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Research article

A study of the equivalence of inference results in the contexts of true and misspecified multivariate general linear models

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Abstract: In practical applications of regression models, we may meet with the situation where a true model is misspecified in some other forms due to certain unforeseeable reasons, so that estimation and statistical inference results obtained under the true and misspecified regression models are not necessarily the same, and therefore, it is necessary to compare these results and to establish certain links between them for the purpose of reasonably explaining and utilizing the misspecified regression models. In this paper, we propose and investigate some comparison and equivalence analysis problems on estimations and predictions under true and misspecified multivariate general linear models. We first give the derivations and presentations of the best linear unbiased predictors (BLUPs) and the best linear unbiased estimators (BLUEs) of unknown parametric matrices under a true multivariate general linear model and its misspecified form. We then derive a variety of necessary and sufficient conditions for the BLUPs/BLUEs under the two competing models to be equal using a series of highly-selective formulas and facts associated with ranks, ranges and generalized inverses of matrices, as well as block matrix operations.

Keywords: block matrix; BLUE; BLUP; misspecified model; MGLM; rank formula
Mathematics Subject Classification: 62F12, 62F30, 62J10

1. Introduction

Throughout, the symbol $\mathbb{R}^{m\times n}$ stands for the collection of all $m \times n$ matrices with real numbers; $\mathbf{M}'$, $r(\mathbf{M})$, and $\mathscr{R}(\mathbf{M})$ stand for the transpose, the rank, and the range (column space) of a matrix $\mathbf{M} \in \mathbb{R}^{m\times n}$, respectively; $\mathbf{I}_m$ denotes the identity matrix of order m. Two symmetric matrices $\mathbf{M}$ and $\mathbf{N}$ of the same size are said to satisfy the inequality $\mathbf{M} \succcurlyeq \mathbf{N}$ in the Löwner partial ordering if $\mathbf{M} - \mathbf{N}$ is nonnegative definite. The Kronecker product of any two matrices $\mathbf{M}$ and $\mathbf{N}$ is defined to be $\mathbf{M} \otimes \mathbf{N} = (m_{ij}\mathbf{N})$. The vectorization operator of a matrix $\mathbf{M} = [\,\mathbf{m}_1, \ldots, \mathbf{m}_n\,]$ is defined to be $\mathrm{vec}(\mathbf{M}) = \overrightarrow{\mathbf{M}} = [\,\mathbf{m}_1', \ldots, \mathbf{m}_n'\,]'$. A well-known property on the vec operator of a triple matrix product is $\overrightarrow{\mathbf{MXN}} = (\mathbf{N}' \otimes \mathbf{M})\overrightarrow{\mathbf{X}}$. The Moore-Penrose generalized inverse of $\mathbf{M} \in \mathbb{R}^{m\times n}$, denoted by $\mathbf{M}^+$, is defined by the unique solution $\mathbf{G}$ to the four matrix equations $\mathbf{MGM} = \mathbf{M}$, $\mathbf{GMG} = \mathbf{G}$, $(\mathbf{MG})' = \mathbf{MG}$ and $(\mathbf{GM})' = \mathbf{GM}$. We also denote by $\mathbf{P_M} = \mathbf{MM}^+$, $\mathbf{M}^{\perp} = \mathbf{E_M} = \mathbf{I}_m - \mathbf{MM}^+$ and $\mathbf{F_M} = \mathbf{I}_n - \mathbf{M}^+\mathbf{M}$ the three orthogonal projectors induced from $\mathbf{M}$, respectively, which will help in briefly denoting calculation processes related to generalized inverses of matrices. Further information about the orthogonal projectors $\mathbf{P_M}$, $\mathbf{E_M}$ and $\mathbf{F_M}$ with their applications in the linear statistical models can be found, e.g., in [11, 21, 22].

In this paper, we reconsider a multivariate general linear model (for short, MGLM):

$$\mathscr{M}: \quad \mathbf{Y} = \mathbf{X}\Theta + \mathbf{\Omega}, \;\; \mathrm{E}(\mathbf{\Omega}) = \mathbf{0}, \;\; \mathrm{Cov}(\overrightarrow{\mathbf{\Omega}}) = \mathrm{Cov}\{\overrightarrow{\mathbf{\Omega}}, \overrightarrow{\mathbf{\Omega}}\} = \mathbf{\Sigma}_2 \otimes \mathbf{\Sigma}_1, \tag{1.1}$$

where it is assumed that $\mathbf{Y} \in \mathbb{R}^{n\times m}$ is an observable random matrix (longitudinal data set), $\mathbf{X} = (z_{ij}) \in \mathbb{R}^{n\times p}$ is a known model matrix of arbitrary rank ($0 \leq r(\mathbf{X}) \leq \min\{n, p\}$), $\Theta = (\theta_{ij}) \in \mathbb{R}^{p\times m}$ is a matrix of fixed but unknown parameters, $\mathbf{\Omega} \in \mathbb{R}^{n\times m}$ is a matrix of randomly distributed error terms, $\mathbf{\Sigma}_1 = (\sigma_{1ij}) \in \mathbb{R}^{n\times n}$ and $\mathbf{\Sigma}_2 = (\sigma_{2ij}) \in \mathbb{R}^{m\times m}$ are two known nonnegative definite matrices of arbitrary ranks and $\mathbf{\Sigma}_2 \otimes \mathbf{\Sigma}_1$ means that $\overrightarrow{\mathbf{\Omega}}$ has a Kronecker product structured covariance matrix.

We now give some general remarks regarding $\mathscr{M}$ in (1.1) and propose a research topic in the context of the model. An MGLM as in (1.1) is a relative direct extension of the most welcome type of univariate general linear models, which means the incorporation of regressing one response variable on a given set of regressors to several response variables on the regressors. This model is also a typical representation of various multivariate regression frameworks yet has been a core issue of study in the theory of multivariate analysis and its applications. Usually in the statistical applications of regression models, we may meet with situations where a true regression model is misspecified in some other forms due to different unforeseeable reasons. In such a case, the estimation and statistical inference results under the true and misspecified models are not necessarily the same, so that we have to face with the work of clearly and reasonably explaining and comparing the results. To illustrate this problem, we typically assume that the model matrix $\mathbf{X}$ in (1.1) is misspecified as $\mathbf{X}_0 \in \mathbb{R}^{n\times q}$, and the covariance matrix $\mathbf{\Sigma}_2 \otimes \mathbf{\Sigma}_1$ in (1.1) is misspecified as $\mathbf{V}_2 \otimes \mathbf{V}_1$. In this case, (1.1) appears in the following misspecified form:

$$\mathscr{N}: \quad \mathbf{Y} = \mathbf{X}_0\Theta_0 + \mathbf{\Omega}_0, \;\; \mathrm{E}(\mathbf{\Omega}_0) = \mathbf{0}, \;\; \mathrm{Cov}(\overrightarrow{\mathbf{\Omega}}_0) = \mathbf{V}_2 \otimes \mathbf{V}_1, \tag{1.2}$$

where it is assumed that $\mathbf{X}_0 \in \mathbb{R}^{n\times q}$ is a known model matrix of arbitrary rank, and $\Theta_0 \in \mathbb{R}^{q\times m}$ is a matrix of fixed but unknown parameters. $\mathbf{\Omega}_0 \in \mathbb{R}^{n\times m}$ is a matrix of randomly distributed error terms, and $\mathbf{V}_1 \in \mathbb{R}^{n\times n}$ and $\mathbf{V}_2 \in \mathbb{R}^{m\times m}$ are two known nonnegative definite matrices of arbitrary ranks. Because $\mathbf{X}_0$, Θ_0 and $\mathbf{V}_2 \otimes \mathbf{V}_1$ in (1.2) can be taken any pre-assumed expressions, a general form as in (1.2) includes almost all misspecified models of (1.1) as its special cases, such as $\mathbf{X}_0\Theta_0 = \mathbf{X}\Theta + \mathbf{W\Gamma}$, $\mathbf{V}_2 \otimes \mathbf{V}_1 = \sigma^2\mathbf{I}_{mn}$, etc.

Before proposing and discussing a number of concrete comparison problems regarding inference results and facts in the contexts of (1.1) and (1.2), we review some relevant methods and techniques that can be conveniently used in the analysis of multivariate general linear models. Recall that the Kronecker products and vec operations of matrices are popular and useful tools in dealing with matrix operations associated with MGLMs. Referring to these operations, we can alternatively represent the two models in (1.1) and (1.2) as the following ordinary linear models:

$$\widehat{\mathscr{M}}: \ \overrightarrow{\mathbf{Y}} = (\mathbf{I}_m \otimes \mathbf{X})\overrightarrow{\Theta} + \overrightarrow{\mathbf{\Omega}}, \ \ \mathrm{E}(\overrightarrow{\mathbf{\Omega}}) = \mathbf{0}, \ \ \mathrm{Cov}(\overrightarrow{\mathbf{\Omega}}) = \mathbf{\Sigma}_2 \otimes \mathbf{\Sigma}_1, \tag{1.3}$$

$$\widehat{\mathscr{N}}: \ \overrightarrow{\mathbf{Y}} = (\mathbf{I}_m \otimes \mathbf{X}_0)\overrightarrow{\Theta}_0 + \overrightarrow{\mathbf{\Omega}}_0, \ \ \mathrm{E}(\overrightarrow{\mathbf{\Omega}}_0) = \mathbf{0}, \ \ \mathrm{Cov}(\overrightarrow{\mathbf{\Omega}}_0) = \mathbf{V}_2 \otimes \mathbf{V}_1. \tag{1.4}$$

Given (1.1), a primary task is to estimate or predict certain functions of the unknown parameter matrices Θ and $\mathbf{\Omega}$ in (1.1). To do so, we construct a set of parametric functions containing both Θ and $\mathbf{\Omega}$ as follows:

$$\mathbf{R} = \mathbf{K}\Theta + \mathbf{J}\mathbf{\Omega}, \ \ \overrightarrow{\mathbf{R}} = (\mathbf{I}_m \otimes \mathbf{K})\overrightarrow{\Theta} + (\mathbf{I}_m \otimes \mathbf{J})\overrightarrow{\mathbf{\Omega}}, \tag{1.5}$$

where it is assumed that $\mathbf{K}$ and $\mathbf{J}$ are $k \times p$ and $k \times n$ matrices, respectively. In this case,

$$\mathrm{E}(\mathbf{R}) = \mathbf{K}\Theta, \ \ \mathrm{Cov}(\overrightarrow{\mathbf{R}}) = (\mathbf{I}_m \otimes \mathbf{J})(\mathbf{\Sigma}_2 \otimes \mathbf{\Sigma}_1)(\mathbf{I}_m \otimes \mathbf{J})', \tag{1.6}$$

$$\mathrm{Cov}\{\overrightarrow{\mathbf{R}}, \overrightarrow{\mathbf{Y}}\} = \mathrm{Cov}\{\overrightarrow{\mathbf{J}\mathbf{\Omega}}, \overrightarrow{\mathbf{\Omega}}\} = (\mathbf{I}_m \otimes \mathbf{J})(\mathbf{\Sigma}_2 \otimes \mathbf{\Sigma}_1). \tag{1.7}$$

When $\mathbf{K} = \mathbf{X}$ and $\mathbf{J} = \mathbf{I}_n$, (1.5) becomes $\mathbf{R} = \mathbf{X}\Theta + \mathbf{\Omega} = \mathbf{Y}$, the observed response matrix. Hence, (1.5) includes all matrix operations in (1.1) as its special cases. Thus, the construction of $\mathbf{R}$ can be used to identify their estimations and predictions of Θ and $\mathbf{\Omega}$, simultaneously. Under the misspecified assumptions in (1.2), a general matrix of parametric functions is given by

$$\mathbf{R}_0 = \mathbf{K}_0\Theta_0 + \mathbf{J}_0\mathbf{\Omega}_0, \ \ \overrightarrow{\mathbf{R}}_0 = (\mathbf{I}_m \otimes \mathbf{K}_0)\overrightarrow{\Theta}_0 + (\mathbf{I}_m \otimes \mathbf{J}_0)\overrightarrow{\mathbf{\Omega}}_0, \tag{1.8}$$

where it is assumed that $\mathbf{K}_0$ and $\mathbf{J}_0$ are $k \times q$ and $k \times n$ matrices, respectively.

