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Research article

Mixed radial-angular bounds for Hardy-type operators on Heisenberg group

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Abstract: In this paper, we study n-dimensional Hardy operator and its dual in mixed radial-angular spaces on Heisenberg group and obtain their sharp bounds by using the rotation method. Furthermore, the sharp bounds of n-dimensional weighted Hardy operator and weighted Cesàro operator are also obtained.

Keywords: Hardy operator; dual operator; weighted Hardy operator; weighted Cesàro operator; mixed radial-angular space; Heisenberg group
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1. Introduction

The classic Hardy operator and its dual operator are defined by

$$H(f)(x) := \frac{1}{x}\int_0^x f(y)dy, \quad H^*(f)(x) := \int_x^\infty \frac{f(y)}{y}dy,$$

for the locally integrable function f on $\mathbb{R}$ and $x \neq 0$. The classic Hardy operator was introduced by Hardy and he showed the following Hardy inequalities

$$\|H(f)\|_{L^p} \leq \frac{p}{p-1}\|f\|_{L^p}, \quad \|H^*(f)\|_{L^p} \leq p\|f\|_{L^p},$$

where $1 < p < \infty$, the constants $\frac{p}{p-1}$, p are best possible.

Faris [6] first extended Hardy-type operator to higher dimension, Christ and Grafakos [2] gave an equivalent version of n-dimensional Hardy operator $\mathcal{H}$ for nonnegative function f on $\mathbb{R}^n$,

$$\mathcal{H}f(x) := \frac{1}{\Omega_n|x|^n}\int_{|y|<|x|} f(y)dy, \quad x \in \mathbb{R}^n\backslash\{0\},$$

where $\Omega_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(1+\frac{n}{2})}$ is the volume of the unit ball in $\mathbb{R}^n$. By a direct computation, the dual operator of $\mathcal{H}$ can be defined by setting, for nonnegative function f on $\mathbb{R}^n$,

$$\mathcal{H}^*(f)(x) := \int_{|y|\geq|x|} \frac{f(y)}{\Omega_n|y|^n}dy, \quad x \in \mathbb{R}^n\backslash\{0\}.$$

Christ and Grafakos [2] proved that the norms of $\mathcal{H}$ and $\mathcal{H}^*$ on $L^p(\mathbb{R}^n)(1 < p < \infty)$ are also $\frac{p}{p-1}$ and p, which is the same as that in the 1-dimensional case and is also independent of the dimension. The sharp weak estimate for $\mathcal{H}$ was obtained by Zhao et al. [19]. For $1 \leq p \leq \infty$,

$$\|\mathcal{H}(f)\|_{L^{p,\infty}} \leq 1 \times \|f\|_{L^p},$$

where 1 is best constant.

In recent years, the research on Hardy operator related issues is receiving increasing attention, Hardy et al. provided us with the early development and application of Hardy's inequalities. In [8, 9, 15], Fu et al. have engaged in many related discuss, which provide convenience for our research.

In this paper, we will investigate the sharp bound for Hardy-type operators in the setting of the Heisenberg group, which plays important role in several branches of mathematics. Now, allow us to introduce some basic knowledge about the Heisenberg group which will be used in the following. The Heisenberg group $\mathbb{H}^n$ is a non-commutative nilpotent Lie group, with the underlying manifold $\mathbb{R}^{2n} \times \mathbb{R}$ with the group law

$$x \circ y = \left(x_1 + y_1, \ldots, x_{2n} + y_{2n}, x_{2n+1} + y_{2n+1} + 2\sum_{j=1}^{n}\left(y_j x_{n+j} - x_j y_{n+j}\right)\right)$$

and

$$\delta_r(x_1, x_2, \ldots, x_{2n}, x_{2n+1}) = (rx_1, rx_2, \ldots, rx_{2n}, r^2 x_{2n+1}), \quad r > 0,$$

where $x = (x_1, \cdots, x_{2n}, x_{2n+1})$, $y = (y_1, \cdots, y_{2n}, y_{2n+1})$. The Haar measure on $\mathbb{H}^n$ coincides with the usual Lebesgue measure on $\mathbb{R}^{2n+1}$. We denote the measure of any measurable set $E \subset \mathbb{H}^n$ by $|E|$. Then

$$|\delta_r(E)| = r^Q|E|, d(\delta_r x) = r^Q dx,$$

where $Q = 2n + 2$ is called the homogeneous dimension of $\mathbb{H}^n$.

