\tilde{u}\|_\infty \int_0^T r_1(t) dt \\
&+ C_2 \Big(\int_0^T |\,\dot{u}(t)\,|^2\, dt\Big)^{\frac{1}{2}} \\
&\leq \frac{\lambda d_2}{4}\|\dot{u}\|_2^2 + \frac{C_1^2 d_1^2 T}{3\lambda d_2}\Big(\int_0^T r_1(t) dt\Big)^2 h_1^2(|\,\bar{u}\,|) + C_1 d_1 K_1 \Big(\frac{T}{12}\Big)^{\frac{\alpha+1}{2}} \int_0^T r_1(t) dt \|\dot{u}\|_2^{\alpha+1} \\
&+ \Big(C_1 C_3 d_1 \Big(\frac{T}{12}\Big)^{\frac{1}{2}} \int_0^T r_1(t) dt + C_2\Big)\|\dot{u}\|_2 .
\end{aligned} \tag{3.2}$$

It follows from the condition (H_2) and Sobolev's inequality that

$$\begin{aligned}
\int_0^T e^{Q(t)}[F_2(u(t)) - F_2(\bar{u})]dt &= \int_0^T e^{Q(t)} \int_0^1 (\nabla F_2(\bar{u} + s\tilde{u}(t)),\tilde{u}(t)) ds dt \\
&= \int_0^T e^{Q(t)} \int_0^1 (\nabla F_2(\bar{u} + s\tilde{u}(t)) - \nabla F_2(\bar{u}),\tilde{u}(t)) ds dt \\
&= \int_0^T e^{Q(t)} \int_0^1 \frac{1}{s}(\nabla F_2(\bar{u} + s\tilde{u}(t)) - \nabla F_2(\bar{u}), s\tilde{u}(t)) ds dt \\
&\leq \int_0^T e^{Q(t)} \int_0^1 \frac{1}{s} k(t) h_2(|\,s\tilde{u}(t)\,|) ds dt \\
&\leq \int_0^T e^{Q(t)} \int_0^1 \frac{1}{s} k(t) h_2(s\|\tilde{u}\|_\infty) ds dt \\
&\leq \int_0^T e^{Q(t)} \int_0^1 \frac{1}{s} k(t) s^2 K_2 \|\tilde{u}\|_\infty^2\, ds dt \\
&\leq K_2 \int_0^T e^{Q(t)} k(t) dt \|\tilde{u}\|_\infty^2 \\
&\leq \frac{d_1 K_2 T}{12} \int_0^T k(t) dt \|\dot{u}\|_2^2 ,
\end{aligned} \tag{3.3}$$

for sufficiently large $\|\tilde{u}\|_\infty$. By Sobolev's inequality, we have

$$\int_0^T e^{Q(t)}(f(t),u(t))dt = \int_0^T e^{Q(t)}(f(t),\tilde{u}(t))dt + \int_0^T e^{Q(t)}(f(t),\bar{u})dt$$
$$\le \int_0^T e^{Q(t)} \mid f(t) \mid dt \|\tilde{u}\|_\infty + \int_0^T e^{Q(t)} \mid f(t) \mid dt \,|\bar{u}\,|$$
$$\le C_4\|\dot{u}\|_2 + \int_0^T e^{Q(t)} \mid f(t) \mid dt \,|\bar{u}\,|. \tag{3.4}$$