Notice that the assumptions in the contexts of (1.1) and (1.2) are apparently different in representations. Thus, this fact means the statistical inference results on $\mathbf{R}$ derived from (1.1) and those on $\mathbf{R}_0$ derived from (1.2) are not necessarily the same, and of course, the findings under (1.2) generally are incorrect conclusions. Even so, there is a possibility that certain calculational results under (1.1) and (1.2) coincide. This possibility prompts statisticians to consider the comparison and relevance problems of inference results under the two models, especially to establish the relationships of predictions/estimations of unknown parameters under the two models. A classic investigation on relationships between true and misspecified linear models was given in [17], while many investigations on comparison problems of predictions/estimations of unknown parameters under true models and their misspecified models can be found in the literature; see, e.g., [2–5, 8, 9, 12, 14–18, 23, 26]. Recently, [30] discussed some kinds of relationships between true models and their misspecified forms under a general linear model, [7] considered the equivalence of predictions/estimations under an MGLM with augmentation, and [31] considered simultaneous prediction issues under an MGLM with future observations.

In this paper, we focus on the problems pertaining to the comparisons of the best linear unbiased predictors (for short, BLUPs) of $\mathbf{R}$ derived from (1.1) and those of $\mathbf{R}_0$ derived from (1.2). The BLUPs now are known as one of the important parametric methods of predicting unknown parameters, which is a core concept and conventional topic in the regression analysis of linear statistical models, and many general and special contributions in relation to BLUPs under linear statistical models were given in the literature. This paper is mainly concerned with the connection analysis of the BLUPs of $\mathbf{R}$ under (1.1) and $\mathbf{R}_0$ under (1.2).

The rest of this paper is organized as follows. In Section 2, we introduce notation and a collection of matrix analysis tools that we shall use to characterize matrix equalities that involve generalized inverses and give the definitions of predictability and the BLUPs of parameter matrix under (1.1). In Section 3, we present some basic estimation and inference theory regarding an MGLM, including analytical expressions of the BLUPs and their mathematical and statistical properties and features in the contexts of (1.1) and (1.2). In Section 4, we derive several groups of identifying conditions for the BLUPs to equal under the true and misspecified MGLMs using a series of precise and analytical tools in matrix theory. Some conclusions and remarks are given in Section 5. The proofs of the main results are given in the Appendix.

2. Notation and some preliminaries

For the purpose of establishing and describing equalities for different predictions/estimations in the context of linear statistical models, we need to adopt a selection of commonly-used matrix rank formulas and equivalent facts about matrix equalities in the following three lemmas, which will underpin the establishments and simplifications of various complicated matrix expressions and matrix equalities that appear in the statistical inference of MGLMs.

Lemma 2.1. *Let $\mathcal{S}$ and $\mathcal{T}$ be two sets composed by matrices of the same size. Then,*

$$\mathcal{S} \cap \mathcal{T} \neq \emptyset \Leftrightarrow \min_{\mathbf{S}\in\mathcal{S},\, \mathbf{T}\in\mathcal{T}} r(\,\mathbf{S} - \mathbf{T}\,) = 0, \tag{2.1}$$

$$\mathcal{S} \subseteq \mathcal{T} \Leftrightarrow \max_{\mathbf{S}\in\mathcal{S}} \min_{\mathbf{T}\in\mathcal{T}} r(\,\mathbf{S} - \mathbf{T}\,) = 0. \tag{2.2}$$

Lemma 2.2. *[13] Let $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{B} \in \mathbb{R}^{m\times k}$, and $\mathbf{C} \in \mathbb{R}^{l\times n}$. Then, the following rank equalities hold:*

$$r[\,\mathbf{A},\ \mathbf{B}\,] = r(\mathbf{A}) + r(\mathbf{E}_{\mathbf{A}}\mathbf{B}) = r(\mathbf{B}) + r(\mathbf{E}_{\mathbf{B}}\mathbf{A}), \tag{2.3}$$

$$r\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix} = r(\mathbf{A}) + r(\mathbf{C}\mathbf{F}_{\mathbf{A}}) = r(\mathbf{C}) + r(\mathbf{A}\mathbf{F}_{\mathbf{C}}), \tag{2.4}$$

$$r\begin{bmatrix} \mathbf{A}\mathbf{A}' & \mathbf{B} \\ \mathbf{B}' & \mathbf{0} \end{bmatrix} = r[\,\mathbf{A},\ \mathbf{B}\,] + r(\mathbf{B}). \tag{2.5}$$

In particular, the following results hold.

(a) $r[\,\mathbf{A},\ \mathbf{B}\,] = r(\mathbf{A}) \Leftrightarrow \mathscr{R}(\mathbf{B}) \subseteq \mathscr{R}(\mathbf{A}) \Leftrightarrow \mathbf{A}\mathbf{A}^{+}\mathbf{B} = \mathbf{B} \Leftrightarrow \mathbf{E}_{\mathbf{A}}\mathbf{B} = \mathbf{0}$.

(b) $r\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix} = r(\mathbf{A}) \Leftrightarrow \mathscr{R}(\mathbf{C}') \subseteq \mathscr{R}(\mathbf{A}') \Leftrightarrow \mathbf{C}\mathbf{A}^{+}\mathbf{A} = \mathbf{C} \Leftrightarrow \mathbf{C}\mathbf{F}_{\mathbf{A}} = \mathbf{0}$.

(c) $r[\,\mathbf{A},\ \mathbf{B}\,] = r(\mathbf{A}) + r(\mathbf{B}) \Leftrightarrow \mathscr{R}(\mathbf{A}) \cap \mathscr{R}(\mathbf{B}) = \{\mathbf{0}\} \Leftrightarrow \mathscr{R}((\mathbf{E}_{\mathbf{A}}\mathbf{B})') = \mathscr{R}(\mathbf{B}') \Leftrightarrow \mathscr{R}((\mathbf{E}_{\mathbf{B}}\mathbf{A})') = \mathscr{R}(\mathbf{A}')$.

(d) $r\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix} = r(\mathbf{A}) + r(\mathbf{C}) \Leftrightarrow \mathscr{R}(\mathbf{A}') \cap \mathscr{R}(\mathbf{C}') = \{\mathbf{0}\} \Leftrightarrow \mathscr{R}(\mathbf{C}\mathbf{F}_{\mathbf{A}}) = \mathscr{R}(\mathbf{C}) \Leftrightarrow \mathscr{R}(\mathbf{A}\mathbf{F}_{\mathbf{C}}) = \mathscr{R}(\mathbf{A})$.

Lemma 2.3. *[25] Let $\mathbf{X} \in \mathbb{R}^{n\times p}$, $\mathbf{X}_0 \in \mathbb{R}^{n\times q}$, and let $\mathbf{\Sigma}_1$, $\mathbf{V}_1 \in \mathbb{R}^{n\times n}$ be two nonnegative definite matrices. Then,*

$$r\begin{bmatrix} \mathbf{X} & \mathbf{X}_0 & \mathbf{\Sigma}_1 \\ \mathbf{0} & \mathbf{0} & \mathbf{X}' \end{bmatrix} = r[\,\mathbf{X},\ \mathbf{X}_0,\ \mathbf{\Sigma}_1\,] + r(\mathbf{X}),\quad r\begin{bmatrix} \mathbf{X} & \mathbf{X}_0 & \mathbf{V}_1 \\ \mathbf{0} & \mathbf{0} & \mathbf{X}_0' \end{bmatrix} = r[\,\mathbf{X},\ \mathbf{X}_0,\ \mathbf{V}_1\,] + r(\mathbf{X}_0), \tag{2.6}$$

$$r\begin{bmatrix} \mathbf{X} & \mathbf{X}_0 & \mathbf{\Sigma}_1 & \mathbf{V}_1 \\ \mathbf{0} & \mathbf{0} & \mathbf{X}' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X}_0' \end{bmatrix} = r[\, \mathbf{X},\ \mathbf{X}_0,\ \mathbf{\Sigma}_1,\ \mathbf{V}_1\,] + r(\mathbf{X}) + r(\mathbf{X}_0). \tag{2.7}$$

Lemma 2.4. *[24,29] Let* $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{B} \in \mathbb{R}^{m\times k}$, *and* $\mathbf{C} \in \mathbb{R}^{l\times n}$. *Then, the maximum and minimum ranks of* $\mathbf{A} - \mathbf{BW}$ *and* $\mathbf{A} - \mathbf{BWC}$ *with respect to the variable matrix* $\mathbf{W}$ *are given by the following analytical formulas:*

$$\max_{\mathbf{W}\in\mathbb{R}^{k\times n}} r(\,\mathbf{A} - \mathbf{BW}\,) = \min\{\, r[\,\mathbf{A},\ \mathbf{B}\,],\ \ n\,\}, \tag{2.8}$$

$$\min_{\mathbf{W}\in\mathbb{R}^{k\times n}} r(\,\mathbf{A} - \mathbf{BW}\,) = r[\,\mathbf{A},\ \mathbf{B}\,] - r(\mathbf{B}), \tag{2.9}$$

$$\max_{\mathbf{W}\in\mathbb{R}^{k\times l}} r(\,\mathbf{A} - \mathbf{BWC}\,) = \min\left\{ r[\,\mathbf{A},\ \mathbf{B}\,],\ \ r\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix} \right\}. \tag{2.10}$$

Below, we give an existing result about the general solution of a basic linear matrix equation.

Lemma 2.5. *[20] The linear matrix equation* $\mathbf{AX} = \mathbf{B}$ *is consistent if and only if* $r[\mathbf{A},\ \mathbf{B}] = r(\mathbf{A})$, *or equivalently,* $\mathbf{AA}^{+}\mathbf{B} = \mathbf{B}$. *In this case, the general solution of the equation can be written in the parametric form* $\mathbf{X} = \mathbf{A}^{+}\mathbf{B} + (\,\mathbf{I} - \mathbf{A}^{+}\mathbf{A}\,)\mathbf{U}$, *where* $\mathbf{U}$ *is an arbitrary matrix.*

Lemma 2.6. *Let* $\mathbf{A} \in \mathbb{R}^{m\times n}$ *and* $\mathbf{B} \in \mathbb{R}^{m\times k}$ *and assume that* $\mathscr{R}(\mathbf{A}) = \mathscr{R}(\mathbf{B})$. *Then,*

$$\mathbf{XA} = \mathbf{0} \Leftrightarrow \mathbf{XB} = \mathbf{0}. \tag{2.11}$$

For the purpose of clearly and analytically solving the matrix minimization problem in (3.1), we need to use the following existing result on a constrained quadratic matrix-valued function minimization problem, which was proved in [27].

Lemma 2.7. *[27] Let*

$$f(\mathbf{X}) = (\,\mathbf{XC} + \mathbf{D}\,)\mathbf{M}(\,\mathbf{XC} + \mathbf{D})' \quad s.t.\ \ \mathbf{XA} = \mathbf{B}, \tag{2.12}$$

where $\mathbf{A} \in \mathbb{R}^{p\times q}$, $\mathbf{B} \in \mathbb{R}^{n\times q}$, $\mathbf{C} \in \mathbb{R}^{p\times m}$, $\mathbf{D} \in \mathbb{R}^{n\times m}$ *are given,* $\mathbf{M} \in \mathbb{R}^{m\times m}$ *is positive semi-definite, and the matrix equation* $\mathbf{XA} = \mathbf{B}$ *is consistent. Then, there always exists a solution* $\mathbf{X}_0$ *of* $\mathbf{X}_0\mathbf{A} = \mathbf{B}$ *such that*

$$f(\mathbf{X}) \succcurlyeq f(\mathbf{X}_0) \tag{2.13}$$

holds for all solutions of $\mathbf{XA} = \mathbf{B}$. *In this case, the matrix* $\mathbf{X}_0$ *satisfying* (2.13) *is determined by the following consistent matrix equation:*

$$\mathbf{X}_0[\,\mathbf{A},\ \mathbf{CMC}'\mathbf{A}^{\perp}\,] = [\,\mathbf{B},\ -\mathbf{DMC}'\mathbf{A}^{\perp}\,], \tag{2.14}$$

while the general expression of $\mathbf{X}_0$ *and the corresponding* $f(\mathbf{X}_0)$ *are given by*

$$\mathbf{X}_0 = \operatorname*{argmin}_{\mathbf{XA}=\mathbf{B}} f(\mathbf{X}) = [\,\mathbf{B},\ -\mathbf{DMC}'\mathbf{A}^{\perp}\,][\,\mathbf{A},\ \mathbf{CMC}'\mathbf{A}^{\perp}\,]^{+} + \mathbf{V}[\,\mathbf{A},\ \mathbf{CMC}'\,]^{\perp}, \tag{2.15}$$

$$f(\mathbf{X}_0) = \min_{\mathbf{XA}=\mathbf{B}} f(\mathbf{X}) = \mathbf{KMK}' - \mathbf{KMC}'(\mathbf{A}^{\perp}\mathbf{CMC}'\mathbf{A}^{\perp})^{+}\mathbf{CMK}', \tag{2.16}$$

$$f(\mathbf{X}) - f(\mathbf{X}_0) = (\mathbf{XCMC}'\mathbf{A}^{\perp} + \mathbf{DMC}'\mathbf{A}^{\perp})(\mathbf{A}^{\perp}\mathbf{CMC}'\mathbf{A}^{\perp})^{+}(\mathbf{XCMC}'\mathbf{A}^{\perp} + \mathbf{DMC}'\mathbf{A}^{\perp})', \tag{2.17}$$

where $\mathbf{K} = \mathbf{BA}^{+}\mathbf{C} + \mathbf{D}$, *and* $\mathbf{V} \in \mathbb{R}^{n\times p}$ *is arbitrary.*

3. Basic theory for the BLUPs of all unknown parameter matrices in an MGLM

In this section, we present a review of the most important theoretical concepts concerning an MGLM. As usual, the unbiasedness of given linear predictions/estimations with respect to certain unknown parameters in an MGLM is an important property, but there are often many possible unbiased predictions/estimations for the same parameters. The classic statistical concept of predictability was originated from [6], while the predictability/estimability concepts of parameters in an MGLM were established in [1, 19]. Under the assumptions in (1.1), the predictability/estimability of the unknown parameters is defined as follows.

