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Research article

Discontinuous solutions of delay fractional integral equation via measures of noncompactness

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Abstract: This article considers the existence and the uniqueness of monotonic solutions of a delay functional integral equation of fractional order in the weighted Lebesgue space $L_1^N(\mathbb{R}^+)$. Our analysis uses a suitable measure of noncompactness, a modified version of Darbo's fixed point theorem, and fractional calculus in the mentioned space. An illustrated example to show the applicability and significance of our outcomes is included.

Keywords: weighted Lebesgue space; delay fractional integral equations; measure of noncompactness (M.N.C.); Carathéodory conditions
Mathematics Subject Classification: 47N20, 47H30, 45G10

1. Introduction

This article investigates and examines the presence and then the uniqueness of a.e. nondecreasing solutions to the problem

$$\begin{cases} z(\theta) = h(\theta) + f_1(\theta, z(\theta-\tau)) + f_3\Big(\theta,\ l(\theta)\int_0^\theta \frac{(\theta-s)^{\alpha-1}}{\Gamma(\alpha)} f_2(s, z(s-\tau))ds\Big),\ \theta\in\mathbb{R}^+ \\ z(\theta) = z_0 \quad \text{on} \quad [-\tau, 0),\ \ 0<\alpha<1 \end{cases} \tag{1.1}$$

in weighted Lebesgue space $L_1^N(\mathbb{R}^+)$, which is a larger space than the classical Lebesgue space $L_1(\mathbb{R}^+)$. This permits us to concentrate on our aims under more general conditions. To attain these aims, we prove a modified version of Darbo's fixed point principle [1] combined with a suitable measure of noncompactness (M.N.C.) in $L_1^N(\mathbb{R}^+)$. We use the notion of sets are compact in measure to prove that our (M.N.C.) is equivalent to the Hausdorff (M.N.C.). The technique used in this article differs from the ones used in [2–4], where we dispense the compactness assumptions.

Moreover, we focus on nondecreasing solutions, which don't belong to $L_1(\mathbb{R}^+)$, so we consider our solutions in the space $L_1^N(\mathbb{R}^+)$ to bypass these difficulties.

Equation (1.1) represents a generalization and extension of the classical, convolution, and fractional integral equations discussed in the former literature [5–11].

The authors in [12] had examined the existence and the uniqueness of a.e. nonincreasing results of some delay-Volterra Hammerstein integral problems

$$z(\theta) = h(\theta) + m(\theta) \cdot g(\theta, z(\theta - \tau)) + \int_0^\theta k(\theta, s) f(s, z(s - \tau)) ds, \ \theta \in \mathbb{R}^+$$

in both $L_1(\mathbb{R}^+)$ and $L_1^{loc}(\mathbb{R}^+)$.

Models involving delay integral or differential equations arise in mathematical biology, physics, medicine, and in models of machine operations (see e.g. [13, 14]).

In [15], Cooke and Kaplan created the following model to describe the noticed periodic epidemics of several infectious diseases with periodic contact levels that vary seasonally

$$z'(\theta) = f(\theta, z(\theta)) - f(\theta - \tau, z(\theta - \tau)),$$

which has been also examined in [16–18].

In [19, 20] the authors studied equations of the type

$$\begin{aligned} z(\theta) &= f(\theta, z(\theta - \tau)) + \int_\theta^\infty H(s, z(s), z(s - \tau)) ds, \\ y(\theta) &= f(\theta, y(\theta - \tau)) + \int_{-\infty}^\theta Q(s, y(s), y(s - \tau)) C(\theta - s) ds + p(\theta), \end{aligned}$$

using contraction mappings and combining Lyapunov's direct method and Krasnoselskii-type fixed point theorem. Many physical and biological models such as electric, pneumatic, and hydraulic networks (see [21, 22]) are described by delayed differential or integral equations with discontinuous functions. For example, in [23] the authors considered the discontinuity solutions for the delay differential equation

$$\begin{aligned} y'(\theta) &= f(\theta, y(\theta), y(\alpha(\theta, y(\theta)))), \ \ \theta \in [0, T], \\ y(\theta) &= \varphi(\theta), \ \ \theta \in [a, 0], \ \ \text{where} \ \ a = \inf_{\theta \geq 0} \alpha(\theta, y(\theta)) \leq 0. \end{aligned}$$

The following Abel integral equation reconstructs the potential $V(z)$ for measurements of the duration of oscillations T of a pendulum,

$$\int_0^E (E - V)^{\frac{-1}{2}} z'(V) \, dV = \frac{T(E)}{\sqrt{8m}},$$

where m and E denote the particle mass and energy, respectively (cf. [5]).

This article is motivated by inspecting and studying the existence and the uniqueness of discontinuous monotonic solutions for a general fractional integral equation in $L_1^N(\mathbb{R}^+)$. We give an example to demonstrate the applicability and significance of our theorems.