The Heisenberg distance derived from the norm

$$|x|_h = \left[\left(\sum_{i=1}^{2n} x_i^2\right)^2 + x_{2n+1}^2\right]^{1/4},$$

where $x = (x_1, x_2, \cdots, x_{2n}, x_{2n+1})$, is given by

$$d(p, q) = d(q^{-1}p, 0) = |q^{-1}p|_h.$$

This distance d is left-invariant in the sense that $d(p, q)$ remains unchanged when p and q are both left-translated by some fixed vector on $\mathbb{H}^n$. Furthermore, d satisfies the triangular inequality [12]

$$d(p, q) \leq d(p, x) + d(x, q), \quad p, x, q \in \mathbb{H}^n.$$

For $r > 0$ and $x \in \mathbb{H}^n$, the ball and sphere with center x and radius r on $\mathbb{H}^n$ are given by

$$B(x, r) = \{y \in \mathbb{H}^n : d(x, y) < r\}$$

and

$$S(x, r) = \{y \in \mathbb{H}^n : d(x, y) = r\},$$

respectively. And we have

$$|B(x, r)| = |B(0, r)| = \Omega_Q r^Q,$$

where

$$\Omega_Q = \frac{2\pi^{n+\frac{1}{2}}\Gamma(n/2)}{(n+1)\Gamma(n)\Gamma((n+1)/2)}$$

is the volume of the unit ball $B(0, 1)$ on $\mathbb{H}^n$, and the area of the unit sphere $\mathbb{S}^{Q-1}$ is $\omega_Q = Q\Omega_Q$ (see [4]). More about Heisenberg group can refer to [7, 11, 16].

The n-dimensional Hardy operator and its dual operator on Heisenberg group is defined by Wu and Fu [18]

$$\mathcal{H}_h f(x) := \frac{1}{\Omega_Q |x|_h^Q} \int_{|y|_h < |x|_h} f(y) dy, \quad \mathcal{H}_h^* f(x) := \int_{|y|_h \geq |x|_h} \frac{f(y)}{\Omega_Q |y|_h^Q} dy, \tag{1.1}$$

where $x \in \mathbb{H}^n \backslash \{0\}$, f be a locally integrable function on $\mathbb{H}^n$. They proved that $\mathcal{H}_h$ and $\mathcal{H}_h^*$ is bounded from $L^p(\mathbb{H}^n)$ to $L^p(\mathbb{H}^n)$, $1 < p \leq \infty$. Moreover,

$$\|\mathcal{H}_h\|_{L^p(\mathbb{H}^n)} = \frac{p}{p-1} \|f\|_{L^p(\mathbb{H}^n)}, \quad \|\mathcal{H}_h^*\|_{L^p(\mathbb{H}^n)} = p \|f\|_{L^p(\mathbb{H}^n)}. \tag{1.2}$$

This is the same as the result on $\mathbb{R}^n$.

In [10, 13], León-Saavedra and González studied the behavior of Cesàro operator, Chu et al. in [3] defined the n-dimensional weighted Hardy operator on Heisenberg group $\mathcal{H}_{hw}$ and n-dimensional weighted Cesàro operator on Heisenberg group $\mathcal{H}_{hw}^*$. Let us recall their definition.

Definition 1. *Let $w : [0, 1] \to [0, \infty)$ be a measurable function. For a measurable function f on $\mathbb{H}^n$, the n-dimensional weighted Hardy operator on Heisenberg group $\mathcal{H}_{hw}$ is defined by*

$$\mathcal{H}_{hw} f(x) := \int_0^1 f(\delta_t x) w(t) dt, \quad x \in \mathbb{H}^n.$$

For a measurable complex-valued function f on $\mathbb{H}^n$, nonnegative function $w : [0, 1] \to (0, \infty)$,

$$\int_0^1 t^{-\frac{Q}{p}} w(t) dt < \infty$$

and

$$\int_0^1 t^{-Q(1-1/p)} w(t) dt < \infty,$$

the n-dimensional weighted Cesàro operator is defined by

$$\mathcal{H}_{hw}^* f(x) := \int_0^1 \frac{f(\delta_{1/t} x)}{t^Q} w(t) dt, \quad x \in \mathbb{H}^n,$$

which satisfies

$$\int_{\mathbb{H}^n} f(x)(\mathcal{H}_{hw}g)(x)dx = \int_{\mathbb{H}^n} g(x)(\mathcal{H}^*_{hw}f)(x)dx,$$

where $f \in L^p(\mathbb{H}^n)$, $g \in L^q(\mathbb{H}^n)$, $1 < p < \infty$, $q = p/(p-1)$, $\mathcal{H}_{hw}$ is bounded on $L^p(\mathbb{H}^n)$ and $\mathcal{H}^*_{hw}$ is bounded on $L^q(\mathbb{H}^n)$.