Thus, by (3.1)–(3.4), we obtain

$$\varphi(u) = \frac{1}{2}\int_0^T e^{Q(t)}(A(t)\dot{u}(t),\dot{u}(t))dt + \int_0^T e^{Q(t)}F(t,u(t))dt + \int_0^T e^{Q(t)}(f(t),u(t))dt$$
$$= \frac{1}{2}\int_0^T e^{Q(t)}(A(t)\dot{u}(t),\dot{u}(t))dt + \int_0^T e^{Q(t)}[F_1(t,u(t)) - F_1(t,\bar{u})]dt$$
$$+ \int_0^T e^{Q(t)}[F_2(u(t)) - F_2(\bar{u})]dt + \int_0^T e^{Q(t)}F(t,\bar{u})dt + \int_0^T e^{Q(t)}(f(t),u(t))dt$$
$$\ge (\frac{\lambda d_2}{4} - \frac{d_1K_2T}{12}\int_0^T k(t)dt)\|\dot{u}\|_2^2 \; - C_1d_1K_1(\frac{T}{12})^{\frac{\alpha+1}{2}}\int_0^T r_1(t)dt\|\dot{u}\|_2^{\alpha+1}$$
$$-(C_1C_3d_1(\frac{T}{12})^{\frac{1}{2}}\int_0^T r_1(t)dt + C_2 + C_4)\|\dot{u}\|_2 + |\,\bar{u}\,|^{2\alpha}(\frac{1}{|\,\bar{u}\,|^{2\alpha}}\int_0^T e^{Q(t)}F(t,\bar{u})dt$$
$$-\frac{C_1^2d_1^2T}{3\lambda d_2}(\int_0^T r_1(t)dt)^2\frac{h_1^2(|\,\bar{u}\,|)}{|\,\bar{u}\,|^{2\alpha}} \; - \int_0^T e^{Q(t)} \mid f(t) \mid dt \frac{1}{|\,\bar{u}\,|^{2\alpha-1}}). \tag{3.5}$$

Since

$$\|u\| \to +\infty \;\; \Leftrightarrow \;\; (|\bar{u}|^2 + \|\dot{u}\|_2^2)^{\frac{1}{2}} \to +\infty\,, \tag{3.6}$$

and $\frac{\lambda d_2}{4} - \frac{d_1K_2T}{12}\int_0^T k(t)dt > 0$ as $\int_0^T k(t)dt < \frac{3\lambda d_2}{d_1K_2T}$, it follows from (H_1), (H_3), (3.5) and (3.6) that $\varphi(u) \to +\infty$ as $\|u\| \to +\infty$, i.e., $\varphi(u)$ is coercive. By Theorem 1.1 and Corollary 1.1 in [3] (i.e., the least action principle), we complete the proof of Theorem 3.1.

Remark 1. The condition $\lim\limits_{s\to+\infty}\frac{h_1(s)}{s^\alpha} \le K_1$ in Theorem 3.1 is weaker than the condition $h(t) \le K_1t^\alpha + K_2$ in Theorem 1.1 in [4] (i.e., Theorem A in the present paper), so that Theorem 3.1 generalizes Theorem 1.1 in [4] even in the case of $F_2(x) = 0$, $q(t) \equiv 0$, $A(t) = I_{N\times N}$ and $f(t) \equiv 0$.

Theorem 3.2. *Let* $F(t,x) = F_1(t,x) + F_2(x)$, *and suppose that* $F_1(t,x)$ *and* $F_2(x)$ *satisfy assumption* (A) *and* (H_2). *If the following conditions hold:*

(H_4) *There exist constants* $C_1 > 0$, $K_1 \ge 0$, $\alpha \in [0,1)$ *and a non-negative function* $h_1 \in C([0,+\infty);[0,+\infty))$ *with the properties:*

(i) $h_1(s) \le h_1(t)$ $\quad\quad \forall s \le t, s,t \in [0,+\infty)$,

(ii) $h_1(s+t) \le C_1(h_1(s) + h_1(t))$ $\quad\quad \forall s,t \in [0,+\infty)$,

(iii) $\lim\limits_{s\to+\infty}\frac{h_1(s)}{s^\alpha} \le K_1$.

Moreover, there exist $r_1, r_2 \in L^1(0,T;R^+)$ *such that*

$$|\nabla F_1(t,x)| \le r_1(t)h_1(|x|) + r_2(t)\,,\; for \;\; x \in R^N,\;\; a.e.\, t \in [0,T]\,.$$

(H_5) $\lim\limits_{|x|\to+\infty} \frac{1}{h_1^2(|x|)} \int_0^T e^{Q(t)} (F(t,x) + |f(t)||x|)dt = -\infty$.

Then problem (1.1) has at least one solution on H_T^1.