2. Preliminaries

Let $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}^+ = [0, \infty)$ and the symbols M.N.C. (M.W.N.C.) refer to the measure of noncompactness (weak noncompactness), respectively. Denote by $L_1^N = L_1^N(\mathbb{R}^+)$ the weighted Lebesgue space, which is the Banach space of all Lebesgue integrable functions z on $\mathbb{R}^+$ related to the norm

$$||z||_{L_1^N} = ||z||_{L_1^N(\mathbb{R}^+)} = \int_0^\infty e^{-N\theta}|z(\theta)|\, d\theta, \quad N > 0.$$

If N = 0 we have classical Lebesgue space L_1 with the standard norm.

Now, we need to recall some operators with their properties on L_1^N, which will be needed in the sequel.

Definition 2.1. [24] Suppose that the function $f(\theta, z) = f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ fulfills the Carathéodory conditions, i.e. it is measurable in θ for any $z \in \mathbb{R}$ and continuous in z for almost all $\theta \in \mathbb{R}^+$. Then, we denote the Nemytskii (Superposition) operator by

$$F_f(z)(\theta) = f(\theta, z(\theta)),\ \theta \in \mathbb{R}^+.$$

Lemma 2.2. *[4] Suppose that the function f fulfills the Carathéodory conditions and*

$$|f(\theta, z)| \le a(\theta) + b \cdot |z|,$$

where $a \in L_1^N$ *and* $b \ge 0$ *for all* $\theta \in \mathbb{R}^+$ *and* $z \in \mathbb{R}$. *Then* $F_f : L_1^N \to L_1^N$ *is continuous.*

Definition 2.3. [25, 26] Let $z \in L_1$, $\alpha \in \mathbb{R}^+$. The Riemann-Liouville (R-L) fractional integral of function z of order α is defined as:

$$I^\alpha z(\theta) = \int_0^\theta \frac{(\theta - s)^{\alpha-1}}{\Gamma(\alpha)} z(s)\, ds, \quad \alpha > 0, \quad \theta > 0,$$

where $\Gamma(\alpha) = \int_0^\theta e^{-\theta}\theta^{\alpha-1} d\theta$.

Lemma 2.4. *[4, 25] For* $\alpha > 0$, *we have*

- *(a) the operator* $I^\alpha : L_1^N \to L_1^N$ *continuously;*
- *(b)* $||I^\alpha z||_{L_1^N} \le \frac{1}{N^\alpha}||z||_{L_1^N}$;
- *(c) the operator* I^α *takes a.e. nondecreasing and nonnegative functions into functions have the same properties.*

Let $J = [a, b]$ and the symbol B_r points to the closed ball has radius r and center at zero element κ. Denote by $S = S(J)$ the set of all measurable functions (in Lebesgue sense) on J. The functions equal a.e. in the set S corresponding to the metric

$$d(z, y) = \inf_{\rho>0}[\rho + meas\{\theta \ :\ |z(\theta) - y(\theta)| \ge \rho\}]$$

construct a complete metric space. Furthermore, the convergence with respect to the metric d is the same as the convergence in measure on J (Proposition 2.14 in [27]).

Remark 2.5. Concerning the case of $\mathbb{R}^+$, as the measure is σ-finite, a notion of *convergence in finite measure* is used and it means, that (z_n) is convergent to z in finite measure iff it converges to z on every set $T \subset \mathbb{R}^+$ of finite measure. We will call the compactness in these spaces "compactness in finite measure" ("compactness in measure").

Remark 2.6. Let $Z \subset L_1^N(J)$ be a bounded set. Suppose that there is a family $(\Omega_c)_{0\le c\le b-a} \subset J$, such that meas$\Omega_c = c$ for every $c \in [0, b-a]$, and for every $z \in Z$, $z(t_1) \ge z(t_2)$, $(t_1 \in \Omega_c, t_2 \notin \Omega_c)$. That family is equimeasurable and the set Z is compact in measure in $L_1^N(J)$. Obviously, by taking $\Omega_c = [0, c) \cup W$ or $\Omega_c = [0, c) \setminus W$, where W is a set of measure zero, such family consists of nondecreasing functions (possibly except for a set W). The functions from this family are called "a.e. nondecreasing" functions. It is clear that the same is true for $\mathbb{R}^+$.

Remark 2.7. Since $\theta \to e^{-N\theta}$ is nonincreasing on $\mathbb{R}^+$ (for $N > 0$), then the pointwise product of this function with monotonic (nondecreasing or nonincreasing) integrable functions do not change their monotonicity properties. Immediately, as in the case of L_1 [6], we get:

Theorem 2.8. *Let $Z \subset L_1^N(J)$ be a bounded set containing functions that are a.e. nonincreasing (or a.e. nondecreasing) on the interval J. Then the set Z represents a compact in measure set in $L_1^N(J)$.*

Next, we will extend these results from bounded domain J to $\mathbb{R}^+$.