Remark 1. *In [3], Chu et al. proved the equality*

$$\mathcal{H}_{hw}f(x) := \int_0^1 f(\delta_t x)w(t)dt = \mathcal{H}_h f(x), \quad x \in \mathbb{H}^n \backslash \{0\},$$

was established when $w(t) = Qt^{Q-1}$ *and* f *is radial function.*

Recently, many operators in harmonic analysis have been proved to be bounded on mixed radial-angular spaces, for instance, Duoandikoetxea and Oruetxebarria [5] built the extrapolation theorems on mixed radial-angular spaces to study the boundedness of a large class of operators which are weighted bounded. In [17], Wei and Yan studied the sharp bounds for n-dimensional Hardy operator and its dual in mixed radial-angular spaces on Euclidean space. Inspired by them, we will investigate the sharp bounds for n-dimensional Hardy operator and its dual operator in mixed radial-angular spaces on Heisenberg groups.

Now, we give the definition of mixed radial-angular spaces on Heisenberg group.

Definition 2. *For any* $n \geq 2, 1 \leq p, \bar{p} \leq \infty$, *the mixed radial-angular space* $L^p_{|x|_h} L^{\bar{p}}_\theta(\mathbb{H}^n)$ *consists of all functions* f *in* $\mathbb{H}^n$ *for which*

$$\|f\|_{L^p_{|x|_h} L^{\bar{p}}_\theta(\mathbb{H}^n)} := \left(\int_0^\infty \left(\int_{\mathbb{S}^{Q-1}} |f(r,\theta)|^{\bar{p}} d\theta \right)^{\frac{p}{\bar{p}}} r^{Q-1} dr \right)^{\frac{1}{p}} < \infty,$$

where $\mathbb{S}^{Q-1}$ *denotes the unit sphere in* $\mathbb{H}^n$.

Next, we will provide the main results of this article.

2. Mixed radial-angular bounds for $\mathcal{H}_h$ and $\mathcal{H}^*_h$

Theorem 1. *Let* $n \geq 2, 1 < p, \bar{p}_1, \bar{p}_2 < \infty$. *Then* $\mathcal{H}_h$ *is bounded from* $L^p_{|x|_h} L^{\bar{p}_1}_\theta(\mathbb{H}^n)$ *to* $L^p_{|x|_h} L^{\bar{p}_2}_\theta(\mathbb{H}^n)$. *Moreover,*

$$\|\mathcal{H}_h\|_{L^p_{|x|_h} L^{\bar{p}_1}_\theta(\mathbb{H}^n) \to L^p_{|x|_h} L^{\bar{p}_2}_\theta(\mathbb{H}^n)} = \frac{p}{p-1} \omega_Q^{1/\bar{p}_2 - 1/\bar{p}_1}.$$

Theorem 2. *Let* $n \geq 2, 1 < p, \bar{p}_1, \bar{p}_2 < \infty$. *Then* $\mathcal{H}^*_h$ *is bounded from* $L^p_{|x|_h} L^{\bar{p}_1}_\theta(\mathbb{H}^n)$ *to* $L^p_{|x|_h} L^{\bar{p}_2}_\theta(\mathbb{H}^n)$. *Moreover,*

$$\|\mathcal{H}^*_h\|_{L^p_{|x|_h} L^{\bar{p}_1}_\theta(\mathbb{H}^n) \to L^p_{|x|_h} L^{\bar{p}_2}_\theta(\mathbb{H}^n)} = p \omega_Q^{1/\bar{p}_2 - 1/\bar{p}_1}.$$

Proof of Theorem 1. Set

$$g(x) = \frac{1}{\omega_Q} \int_{\mathbb{S}^{Q-1}} f(\delta_{|x|_h} \theta) d\theta, \quad x \in \mathbb{H}^n, \tag{2.1}$$

then g is a radial function. Moreover, we have

$$\begin{aligned}\|g\|_{L^p_{|x|_h}L^{\bar{p}_1}_\theta(\mathbb{H}^n)} &= \left(\int_0^\infty \left(\int_{\mathbb{S}^{Q-1}} |g(r,\theta)|^{\bar{p}_1} d\theta\right)^{p/\bar{p}_1} r^{Q-1}dr\right)^{1/p}\\ &= \left(\int_0^\infty \left(\omega_Q |g(r)|^{\bar{p}_1}\right)^{p/\bar{p}_1} r^{Q-1}dr\right)^{1/p}\\ &= \omega_Q^{1/\bar{p}_1}\left(\int_0^\infty |g(r)|^p r^{Q-1}dr\right)^{1/p},\end{aligned}$$