Definition 3.1. *Let the parametric matrix* $\mathbf{R}$ *be as given in* (1.5).

(a) The matrix $\mathbf{R}$ *is said to be predictable under* (1.1) *if there exists a* $k \times n$ *matrix* $\mathbf{L}$ *such that* $\mathrm{E}(\mathbf{L}\mathbf{Y} - \mathbf{R}) = \mathbf{0}$.

(b) The vector $\overrightarrow{\mathbf{R}}$ *is said to be predictable under* (1.3) *if there exists an* $mk \times mn$ *matrix* $\mathbf{L}$ *such that* $\mathrm{E}(\mathbf{L}\overrightarrow{\mathbf{Y}} - \overrightarrow{\mathbf{R}}) = \mathbf{0}$.

Definition 3.2. *Let the parametric matrix* $\mathbf{R}$ *be as given in* (1.5).

(a) Given that $\mathbf{R}$ *is predictable under* (1.1)*, if there exists a matrix* $\mathbf{L}_0$ *such that*

$$\mathrm{Cov}(\,\overrightarrow{\mathbf{L}_0\mathbf{Y}} - \overrightarrow{\mathbf{R}}\,) = \min \ \text{ s.t. } \mathrm{E}(\mathbf{L}_0\mathbf{Y} - \mathbf{R}) = \mathbf{0} \tag{3.1}$$

holds in the Löwner partial ordering, the linear statistic $\mathbf{L}_0\mathbf{Y}$ *is defined to be the best linear unbiased predictor (BLUP) of* $\mathbf{R}$ *under* (1.1)*, and is described by*

$$\mathbf{L}_0\mathbf{Y} = \mathrm{BLUP}_{\mathscr{M}}(\mathbf{R}) = \mathrm{BLUP}_{\mathscr{M}}(\mathbf{K}\Theta + \mathbf{J}\mathbf{\Omega}). \tag{3.2}$$

If $\mathbf{J} = \mathbf{0}$ *or* $\mathbf{K} = \mathbf{0}$ *in* (1.5)*, the* $\mathbf{L}_0\mathbf{Y}$ *satisfying* (3.1) *is called the best linear unbiased estimator (BLUE) of* $\mathbf{K}\Theta$ *and the BLUP of* $\mathbf{J}\mathbf{\Omega}$ *under* (1.1)*, respectively, and is described by*

$$\mathbf{L}_0\mathbf{Y} = \mathrm{BLUE}_{\mathscr{M}}(\mathbf{K}\Theta), \ \ \mathbf{L}_0\mathbf{Y} = \mathrm{BLUP}_{\mathscr{M}}(\mathbf{J}\mathbf{\Omega}), \tag{3.3}$$

respectively.

(b) Given that $\overrightarrow{\mathbf{R}}$ *is predictable under* (1.3)*, if there exists a matrix* $\mathbf{L}_0$ *such that*

$$\mathrm{Cov}(\,\mathbf{L}_0\overrightarrow{\mathbf{Y}} - \overrightarrow{\mathbf{R}}\,) = \min \ \text{ s.t. } \mathrm{E}(\mathbf{L}_0\overrightarrow{\mathbf{Y}} - \overrightarrow{\mathbf{R}}) = \mathbf{0} \tag{3.4}$$

hold in the Löwner partial ordering, the linear statistic $\mathbf{L}_0\overrightarrow{\mathbf{Y}}$ *is defined to be the BLUP of* $\overrightarrow{\mathbf{R}}$ *under* (1.3) *and is described by*

$$\mathbf{L}_0\overrightarrow{\mathbf{Y}} = \mathrm{BLUP}_{\overrightarrow{\mathscr{M}}}(\overrightarrow{\mathbf{R}}) = \mathrm{BLUP}_{\overrightarrow{\mathscr{M}}}((\mathbf{I}_m \otimes \mathbf{K})\overrightarrow{\Theta} + (\mathbf{I}_m \otimes \mathbf{J})\overrightarrow{\mathbf{\Omega}}). \tag{3.5}$$

If $\mathbf{J} = \mathbf{0}$ *or* $\mathbf{K} = \mathbf{0}$ *in* (1.5)*, the* $\mathbf{L}_0\overrightarrow{\mathbf{Y}}$ *satisfying* (3.4) *is called the BLUE of* $(\mathbf{I}_m \otimes \mathbf{K})\overrightarrow{\Theta}$ *and the BLUP of* $(\mathbf{I}_m \otimes \mathbf{J})\overrightarrow{\mathbf{\Omega}}$ *under* (1.3)*, respectively, and is denoted by*

$$\mathbf{L}_0\overrightarrow{\mathbf{Y}} = \mathrm{BLUE}_{\overrightarrow{\mathscr{M}}}((\mathbf{I}_m \otimes \mathbf{K})\overrightarrow{\Theta}), \ \ \mathbf{L}_0\overrightarrow{\mathbf{Y}} = \mathrm{BLUP}_{\overrightarrow{\mathscr{M}}}((\mathbf{I}_m \otimes \mathbf{J})\overrightarrow{\mathbf{\Omega}}), \tag{3.6}$$

respectively.

Admittedly, BLUPs/BLUEs of unknown parameters were common concepts and principles in the statistical inference of parametric models, which were highly appraised and regarded in the domain of regression analysis due to their simple and optimal performances and properties, while the study of BLUPs/BLUEs and the related issues were core parts in the research field of statistics and applications. As a fundamental and theoretical tool in matrix theory, the analytical solution of the constrained quadratic matrix-valued function optimization problem in the Löwner partial ordering in Lemma 2.7 was used to derive a number of exact and analytical formulas for calculating BLUPs/BLUEs and their properties under various linear regression frameworks; see, e.g., [5, 7, 10, 28, 30, 31].

In this section, we present a sequence of existing formulas, results, and facts about the predictability/estimability and the analytical formulas of the BLUPs of $\mathbf{R}$ and $\mathbf{R}_0$ in (1.5) and (1.8). Recall that the unbiasedness of predictions/estimations and the lowest dispersion matrices, as formulated in (3.1), are two of the most intrinsic requirements in statistical inference in the context of regression models, which can conveniently be interpreted as some special cases of mathematical optimization problems with regard to constrained quadratic matrix-valued functions in the Löwner partial ordering.

Due to the linear nature of an MGLM, we see from (1.1) and (1.5) that $\mathbf{LY}-\mathbf{R}$ and $\overrightarrow{\mathbf{LY}-\mathbf{R}}$ can be rewritten as

$$\mathbf{LY}-\mathbf{R} = \mathbf{LX}\Theta + \mathbf{L\Omega} - \mathbf{K}\Theta - \mathbf{J\Omega} = (\mathbf{LX}-\mathbf{K})\Theta + (\mathbf{L}-\mathbf{J})\mathbf{\Omega}, \tag{3.7}$$

$$\overrightarrow{\mathbf{LY}-\mathbf{R}} = (\mathbf{I}_m \otimes (\mathbf{LX}-\mathbf{K}))\overrightarrow{\Theta} + (\mathbf{I}_m \otimes (\mathbf{L}-\mathbf{J}))\overrightarrow{\mathbf{\Omega}}. \tag{3.8}$$

Hence, the expectations of $\mathbf{LY}-\mathbf{R}$ and $\overrightarrow{\mathbf{LY}-\mathbf{R}}$ can be expressed as

$$\mathrm{E}(\mathbf{LY}-\mathbf{R}) = (\mathbf{LX}-\mathbf{K})\Theta, \ \ \mathrm{E}(\overrightarrow{\mathbf{LY}-\mathbf{R}}) = (\mathbf{I}_m \otimes (\mathbf{LX}-\mathbf{K}))\overrightarrow{\Theta}. \tag{3.9}$$

The covariance matrix of $\overrightarrow{\mathbf{LY}-\mathbf{R}}$ can be expressed as

$$\begin{aligned}\mathrm{Cov}(\overrightarrow{\mathbf{LY}-\mathbf{R}}) &= (\mathbf{I}_m \otimes (\mathbf{L}-\mathbf{J}))\mathrm{Cov}(\overrightarrow{\mathbf{\Omega}})(\mathbf{I}_m \otimes (\mathbf{L}-\mathbf{J}))' \\ &= (\mathbf{I}_m \otimes (\mathbf{L}-\mathbf{J}))(\mathbf{\Sigma}_2 \otimes \mathbf{\Sigma}_1)(\mathbf{I}_m \otimes (\mathbf{L}-\mathbf{J}))' \\ &= \mathbf{\Sigma}_2 \otimes (\mathbf{L}-\mathbf{J})\mathbf{\Sigma}_1(\mathbf{L}-\mathbf{J})' \triangleq \mathbf{\Sigma}_2 \otimes f(\mathbf{L}),\end{aligned} \tag{3.10}$$

where $f(\mathbf{L}) = (\mathbf{L}-\mathbf{J})\mathbf{\Sigma}_1(\mathbf{L}-\mathbf{J})'$.

Concerning the predictability and the BLUP of $\mathbf{R}$ in (1.5), we use the following existing results.

Theorem 3.1. *[7] Let* $\mathbf{R}$ *be as given in* (1.5). *Then, the following statements are equivalent*:

(a) $\mathbf{R}$ *is predictable by* $\mathbf{Y}$ *in* (1.1).
(b) $\mathscr{R}(\mathbf{I}_m \otimes \mathbf{K}') \subseteq \mathscr{R}(\mathbf{I}_m \otimes \mathbf{X}')$.
(c) $\mathscr{R}(\mathbf{K}') \subseteq \mathscr{R}(\mathbf{X}')$.