Corollary 2.9. *Let $Z \subset L_1^N$ be a bounded set containing functions that are a.e. nondecreasing (or a.e. nonincreasing) on $\mathbb{R}^+$. Then the set Z represents a compact in measure set in L_1^N.*

Proof. Let $L_1^N(T)$ be the space for σ-finite measure space T and then there exists some equivalent finite measure ν ($\nu(\mathbb{R}^+) < \infty$) [27, Corollary 2.20 and Proposition 2.1.].

Therefore, the convergence of sequences in S is equivalent to the metric d and $d_\nu(z, y) = \inf_{\rho>0}[\rho + \nu\{\theta : |z(\theta) - y(\theta)| \ge \rho\}]$ [28, Proposition 2.2]. Let $(z_n) \subset Z$ be an arbitrary bounded sequence.

As a subset of a metric space $Z = (L_1^N(\mathbb{R}^+), d_\nu)$ that sequence is compact in this metric space (Theorem 2.8). Then there exists a subsequence (z_{n_k}) of (z_n) that is convergent in the space Z to some z, i.e.

$$d_\nu(z_{n_k}, z) \overset{k\to\infty}{\longrightarrow} 0.$$

As said before these two metrics have the same convergent sequences, then

$$d(z_{n_k}, z) \overset{k\to\infty}{\longrightarrow} 0.$$

Then, the set Z is compact in finite measure in L_1^N. □

Remark 2.10. Let Q_r be the set of all functions $z \in L_1^N$ that is a.e. nondecreasing and a.e. positive on $\mathbb{R}^+$. Then Q_r is closed, nonempty, convex and bounded subset of L_1^N, such that $\|z\|_{L_1^N} < r,\ r > 0$. Moreover, the set Q_r is compact in measure (cf. [6] and [29, Lemma 4.10]).

Definition 2.11. [1] Let $Z \neq \emptyset$ be a bounded subset of a Banach space E. The Hausdorff M.N.C. $\chi(Z)$ is given by

$$\chi(Z) = \inf\{r > 0 : \text{there exists a finite subset Y of E, such that } Z \subset Y + B_r\}.$$

Moreover, the De Blasi M.W.N.C. β is given by [30]:

$$\beta(Z) = inf\{r > 0 : \text{ There exists a weakly compact subset Y of E, such that } Z \subset Y + B_r\}.$$

Theorem 2.12. *[2, 3] Let $\emptyset \neq Z \subset L_1^N$ be a bounded set and $\varepsilon > 0$, we have*

$$c^T(Z) = \lim_{\varepsilon \to 0} \sup_{z \in Z} \left\{ \sup \left\{ \int_D e^{-N\theta} |z(\theta)| \, d\theta : D \subset [0, T], \ meas(D) \leq \varepsilon \right\} \right\}$$

$$c(Z) = \lim_{T \to \infty} c^T(Z) \tag{2.1}$$

and

$$d(Z) = \lim_{T \to \infty} \sup \left\{ \int_T^\infty e^{-N\theta} |z(\theta)| \, d\theta : z \in Z \right\}.$$

Then

$$\gamma(Z) = c(Z) + d(Z) \tag{2.2}$$

forms a M.W.N.C. on the space L_1^N.

Next, we will demonstrate that M.W.N.C. γ and M.N.C. χ are equivalent, which is important for establishing our main findings.

Theorem 2.13. *If $\emptyset \neq Z \subset L_1^N$ is a bounded and compact in measure set, then*

$$\chi(Z) \leq \gamma(Z) \leq 2\chi(Z).$$

Proof. Let $\chi(Z) = r$ and $\varepsilon > 0$ be arbitrary. Then we can obtain a finite set $Y \subset L_1^N$, such that $Z \subset Y + (r + \varepsilon)B_1$. By the properties of γ, we have

$$\gamma(Z) \leq \gamma(Y) + (r + \varepsilon)\mu(B_1) = 2(r + \varepsilon)$$

and since ε is arbitrary, we get

$$\gamma(Z) \leq 2\chi(Z). \tag{2.3}$$

Moreover, let $\emptyset \neq Z \subset L_1^N$ be compact in finite measure. Suppose that $\chi(Z) = r$ and $c(Z) = r_1, \ d(Z) = r_2$, where $r_1 + r_2 = r$. Fix an arbitrary $\eta > 0$. Then for any measurable subset $D \subset [0, T]$, such that *meas* $D < \varepsilon$,

$$\|z \cdot \chi_D\|_{L_1^N} \leq r_1 + \eta \tag{2.4}$$

for any $z \in Z$ there exist $T > 0$ and $\varepsilon > 0$, such that

$$\sup_{z \in Z} \|z \cdot \chi_{[T,\infty)}\|_{L_1^N} \leq r_2 + \eta. \tag{2.5}$$

Now, for $z \in Z$ and an arbitrary $h \geq 0$ be arbitrary, we symbolize

$$\Omega(z, h) = \{\theta \in [0, T] : |z(\theta)| \geq h\}.$$

Since Z is bounded, we deduce

$$\lim_{h \to \infty} \{\sup[meas \Omega(z, h) : z \in Z]\} = 0.$$

By this consideration, we can select $h_0 \geq 0$, such that

$$meas\Omega(z, h_0) \leq \varepsilon$$

for any $z \in Z$. Then, by using (2.4), we have

$$\|z \cdot \chi_{\Omega(z,h_0)}\|_{L_1^N} \leq r_1 + \eta \tag{2.6}$$

for an arbitrary $z \in Z$.