where $g(r)$ can be defined as $g(r) = g(x)$ for any $x \in \mathbb{H}^n$ with $|x|_h = r$ since g is a radial function. By using Hölder inequality, we have

$$\begin{aligned}\|g\|_{L^p_{|x|_h}L^{\bar{p}_1}_\theta(\mathbb{H}^n)} &= \omega_Q^{1/\bar{p}_1}\left(\int_0^\infty \left|\frac{1}{\omega_Q}\int_{\mathbb{S}^{Q-1}} f(\delta_r\theta)d\theta\right|^p r^{Q-1}dr\right)^{1/p}\\ &= \omega_Q^{1/\bar{p}_1-1}\left(\int_0^\infty \left|\int_{\mathbb{S}^{Q-1}} f(\delta_r\theta)d\theta\right|^p r^{Q-1}dr\right)^{1/p}\\ &\leq \omega_Q^{1/\bar{p}_1-1}\left(\int_0^\infty \left(\int_{\mathbb{S}^{Q-1}} |f(\delta_r\theta)|^{\bar{p}_1}d\theta\right)^{p/\bar{p}_1}\left(\int_{\mathbb{S}^{Q-1}} d\theta\right)^{p/\bar{p}_1'} r^{Q-1}dr\right)^{1/p}\\ &= \left(\int_0^\infty \left(\int_{\mathbb{S}^{Q-1}} |f(\delta_r\theta)|^{\bar{p}_1}d\theta\right)^{p/\bar{p}_1} r^{Q-1}dr\right)^{1/p}\\ &= \|f\|_{L^p_{|x|_h}L^{\bar{p}_1}_\theta(\mathbb{H}^n)}.\end{aligned}$$

Next, we use another form of Hardy operator

$$\mathcal{H}_h(f)(x) = \frac{1}{|B(0,|x|_h)|}\int_{B(0,|x|_h)} f(y)dy, \quad x \in \mathbb{H}^n\backslash\{0\}.$$

By change of variables, we can get

$$\begin{aligned}\mathcal{H}_h g(x) &= \frac{1}{|B(0,|x|_h)|}\int_{B(0,|x|_h)}\left(\frac{1}{\omega_Q}\int_{\mathbb{S}^{Q-1}} f(\delta_{|x|_h}\theta)d\theta\right)dy\\ &= \frac{1}{|B(0,|x|_h)|}\int_0^{|x|_h}\int_{|y'|_h=1}\left(\frac{1}{\omega_Q}\int_{\mathbb{S}^{Q-1}} f(\delta_r\theta)d\theta\right)r^{Q-1}dy'dr\\ &= \frac{1}{|B(0,|x|_h)|}\int_0^{|x|_h}\int_{|y'|_h=1} f(\delta_r\theta)r^{Q-1}d\theta dr\\ &= \mathcal{H}_h f(x).\end{aligned}$$

Thus, we have obtained

$$\frac{\|\mathcal{H}_h(f)\|_{L^p_{|x|_h}L^{\bar{p}_2}_\theta(\mathbb{H}^n)}}{\|f\|_{L^p_{|x|_h}L^{\bar{p}_1}_\theta(\mathbb{H}^n)}} \leq \frac{\|\mathcal{H}_h(g)\|_{L^p_{|x|_h}L^{\bar{p}_2}_\theta(\mathbb{H}^n)}}{\|g\|_{L^p_{|x|_h}L^{\bar{p}_1}_\theta(\mathbb{H}^n)}},$$

which implies the operator $\mathcal{H}$ and its restriction to radial function have same norm from $L^p_{|x|_h}L^{\bar{p}_1}_\theta$ to $L^p_{|x|_h}L^{\bar{p}_2}_\theta$. Without loss of generality, we can assume that f is a radial function in the rest of proof. Consequently, we have

$$\begin{aligned}
\|\mathcal{H}_h f\|_{L^p_{|x|_h}L^{\bar{p}_2}_\theta(\mathbb{H}^n)} &= \left(\int_0^\infty \left(\int_{\mathbb{S}^{Q-1}} |\mathcal{H}_h(f)(r,\theta)|^{\bar{p}_2} d\theta\right)^{p/\bar{p}_2} r^{Q-1}dr\right)^{\frac{1}{p}} \\
&= \left(\int_0^\infty \left(\int_{\mathbb{S}^{Q-1}} |\mathcal{H}_h(f)(r)|^{\bar{p}_2} d\theta\right)^{p/\bar{p}_2} r^{Q-1}dr\right)^{1/p} \\
&= \omega_Q^{1/\bar{p}_2}\left(\int_0^\infty |\mathcal{H}_h(f)(r)|^p r^{Q-1}dr\right)^{1/p},
\end{aligned}$$