Theorem 3.2. *[7, 31] Assume* $\mathbf{R}$ *in* (1.5) *is predictable. Then*,

$$\mathrm{Cov}(\overrightarrow{\mathbf{LY}-\mathbf{R}}) = \min \ \ s.t., \ \mathrm{E}(\mathbf{LY}-\mathbf{R}) = \mathbf{0} \Leftrightarrow \mathbf{L}[\,\mathbf{X}, \mathbf{\Sigma}_1\mathbf{X}^{\perp}\,] = [\,\mathbf{K}, \mathbf{J\Sigma}_1\mathbf{X}^{\perp}\,]. \tag{3.11}$$

The matrix equation in (3.11), *called the BLUP equation associated with* $\mathbf{R}$, *is consistent as well*, *i.e.*,

$$[\,\mathbf{K},\ \mathbf{J}\boldsymbol{\Sigma}_1\mathbf{X}^{\perp}\,][\,\mathbf{X},\ \boldsymbol{\Sigma}_1\mathbf{X}^{\perp}\,]^{+}[\,\mathbf{X},\ \boldsymbol{\Sigma}_1\mathbf{X}^{\perp}\,] = [\,\mathbf{K},\ \mathbf{J}\boldsymbol{\Sigma}_1\mathbf{X}^{\perp}\,] \tag{3.12}$$

holds under Theorem 3.1(c), *while the general expressions of* $\mathbf{L}$ *and the corresponding* $\mathrm{BLUP}_{\mathscr{M}}(\mathbf{R})$ *can be written as*

$$\mathrm{BLUP}_{\mathscr{M}}(\mathbf{R}) = \mathbf{P}_{\mathbf{K};\mathbf{J};\mathbf{X};\boldsymbol{\Sigma}_1}\mathbf{Y} = ([\,\mathbf{K},\ \mathbf{J}\boldsymbol{\Sigma}_1\mathbf{X}^{\perp}\,][\,\mathbf{X},\ \boldsymbol{\Sigma}_1\mathbf{X}^{\perp}\,]^{+} + \mathbf{U}[\,\mathbf{X},\ \boldsymbol{\Sigma}_1\mathbf{X}^{\perp}\,]^{\perp})\,\mathbf{Y}, \tag{3.13}$$

where $\mathbf{U} \in \mathbb{R}^{k\times n}$ *is arbitrary. In particular*,

$$\mathrm{BLUE}_{\mathscr{M}}(\mathbf{K}\Theta) = ([\,\mathbf{K},\ \mathbf{0}\,][\,\mathbf{X},\ \boldsymbol{\Sigma}_1\mathbf{X}^{\perp}\,]^{+} + \mathbf{U}[\,\mathbf{X},\ \boldsymbol{\Sigma}_1\mathbf{X}^{\perp}\,]^{\perp})\,\mathbf{Y}, \tag{3.14}$$

$$\mathrm{BLUP}_{\mathscr{M}}(\mathbf{J}\boldsymbol{\Omega}) = ([\,\mathbf{0},\ \mathbf{J}\boldsymbol{\Sigma}_1\mathbf{X}^{\perp}\,][\,\mathbf{X},\ \boldsymbol{\Sigma}_1\mathbf{X}^{\perp}\,]^{+} + \mathbf{U}[\,\mathbf{X},\ \boldsymbol{\Sigma}_1\mathbf{X}^{\perp}\,]^{\perp})\,\mathbf{Y}, \tag{3.15}$$

where $\mathbf{U} \in \mathbb{R}^{k\times n}$ *is arbitrary. Further*, *the following results hold.*

(a) ([21, p. 123]) $r[\,\mathbf{X},\ \boldsymbol{\Sigma}_1\mathbf{X}^{\perp}\,] = r[\,\mathbf{X},\ \boldsymbol{\Sigma}_1\,]$, $\mathscr{R}[\,\mathbf{X},\ \boldsymbol{\Sigma}_1\mathbf{X}^{\perp}\,] = \mathscr{R}[\,\mathbf{X},\ \boldsymbol{\Sigma}_1\,]$, *and* $\mathscr{R}(\mathbf{X}) \cap \mathscr{R}(\boldsymbol{\Sigma}_1\mathbf{X}^{\perp}) = \{\mathbf{0}\}$.

(b) $\mathbf{L} = \mathbf{P}_{\mathbf{K};\mathbf{J};\mathbf{X};\boldsymbol{\Sigma}_1}$ *is unique if and only if* $r[\,\mathbf{X},\ \boldsymbol{\Sigma}_1\,] = n$.

(c) $\mathrm{BLUP}_{\mathscr{M}}(\mathbf{R})$ *is unique if and only if* $\mathscr{R}(\mathbf{Y}) \subseteq \mathscr{R}[\,\mathbf{X},\ \boldsymbol{\Sigma}_1\,]$.

(d) The expectation, *the dispersion matrices of* $\overrightarrow{\mathrm{BLUP}_{\mathscr{M}}(\mathbf{R})}$ *and* $\overrightarrow{\mathbf{R}} - \overrightarrow{\mathrm{BLUP}_{\mathscr{M}}(\mathbf{R})}$ *and covariance matrix between* $\overrightarrow{\mathrm{BLUP}_{\mathscr{M}}(\mathbf{R})}$ *and* $\overrightarrow{\mathbf{R}}$ *are unique*, *and they are given by*

$$\mathrm{E}(\mathrm{BLUP}_{\mathscr{M}}(\mathbf{R}) - \mathbf{R}) = \mathbf{0}, \tag{3.16}$$

$$\begin{aligned}&\mathrm{Cov}(\overrightarrow{\mathrm{BLUP}_{\mathscr{M}}(\mathbf{R})})\\ &\ = \boldsymbol{\Sigma}_2 \otimes ([\,\mathbf{K},\ \mathbf{J}\boldsymbol{\Sigma}_1\mathbf{X}^{\perp}\,][\,\mathbf{X},\ \boldsymbol{\Sigma}_1\mathbf{X}^{\perp}\,]^{+})\boldsymbol{\Sigma}_1([\,\mathbf{K},\ \mathbf{J}\boldsymbol{\Sigma}_1\mathbf{X}^{\perp}\,[\,\mathbf{X},\ \boldsymbol{\Sigma}_1\mathbf{X}^{\perp}\,]^{+})',\end{aligned} \tag{3.17}$$

$$\mathrm{Cov}\{\overrightarrow{\mathrm{BLUP}_{\mathscr{M}}(\mathbf{R})},\ \overrightarrow{\mathbf{R}}\} = \boldsymbol{\Sigma}_2 \otimes [\,\mathbf{K},\ \mathbf{J}\boldsymbol{\Sigma}_1\mathbf{X}^{\perp}\,][\,\mathbf{X},\ \boldsymbol{\Sigma}_1\mathbf{X}^{\perp}\,]^{+}\boldsymbol{\Sigma}_1\mathbf{J}', \tag{3.18}$$

$$\begin{aligned}&\mathrm{Cov}(\overrightarrow{\mathbf{R}}) - \mathrm{Cov}(\overrightarrow{\mathrm{BLUP}_{\mathscr{M}}(\mathbf{R})}) = \boldsymbol{\Sigma}_2 \otimes \mathbf{J}\boldsymbol{\Sigma}_1\mathbf{J}'\\ &\quad - \boldsymbol{\Sigma}_2 \otimes ([\,\mathbf{K},\ \mathbf{J}\boldsymbol{\Sigma}_1\mathbf{X}^{\perp}\,][\,\mathbf{X},\ \boldsymbol{\Sigma}_1\mathbf{X}^{\perp}\,]^{+})\boldsymbol{\Sigma}_1([\,\mathbf{K},\ \mathbf{J}\boldsymbol{\Sigma}_1\mathbf{X}^{\perp}\,][\,\mathbf{X},\ \boldsymbol{\Sigma}_1\mathbf{X}^{\perp}\,]^{+})',\end{aligned} \tag{3.19}$$

$$\begin{aligned}\mathrm{Cov}(\overrightarrow{\mathbf{R}} - \overrightarrow{\mathrm{BLUP}_{\mathscr{M}}(\mathbf{R})}) &= \boldsymbol{\Sigma}_2 \otimes ([\,\mathbf{K},\ \mathbf{J}\boldsymbol{\Sigma}_1\mathbf{X}^{\perp}\,][\,\mathbf{X},\ \boldsymbol{\Sigma}_1\mathbf{X}^{\perp}\,]^{+} - \mathbf{J})\boldsymbol{\Sigma}_1\\ &\quad \times ([\,\mathbf{K},\ \mathbf{J}\boldsymbol{\Sigma}_1\mathbf{X}^{\perp}\,][\,\mathbf{X},\ \boldsymbol{\Sigma}_1\mathbf{X}^{\perp}\,]^{+} - \mathbf{J})'.\end{aligned} \tag{3.20}$$

(e) $\mathrm{BLUP}_{\mathscr{M}}(\mathbf{R})$, $\mathrm{BLUE}_{\mathscr{M}}(\mathbf{K}\Theta)$ *and* $\mathrm{BLUP}_{\mathscr{M}}(\mathbf{J}\boldsymbol{\Omega})$ *satisfy*

$$\mathrm{BLUP}_{\mathscr{M}}(\mathbf{R}) = \mathrm{BLUE}_{\mathscr{M}}(\mathbf{K}\Theta) + \mathrm{BLUP}_{\mathscr{M}}(\mathbf{J}\boldsymbol{\Omega}), \tag{3.21}$$

$$\mathrm{Cov}\{\overrightarrow{\mathrm{BLUE}_{\mathscr{M}}(\mathbf{K}\Theta)},\ \overrightarrow{\mathrm{BLUP}_{\mathscr{M}}(\mathbf{J}\boldsymbol{\Omega})}\} = \mathbf{0}, \tag{3.22}$$

$$\mathrm{Cov}(\overrightarrow{\mathrm{BLUP}_{\mathscr{M}}(\mathbf{R})}) = \mathrm{Cov}(\overrightarrow{\mathrm{BLUE}_{\mathscr{M}}(\mathbf{K}\Theta)}) + \mathrm{Cov}(\overrightarrow{\mathrm{BLUP}_{\mathscr{M}}(\mathbf{J}\boldsymbol{\Omega})}). \tag{3.23}$$

(f) $\mathrm{BLUP}_{\mathscr{M}}(\mathbf{T}\mathbf{R}) = \mathbf{T}\mathrm{BLUP}_{\mathscr{M}}(\mathbf{R})$ *holds for any matrix* $\mathbf{T} \in \mathbb{R}^{t\times k}$.

Concerning the BLUE of the mean matrix $\mathbf{X}\Theta$ and the BLUP of the error matrix $\boldsymbol{\Omega}$ in (1.1), we have the following result.

Corollary 3.1. *The mean matrix* $\mathbf{X}\Theta$ *in* (1.1) *is always estimable*, *and the following results hold.*

(a) *The general expression of* $\mathrm{BLUE}_{\mathscr{M}}(\mathbf{X}\Theta)$ *can be written as*

$$\mathrm{BLUE}_{\mathscr{M}}(\mathbf{X}\Theta) = ([\,\mathbf{X},\ \mathbf{0}\,][\,\mathbf{X},\ \mathbf{\Sigma}_1\mathbf{X}^{\perp}\,]^{+} + \mathbf{U}[\,\mathbf{X},\ \mathbf{\Sigma}_1\mathbf{X}^{\perp}\,]^{\perp})\mathbf{Y} \tag{3.24}$$

with

$$\mathrm{E}(\mathrm{BLUE}_{\mathscr{M}}(\mathbf{X}\Theta)) = \mathbf{X}\Theta, \tag{3.25}$$

$$\mathrm{Cov}(\overrightarrow{\mathrm{BLUE}_{\mathscr{M}}(\mathbf{X}\Theta)}) = \mathbf{\Sigma}_2 \otimes ([\,\mathbf{X},\ \mathbf{0}\,][\,\mathbf{X},\ \mathbf{\Sigma}_1\mathbf{X}^{\perp}\,]^{+})\mathbf{\Sigma}_1([\,\mathbf{X},\ \mathbf{0}\,][\,\mathbf{X},\ \mathbf{\Sigma}_1\mathbf{X}^{\perp}\,]^{+})', \tag{3.26}$$

where $\mathbf{U} \in \mathbb{R}^{n\times n}$ *is arbitrary.*

(b) *The general expression of* $\mathrm{BLUP}_{\mathscr{M}}(\mathbf{\Omega})$ *can be written as*

$$\begin{aligned}\mathrm{BLUP}_{\mathscr{M}}(\mathbf{\Omega}) &= ([\,\mathbf{0},\ \mathbf{\Sigma}_1\mathbf{X}^{\perp}\,][\,\mathbf{X},\ \mathbf{\Sigma}_1\mathbf{X}^{\perp}\,]^{+} + \mathbf{U}[\,\mathbf{X},\ \mathbf{\Sigma}_1\mathbf{X}^{\perp}\,]^{\perp})\mathbf{Y}\\ &= (\mathbf{\Sigma}_1(\mathbf{X}^{\perp}\mathbf{\Sigma}_1\mathbf{X}^{\perp})^{+} + \mathbf{U}[\,\mathbf{X},\ \mathbf{\Sigma}_1\mathbf{X}^{\perp}\,]^{\perp})\mathbf{Y}\end{aligned} \tag{3.27}$$

with

$$\mathrm{Cov}\{\overrightarrow{\mathrm{BLUP}_{\mathscr{M}}(\mathbf{\Omega})},\ \overrightarrow{\mathbf{\Omega}}\} = \mathrm{Cov}(\overrightarrow{\mathrm{BLUP}_{\mathscr{M}}(\mathbf{\Omega})}) = \mathbf{\Sigma}_2 \otimes \mathbf{\Sigma}_1(\mathbf{X}^{\perp}\mathbf{\Sigma}_1\mathbf{X}^{\perp})^{+}\mathbf{\Sigma}_1, \tag{3.28}$$

$$\mathrm{Cov}(\overrightarrow{\mathbf{\Omega}} - \overrightarrow{\mathrm{BLUP}_{\mathscr{M}}(\mathbf{\Omega})}) = \mathrm{Cov}(\overrightarrow{\mathbf{\Omega}}) - \mathrm{Cov}(\overrightarrow{\mathrm{BLUP}_{\mathscr{M}}(\mathbf{\Omega})}) = \mathbf{\Sigma}_2 \otimes \mathbf{\Sigma}_1 - \mathbf{\Sigma}_2 \otimes \mathbf{\Sigma}_1(\mathbf{X}^{\perp}\mathbf{\Sigma}_1\mathbf{X}^{\perp})^{+}\mathbf{\Sigma}_1, \tag{3.29}$$

where $\mathbf{U} \in \mathbb{R}^{n\times n}$ *is arbitrary.*

(c) *The three matrices* $\mathbf{Y}$, $\mathrm{BLUE}_{\mathscr{M}}(\mathbf{X}\Theta)$, *and* $\mathrm{BLUP}_{\mathscr{M}}(\mathbf{\Omega})$ *satisfy*

$$\mathbf{Y} = \mathrm{BLUE}_{\mathscr{M}}(\mathbf{X}\Theta) + \mathrm{BLUP}_{\mathscr{M}}(\mathbf{\Omega}), \tag{3.30}$$

$$\mathrm{Cov}\{\overrightarrow{\mathrm{BLUE}_{\mathscr{M}}(\mathbf{X}\Theta)},\ \overrightarrow{\mathrm{BLUP}_{\mathscr{M}}(\mathbf{\Omega})}\} = \mathbf{0}, \tag{3.31}$$

$$\mathrm{Cov}(\overrightarrow{\mathbf{Y}}) = \mathrm{Cov}(\overrightarrow{\mathrm{BLUE}_{\mathscr{M}}(\mathbf{X}\Theta)}) + \mathrm{Cov}(\overrightarrow{\mathrm{BLUP}_{\mathscr{M}}(\mathbf{\Omega})}). \tag{3.32}$$

Referring to Theorem 3.2, we obtain the BLUP of $\mathbf{R}_0$ in (1.8) as follows.