Next, for any $z \in Z$ we denote by z_{h_0} the function

$$z_{h_0}(\theta) = \begin{cases} 0 & \text{for} \quad \theta \geq T \\ \theta & \text{for} \quad \theta \in [0, T] - \Omega(z, h_0) \\ h_0 \; sign \; z(\theta) & \text{for} \quad \theta \in \Omega(z, h_0). \end{cases}$$

Since Z is compact in finite measure, which indicates that $Z_{h_0} = \{z_{h_0} : z \in Z\}$ is also compact in finite measure. It is clear that $c(Z_{h_0}) = d(Z_{h_0}) = 0$ which indicates that $\gamma(Z_{h_0}) = 0$. Then the set Z_{h_0} is compact in L_1^N.

Consequently

$$\chi(Z_{h_0}) = 0.$$

Now, applying (2.5) we infer

$$\begin{aligned} \|z - z_{h_0}\|_{L_1^N} &= \|(z - z_{h_0}) \cdot \chi_{[0,T)}\|_{L_1^N} + \|z \cdot \chi_{[T,\infty)}\|_{L_1^N} \\ &\leq \|(z - z_{h_0}) \cdot \chi_{[0,T)}\|_{L_1^N} + r_2 + \eta. \end{aligned} \tag{2.7}$$

Moreover, by (2.6) we get

$$\|(z - z_{h_0}) \cdot \chi_{[0,T)}\|_{L_1^N} \leq \|z \cdot \chi_{\Omega(z,h_0)}\|_{L_1^N} \leq r_1 + \eta.$$

Thus, considering (2.7) we infer

$$\|z - z_{h_0}\|_{L_1^N} \leq r_1 + r_2 + 2\eta$$

and consequently

$$Z \subset Z_{h_0} + B_{r+2\eta}.$$

Thus

$$\chi(Z) \leq (r + 2\eta)\chi(B_1) = r + 2\eta$$

and since η is arbitrary, we have

$$\chi(Z) \leq \gamma(Z).$$

This inequality in conjunction with (2.3) fulfills the proof. □

It allows us to prove the next modified version of the Darbo-type fixed point hypothesis.

Corollary 2.14. *Let $\emptyset \neq Q \subset L_1^N$ be a convex, bounded, and closed set. Also, assume Q consists of functions which are a.e. positive and a.e. nondecreasing (or a.e. nonincreasing) on $\mathbb{R}^+$. Suppose $H : Q \to Q$ is a continuous operator and takes a.e. positive and a.e. nondecreasing (or a.e. nonincreasing) functions on $\mathbb{R}^+$ into functions of the same type. Finally, suppose there exists $0 \le k < \frac{1}{2}$ with*

$$\gamma(H(Z)) \le 2k\gamma(Z)$$

for any set $\emptyset \neq Z \subset Q$. Then H has at least one fixed point in Q.

Proof. Let Z be a subset of Q. Note from Remark 2.10 that Z and $H(Z)$ are compact in measure in L_1^N. Then from Theorem 2.13, we have

$$\begin{aligned} \chi(HZ) &\le \gamma(HZ) \le 2\chi(HZ) \le 2k\chi(Z) \le 2k\gamma(Z) \\ &\Rightarrow \gamma(HZ) \le 2k\gamma(Z). \end{aligned}$$

The above estimation with $0 \le k < \frac{1}{2}$ completes the proof. □

3. Main results

In what follows, we will examine the presence and the uniqueness of the solutions for Eq (1.1). Allow us to rewrite (1.1) in the operator form

$$\begin{cases} z(\theta) = (Hz)(\theta) = h(\theta) + F_{f_1}(z_\tau)(\theta) + F_{f_3}\Big(l \cdot (I^\alpha F_{f_2})(z_\tau)\Big)(\theta), \quad \theta \in \mathbb{R}^+, \\ \\ z(\theta) = z_0, \quad \theta \in [-\tau, 0), \ 0 < \alpha < 1, \end{cases} \tag{3.1}$$

where $z_\tau(\theta) = z(\theta - \tau)$, $\tau < \theta$, I^α is defined in Definition 2.3 and $F_{f_i}, i = 1, 2, 3$ are superposition operators as in Definition 2.1. Note that, for any integrable function z, a function z_τ is integrable too.

3.1. Presence of the solutions

The next presented assumptions are more general than the ones considered earlier, for example, all growth and bound conditions are expressed in terms of functions from L_1^N.