where $\mathcal{H}_h(f)(r)$ can be defined as $\mathcal{H}_h(f)(r) = \mathcal{H}_h(f)(x)$ for any $|x|_h = r$. Using Minkowski's inequality, we can get

$$\begin{aligned}
\|\mathcal{H}_h f\|_{L^p_{|x|_h}L^{\bar{p}_2}_\theta(\mathbb{H}^n)} &= \omega_Q^{1/\bar{p}_2}\left(\int_0^\infty \left|\frac{1}{\Omega_Q}\int_{B(0,1)} f(\delta_r y)dy\right|^p r^{Q-1}dr\right)^{1/p} \\
&= \frac{\omega_Q^{1/\bar{p}_2}}{\Omega_Q}\left(\int_0^\infty \left|\int_{B(0,1)} f(\delta_r y)dy\right|^p r^{Q-1}dr\right)^{1/p} \\
&\le \frac{\omega_Q^{1/\bar{P}_2}}{\Omega_Q}\int_{B(0,1)}\left(\int_0^\infty |f(\delta_{|y|_h} r)|^p r^{Q-1}dr\right)^{1/p} dy \\
&= \frac{\omega_Q^{1/\bar{P}_2}}{\Omega_Q}\int_{B(0,1)}\left(\int_0^\infty |f(r)|^p r^{Q-1}dr\right)^{1/p} |y|_h^{-Q/p}dy \\
&= \frac{\omega_Q^{1/\bar{p}_2-1/\bar{p}_1}}{\Omega_Q}\int_{B(0,1)}\left(\int_0^\infty \omega_Q^{p/\bar{p}_1}|f(r)|^p r^{Q-1}dr\right)^{1/p} |y|_h^{-Q/p}dy \\
&= \frac{\omega_Q^{1/\bar{p}_2-1/\bar{p}_1}}{\Omega_Q}\int_{B(0,1)} |y|_h^{-Q/p}dy\|f\|_{L^p_{|x|_h}L^{\bar{p}_1}_\theta} \\
&= \frac{p}{p-1}\omega_Q^{1/\bar{p}_2-1/\bar{p}_1}\|f\|_{L^p_{|x|_h}L^{\bar{p}_1}_\theta(\mathbb{H}^n)}.
\end{aligned}$$

Therefore, we have

$$\|\mathcal{H}_h f\|_{L^p_{|x|_h}L^{\bar{p}_2}_\theta(\mathbb{H}^n)} \le \frac{p}{p-1}\omega_Q^{1/\bar{p}_2-1/\bar{p}_1}\|f\|_{L^p_{|x|_h}L^{\bar{p}_1}_\theta(\mathbb{H}^n)}. \tag{2.2}$$

On the other hand, for $0 < \epsilon < 1$, take

$$f_\epsilon(x) = \begin{cases} 0, & |x|_h \le 1, \\ |x|_h^{-\left(\frac{Q}{p}+\epsilon\right)} & |x|_h > 1 \end{cases}.$$

Then we can obtain

$$\|f_\epsilon\|_{L^p_{|x|_h}L^{\bar{p}_1}_\theta} = \frac{\omega_Q^{1/\bar{p}_1}}{(p\epsilon)^{1/p}}$$

and

$$\mathcal{H}_h(f_\epsilon)(x) = \begin{cases} 0, & |x|_h \le 1, \\ \Omega_Q^{-1}|x|_h^{-\frac{Q}{p}-\epsilon} \int_{|x|_h^{-1}<|y|_h<1} |y|_h^{-\frac{Q}{p}-\epsilon} dy, & |x|_h > 1 \end{cases}.$$