Proof. We will use the Saddle Point Theorem 4.7 in [3] to prove Theorem 3.2. First, we prove that the functional $\varphi(u)$ satisfies the (PS) condition. Suppose that $\{u_n\}$ is a (PS) sequence for $\varphi(u)$, that is, $\lim\limits_{n\to\infty} \varphi'(u_n) = 0$ and $\varphi(u_n)$ is bounded. By a way similar to (3.2)–(3.4), we have

$$\left|\int_0^T e^{Q(t)} (\nabla F_1(t,u_n(t)), \tilde{u}_n(t)) dt\right|$$

$$\leq \frac{\lambda d_2}{4}\|\dot{u}_n\|_2^2 + C_1 d_1 K_1 (\frac{T}{12})^{\frac{\alpha+1}{2}} \int_0^T r_1(t)dt \|\dot{u}_n\|_2^{\alpha+1} + (C_1 C_3 d_1 (\frac{T}{12})^{\frac{1}{2}} \int_0^T r_1(t)dt + C_2)\|\dot{u}_n\|_2 + \frac{C_1^2 d_1^2 T}{3\lambda d_2} (\int_0^T r_1(t)dt)^2 h_1^2(|\bar{u}_n|),$$

$$\begin{aligned}
\int_0^T e^{Q(t)} (\nabla F_2(u_n(t)), \tilde{u}_n(t)) dt &= \int_0^T e^{Q(t)} (\nabla F_2(\bar{u}_n + \tilde{u}_n(t)), \tilde{u}_n(t)) dt \\
&= \int_0^T e^{Q(t)} (\nabla F_2(\bar{u}_n + \tilde{u}_n(t)) - \nabla F_2(\bar{u}_n), \tilde{u}_n(t)) dt \\
&\leq \int_0^T e^{Q(t)} k(t) h_2(|\tilde{u}_n(t)|) dt \\
&\leq \int_0^T e^{Q(t)} k(t) h_2(\|\tilde{u}_n\|_\infty) dt \\
&\leq \int_0^T e^{Q(t)} k(t) (K_2 \|\tilde{u}_n\|_\infty^2 + C_5) dt \\
&\leq K_2 \int_0^T e^{Q(t)} k(t) dt \|\tilde{u}_n\|_\infty^2 + C_6 \\
&\leq \frac{d_1 K_2 T}{12} \int_0^T k(t) dt \|\dot{u}_n\|_2^2 + C_6
\end{aligned}$$

and

$$\int_0^T e^{Q(t)} (f(t), \tilde{u}_n(t)) dt \leq \int_0^T e^{Q(t)} |f(t)| dt \|\tilde{u}_n\|_\infty \leq C_4 \|\dot{u}_n\|_2.$$

Hence,

$$\begin{aligned}
\|\tilde{u}_n\| &\geq (\varphi'(u_n), \tilde{u}_n) \\
&= \int_0^T e^{Q(t)} (A(t)\dot{u}_n(t), \dot{u}_n(t)) dt \\
&+ \int_0^T e^{Q(t)} (\nabla F_1(t,u_n(t)), \tilde{u}_n(t)) dt + \int_0^T e^{Q(t)} (\nabla F_2(u_n(t)), \tilde{u}_n(t)) dt + \int_0^T e^{Q(t)} (f(t), \tilde{u}_n(t)) dt \\
&\geq \lambda d_2 \|\dot{u}_n\|_2^2 - \frac{\lambda d_2}{4}\|\dot{u}_n\|_2^2 - \frac{C_1^2 d_1^2 T}{3\lambda d_2} (\int_0^T r_1(t)dt)^2 h_1^2(|\bar{u}_n|) - C_1 d_1 K_1 (\frac{T}{12})^{\frac{\alpha+1}{2}} \int_0^T r_1(t)dt \|\dot{u}_n\|_2^{\alpha+1} \\
&- (C_1 C_3 d_1 (\frac{T}{12})^{\frac{1}{2}} \int_0^T r_1(t)dt + C_2)\|\dot{u}_n\|_2 - \frac{d_1 K_2 T}{12} \int_0^T k(t)dt \|\dot{u}_n\|_2^2 - C_4 \|\dot{u}_n\|_2 - C_6 \\
&= (\frac{3\lambda d_2}{4} - \frac{d_1 K_2 T}{12} \int_0^T k(t)dt) \|\dot{u}_n\|_2^2 - \frac{C_1^2 d_1^2 T}{3\lambda d_2} (\int_0^T r_1(t)dt)^2 h_1^2(|\bar{u}_n|) \\
&- C_1 d_1 K_1 (\frac{T}{12})^{\frac{\alpha+1}{2}} \int_0^T r_1(t)dt \|\dot{u}_n\|_2^{\alpha+1} - (C_1 C_3 d_1 (\frac{T}{12})^{\frac{1}{2}} \int_0^T r_1(t)dt + C_2 + C_4)\|\dot{u}_n\|_2 - C_6.
\end{aligned}$$