(i) Suppose that, $l, h : \mathbb{R}^+ \to \mathbb{R}^+$ are a.e. nondecreasing functions and l is a bounded function, such that $\sup_{\theta \in \mathbb{R}^+} |l(\theta)| \le M$ and $h \in L_1^N$.

(ii) Assume that the functions $f_i : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$, $i = 1, 2, 3$ fulfill Carathéodory conditions. Moreover, each $f_i(\theta, z) \ge 0$ for a.e. $(\theta, z) \in \mathbb{R}^+ \times \mathbb{R}$ and $f_i, i = 1, 2, 3$ are supposed to be nondecreasing concerning the two variables θ and z, independently.

(iii) There exist positive functions $a_i \in L_1^N$ and constants $b_i \ge 0$, such that

$$|f_i(\theta, z)| \le a_i(\theta) + b_i|z|, \ i = 1, 2, 3 \tag{3.2}$$

for almost all $\theta \in \mathbb{R}^+$ and all $z \in \mathbb{R}$.

(iv) There exists a constant $N > 0$, such that

$$\left(b_1 + \frac{Mb_2b_3}{N^\alpha}\right) < \frac{1}{2}.$$

Theorem 3.1. *Suppose assumptions (i)–(iv) are satisfied. Then* (1.1) *has at least one solution* $z \in L_1^N$ *that is a.e. nondecreasing on* $\mathbb{R}^+$.

Proof. From assumptions (ii), (iii), and Lemma 2.2, we have that $F_{f_i}, i = 1, 2, 3$ map L_1^N into itself continuously. Since I^α maps L_1^N into itself and is continuous, then by utilizing assumption (i), we indicate that the operator $H : L_1^N \to L_1^N$ and it is continuous. Using (3.1) with assumptions (i)–(iii) and Lemma 2.4, we have for $z \in L_1^N$ that

$$\begin{aligned}
\|Hz\|_{L_1^N} &\leq \|h\|_{L_1^N} + \|F_{f_1}(z_\tau)\|_{L_1^N} + \left\| F_{f_3}\Big(l \cdot (I^\alpha F_{f_2})(z_\tau)\Big)\right\|_{L_1^N} \\
&\leq \|h\|_{L_1^N} + \Big\|a_1 + b_1|z_\tau|\Big\|_{L_1^N} + \Big\|a_3 + b_3\Big|l \cdot (I^\alpha F_{f_2})(z_\tau)\Big|\ \Big\|_{L_1^N} \\
&\leq \|h\|_{L_1^N} + \|a_1\|_{L_1^N} + b_1\|z_\tau\|_{L_1^N} + \|a_3\|_{L_1^N} + b_3\frac{M}{N^\alpha}\Big\|F_{f_2}(z_\tau)\Big\|_{L_1^N} \\
&\leq \|h\|_{L_1^N} + \|a_1\|_{L_1^N} + \|a_3\|_{L_1^N} + b_1\|z_\tau\|_{L_1^N} + \frac{Mb_3}{N^\alpha}\Big\|a_2 + b_2|z_\tau|\Big\|_{L_1^N} \\
&\leq \|h\|_{L_1^N} + \|a_1\|_{L_1^N} + \|a_3\|_{L_1^N} + b_1\int_0^\infty e^{-N\theta}\,|z(\theta-\tau)|\,d\theta + \frac{Mb_3}{N^\alpha}\Big(\|a_2\|_{L_1^N} + b_2\|z_\tau\|_{L_1^N}\Big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|Hz\|_{L_1^N} &\leq \|h\|_{L_1^N} + \|a_1\|_{L_1^N} + \|a_3\|_{L_1^N} + b_1\int_0^\tau e^{-N\theta}|z(\theta-\tau)|\,d\theta + b_1\int_\tau^\infty e^{-N\theta}\,|z(\theta-\tau)|\,d\theta \\
&\quad + \frac{Mb_3}{N^\alpha}\|a_2\|_{L_1^N} + \frac{Mb_2b_3}{N^\alpha}\Big(\int_0^\tau e^{-N\theta}|z(\theta-\tau)|\,d\theta + \int_\tau^\infty e^{-N\theta}\,|z(\theta-\tau)|\,d\theta\Big).
\end{aligned}$$