So, we have

$$\begin{aligned}
\|\mathcal{H}_h(f_\epsilon)\|_{L^p_{|x|_h}L^{\bar{p}_2}_\theta(\mathbb{H}^n)} &= \frac{\omega_Q^{1/\bar{p}_2}}{\Omega_Q}\left(\int_{r>1}\left|r^{-\frac{Q}{p}-\epsilon}\int_{r^{-1}<|y|_h<1}|y|_h^{-\frac{Q}{p}-\epsilon}dy\right|^p r^{Q-1}dr\right)^{1/p} \\
&\ge \frac{\omega_Q^{1/\bar{p}_2}}{\Omega_Q}\left(\int_{r>\frac{1}{\epsilon}}\left|r^{-\frac{Q}{p}-\epsilon}\int_{\epsilon<|y|_h<1}|y|_h^{-\frac{Q}{p}-\epsilon}dy\right|^p r^{Q-1}dr\right)^{1/p} \\
&= \frac{\omega_Q^{1/\bar{p}_2}}{\Omega_Q}\left(\int_{r>\frac{1}{\epsilon}} r^{-p\epsilon-Q}dr\right)^{1/p}\int_{\epsilon<|y|_h<1}|y|_h^{-\frac{Q}{p}-\epsilon}dy \\
&= \frac{\omega_Q^{1+1/\bar{p}_2}}{\Omega_Q}\left(\int_{r>\frac{1}{\epsilon}} r^{-p\epsilon-Q}dr\right)^{1/p}\int_\epsilon^1 r^{Q-1-\frac{Q}{p}-\epsilon}dr \\
&= \epsilon^\epsilon\frac{1-\epsilon^{Q-\frac{Q}{p}-\epsilon}}{1-\frac{1}{p}-\frac{\epsilon}{Q}}\omega_Q^{1/\bar{p}_2-1/\bar{p}_1}\|f_\epsilon\|_{L^p_{|x|_h}L^{\bar{p}_1}_\theta}.
\end{aligned}$$

Thus, we have obtained

$$\|\mathcal{H}_h\|_{L^p_{|x|_h}L^{\bar{p}_1}_\theta(\mathbb{H}^n)\to L^p_{|x|_h}L^{\bar{p}_2}_\theta(\mathbb{H}^n)} \ge \epsilon^\epsilon\frac{1-\epsilon^{Q-\frac{Q}{p}-\epsilon}}{1-\frac{1}{p}-\frac{\epsilon}{Q}}\omega_Q^{1/\bar{p}_2-1/\bar{p}_1}\|f_\epsilon\|_{L^p_{|x|_h}L^{\bar{p}_1}_\theta}.$$

Since $\epsilon^\epsilon \to 1$ as $\epsilon \to 0$, by letting $\epsilon \to 0$, we have

$$\|\mathcal{H}_h\|_{L^p_{|x|_h}L^{\bar{p}_1}_\theta(\mathbb{H}^n} \ge \frac{p}{p-1}\omega_Q^{1/\bar{p}_2-1/\bar{p}_1}\|f\|_{L^p_{|x|_h}L^{\bar{p}_1}_\theta(\mathbb{H}^n)}. \tag{2.3}$$

Combining (2.2) and (2.3), we can get

$$\|\mathcal{H}_h f\|_{L^p_{|x|_h}L^{\bar{p}_2}_\theta(\mathbb{H}^n)} = \frac{p}{p-1}\omega_Q^{1/\bar{p}_2-1/\bar{p}_1}\|f\|_{L^p_{|x|_h}L^{\bar{p}_1}_\theta(\mathbb{H}^n)}.$$

This completes the proof of Theorem 1. □

Proof of Theorem 2. The proof of Theorem 2 is similar to prove of Theorem 1, we omit the details. □

3. Mixed radial-angular bounds for $\mathcal{H}_{hw}$ and $\mathcal{H}^*_{hw}$

Theorem 3. *Let $w : [0,1] \to (0,\infty)$ be a function, $n \ge 2, 1 < p, \bar{p}_1, \bar{p}_2 < \infty$. Then the n-dimensional weighted Hardy operator on Heisenberg group $\mathcal{H}_{hw}$ is bounded from $L^p_{|x|_h}L^{\bar{p}_1}_\theta(\mathbb{H}^n)$ to $L^p_{|x|_h}L^{\bar{p}_2}_\theta(\mathbb{H}^n)$ if and only if*

$$\int_0^1 t^{-\frac{Q}{p}}w(t)dt < \infty.$$

Moreover,

$$\|\mathcal{H}_{hw}\|_{L^p_{|x|_h}L^{\bar{p}_1}_\theta(\mathbb{H}^n)\to L^p_{|x|_h}L^{\bar{p}_2}_\theta(\mathbb{H}^n)} = \omega_Q^{1/\bar{p}_2-1/\bar{p}_1}\int_0^1 t^{-\frac{Q}{p}}w(t)dt.$$