It follows from (2.1) that

$$\|\tilde{u}_n\| \le (1+\frac{T^2}{4\pi^2})^{\frac{1}{2}}(\int_0^T |\dot{u}_n(t)|^2 dt)^{\frac{1}{2}}.$$

Thus,

$$\|\dot{u}_n\|_2 \le C_7 h_1(|\bar{u}_n|) + C_8, \text{ for sufficiently large } n. \tag{3.7}$$

By (3.2)–(3.4) and (H_5), we have

$$\begin{aligned}
\varphi(u_n) &= \frac{1}{2}\int_0^T e^{Q(t)}(A(t)\dot{u}_n(t),\dot{u}_n(t))dt + \int_0^T e^{Q(t)}F(t,u_n(t))dt + \int_0^T e^{Q(t)}(f(t),u_n(t))dt \\
&= \frac{1}{2}\int_0^T e^{Q(t)}(A(t)\dot{u}_n(t),\dot{u}_n(t))dt + \int_0^T e^{Q(t)}[F_1(t,u_n(t)) - F_1(t,\bar{u}_n)]dt \\
&\quad + \int_0^T e^{Q(t)}[F_2(u_n(t)) - F_2(\bar{u}_n)]dt + \int_0^T e^{Q(t)}F(t,\bar{u}_n)dt + \int_0^T e^{Q(t)}(f(t),u_n(t))dt \\
&\le (\frac{d_1 a^N}{2} + \frac{\lambda d_2}{4} + \frac{d_1 K_2 T}{12}\int_0^T k(t)dt)\|\dot{u}_n\|_2^2 + C_1 d_1 K_1 (\frac{T}{12})^{\frac{\alpha+1}{2}}\int_0^T r_1(t)dt\|\dot{u}_n\|_2^{\alpha+1} \\
&\quad + (C_1 C_3 d_1 (\frac{T}{12})^{\frac{1}{2}}\int_0^T r_1(t)dt + C_2 + C_4)\|\dot{u}_n\|_2 + \frac{C_1^2 d_1^2 T}{3\lambda d_2}(\int_0^T r_1(t)dt)^2 h_1^2(|\bar{u}_n|) \\
&\quad + \int_0^T e^{Q(t)}|f(t)|dt|\bar{u}_n| + \int_0^T e^{Q(t)}F(t,\bar{u}_n)dt \\
&\le C_9 h_1^2(|\bar{u}_n|) + C_{10}h_1^{\alpha+1}(|\bar{u}_n|) + C_{11}h_1(|\bar{u}_n|) + \int_0^T e^{Q(t)}|f(t)|dt|\bar{u}_n| + \int_0^T e^{Q(t)}F(t,\bar{u}_n)dt \\
&= h_1^2(|\bar{u}_n|)(\frac{1}{h_1^2(|\bar{u}_n|)}\int_0^T e^{Q(t)}(F(t,\bar{u}_n) + |f(t)||\bar{u}_n|)dt + C_{10}\frac{1}{h_1^{1-\alpha}(|\bar{u}_n|)} + C_{11}\frac{1}{h_1(|\bar{u}_n|)} + C_9) \\
&\to -\infty \text{ as } |\bar{u}_n| \to +\infty,
\end{aligned}$$

which contradicts the boundedness of $\varphi(u_n)$. Therefore, $\{|\bar{u}_n|\}$ is bounded, and then $\{u_n\}$ is bounded by (3.7). We conclude that the (PS) condition is satisfied.

Next, we only need to prove the following conditions:

(l_1) $\varphi(u) \to -\infty$ as $|u| \to +\infty$ in R^N;

(l_2) $\varphi(u) \to +\infty$ as $\|u\| \to +\infty$ in $\tilde{H}_T^1$.