Put $\theta - \tau = u$ and since $e^{-N(u+\tau)} \leq e^{-Nu}$, we obtain

$$\begin{aligned}
\|Hz\|_{L_1^N} &\leq \|h\|_{L_1^N} + \|a_1\|_{L_1^N} + \|a_3\|_{L_1^N} + \frac{Mb_3}{N^\alpha}\|a_2\|_{L_1^N} \\
&\quad + \Big(b_1 + \frac{Mb_2b_3}{N^\alpha}\Big)\int_{-\tau}^0 e^{-Nu}|z(u)|\,du + \Big(b_1 + \frac{Mb_2b_3}{N^\alpha}\Big)\int_0^\infty e^{-Nu}|z(u)|\,du \\
&\leq \|h\|_{L_1^N} + \|a_1\|_{L_1^N} + \|a_3\|_{L_1^N} + \frac{Mb_3}{N^\alpha}\|a_2\|_{L_1^N} + \Big(b_1 + \frac{Mb_2b_3}{N^\alpha}\Big)\int_{-\tau}^0 e^{-Nu}|z_0|\,du \\
&\quad + \Big(b_1 + \frac{Mb_2b_3}{N^\alpha}\Big)\|z\|_{L_1^N} \\
&\leq \|h\|_{L_1^N} + \|a_1\|_{L_1^N} + \|a_3\|_{L_1^N} + \frac{Mb_3}{N^\alpha}\|a_2\|_{L_1^N} + \Big(b_1 + \frac{Mb_2b_3}{N^\alpha}\Big)\frac{|z_0|e^{N\tau}}{N} \\
&\quad + \Big(b_1 + \frac{Mb_2b_3}{N^\alpha}\Big)\|z\|_{L_1^N}.
\end{aligned}$$

Thus if $z \in B_r = \{m \in L_1^N : \|m\|_{L_1^N} \leq r\}$ (r is given below) we have

$$\begin{aligned}\|Hz\|_{L_1^N} &\leq \|h\|_{L_1^N} + \|a_1\|_{L_1^N} + \|a_3\|_{L_1^N} + \frac{Mb_3}{N^\alpha}\|a_2\|_{L_1^N} + \left(b_1 + \frac{Mb_2b_3}{N^\alpha}\right)\frac{|z_0|e^{N\tau}}{N} \\ &\quad + \left(b_1 + \frac{Mb_2b_3}{N^\alpha}\right)\cdot r = r,\end{aligned}$$

where

$$r = \frac{\|h\|_{L_1^N} + \|a_1\|_{L_1^N} + \|a_3\|_{L_1^N} + \frac{Mb_3}{N^\alpha}\|a_2\|_{L_1^N} + \left(b_1 + \frac{Mb_2b_3}{N^\alpha}\right)\frac{|z_0|e^{N\tau}}{N}}{1 - \left(b_1 + \frac{Mb_2b_3}{N^\alpha}\right)}.$$

Thus $H : B_r \to B_r$ is continuous.

Let us denote by $Q_r \subset B_r$ the set of all positive and a.e. nondecreasing functions on $\mathbb{R}^+$. The set Q_r is bounded, nonempty, closed, convex, and compact in measure in regards to Remark 2.10.

Now, we shall demonstrate that H preserves the positivity and the monotonicity of functions. Choose $z \in Q_r$. Then $z(\theta)$ is positive and a.e. nondecreasing on $\mathbb{R}^+$ and thus each f_i is of the same type according to assumption (ii). In addition, I^α is positive and a.e. nondecreasing on $\mathbb{R}^+$. Thus by assumption (i) we infer that (Hz) is positive and a.e. nondecreasing on $\mathbb{R}^+$. Then $H : Q_r \to Q_r$ is continuous.

In what follows, let us fix a nonempty subset Z of Q_r. For $z \in Z$ and fix arbitrary $\varepsilon > 0$, such that for any $D \subset \mathbb{R}^+$ with $meas(D) \leq \varepsilon$, we have

$$\begin{aligned}&\int_D e^{-N\theta}|(Hz)(\theta)|d\theta \\ \leq\ &\int_D e^{-N\theta}|h(\theta)|d\theta + \int_D e^{-N\theta}|f_1(\theta, z(\theta-\tau))|d\theta + \int_D e^{-N\theta}\left|f_3\left(\theta, l(\theta)\int_0^\theta \frac{(\theta-s)^{\alpha-1}}{\Gamma(\alpha)} f_2(s, z(s-\tau))ds\right)\right|d\theta \\ \leq\ &\int_D e^{-N\theta}|h(\theta)|\,d\theta + \int_D e^{-N\theta}|f_1(\theta, z(\theta))|\,d\theta + \int_D e^{-N\theta}\left|f_3\left(\theta,\ l(\theta)\int_0^\theta \frac{(\theta-s)^{\alpha-1}}{\Gamma(\alpha)} f_2(s, z(s))\,ds\right)\right|d\theta \\ \leq\ &\|h\|_{L_1^N(D)} + \int_D e^{-N\theta}\Big(a_1(\theta) + b_1|z(\theta)|\Big)\,d\theta \\ &\quad + \int_D e^{-N\theta}\left(a_3(\theta) + Mb_3\int_0^\theta \frac{(\theta-s)^{\alpha-1}}{\Gamma(\alpha)}\Big(a_2(s) + b_2|z(s)|\Big)\,ds\right)d\theta \\ \leq\ &\|h\|_{L_1^N(D)} + \|a_1\|_{L_1^N(D)} + \|a_3\|_{L_1^N(D)} + b_1\int_D e^{-N\theta}|z(\theta)|\,d\theta + \frac{Mb_3}{N^\alpha}\int_D e^{-Ns}\Big(a_2(s) + b_2|z(s)|\Big)\,ds \\ \leq\ &\|h\|_{L_1^N(D)} + \|a_1\|_{L_1^N(D)} + \|a_3\|_{L_1^N(D)} + \frac{Mb_3}{N^\alpha}\|a_2\|_{L_1^N(D)} + \left(b_1 + \frac{Mb_2b_3}{N^\alpha}\right)\int_D e^{-N\theta}|z(\theta)|\,d\theta,\end{aligned}$$