Theorem 3.3. *Assume that* $\mathbf{R}_0$ *is as given in* (1.8). *Then*, *the matrix equation*

$$\mathbf{L}_0[\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,] = [\,\mathbf{K}_0,\ \mathbf{J}_0\mathbf{V}_1\mathbf{X}_0^{\perp}\,] \tag{3.33}$$

is solvable for $\mathbf{L}_0$ *if and only if* $\mathscr{R}(\mathbf{K}_0') \subseteq \mathscr{R}(\mathbf{X}_0')$. *In this case*, *the general solution of the equation*, *denoted by* $\mathbf{L}_0 = \mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}$, *and the corresponding BLUP of* $\mathbf{R}_0$ *under the misspecified model in* (1.2) *are given by*

$$\mathrm{BLUP}_{\mathscr{N}}(\mathbf{R}_0) = \mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}\mathbf{Y} = ([\,\mathbf{K}_0,\ \mathbf{J}_0\mathbf{V}_1\mathbf{X}_0^{\perp}\,][\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{+} + \mathbf{U}_0[\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{\perp})\,\mathbf{Y}, \tag{3.34}$$

where $\mathbf{U}_0 \in \mathbb{R}^{k\times n}$ *is arbitrary. In particular,*

$$\mathrm{BLUE}_{\mathscr{N}}(\mathbf{K}_0\Theta_0) = ([\,\mathbf{K}_0,\ \mathbf{0}\,][\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{+} + \mathbf{U}_0[\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{\perp})\,\mathbf{Y}, \tag{3.35}$$

$$\mathrm{BLUP}_{\mathscr{N}}(\mathbf{J}_0\mathbf{\Omega}_0) = ([\,\mathbf{0},\ \mathbf{J}_0\mathbf{V}_1\mathbf{X}_0^{\perp}\,][\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{+} + \mathbf{U}_0[\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{\perp})\,\mathbf{Y}, \tag{3.36}$$

where $\mathbf{U}_0 \in \mathbb{R}^{k\times n}$ *is arbitrary. Under the assumptions in* (1.1)*,*

$$\mathrm{E}(\mathrm{BLUP}_{\mathscr{N}}(\mathbf{R}_0)) = \mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}\mathbf{X}\Theta, \tag{3.37}$$

$$\mathrm{Cov}(\overrightarrow{\mathrm{BLUP}_{\mathscr{N}}(\mathbf{R}_0)}) = \mathbf{\Sigma}_2 \otimes (\mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}\mathbf{\Sigma}_1\mathbf{P}'_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}), \tag{3.38}$$

where both $\mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}\mathbf{X}$ *and* $\mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}\mathbf{\Sigma}_1$ *are not necessarily unique. Further, the following results hold*:

(a) $\mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}\mathbf{X}$ *is unique if and only if* $\mathscr{R}(\mathbf{X}) \subseteq \mathscr{R}[\mathbf{X}_0, \mathbf{V}_1]$.
(b) $\mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}\mathbf{\Sigma}_1$ *is unique if and only if* $\mathscr{R}(\mathbf{\Sigma}_1) \subseteq \mathscr{R}[\mathbf{X}_0, \mathbf{V}_1]$.
(c) $\mathrm{BLUP}_{\mathscr{N}}(\mathbf{R}_0)$ *is unique if and only if* $\mathscr{R}(\mathbf{Y}) \subseteq \mathscr{R}[\mathbf{X}_0, \mathbf{V}_1]$.

4. Comparison results

In this section, we address the following eight problems:

(I) $\{\mathbf{P}_{\mathbf{K};\mathbf{J};\mathbf{X};\mathbf{\Sigma}_1}\} \cap \{\mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}\} \neq \emptyset$, so that $\{\mathrm{BLUP}_{\mathscr{M}}(\mathbf{R})\} \cap \{\mathrm{BLUP}_{\mathscr{N}}(\mathbf{R}_0)\} \neq \emptyset$ holds definitely,
(II) $\{\mathbf{P}_{\mathbf{K};\mathbf{J};\mathbf{X};\mathbf{\Sigma}_1}\} \subseteq \{\mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}\}$, so that $\{\mathrm{BLUP}_{\mathscr{M}}(\mathbf{R})\} \subseteq \{\mathrm{BLUP}_{\mathscr{N}}(\mathbf{R}_0)\}$ holds definitely,
(III) $\{\mathbf{P}_{\mathbf{K};\mathbf{J};\mathbf{X};\mathbf{\Sigma}_1}\} \supseteq \{\mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}\}$, so that $\{\mathrm{BLUP}_{\mathscr{M}}(\mathbf{R})\} \supseteq \{\mathrm{BLUP}_{\mathscr{N}}(\mathbf{R}_0)\}$ holds definitely,
(IV) $\{\mathbf{P}_{\mathbf{K};\mathbf{J};\mathbf{X};\mathbf{\Sigma}_1}\} = \{\mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}\}$, so that $\{\mathrm{BLUP}_{\mathscr{M}}(\mathbf{R})\} = \{\mathrm{BLUP}_{\mathscr{N}}(\mathbf{R}_0)\}$ holds definitely,
(V) $\{\mathrm{BLUP}_{\mathscr{M}}(\mathbf{R})\} \cap \{\mathrm{BLUP}_{\mathscr{N}}(\mathbf{R}_0)\} \neq \emptyset$ holds with probability 1,
(VI) $\{\mathrm{BLUP}_{\mathscr{M}}(\mathbf{R})\} \subseteq \{\mathrm{BLUP}_{\mathscr{N}}(\mathbf{R}_0)\}$ holds with probability 1,
(VII) $\{\mathrm{BLUP}_{\mathscr{M}}(\mathbf{R})\} \supseteq \{\mathrm{BLUP}_{\mathscr{N}}(\mathbf{R}_0)\}$ holds with probability 1,
(VIII) $\{\mathrm{BLUP}_{\mathscr{M}}(\mathbf{R})\} = \{\mathrm{BLUP}_{\mathscr{N}}(\mathbf{R}_0)\}$ holds with probability 1,

for the BLUPs defined and obtained in the preceding section.

In order to obtain satisfactory conclusions about the above BLUPs' equalities problems, we first present three manifest rules for delineating equalities of different linear statistics under multivariate linear models, and we then go on to describe some enabling methods to establish equalities between two linear statistics. Assume that two linear statistics $\mathbf{G}_1\mathbf{Y}$ and $\mathbf{G}_2\mathbf{Y}$ are given under (1.1). When establishing equalities between two linear statistics, the following three cases should be addressed for the purpose of delineating equalities of estimators formulated above.

Definition 4.1. Let $\mathbf{Y}$ be as given in (1.1), and let $\mathbf{G}_1,\ \mathbf{G}_2 \in \mathbb{R}^{k\times n}$.

(I) The equality $\mathbf{G}_1\mathbf{Y} = \mathbf{G}_2\mathbf{Y}$ is said to hold definitely if $\mathbf{G}_1 = \mathbf{G}_2$.
(II) The equality $\mathbf{G}_1\mathbf{Y} = \mathbf{G}_2\mathbf{Y}$ is said to hold with probability 1 if both $\mathrm{E}(\mathbf{G}_1\mathbf{Y} - \mathbf{G}_2\mathbf{Y}) = \mathbf{0}$ and $\mathrm{Cov}((\mathbf{I}_m \otimes \mathbf{G}_1)\overrightarrow{\mathbf{Y}} - (\mathbf{I}_m \otimes \mathbf{G}_2)\overrightarrow{\mathbf{Y}}) = \mathbf{0}$.
(III) $\mathbf{G}_1\mathbf{Y}$ and $\mathbf{G}_2\mathbf{Y}$ are said to have the same expectation matrices and dispersion matrices if both $\mathrm{E}(\mathbf{G}_1\mathbf{Y}) = \mathrm{E}(\mathbf{G}_2\mathbf{Y})$ and $\mathrm{Cov}[(\mathbf{I}_m \otimes \mathbf{G}_1)\overrightarrow{\mathbf{Y}}] = \mathrm{Cov}((\mathbf{I}_m \otimes \mathbf{G}_2)\overrightarrow{\mathbf{Y}})$ hold.

These three types of equalities are not necessarily equivalent since they are defined from different criteria for purposes of comparison and contrast. These facts show that equalities of linear statistics under (1.1) can all be characterized by the corresponding linear and quadratic matrix equations. In particular, under the assumption in (1.1),

$$\mathrm{E}(\,\mathbf{G}_1\mathbf{Y} - \mathbf{G}_2\mathbf{Y}\,) = \mathbf{0} \ \text{ and } \ \mathrm{Cov}((\mathbf{I}_m \otimes \mathbf{G}_1)\overrightarrow{\mathbf{Y}} - (\mathbf{I}_m \otimes \mathbf{G}_2)\overrightarrow{\mathbf{Y}}) = \mathbf{0}$$

$$\Leftrightarrow \mathbf{G}_1\mathbf{X} - \mathbf{G}_2\mathbf{X} = \mathbf{0} \text{ and } [(\mathbf{I}_m \otimes \mathbf{G}_1) - (\mathbf{I}_m \otimes \mathbf{G}_2)](\mathbf{\Sigma}_2 \otimes \mathbf{\Sigma}_1)[(\mathbf{I}_m \otimes \mathbf{G}_1) - (\mathbf{I}_m \otimes \mathbf{G}_2)]' = \mathbf{0}$$
$$\Leftrightarrow \mathbf{G}_1\mathbf{X} = \mathbf{G}_2\mathbf{X} \text{ and } [(\mathbf{I}_m \otimes \mathbf{G}_1) - (\mathbf{I}_m \otimes \mathbf{G}_2)](\mathbf{\Sigma}_2 \otimes \mathbf{\Sigma}_1) = \mathbf{0}$$
$$\Leftrightarrow \mathbf{G}_1\mathbf{X} = \mathbf{G}_2\mathbf{X} \text{ and } \mathbf{\Sigma}_2 \otimes (\mathbf{G}_1 - \mathbf{G}_2)\mathbf{\Sigma}_1 = \mathbf{0}. \tag{4.1}$$

Because $\mathbf{\Sigma}_2$ is a non-zero matrix, the equality $(\mathbf{G}_1 - \mathbf{G}_2)\mathbf{\Sigma}_1 = \mathbf{0}$ holds. Combining the two equalities in (4.1), obtain

$$(\,\mathbf{G}_1 - \mathbf{G}_2\,)[\,\mathbf{X}, \mathbf{\Sigma}_1\,] = \mathbf{0}. \tag{4.2}$$

Applying Lemma 2.6 to it yields the following result.