In fact, by (H_5), we obtain

$$\begin{aligned}
\varphi(u) &= \int_0^T e^{Q(t)}F(t,u)dt + \int_0^T e^{Q(t)}(f(t),u)dt \\
&\le \int_0^T e^{Q(t)}(F(t,u) + |f(t)||u|)dt \to -\infty \text{ as } |u| \to +\infty \text{ in } R^N.
\end{aligned}$$

Thus (l_1) is proved.

For $u \in \tilde{H}_T^1$, as we have argued in (3.2) and (3.3), we have

$$\begin{aligned}
\left|\int_0^T e^{Q(t)}[F_1(t,\tilde{u}(t)) - F_1(t,0)]dt\right| &= \left|\int_0^T e^{Q(t)}\int_0^1 (\nabla F_1(t,s\tilde{u}(t)),\tilde{u}(t))dsdt\right| \\
&\le \int_0^T\int_0^1 e^{Q(t)}r_1(t)h_1(|s\tilde{u}(t)|)\cdot|\tilde{u}(t)|dsdt + \int_0^T\int_0^1 e^{Q(t)}r_2(t)|\tilde{u}(t)|dsdt \\
&\le d_1 h_1(\|\tilde{u}\|_\infty)\|\tilde{u}\|_\infty\int_0^T r_1(t)dt + C_2(\int_0^T |\dot{u}(t)|^2 dt)^{\frac{1}{2}}
\end{aligned}$$

$$\leq d_1(K_1\|\tilde{u}\|_\infty^\alpha + C_3)\|\tilde{u}\|_\infty \int_0^T r_1(t)dt + C_2(\int_0^T |\dot{u}(t)|^2\, dt)^{\frac{1}{2}}$$
$$\leq d_1K_1(\frac{T}{12})^{\frac{\alpha+1}{2}}\int_0^T r_1(t)dt\|\dot{u}\|_2^{\alpha+1} + (C_3d_1(\frac{T}{12})^{\frac{1}{2}}\int_0^T r_1(t)dt + C_2)\|\dot{u}\|_2$$

and

$$\int_0^T e^{Q(t)}[F_2(u(t)) - F_2(0)]dt \leq \frac{d_1K_2T}{12}\int_0^T k(t)dt\|\dot{u}\|_2^2,$$

which implies that

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\int_0^T e^{Q(t)}(A(t)\dot{u}(t),\dot{u}(t))dt + \int_0^T e^{Q(t)}F(t,u(t))dt + \int_0^T e^{Q(t)}(f(t),u(t))dt \\
&= \frac{1}{2}\int_0^T e^{Q(t)}(A(t)\dot{u}(t),\dot{u}(t))dt + \int_0^T e^{Q(t)}[F_1(t,u(t)) - F_1(t,0)]dt \\
&+ \int_0^T e^{Q(t)}[F_2(u(t)) - F_2(0)]dt + \int_0^T e^{Q(t)}F(t,0)dt + \int_0^T e^{Q(t)}(f(t),u(t))dt \\
&\geq (\frac{\lambda d_2}{4} - \frac{d_1K_2T}{12}\int_0^T k(t)dt)\|\dot{u}\|_2^2 - d_1K_1(\frac{T}{12})^{\frac{\alpha+1}{2}}\int_0^T r_1(t)dt\|\dot{u}\|_2^{\alpha+1} \\
&- (C_3d_1(\frac{T}{12})^{\frac{1}{2}}\int_0^T r_1(t)dt + C_2 + C_4)\|\dot{u}\|_2 + \int_0^T e^{Q(t)}F(t,0)dt, \qquad (3.8)
\end{aligned}$$

for all $u \in \tilde{H}_T^1$.

By Wirtinger's inequality, one has

$$\|u\| \to +\infty \iff \|\dot{u}\|_2 \to +\infty,\ u \in \tilde{H}_T^1.$$

We know that $\frac{\lambda d_2}{4} - \frac{d_1K_2T}{12}\int_0^T k(t)dt > 0$ as $\int_0^T k(t)dt < \frac{3\lambda d_2}{d_1K_2T}$. It follows from (3.8) that $\varphi(u) \to +\infty$ as $\|u\| \to +\infty$ in $\tilde{H}_T^1$, that is, (l_2) is proved.