where the notation $\|\cdot\|_{L_1^N(D)}$ refers to the operator norm which maps the space $L_1^N(D) \to L_1^N(D)$. Since $h, a_i \in L_1^N,\ i = 1,2,3$, we have

$$\lim_{\varepsilon\to 0}\left\{\sup_{z\in Z}\left\{\sup\left[\|h\|_{L_1^N(D)} + \|a_1\|_{L_1^N(D)} + \|a_3\|_{L_1^N(D)} + \frac{Mb_3}{N^\alpha}\|a_2\|_{L_1^N(D)} : D \subset \mathbb{R}^+, meas(D) \leq \varepsilon\right]\right\}\right\} = 0.$$

Thus, by using Definition (2.1), we have

$$c(HZ) \leq \left(b_1 + \frac{Mb_2b_3}{N^\alpha}\right)\cdot c(Z). \tag{3.3}$$

For $T > 0$ and $z \in Z$, we have the following estimate

$$\begin{aligned}\int_T^{\infty} e^{-N\theta}|(Hz)(\theta)|d\theta \;&\leq\; \|h\|_{L_1^N(T)} + \|a_1\|_{L_1^N(T)} + \|a_3\|_{L_1^N(T)} + \frac{Mb_3}{N^{\alpha}}\|a_2\|_{L_1^N(T)} \\ &\quad + \Big(b_1 + \frac{Mb_2b_3}{N^{\alpha}}\Big)\int_T^{\infty} e^{-N\theta}|z(\theta)|\,d\theta,\end{aligned}$$

where the notation $\|\cdot\|_{L_1^N(T)}$ refers to the operator norm which maps the space $L_1^N[T,\infty) \to L_1^N[T,\infty)$. As $T \to \infty$ we get

$$d(HZ) \leq \Big(b_1 + \frac{Mb_2b_3}{N^{\alpha}}\Big)\cdot d(Z). \tag{3.4}$$

Joining (3.3) and (3.4), and by recalling Definition (2.2), we have

$$\gamma(HZ) \leq \Big(b_1 + \frac{Mb_2b_3}{N^{\alpha}}\Big)\cdot \gamma(Z).$$

From assumption (iv) (and the properties of H on Q_r) we may apply Corollary 2.14 which fulfills the proof. □

3.2. Uniqueness of the solutions

Next, we examine the uniqueness of solution for Eq (1.1).

Theorem 3.2. *Let assumptions of Theorem 3.1 be fulfilled, but replace (3.2) by the following one:*

(v) There exist constants $b_i \geq 0$ and positive functions $a_i \in L_1^N$, such that

$$|f_i(\theta,0)| \leq |a_i(\theta)| \quad \text{and} \quad |f_i(\theta,z) - f_i(\theta,y)| \leq b_i|z-y|, \quad i = 1,2,3, \quad z,y \in Q_r,$$

where Q_r is given in Theorem 3.1.

Then (1.1) *has a unique integrable solution in the set Q_r.*

Proof. By using the above suppositions, we get

$$\begin{aligned}\Big||f_i(\theta,z)| - |f_i(\theta,0)|\Big| \;&\leq\; |f_i(\theta,z) - f_i(\theta,0)| \leq b_i|z| \\ \Rightarrow |f_i(\theta,z)| \;&\leq\; |f_i(\theta,0)| + b_i|z| \leq a_i(\theta) + b_i|z|, \; i = 1,2,3.\end{aligned}$$

Then all assumptions of Theorem 3.1 are fulfilled and therefore Eq (1.1) has at least one integrable solution $z \in L_1^N$.

Next, let z and y be any two distinct solutions of Eq (1.1), we have

$$\begin{aligned}\|z-y\|_{L_1^N} \;&\leq\; \Big\|f_1(\theta, z(\theta-\tau)) - f_1(\theta, y(\theta-\tau))\Big\|_{L_1^N} + \Big\|f_3\Big(\theta,\ l(\theta)\int_0^{\theta}\frac{(\theta-s)^{\alpha-1}}{\Gamma(\alpha)}f_2(s, z(s-\tau))\,ds\Big) \\ &\quad - f_3\Big(\theta,\ l(\theta)\int_0^{\theta}\frac{(\theta-s)^{\alpha-1}}{\Gamma(\alpha)}f_2(s, y(s-\tau))\,ds\Big)\Big\|_{L_1^N} \\ &\leq\; b_1\Big\|z(\theta-\tau) - y(\theta-\tau)\Big\|_{L_1^N}\end{aligned}$$