Lemma 4.1. *Let* $\mathbf{Y}$ *be as given in* (1.1), *and let* $\mathbf{G}_1, \ \mathbf{G}_2 \in \mathbb{R}^{k\times n}$. *Then*,

$$\mathbf{G}_1\mathbf{Y} = \mathbf{G}_2\mathbf{Y} \textit{ holds with probability } 1$$
$$\Leftrightarrow (\,\mathbf{G}_1 - \mathbf{G}_2\,)[\,\mathbf{X}, \mathbf{\Sigma}_1\,] = \mathbf{0} \Leftrightarrow (\,\mathbf{G}_1 - \mathbf{G}_2\,)[\,\mathbf{X}, \mathbf{\Sigma}_1\mathbf{X}^{\perp}\,] = \mathbf{0}. \tag{4.3}$$

Further, let $\{\mathbf{G}_1\}$ *and* $\{\mathbf{G}_2\}$ *be two sets of matrices of the same size. Then, the following results hold.*

(a) $\{\mathbf{G}_1\mathbf{Y}\} \cap \{\mathbf{G}_2\mathbf{Y}\} \neq \emptyset$ *holds with probability* 1 *if and only if*

$$\min_{\mathbf{G}_1\in\{\mathbf{G}_1\},\,\mathbf{G}_2\in\{\mathbf{G}_2\}} r((\,\mathbf{G}_1 - \mathbf{G}_2\,)[\,\mathbf{X}, \mathbf{\Sigma}_1\mathbf{X}^{\perp}\,]) = 0. \tag{4.4}$$

(b) $\{\mathbf{G}_1\mathbf{Y}\} \subseteq \{\mathbf{G}_2\mathbf{Y}\}$ *holds with probability* 1 *if and only if*

$$\max_{\mathbf{G}_1\in\{\mathbf{G}_1\}}\ \min_{\mathbf{G}_2\in\{\mathbf{G}_2\}} r((\,\mathbf{G}_1 - \mathbf{G}_2\,)[\,\mathbf{X}, \mathbf{\Sigma}_1\mathbf{X}^{\perp}\,]) = 0. \tag{4.5}$$

Because the coefficient matrices $\mathbf{P}_{\mathbf{K};\mathbf{J};\mathbf{X};\mathbf{\Sigma}_1}$ and $\mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}$ in (3.13) and (3.34) are not necessarily unique, we use

$$\{\mathbf{P}_{\mathbf{K};\mathbf{J};\mathbf{X};\mathbf{\Sigma}_1}\}, \qquad \{\mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}\}, \tag{4.6}$$
$$\{\mathrm{BLUP}_{\mathscr{M}}(\mathbf{R})\} = \{\mathbf{P}_{\mathbf{K};\mathbf{J};\mathbf{X};\mathbf{\Sigma}_1}\mathbf{Y}\}, \quad \{\mathrm{BLUP}_{\mathscr{N}}(\mathbf{R}_0)\} = \{\mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}\mathbf{Y}\} \tag{4.7}$$

to denote the collections of all coefficient matrices and the corresponding BLUPs. Under the assumption that (1.1) is a true model, the predictor $\mathrm{BLUP}_{\mathscr{N}}(\mathbf{R}_0)$ in (3.34) is not necessarily unbiased for $\mathbf{R}$. Concerning the expectation of $\mathrm{BLUP}_{\mathscr{N}}(\mathbf{R}_0)$, we have the following result.

Theorem 4.1. *Let* $\mathrm{BLUP}_{\mathscr{M}}(\mathbf{R})$ *and* $\mathbf{P}_{\mathbf{K};\mathbf{J};\mathbf{X};\mathbf{\Sigma}_1}$ *be as given in* (3.13), *let* $\mathrm{BLUP}_{\mathscr{N}}(\mathbf{R}_0)$ *and* $\mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}$ *be as given in* (3.34), *and define*

$$\mathbf{M}_1 = \begin{bmatrix} \mathbf{X} & \mathbf{X}_0 & \mathbf{V}_1 \\ \mathbf{0} & \mathbf{0} & \mathbf{X}_0' \end{bmatrix} \ \textit{and} \ \mathbf{N}_1 = [\,\mathbf{K}, \mathbf{K}_0, \mathbf{J}_0\mathbf{V}_1\,].$$

Then, we have the following results:

(a) The following two statements are equivalent:

(i) There exists a $\mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}$ *such that* $\mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}\mathbf{X} = \mathbf{K}$.

(ii) $\mathscr{R}(\mathbf{N}_1') \subseteq \mathscr{R}(\mathbf{M}_1')$.

In this case, the general expression of $\mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}$ *of* $\mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}\mathbf{X} = \mathbf{K}$ *is*

$$\begin{aligned}\mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1} &= [\,\mathbf{K}_0,\ \mathbf{J}_0\mathbf{V}_1\mathbf{X}_0^{\perp}\,][\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{+}\\ &\quad + (\,\mathbf{K} - [\,\mathbf{K}_0,\ \mathbf{J}_0\mathbf{V}_1\mathbf{X}_0^{\perp}\,][\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{+}\mathbf{X}\,)([\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{\perp}\mathbf{X})^{+}[\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{\perp}\\ &\quad + \mathbf{W}(\,\mathbf{I}_n - ([\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{\perp}\mathbf{X})([\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{\perp}\mathbf{X})^{+}\,)[\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{\perp}, \qquad (4.8)\end{aligned}$$

where the matrix $\mathbf{W}$ *is arbitrary. Correspondingly,* $\mathrm{BLUP}_{\mathscr{N}}(\mathbf{R}_0) = \mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}\mathbf{Y}$ *satisfies* $\mathrm{E}(\mathrm{BLUP}_{\mathscr{N}}(\mathbf{R}_0) - \mathrm{BLUP}_{\mathscr{M}}(\mathbf{R})) = \mathbf{0}$, *namely,* $\mathrm{BLUP}_{\mathscr{N}}(\mathbf{R}_0)$ *and* $\mathrm{BLUP}_{\mathscr{M}}(\mathbf{R})$ *have the same expectation.*

(b) The following two statements are equivalent:

(i) All $\mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}$ *satisfy* $\mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}\mathbf{X} = \mathbf{K}$.

(ii) $\mathscr{R}(\mathbf{N}_1') \subseteq \mathscr{R}(\mathbf{M}_1')$ *and* $\mathscr{R}(\mathbf{X}) \subseteq \mathscr{R}[\,\mathbf{X}_0,\ \mathbf{V}_1\,]$.

Correspondingly, all $\mathrm{BLUP}_{\mathscr{N}}(\mathbf{R}_0) = \mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}\mathbf{Y}$ *satisfy* $\mathrm{E}(\mathrm{BLUP}_{\mathscr{N}}(\mathbf{R}_0) - \mathrm{BLUP}_{\mathscr{M}}(\mathbf{R})) = \mathbf{0}$.

Theorem 4.2. *Let* $\mathrm{BLUP}_{\mathscr{M}}(\mathbf{R})$ *and* $\mathbf{P}_{\mathbf{K};\mathbf{J};\mathbf{X};\boldsymbol{\Sigma}_1}$ *be as given in* (3.13), *let* $\mathrm{BLUP}_{\mathscr{N}}(\mathbf{R}_0)$ *and* $\mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}$ *be as given in* (3.34), *and define*

$$\mathbf{M}_2 = \begin{bmatrix} \mathbf{X} & \mathbf{X}_0 & \boldsymbol{\Sigma}_1 & \mathbf{V}_1 \\ \mathbf{0} & \mathbf{0} & \mathbf{X}' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X}_0' \end{bmatrix} \ \text{and}\ \ \mathbf{N}_2 = [\,\mathbf{K},\ \mathbf{K}_0,\ \mathbf{J}\boldsymbol{\Sigma}_1,\ \mathbf{J}_0\mathbf{V}_1\,]. \qquad (4.9)$$

Then, the following results hold.

(a) There exist $\mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}$ *and* $\mathbf{P}_{\mathbf{K};\mathbf{J};\mathbf{X};\boldsymbol{\Sigma}_1}$ *such that* $\mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1} = \mathbf{P}_{\mathbf{K};\mathbf{J};\mathbf{X};\boldsymbol{\Sigma}_1}$ *if and only if* $\mathscr{R}(\mathbf{M}_2') \supseteq \mathscr{R}(\mathbf{N}_2')$. *In this case,* $\{\mathrm{BLUP}_{\mathscr{M}}(\mathbf{R})\} \cap \{\mathrm{BLUP}_{\mathscr{N}}(\mathbf{R}_0)\} \neq \emptyset$ *holds definitely.*

(b) $\{\mathbf{P}_{\mathbf{K};\mathbf{J};\mathbf{X};\boldsymbol{\Sigma}_1}\} \supseteq \{\mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}\}$ *if and only if* $\mathscr{R}(\mathbf{M}_2') \supseteq \mathscr{R}(\mathbf{N}_2')$ *and* $\mathscr{R}[\,\mathbf{X},\ \boldsymbol{\Sigma}_1\,] \subseteq \mathscr{R}[\,\mathbf{X}_0,\ \mathbf{V}_1\,]$. *In this case,* $\{\mathrm{BLUP}_{\mathscr{M}}(\mathbf{R})\} \supseteq \{\mathrm{BLUP}_{\mathscr{N}}(\mathbf{R}_0)\}$ *holds definitely.*

(c) $\{\mathbf{P}_{\mathbf{K};\mathbf{J};\mathbf{X};\boldsymbol{\Sigma}_1}\} \subseteq \{\mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}\}$ *if and only if* $\mathscr{R}(\mathbf{M}_2') \supseteq \mathscr{R}(\mathbf{N}_2')$ *and* $\mathscr{R}[\,\mathbf{X},\ \boldsymbol{\Sigma}_1\,] \supseteq \mathscr{R}[\,\mathbf{X}_0,\ \mathbf{V}_1\,]$. *In this case,* $\{\mathrm{BLUP}_{\mathscr{M}}(\mathbf{R})\} \subseteq \{\mathrm{BLUP}_{\mathscr{N}}(\mathbf{R}_0)\}$ *holds definitely.*

(d) $\{\mathbf{P}_{\mathbf{K};\mathbf{J};\mathbf{X};\boldsymbol{\Sigma}_1}\} = \{\mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}\}$ *if and only if* $\mathscr{R}(\mathbf{M}_2') \supseteq \mathscr{R}(\mathbf{N}_2')$ *and* $\mathscr{R}[\,\mathbf{X},\ \boldsymbol{\Sigma}_1\,] = \mathscr{R}[\,\mathbf{X}_0,\ \mathbf{V}_1\,]$. *In this case,* $\{\mathrm{BLUP}_{\mathscr{M}}(\mathbf{R})\} = \{\mathrm{BLUP}_{\mathscr{N}}(\mathbf{R}_0)\}$ *holds definitely.*

Theorem 4.3. *Let* $\mathrm{BLUP}_{\mathscr{M}}(\mathbf{R})$ *and* $\mathbf{P}_{\mathbf{K};\mathbf{J};\mathbf{X};\boldsymbol{\Sigma}_1}$ *be as given in* (3.13), *let* $\mathrm{BLUP}_{\mathscr{N}}(\mathbf{R}_0)$ *and* $\mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}$ *be as given in* (3.34), *and let* $\mathbf{M}_2$ *and* $\mathbf{N}_2$ *be as given in* (4.9). *Then, the following three statements are equivalent:*

(a) $\{\mathrm{BLUP}_{\mathscr{M}}(\mathbf{R})\} \cap \{\mathrm{BLUP}_{\mathscr{N}}(\mathbf{R}_0)\} \neq \emptyset$ *holds with probability* 1.

(b) $\{\mathrm{BLUP}_{\mathscr{M}}(\mathbf{R})\} \subseteq \{\mathrm{BLUP}_{\mathscr{N}}(\mathbf{R}_0)\}$ *holds with probability* 1.

(c) $\mathscr{R}(\mathbf{M}_2') \supseteq \mathscr{R}(\mathbf{N}_2')$.

Combining Theorems 4.2 and 4.3, we obtain the following result.

Corollary 4.1. *Let* $\mathrm{BLUP}_{\mathscr{M}}(\mathbf{R})$ *and* $\mathbf{P}_{\mathbf{K};\mathbf{J};\mathbf{X};\boldsymbol{\Sigma}_1}$ *be as given in* (3.13), *let* $\mathrm{BLUP}_{\mathscr{N}}(\mathbf{R}_0)$ *and* $\mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}$ *be as given in* (3.34), *and let* $\mathbf{M}_2$ *and* $\mathbf{N}_2$ *be as given in* (4.9). *Then, the following five statements are equivalent:*

(a) There exist $\mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}$ *and* $\mathbf{P}_{\mathbf{K};\mathbf{J};\mathbf{X};\boldsymbol{\Sigma}_1}$ *such that* $\mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1} = \mathbf{P}_{\mathbf{K};\mathbf{J};\mathbf{X};\boldsymbol{\Sigma}_1}$.