By making use of the Saddle Point Theorem 4.7 in [3], we prove that problem (1.1) has at least one solution on H_T^1.

Remark 2. The condition $\lim_{s\to+\infty}\frac{h_1(s)}{s^\alpha} \leq K_1$ in Theorem 3.2 is weaker than the condition $h(t) \leq K_1t^\alpha + K_2$ in Theorem 1.2 in [4] (i.e., Theorem B in the present paper), so that Theorem 3.2 generalizes Theorem 1.2 in [4] even in the case of $F_2(x) = 0$, $q(t) \equiv 0$, $A(t) = I_{N\times N}$ and $f(t) \equiv 0$.

4. Examples

In this section, we give two examples to illustrate the feasibility and effectiveness of our main conclusions.

Example 4.1. Let $A(t)=\begin{pmatrix} \lambda+t & 0 & 0 & \cdots & 0 \\ 0 & \lambda+t & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda+t \end{pmatrix}_{N\times N}$, where $\lambda>0$.

Let $F_1(t,x)=(\frac{d_2}{3d_1}T-t)\ln^{\frac{3}{2}}(1+|x|^2)$, $F_2(x)=\frac{\lambda d_2}{d_1K_2T^2}|x|^2$, $h_1(|x|)=\ln^{\frac{1}{2}}(1+|x|^2)$.

Then,

$$\ln^{\frac{1}{2}}[1+(s+t)^2]\le 2(\ln^{\frac{1}{2}}(1+s^2)+\ln^{\frac{1}{2}}(1+t^2)),\qquad \forall s,t\in[0,+\infty),$$

i.e.,

$$h_1(s+t)\le C_1(h_1(s)+h_1(t)),\qquad \forall s,t\in[0,+\infty)$$

and

$$\lim_{s\to+\infty}\frac{h_1(s)}{s^\alpha}=\lim_{s\to+\infty}\frac{\ln^{\frac{1}{2}}(1+s^2)}{s^\alpha}=0\le K_1\cdot$$

Moreover, we obtained the following results through simple derivation:

① $$|\nabla F_1(t,x)|=\frac{3}{2}|\frac{d_2}{3d_1}T-t|\ln^{\frac{1}{2}}(1+|x|^2)\frac{1}{1+|x|^2}\cdot 2|x|$$
$$\le\frac{3}{2}|\frac{d_2}{3d_1}T-t|\ln^{\frac{1}{2}}(1+|x|^2)$$
$$=r_1(t)h_1(|x|)+r_2(t),$$

② $(\nabla F_2(x)-\nabla F_2(y),x-y)=\frac{2\lambda d_2}{d_1K_2T^2}|x-y|^2=k(t)h_2(|x-y|)$, $(\int_0^T k(t)dt<\frac{3\lambda d_2}{d_1K_2T})$,

③ $\limsup_{s\to+\infty}\frac{h_2(s)}{s^2}=1\le K_2$,

④ $$\lim_{|x|\to+\infty}\frac{1}{|x|^{2\alpha}}\int_0^T e^{Q(t)}F(t,x)dt$$
$$=\lim_{|x|\to\infty}\frac{1}{|x|^{2\alpha}}\int_0^T e^{Q(t)}(\frac{d_2}{3d_1}T-t)\ln^{\frac{3}{2}}(1+|x|^2)dt+\lim_{|x|\to\infty}\frac{1}{|x|^{2\alpha}}\int_0^T e^{Q(t)}\frac{\lambda d_2}{d_1K_2T^2}|x|^2dt$$
$$\ge\int_0^T e^{Q(t)}(\frac{d_2}{3d_1}T-t)dt\cdot\lim_{|x|\to\infty}\frac{\ln^{\frac{3}{2}}(1+|x|^2)}{|x|^{2\alpha}}+\frac{\lambda d_2^2}{d_1K_2T}\lim_{|x|\to\infty}|x|^{2-2\alpha}$$
$$=+\infty,$$

⑤ $\lambda|x|^2\le(\lambda+t)|x|^2=(A(t)x,x)$, for all $x\in R^N$ and $a.e.\,t\in[0,T]$.