$$
\begin{aligned}
&+b_3\Big\| |l(\theta)| \int_0^\theta \frac{(\theta-s)^{\alpha-1}}{\Gamma(\alpha)} |f_2(s,z(s-\tau)) - f_2(s,y(s-\tau))|\, ds \Big\|_{L_1^N} \\
\leq\ & b_1 \Big\| z(\theta-\tau) - y(\theta-\tau) \Big\|_{L_1^N} + \frac{Mb_2b_3}{N^\alpha} \Big\| z(\theta-\tau) - y(\theta-\tau) \Big\|_{L_1^N} \\
\leq\ & b_1 \int_0^\tau e^{-N\theta} |z(\theta-\tau) - y(\theta-\tau)|\, d\theta + b_1 \int_\tau^\infty e^{-N\theta} |z(\theta-\tau) - y(\theta-\tau)|\, d\theta \\
&+ \frac{Mb_2b_3}{N^\alpha} \Big(\int_0^\tau e^{-N\theta} |z(\theta-\tau) - y(\theta-\tau)|\, d\theta + \int_\tau^\infty e^{-N\theta} |z(\theta-\tau) - y(\theta-\tau)|\, d\theta \Big).
\end{aligned}
$$

Put $\theta - \tau = u$ with $e^{-N(u+\tau)} \leq e^{-Nu}$ and since $z(u) = y(u) = z_0$ on $[-\tau, 0)$, we have

$$
\begin{aligned}
\|z-y\|_{L_1^N} &\leq b_1 \int_0^\infty e^{-Nu} |z(u) - y(u)|\, du + \frac{Mb_2b_3}{N^\alpha} \int_0^\infty e^{-Nu} |z(u) - y(u)|\, du \Big) \\
&\leq \Big(b_1 + \frac{Mb_2b_3}{N^\alpha} \Big) \|z-y\|_{L_1^N}.
\end{aligned}
$$

From the above inequality with $\left(b_1 + \frac{Mb_2b_3}{N^\alpha}\right) < \frac{1}{2}$, we deduce that $z = y$, which completes the proof. □

4. Example

Next, we give an example to demonstrate the applicability and significance of our theorems.

Example 4.1. Consider the following integral equation

$$
\begin{aligned}
z(\theta) &= e^\theta + \Big(\frac{1+\theta^2}{2} + \frac{1}{20} z(\theta-\tau) \Big) \\
&\quad + \Big(e^{2\theta} + \frac{\sin\theta}{20} \int_0^\theta \frac{(\theta-s)^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} \Big(e^s + \frac{1}{20} z(s-\tau) \Big)\, ds \Big), \ \ \theta \in \mathbb{R}^+. \qquad (4.1)
\end{aligned}
$$

Equation (4.1) is a special case of Eq (1.1), where

$$
h(\theta) = e^\theta,\ l(\theta) = \sin\theta,\ f_1(\theta, z) = \frac{1+\theta^2}{2} + \frac{1}{20} z,\ f_2(\theta, z) = e^\theta + \frac{1}{20} z,
$$

and

$$
f_3(\theta, z) = e^{2\theta} + \frac{1}{20} z,
$$

such that

(1) $a_1(\theta) = \frac{1+\theta^2}{2},\ a_2(\theta) = e^\theta,\ a_3(\theta) = e^{2\theta},\ b_1 = b_2 = b_3 = \frac{1}{20}$,
(2) $\sup_{\theta\in\mathbb{R}^+} |l(\theta)| = \sup_{\theta\in\mathbb{R}^+} |\sin\theta| \leq 1 = M$,
(3) for $N > 0$, we have

$$
w = \Big(b_1 + \frac{Mb_2b_3}{N^\alpha} \Big) = \frac{1}{20} \Big(1 + \frac{1}{20\sqrt{N}} \Big) = \frac{20\sqrt{N} + 1}{400\sqrt{N}} < \frac{1}{2}.
$$

By recalling Theorem 3.1, we can indicate that (4.1) has at least one integrable solution a.e. nondecreasing on $\mathbb{R}^+$.

Moreover, we have

(1) $|f_1(\theta, 0)| = \frac{1+\theta^2}{2}$, $|f_2(\theta, 0)| = e^\theta$ and $|f_3(\theta, 0)| = e^{2\theta}$.
(2) $|f_i(\theta, z) - f_i(\theta, y)| \leq \frac{1}{20}|z - y|$, $i = 1, 2, 3$.

By recalling Theorem 3.2, Eq (4.1) has a unique integrable solution $z \in L_1^N$.

5. Conclusions

We examine the existence and the uniqueness of monotonic solutions of a delay integral equation of fractional order in the weighted Lebesgue space $L_1^N(\mathbb{R}^+)$, which is a larger space than the classical Lebesgue space $L_1(\mathbb{R}^+)$. Our analysis uses a suitable measure of noncompactness (M.N.C.), a modified version of Darbo's fixed point theorem, and fractional calculus in the mentioned space. To show the applicability and significance of our outcomes, we present an illustrated example in that space.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this article.

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