(b) $\{\mathrm{BLUP}_{\mathscr{M}}(\mathbf{R})\} \cap \{\mathrm{BLUP}_{\mathscr{N}}(\mathbf{R}_0)\} \neq \emptyset$ *holds definitely.*
(c) $\{\mathrm{BLUP}_{\mathscr{M}}(\mathbf{R})\} \cap \{\mathrm{BLUP}_{\mathscr{N}}(\mathbf{R}_0)\} \neq \emptyset$ *holds with probability* 1.
(d) $\{\mathrm{BLUP}_{\mathscr{M}}(\mathbf{R})\} \subseteq \{\mathrm{BLUP}_{\mathscr{N}}(\mathbf{R}_0)\}$ *holds with probability* 1.
(e) $\mathscr{R}(\mathbf{M}_2') \supseteq \mathscr{R}(\mathbf{N}_2')$.

Theorem 4.4. *Let* $\mathrm{BLUP}_{\mathscr{M}}(\mathbf{R})$ *and* $\mathbf{P}_{\mathbf{K};\mathbf{J};\mathbf{X};\mathbf{\Sigma}_1}$ *be as given in* (3.13), *let* $\mathrm{BLUP}_{\mathscr{N}}(\mathbf{R}_0)$ *and* $\mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}$ *be as given in* (3.34), *and let* $\mathbf{M}_2$ *and* $\mathbf{N}_2$ *be as given in* (4.9). *Then, the following three statements are equivalent*:

(a) $\{\mathrm{BLUP}_{\mathscr{M}}(\mathbf{R})\} \supseteq \{\mathrm{BLUP}_{\mathscr{N}}(\mathbf{R}_0)\}$ *holds with probability* 1.
(b) $\{\mathrm{BLUP}_{\mathscr{M}}(\mathbf{R})\} = \{\mathrm{BLUP}_{\mathscr{N}}(\mathbf{R}_0)\}$ *holds with probability* 1.
(c) $\mathscr{R}(\mathbf{M}_2') \supseteq \mathscr{R}(\mathbf{N}_2')$ *and* $\mathscr{R}[\,\mathbf{X},\ \mathbf{\Sigma}_1\,] \subseteq \mathscr{R}[\,\mathbf{X}_0,\ \mathbf{V}_1\,]$.

Combining Theorems 4.2 and 4.4, we obtain the following result.

Corollary 4.2. *Let* $\mathrm{BLUP}_{\mathscr{M}}(\mathbf{R})$ *and* $\mathbf{P}_{\mathbf{K};\mathbf{J};\mathbf{X};\mathbf{\Sigma}_1}$ *let be as given in* (3.13), *let* $\mathrm{BLUP}_{\mathscr{N}}(\mathbf{R}_0)$ *and* $\mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}$ *be as given in* (3.34), *and let* $\mathbf{M}_2$ *and* $\mathbf{N}_2$ *be as given in* (4.9). *Then, the following four statements are equivalent*:

(a) $\{\mathrm{BLUP}_{\mathscr{M}}(\mathbf{R})\} \supseteq \{\mathrm{BLUP}_{\mathscr{N}}(\mathbf{R}_0)\}$ *holds definitely.*
(b) $\{\mathrm{BLUP}_{\mathscr{M}}(\mathbf{R})\} \supseteq \{\mathrm{BLUP}_{\mathscr{N}}(\mathbf{R}_0)\}$ *holds with probability* 1.
(c) $\{\mathrm{BLUP}_{\mathscr{M}}(\mathbf{R})\} = \{\mathrm{BLUP}_{\mathscr{N}}(\mathbf{R}_0)\}$ *holds with probability* 1.
(d) $\mathscr{R}(\mathbf{M}_2') \supseteq \mathscr{R}(\mathbf{N}_2')$ *and* $\mathscr{R}[\,\mathbf{X},\ \mathbf{\Sigma}_1\,] \subseteq \mathscr{R}[\,\mathbf{X}_0,\ \mathbf{V}_1\,]$.

5. Conclusions

The problems on misspecifications and comparisons of linear statistical models are certain specific subjects in the estimation and inference theory of regression models, which include a diversity of concrete issues for discrimination and consideration. As one such problem, we offered in the preceding sections an overview and analysis of the equivalence problems of BLUPs/BLUEs under a pair of true and misspecified MGLMs through the effective uses of various precise and analytical methods and techniques in linear algebra and matrix theory. It is not difficult to understand the resulting facts from mathematical and statistical aspects, and thereby we can take them as a group of theoretical contributions in the statistical inference under certain general model assumptions. This specific study also shows that there are many deep and connotative problems in the classic regression frameworks that we can put forward from theoretical and applied points of view and can reasonably solve them by various known and novel ideas, methods and techniques in different branches of mathematical theory. Specifically, the resulting facts once again illustrate the crucial role and influence of the matrix algebra in dealing with statistical inference problems with regard to parametric models.

Finally, we would like to point out that more intriguing and sophisticated formulas, equalities, and inequalities associated with predictions/estimations, as demonstrated in the preceding sections, can be derived with some efforts under multivariate linear models with various specified assumptions, which will help in building a more solid theoretical and methodological foundation in the framework of parametric regressions.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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Appendix

Proof of Theorem 2.6. It follows from Lemma 2.5 that $\mathscr{R}(\mathbf{A}) = \mathscr{R}(\mathbf{B})$ implies $\mathbf{AA}^{+}\mathbf{B} = \mathbf{B}$ and $\mathbf{BB}^{+}\mathbf{A} = \mathbf{A}$. Therefore,

$$\mathbf{XA} = \mathbf{0} \Rightarrow \mathbf{XAA}^{+}\mathbf{B} = \mathbf{0} \Rightarrow \mathbf{XB} = \mathbf{0},$$
$$\mathbf{XB} = \mathbf{0} \Rightarrow \mathbf{XBB}^{+}\mathbf{A} = \mathbf{0} \Rightarrow \mathbf{XA} = \mathbf{0},$$

as required for (2.11). □

Proof of Theorem 4.1. From (3.34), the equation $\mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}\mathbf{X} = \mathbf{K}$ can be written as

$$[\,\mathbf{K}_0,\ \mathbf{J}_0\mathbf{V}_1\mathbf{X}_0^{\perp}\,][\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{+}\mathbf{X} + \mathbf{U}_0[\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{\perp}\mathbf{X} = \mathbf{K}. \tag{A.1}$$

By Lemma 2.5, the equation is solvable for $\mathbf{U}_0$ if and only if

$$r\begin{bmatrix} \mathbf{K} - [\,\mathbf{K}_0,\ \mathbf{J}_0\mathbf{V}_1\mathbf{X}_0^{\perp}\,][\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{+}\mathbf{X} \\ [\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{\perp}\mathbf{X} \end{bmatrix} = r(\,[\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{\perp}\mathbf{X}\,). \tag{A.2}$$

It is necessary to simplify the rank equality by the formulas in Section 2, and there will be reasonable and detailed calculation steps needed to remove the generalized inverses on both sides of (A.2). We proceed to this goal by applying (2.3), (2.4) and then simplifying by elementary block matrix operations to both sides of (A.2):

$$\begin{aligned}
& r\begin{bmatrix} \mathbf{K} - [\,\mathbf{K}_0,\ \mathbf{J}_0\mathbf{V}_1\mathbf{X}_0^{\perp}\,][\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{+}\mathbf{X} \\ [\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{\perp}\mathbf{X} \end{bmatrix} \\
&= r\begin{bmatrix} \mathbf{K} - [\,\mathbf{K}_0,\ \mathbf{J}_0\mathbf{V}_1\mathbf{X}_0^{\perp}\,][\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{+}\mathbf{X} & \mathbf{0} \\ \mathbf{X} & [\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,] \end{bmatrix} - r[\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,] \\
&= r\begin{bmatrix} \mathbf{K} - [\,\mathbf{K}_0,\ \mathbf{J}_0\mathbf{V}_1\mathbf{X}_0^{\perp}\,][\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{+}\mathbf{X} & \mathbf{0} \\ \mathbf{X} & [\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,] \end{bmatrix} - r[\,\mathbf{X}_0,\ \mathbf{V}_1\,] \\
&= r\begin{bmatrix} \mathbf{K} & [\,\mathbf{K}_0,\ \mathbf{J}_0\mathbf{V}_1\mathbf{X}_0^{\perp}\,] \\ \mathbf{X} & [\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,] \end{bmatrix} - r[\,\mathbf{X}_0,\ \mathbf{V}_1\,] \\
&= r\begin{bmatrix} \mathbf{X} & \mathbf{X}_0 & \mathbf{V}_1 \\ \mathbf{0} & \mathbf{0} & \mathbf{X}_0' \\ \mathbf{K} & \mathbf{K}_0 & \mathbf{J}_0\mathbf{V}_1 \end{bmatrix} - r(\mathbf{X}_0) - r[\,\mathbf{X}_0,\ \mathbf{V}_1\,]
\end{aligned}$$

$$= r\begin{bmatrix}\mathbf{M}_1\\ \mathbf{N}_1\end{bmatrix} - r(\mathbf{X}_0) - r[\,\mathbf{X}_0,\ \mathbf{V}_1\,],$$

and

$$r(\,[\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{\perp}\mathbf{X}\,) = r[\,\mathbf{X},\ \mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,] - r[\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,] = r[\,\mathbf{X},\ \mathbf{X}_0,\ \mathbf{V}_1\,] - r[\,\mathbf{X}_0,\ \mathbf{V}_1\,].$$

Substituting the rank equalities into (A.2) and then simplifying, we obtain $r\begin{bmatrix}\mathbf{M}_1\\ \mathbf{N}_1\end{bmatrix} = r(\mathbf{M}_1)$, that is, $\mathscr{R}(\mathbf{N}_1') \subseteq \mathscr{R}(\mathbf{M}_1')$ holds by Lemma 2.2(b), thus establishing the equivalence of (i) and (ii) in (a). In this case, the general solution of (A.1) by Lemma 2.5 is

$$\begin{aligned}\mathbf{U}_0 = {} & (\mathbf{K} - [\,\mathbf{K}_0,\ \mathbf{J}_0\mathbf{V}_1\mathbf{X}_0^{\perp}\,][\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{+}\mathbf{X})\,([\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{\perp}\mathbf{X})^{+}\\ & + \mathbf{W}[\,\mathbf{I}_n - ([\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{\perp}\mathbf{X})([\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{\perp}\mathbf{X})^{+}\,],\end{aligned} \tag{A.3}$$

where $\mathbf{W}$ is arbitrary. Substitution of (A.3) into $\mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}$ in (3.34) gives (4.8).

Equation (A.1) holds for all $\mathbf{U}_0$ if and only if $\begin{bmatrix}\mathbf{K} - [\,\mathbf{K}_0,\ \mathbf{J}_0\mathbf{V}_1\mathbf{X}_0^{\perp}\,][\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{+}\mathbf{X}\\ [\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{\perp}\mathbf{X}\end{bmatrix} = \mathbf{0}$, that is,

$$r\begin{bmatrix}\mathbf{M}_1\\ \mathbf{N}_1\end{bmatrix} = r[\,\mathbf{X}_0,\ \mathbf{V}_1\,] + r(\mathbf{X}_0). \tag{A.4}$$

Also note from (2.3) and (2.6) that

$$r(\mathbf{M}_1) = r[\mathbf{X},\ \mathbf{X}_0,\ \mathbf{V}_1] + r(\mathbf{X}_0) \geqslant r[\mathbf{X}_0,\ \mathbf{V}_1] + r(\mathbf{X}_0), \tag{A.5}$$

$$r\begin{bmatrix}\mathbf{M}_1\\ \mathbf{N}_1\end{bmatrix} \geqslant r(\mathbf{M}_1) \geqslant r[\,\mathbf{X}_0,\ \mathbf{V}_1\,] + r(\mathbf{X}_0). \tag{A.6}$$

Combining (A.4) and (A.6) yields

$$r\begin{bmatrix}\mathbf{M}_1\\ \mathbf{N}_1\end{bmatrix} = r(\mathbf{M}_1) = r[\,\mathbf{X}_0,\ \mathbf{V}_1\,] + r(\mathbf{X}_0),$$

or equivalently, $\mathscr{R}(\mathbf{N}_1') \subseteq \mathscr{R}(\mathbf{M}_1')$ and $\mathscr{R}(\mathbf{X}) \subseteq \mathscr{R}[\,\mathbf{X}_0,\ \mathbf{V}_1\,]$ hold, thus establishing the equivalence of (i) and (ii) in (b). □