It is easy to see that $A(t)$, $F_1(t,x)$, $F_2(x)$ and $h_1(|x|)$ satisfy all conditions of Theorem 3.1. Hence, problem (1.1) has at least one solution on H_T^1.

Example 4.2. Let $A(t)=\begin{pmatrix} \lambda+t & 0 & 0 & \cdots & 0 \\ 0 & \lambda+t & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda+t \end{pmatrix}_{N\times N}$, where $\lambda>0$.

Let $F_1(t,x)=(\frac{d_2}{3d_1}T-t)\ln^{\frac{3}{2}}(1+|x|^2)$, $F_2(x)=-\frac{\lambda d_2}{d_1K_2T^2}|x|^2$, $h_1(|x|)=\ln^{\frac{1}{2}}(1+|x|^2)$.

Then,

$$\begin{aligned}
&\lim_{|x|\to+\infty}\frac{1}{h_1^2(|x|)}\int_0^T e^{Q(t)}(F(t,x)+|f(t)||x|)dt \\
&=\lim_{|x|\to\infty}\ln^{\frac{1}{2}}(1+|x|^2)\int_0^T e^{Q(t)}(\frac{d_2}{3d_1}T-t)dt-\lim_{|x|\to\infty}\frac{|x|^2}{\ln(1+|x|^2)}\int_0^T e^{Q(t)}\frac{\lambda d_2}{d_1K_2T^2}dt+\lim_{|x|\to\infty}\frac{|x|}{\ln(1+|x|^2)}\int_0^T e^{Q(t)}|f(t)|dt \\
&\le -\frac{d_2T^2}{6}\lim_{|x|\to\infty}\ln^{\frac{1}{2}}(1+|x|^2)-\frac{\lambda d_2^2}{d_1K_2T}\lim_{|x|\to\infty}\frac{|x|^2}{\ln(1+|x|^2)}+\int_0^T e^{Q(t)}|f(t)|dt\cdot\lim_{|x|\to\infty}\frac{|x|}{\ln(1+|x|^2)} \\
&=-\infty.
\end{aligned}$$

The derivation of other conditions for Theorem 3.2 is the same as Example 4.1.

It is clear that $A(t)$, $F_1(t,x)$, $F_2(x)$ and $h_1(|x|)$ satisfy all conditions of Theorem 3.2. Therefore, problem (1.1) has at least one solution on H_T^1.

5. Conclusions

In this paper, we study the existence of periodic solutions of the forced damped vibration systems [i.e., (1.1)] by using the variational method and the critical point theory.

First, we provide an expression for functional $\varphi(u)$ and further prove that the functional $\varphi(u)$ is continuously differentiable and weak lower semi-continuous.

Then, we prove that the critical point of $\varphi(u)$ is a solution of problem (1.1) in the sense of weak-derivative. Moreover, we directly derive the variational principle of problem (1.1) via the semi-inverse method.

Finally, it is proved that problem (1.1) has at least one solution under the given sufficient conditions through the least action principle and the saddle point theorem.

In the future, we can continue to study this problem by looking for new sufficient conditions.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

There is no conflict of interest.