Theorem 4. *Let $w : [0,1] \to (0,\infty)$ be a function, $n \geq 2, 1 < p, \bar{p}_1, \bar{p}_2 < \infty$. Then the n-dimensional weighted Cesàro operator on Heisenberg group $\mathcal{H}^*_{hw}$ is bounded from $L^p_{|x|_h} L^{\bar{p}_1}_\theta(\mathbb{H}^n)$ to $L^p_{|x|_h} L^{\bar{p}_2}_\theta(\mathbb{H}^n)$ if and only if*

$$\int_0^1 t^{-Q(1-1/p)} w(t) dt < \infty.$$

Moreover,

$$\|\mathcal{H}^*_{hw}\|_{L^p_{|x|_h} L^{\bar{p}_1}_\theta(\mathbb{H}^n) \to L^p_{|x|_h} L^{\bar{p}_2}_\theta(\mathbb{H}^n)} = \omega_Q^{1/\bar{p}_2 - 1/\bar{p}_1} \int_0^1 t^{-Q(1-1/p)} w(t) dt.$$

The proof methods for Theorems 3 and 4 are the same, and similar to the proof method for Theorem 1. But as a special case, here we will give the proof of Theorem 4.

Proof of Theorem 4. Inspired by proof of Theorem 1, we have

$$\|\mathcal{H}^*_{hw}\|_{L^p_{|x|_h} L^{\bar{p}_2}_\theta(\mathbb{H}^n)} = \omega_Q^{1/\bar{p}_2} \left(\int_0^\infty |\mathcal{H}_{hw}(f)(r)|^p r^{Q-1} dr \right)^{1/p},$$

where $\mathcal{H}^*_{hw}(f)(r)$ can be defined as $\mathcal{H}^*_{hw}(f)(r) = \mathcal{H}^*_{hw}(f)(x)$ for any $|x|_h = r$. Using Minkowski's inequality, we can get that

$$\begin{aligned}
\|\mathcal{H}^*_{hw}\|_{L^p_{|x|_h} L^{\bar{p}_2}_\theta(\mathbb{H}^n)} &= \omega_Q^{1/\bar{p}_2} \left(\int_0^\infty \left| \int_0^1 \frac{f(\delta_{1/r} t)}{t^Q} w(t) dt \right|^p r^{Q-1} dr \right)^{1/p} \\
&\leq \omega_Q^{1/\bar{p}_2} \int_0^1 \left(\int_0^\infty |f(\delta_{1/t} r)|^p r^{Q-1} dr \right)^{1/p} t^{-Q} w(t) dt \\
&= \omega_Q^{1/\bar{p}_2} \int_0^1 \left(\int_0^\infty |f(r)|^p r^{Q-1} dr \right)^{1/p} t^{-Q+Q/p} w(t) dt \\
&= \omega_Q^{1/\bar{p}_2 - 1/\bar{p}_1} \int_0^1 \left(\int_0^\infty \omega_Q^{p/\bar{p}_1} |f(r)|^p r^{Q-1} dr \right)^{1/p} t^{-Q+Q/p} w(t) dt \\
&= \omega_Q^{1/\bar{p}_2 - 1/\bar{p}_1} \int_0^1 t^{-Q(1-1/p)} w(t) dt \|f\|_{L^p_{|x|_h} L^{\bar{p}_1}_\theta}.
\end{aligned}$$

Therefore, we have

$$\|\mathcal{H}^*_{hw}\|_{L^p_{|x|_h} L^{\bar{p}_2}_\theta(\mathbb{H}^n)} \leq \omega_Q^{1/\bar{p}_2 - 1/\bar{p}_1} \int_0^1 t^{-Q(1-1/p)} w(t) dt \|f\|_{L^p_{|x|_h} L^{\bar{p}_1}_\theta}.$$

On the other, taking

$$C = \|\mathcal{H}^*_{hw}\|_{L^p_{|x|_h} L^{\bar{p}_2}_\theta(\mathbb{H}^n) \to L^p_{|x|_h} L^{\bar{p}_1}_\theta(\mathbb{H}^n)} < \infty$$

and for $f \in L^p_{|x|_h} L^{\bar{p}_2}_\theta(\mathbb{H}^n)$, we obtain

$$\|\mathcal{H}^*_{hw}\|_{L^p_{|x|_h} L^{\bar{p}_2}_\theta(\mathbb{H}^n)} \leq C \|f\|_{L^p_{|x|_h} L^{\bar{p}_1}_\theta(\mathbb{H}^n)}.$$

For any $\epsilon > 0$, taking

$$f_\epsilon(t) = \begin{cases} 0, & |x|_h \leq 1, \\ |x|_h^{-\left(\frac{Q}{p} + \epsilon\right)} & |x|_h > 1 \end{cases},$$

then we have

$$\|f_\epsilon\|^p_{L^p_{|x|_h}L^{\bar{p}_1}_\theta(\mathbb{H}^n)} = \frac{\omega_Q^{p/\bar{p}_1}}{p\epsilon}$$

and

$$\mathcal{H}^*_{hw}(f_\epsilon)(x) = \begin{cases} 0, & |x|_h \le 1, \\ |x|_h^{-\frac{Q}{p}-\epsilon} \int_{|x|_h^{-1}<t<1} t^{\frac{Q}{p}+\epsilon-Q} w(t)dt, & |x|_h > 1 \end{cases},$$

where $\mathcal{H}^*_{hw}(f_\epsilon)(x)$ satisfied $\mathcal{H}^*_{hw}(f_\epsilon)(x) = \mathcal{H}^*_{hw}(f_\epsilon)(r)$ for any $|x|_h = r$.

So we have

$$\begin{aligned} C^p\|f_\epsilon\|^p_{L^p_{|x|_h}L^{\bar{p}_1}_\theta} &\ge \|\mathcal{H}^*_{hw}\|^p_{L^p_{|x|_h}L^{\bar{p}_2}_\theta} \\ &= \omega_Q^{p/\bar{p}_2} \int_{r>1} \left| r^{-\frac{Q}{p}-\epsilon} \int_{r^{-1}<t<1} t^{\frac{Q}{p}+\epsilon-Q} w(t)dt \right|^p r^{Q-1}dr \\ &\ge \omega_Q^{p/\bar{p}_2} \int_{r>\frac{1}{\epsilon}} \left| r^{-\frac{Q}{p}-\epsilon} \int_{\epsilon<t<1} t^{\frac{Q}{p}+\epsilon-Q} w(t)dt \right|^p r^{Q-1}dr \\ &= \omega_Q^{p/\bar{p}_2} \int_{r>\frac{1}{\epsilon}} r^{-p\epsilon-Q}dr \left(\int_{\epsilon<t<1} t^{\frac{Q}{p}+\epsilon-Q} w(t)dt \right)^p \\ &= \omega_Q^{p/\bar{p}_2} \int_{|x|_h>\frac{1}{\epsilon}} |x|_h^{-p\epsilon-Q}dx \left(\int_{\epsilon<t<1} t^{\frac{Q}{p}+\epsilon-Q} w(t)dt \right)^p. \end{aligned}$$

By change of variable $|x|_h = \delta_{1/\epsilon}|y|_h$, we have

$$\begin{aligned} C^p\|f_\epsilon\|^p_{L^p_{|x|_h}L^{\bar{p}_1}} &\ge \omega_Q^{p/\bar{p}_2} \int_{|y|_h>1} |y|_h^{-p\epsilon-Q}\epsilon^{\epsilon p}dy \left(\int_{\epsilon<t<1} t^{\frac{Q}{p}+\epsilon-Q} w(t)dt \right)^p \\ &= \left(\omega_Q^{1/\bar{p}_2-1/\bar{p}_1} \epsilon^\epsilon \int_{1<t<\epsilon} t^{\frac{Q}{p}+\epsilon-Q} w(t)dt \right)^p \|f_\epsilon\|_{L^p_{|x|_h}L^{\bar{p}_1}_\theta(\mathbb{H}^n)}. \end{aligned}$$

This implies that

$$\epsilon^\epsilon \int_{1<t<\epsilon} t^{\frac{Q}{p}+\epsilon-Q} w(t)dt \le C.$$

Let $\epsilon \to 0$, we have

$$\int_0^1 t^{\frac{Q}{p}-Q} w(t)dt \le C.$$

Thus, we have finished the proof of Theorem 4.

It should be noted that operators $\mathcal{H}_{hw}$ and $\mathcal{H}^*_{hw}$ are very special cases of a general Hausdorff operator over locally compact groups, introduced in [14]. □

4. Conclusions

In this article, we investigated the sharp bound for Hardy-type operators in the setting of the Heisenberg group, which plays important role in several branches of mathematics. Firstly, we studied n-dimensional Hardy operator and its dual in mixed radial-angular spaces on Heisenberg group and obtain their sharp bounds by using the rotation method. Furthermore, the sharp bounds of n-dimensional weighted Hardy operator and weighted Cesàro operator are also obtained.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest and competing interests. All procedures were in accordance with the ethical standards of the institutional research committee and with the 1964 Helsinki declaration and its later amendments or comparable ethical standards. All authors contributed equally to this work. The manuscript is approved by all authors for publication. Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.

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