References

1. X. P. Wu, C. L. Tang, Periodic solutions of a class of non-autonomous second order systems, *J. Math. Anal. Appl.*, **236** (1999), 227–235. https://doi.org/10.1006/jmaa.1999.6408
2. P. H. Rabinowitz, On subharmonic solutions of Hamiltonian systems, *Commun. Pure Appl. Math.*, **33** (1980), 609–633. https://doi.org/10.1002/cpa.3160330504
3. J. Mawhin, M. Willem, *Critical point theory and Hamiltonian systems*, New York: Springer, 1989. https://doi.org/10.1007/978-1-4757-2061-7
4. Z. Wang, J. Zhang, Periodic solutions of a class of second order non-autonomous Hamiltonian systems, *Nonlinear Anal-Theor.*, **72** (2010), 4480–4487. https://doi.org/10.1016/j.na.2010.02.023
5. Z. Wang, J. Zhang, New existence results on periodic solutions of non-autonomous second order Hamiltonian systems, *Appl. Math. Lett.*, **79** (2018), 43–50. https://doi.org/10.1016/j.aml.2017.11.016
6. C. L. Tang, Periodic solutions for nonautonomous second order systems with sublinear nonlinearity, *Proc. Amer. Math. Soc.*, **126** (1998), 3263–3270. https://doi.org/10.1090/S0002-9939-98-04706-6
7. G. Bonanno, R. Livrea, M. Schechter, Some notes on a superlinear second order Hamiltonian system, *manuscripta math.*, **154** (2017), 59–77. https://doi.org/10.1007/s00229-016-0903-6
8. J. Pipan, M. Schechter, Non-autonomous second order Hamiltonian systems, *J. Differ. Equ.*, **257** (2014), 351–373. https://doi.org/10.1016/j.jde.2014.03.016
9. N. Aizmahin, T. An, The existence of periodic solutions of non-autonomous second-order Hamiltonian systems, *Nonlinear Anal-Theor.*, **74** (2011), 4862–4867. https://doi.org/10.1016/j.na.2011.04.060
10. J. Ma, C. L. Tang, Periodic solutions for some nonautonomous second order systems, *J. Math. Anal. Appl.*, **275** (2002), 482–494. https://doi.org/10.1016/S0022-247X(02)00636-4
11. C. Tang, Periodic solutions of non-autonomous second order systems, *J. Math. Anal. Appl.*, **202** (1996), 465–469. https://doi.org/10.1006/jmaa.1996.0327
12. Z. Wang, J. Zhang, M. Chen, A unified approach to periodic solutions for a class of non-autonomous second order Hamiltonian systems, *Nonlinear Anal.-Real*, **58** (2021), 103218. https://doi.org/10.1016/j.nonrwa.2020.103218
13. X. Wu, S. Chen, K. Teng, On variational methods for a class of damped vibration problems, *Nonlinear Anal-Theor.*, **68** (2008), 1432–1441. https://doi.org/10.1016/j.na.2006.12.043
14. Z. Wang, J. Zhang, Existence of periodic solutions for a class of damped vibration problems, *C. R. Math.*, **356** (2018), 597–612. https://doi.org/10.1016/j.crma.2018.04.014
15. Q. B. Yin, Y. Guo, D. Wu, X. B. Shu, Existence and multiplicity of mild solutions for first-order Hamilton random impulsive differential equations with Dirichlet boundary conditions, *Qual. Theory Dyn. Syst.*, **22** (2023), 47. https://doi.org/10.1007/s12346-023-00748-5

16. S. Wang, X. B. Shu, L. Shu, Existence of solutions to a class of damped random impulsive differential equations under Dirichlet boundary value conditions, *AIMS Mathematics*, **7** (2022), 7685–7705. https://doi.org/10.3934/math.2022431
17. Y. Guo, X. B. Shu, Q. Yin, Existence of solutions for first-order Hamiltonian random impulsive differential equations with Dirichlet boundary conditions, *Discrete Cont. Dyn.-B*, **27** (2022), 4455–4471. https://doi.org/10.3934/dcdsb.2021236
18. H. J. Ma, Simplified Hamiltonian-based frequency-amplitude formulation for nonlinear vibration systems, *FACTA Univ.-Ser: Mech.*, **20** (2022), 445–455. https://doi.org/10.22190/FUME220420023M
19. F. Faraci, R. Livrea, Infinitely many periodic solutions for a second-order non-autonomous system, *Nonlinear Anal-Theor.*, **54** (2003), 417–429. https://doi.org/10.1016/S0362-546X(03)00099-3
20. J. H. He, Semi-inverse method of establishing generalized variational principles for fluid mechanics with emphasis on turbomachinery aerodynamics, *Int. J. Turbo Jet Eng.*, **14** (1997), 23–28. https://doi.org/10.1515/TJJ.1997.14.1.23
21. J. H. He, A classical variational model for micropolar elastodynamics, *Int. J. Nonlinear Sci. Numer. Simulat.*, **1** (2000), 133–138. https://doi.org/10.1515/IJNSNS.2000.1.2.133
22. J. H. He, Hamilton principle and generalized variational principles of linear thermopiezoelectricity, *J. Appl. Mech.*, **68** (2001), 666–667. https://doi.org/10.1115/1.1352067