Proof of Theorem 4.2. From (3.13) and (3.34), the general expression of the difference $\mathbf{P}_{\mathbf{K};\mathbf{J};\mathbf{X};\mathbf{\Sigma}_1} - \mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}$ can be written as

$$\mathbf{P}_{\mathbf{K};\mathbf{J};\mathbf{X};\mathbf{\Sigma}_1} - \mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1} = \mathbf{G} + \mathbf{U}[\,\mathbf{X},\ \mathbf{\Sigma}_1\mathbf{X}^{\perp}\,]^{\perp} - \mathbf{U}_0[\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{\perp}, \tag{A.7}$$

where $\mathbf{G} = [\,\mathbf{K},\ \mathbf{J}\mathbf{\Sigma}_1\mathbf{X}^{\perp}\,][\,\mathbf{X},\ \mathbf{\Sigma}_1\mathbf{X}^{\perp}\,]^{+} - [\,\mathbf{K}_0,\ \mathbf{J}_0\mathbf{V}_1\mathbf{X}_0^{\perp}\,][\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{+}$, and $\mathbf{U},\ \mathbf{U}_0 \in \mathbb{R}^{k\times n}$ are arbitrary. Applying (2.9) to (A.7), we obtain

$$\begin{aligned}\min_{\mathbf{P}_{\mathbf{K};\mathbf{J};\mathbf{X};\mathbf{\Sigma}_1},\,\mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}} r(\,\mathbf{P}_{\mathbf{K};\mathbf{J};\mathbf{X};\mathbf{\Sigma}_1} - \mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}\,) &= \min_{\mathbf{U},\,\mathbf{U}_0} r(\,\mathbf{G} + \mathbf{U}[\,\mathbf{X},\ \mathbf{\Sigma}_1\mathbf{X}^{\perp}\,]^{\perp} - \mathbf{U}_0[\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{\perp}\,)\\ &= r\begin{bmatrix}\mathbf{G}\\ [\,\mathbf{X},\ \mathbf{\Sigma}_1\mathbf{X}^{\perp}\,]^{\perp}\\ [\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{\perp}\end{bmatrix} - r\begin{bmatrix}[\,\mathbf{X},\ \mathbf{\Sigma}_1\mathbf{X}^{\perp}\,]^{\perp}\\ [\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{\perp}\end{bmatrix}.\end{aligned} \tag{A.8}$$

It is easy to show by (2.3), (2.4) and elementary block matrix operations that

$$
\begin{aligned}
& r\begin{bmatrix} \mathbf{G} \\ [\,\mathbf{X},\ \boldsymbol{\Sigma}_1\mathbf{X}^{\perp}\,]^{\perp} \\ [\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{\perp} \end{bmatrix} \\
=& r\begin{bmatrix} [\,\mathbf{K},\ \mathbf{J}\boldsymbol{\Sigma}_1\mathbf{X}^{\perp}\,][\,\mathbf{X},\ \boldsymbol{\Sigma}_1\mathbf{X}^{\perp}\,]^{+} - [\,\mathbf{K}_0,\ \mathbf{J}_0\mathbf{V}_1\mathbf{X}_0^{\perp}\,][\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{+} & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_n & [\,\mathbf{X},\ \boldsymbol{\Sigma}_1\mathbf{X}^{\perp}\,] & \mathbf{0} \\ \mathbf{I}_n & \mathbf{0} & [\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,] \end{bmatrix} \\
& -\ r[\,\mathbf{X},\ \boldsymbol{\Sigma}_1\mathbf{X}^{\perp}\,] - r[\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,] \\
=& r\begin{bmatrix} \mathbf{0} & -[\,\mathbf{K},\ \mathbf{J}\boldsymbol{\Sigma}_1\mathbf{X}^{\perp}\,] & [\,\mathbf{K}_0,\ \mathbf{J}_0\mathbf{V}_1\mathbf{X}_0^{\perp}\,] \\ \mathbf{I}_n & [\,\mathbf{X},\ \boldsymbol{\Sigma}_1\mathbf{X}^{\perp}\,] & \mathbf{0} \\ \mathbf{I}_n & \mathbf{0} & [\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,] \end{bmatrix} - r[\,\mathbf{X},\ \boldsymbol{\Sigma}_1\,] - r[\,\mathbf{X}_0,\ \mathbf{V}_1\,] \\
=& r\begin{bmatrix} \mathbf{0} & -[\,\mathbf{K},\ \mathbf{J}\boldsymbol{\Sigma}_1\mathbf{X}^{\perp}\,] & [\,\mathbf{K}_0,\ \mathbf{J}_0\mathbf{V}_1\mathbf{X}_0^{\perp}\,] \\ \mathbf{0} & [\,\mathbf{X},\ \boldsymbol{\Sigma}_1\mathbf{X}^{\perp}\,] & -[\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,] \\ \mathbf{I}_n & \mathbf{0} & \mathbf{0} \end{bmatrix} - r[\,\mathbf{X},\ \boldsymbol{\Sigma}_1\,] - r[\,\mathbf{X}_0,\ \mathbf{V}_1\,] \\
=& r\begin{bmatrix} [\,\mathbf{K},\ \mathbf{J}\boldsymbol{\Sigma}_1\mathbf{X}^{\perp}\,] & [\,\mathbf{K}_0,\ \mathbf{J}_0\mathbf{V}_1\mathbf{X}_0^{\perp}\,] \\ [\,\mathbf{X},\ \boldsymbol{\Sigma}_1\mathbf{X}^{\perp}\,] & [\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,] \end{bmatrix} + n - r[\,\mathbf{X},\ \boldsymbol{\Sigma}_1\,] - r[\,\mathbf{X}_0,\ \mathbf{V}_1\,] \\
=& r\begin{bmatrix} \mathbf{X} & \mathbf{X}_0 & \boldsymbol{\Sigma}_1 & \mathbf{V}_1 \\ \mathbf{0} & \mathbf{0} & \mathbf{X}' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X}_0' \\ \mathbf{K} & \mathbf{K}_0 & \mathbf{J}\boldsymbol{\Sigma}_1 & \mathbf{J}_0\mathbf{V}_1 \end{bmatrix} + n - r(\mathbf{X}) - r(\mathbf{X}_0) - r[\,\mathbf{X},\ \boldsymbol{\Sigma}_1\,] - r[\,\mathbf{X}_0,\ \mathbf{V}_1\,] \\
=& r\begin{bmatrix} \mathbf{M}_2 \\ \mathbf{N}_2 \end{bmatrix} + n - r(\mathbf{X}) - r(\mathbf{X}_0) - r[\,\mathbf{X},\ \boldsymbol{\Sigma}_1\,] - r[\,\mathbf{X}_0,\ \mathbf{V}_1\,],
\end{aligned} \tag{A.9}
$$

and

$$
\begin{aligned}
r\begin{bmatrix} [\,\mathbf{X},\ \boldsymbol{\Sigma}_1\mathbf{X}^{\perp}\,]^{\perp} \\ [\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{\perp} \end{bmatrix} &= r\begin{bmatrix} \mathbf{I}_n & [\,\mathbf{X},\ \boldsymbol{\Sigma}_1\mathbf{X}^{\perp}\,] & \mathbf{0} \\ \mathbf{I}_n & \mathbf{0} & [\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,] \end{bmatrix} - r[\,\mathbf{X},\ \boldsymbol{\Sigma}_1\,] - r[\,\mathbf{X}_0,\ \mathbf{V}_1\,] \\
&= r\begin{bmatrix} \mathbf{0} & [\,\mathbf{X},\ \boldsymbol{\Sigma}_1\mathbf{X}^{\perp}\,] & -[\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,] \\ \mathbf{I}_n & \mathbf{0} & \mathbf{0} \end{bmatrix} - r[\,\mathbf{X},\ \boldsymbol{\Sigma}_1\,] - r[\,\mathbf{X}_0,\ \mathbf{V}_1\,] \\
&= r\begin{bmatrix} \mathbf{X} & \mathbf{X}_0 & \boldsymbol{\Sigma}_1 & \mathbf{V}_1 \\ \mathbf{0} & \mathbf{0} & \mathbf{X}' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X}_0' \end{bmatrix} + n - r(\mathbf{X}) - r(\mathbf{X}_0) - r[\,\mathbf{X},\ \boldsymbol{\Sigma}_1\,] - r[\,\mathbf{X}_0,\ \mathbf{V}_1\,] \\
&= r(\mathbf{M}_2) + n - r(\mathbf{X}) - r(\mathbf{X}_0) - r[\,\mathbf{X},\ \boldsymbol{\Sigma}_1\,] - r[\,\mathbf{X}_0,\ \mathbf{V}_1\,].
\end{aligned} \tag{A.10}
$$

Substitution of (A.9) and (A.10) into (A.8) gives

$$
\min_{\mathbf{P}_{\mathbf{K};\mathbf{J};\mathbf{X};\boldsymbol{\Sigma}_1},\,\mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}} r(\,\mathbf{P}_{\mathbf{K};\mathbf{J};\mathbf{X};\boldsymbol{\Sigma}_1} - \mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}\,) = r\begin{bmatrix} \mathbf{M}_2 \\ \mathbf{N}_2 \end{bmatrix} - r(\mathbf{M}_2). \tag{A.11}
$$

Setting the right-hand side of (A.11) equal to zero and applying Lemma 2.2(b) yields the first statement in (a). Applying (2.9) to (A.7) gives rise to

$$
\min_{\mathbf{P}_{\mathbf{K};\mathbf{J};\mathbf{X};\boldsymbol{\Sigma}_1}} r(\,\mathbf{P}_{\mathbf{K};\mathbf{J};\mathbf{X};\boldsymbol{\Sigma}_1} - \mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}\,) = \min_{\mathbf{U}} r(\,\mathbf{G} + \mathbf{U}[\,\mathbf{X},\ \boldsymbol{\Sigma}_1\mathbf{X}^{\perp}\,]^{\perp} - \mathbf{U}_0[\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{\perp}\,)
$$

$$= r\begin{bmatrix} \mathbf{G} - \mathbf{U}_0[\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{\perp} \\ [\,\mathbf{X},\ \mathbf{\Sigma}_1\mathbf{X}^{\perp}\,]^{\perp} \end{bmatrix} - r([\,\mathbf{X},\ \mathbf{\Sigma}_1\mathbf{X}^{\perp}\,]^{\perp})$$

$$= r\begin{bmatrix} \mathbf{G} - \mathbf{U}_0[\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{\perp} \\ [\,\mathbf{X},\ \mathbf{\Sigma}_1\mathbf{X}^{\perp}\,]^{\perp} \end{bmatrix} + r[\,\mathbf{X},\ \mathbf{\Sigma}_1\,] - n. \tag{A.12}$$

Further by (2.10),

$$\begin{aligned}
&\max_{\mathbf{U}_0} r\begin{bmatrix} \mathbf{G} - \mathbf{U}_0[\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{\perp} \\ [\,\mathbf{X},\ \mathbf{\Sigma}_1\mathbf{X}^{\perp}\,]^{\perp} \end{bmatrix} \\
&= \max_{\mathbf{U}_0} r\left(\begin{bmatrix} \mathbf{G} \\ [\,\mathbf{X},\ \mathbf{\Sigma}_1\mathbf{X}^{\perp}\,]^{\perp} \end{bmatrix} - \begin{bmatrix} \mathbf{I}_k \\ \mathbf{0} \end{bmatrix}\mathbf{U}_0[\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{\perp}\right) \\
&= \min\left\{ r\begin{bmatrix} \mathbf{G} \\ [\,\mathbf{X},\ \mathbf{\Sigma}_1\mathbf{X}^{\perp}\,]^{\perp} \\ [\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{\perp} \end{bmatrix},\ r\begin{bmatrix} \mathbf{G} & \mathbf{I}_k \\ [\,\mathbf{X},\ \mathbf{\Sigma}_1\mathbf{X}^{\perp}\,]^{\perp} & \mathbf{0} \end{bmatrix} \right\} \\
&= \min\left\{ r\begin{bmatrix} \mathbf{G} \\ [\,\mathbf{X},\ \mathbf{\Sigma}_1\mathbf{X}^{\perp}\,]^{\perp} \\ [\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{\perp} \end{bmatrix},\ k + n - r[\,\mathbf{X},\ \mathbf{\Sigma}_1\,] \right\} \\
&= \min\left\{ r\begin{bmatrix} \mathbf{M}_2 \\ \mathbf{N}_2 \end{bmatrix} + n - r(\mathbf{X}) - r(\mathbf{X}_0) - r[\,\mathbf{X},\ \mathbf{\Sigma}_1\,] - r[\,\mathbf{X}_0,\ \mathbf{V}_1\,],\ \ k + n - r[\,\mathbf{X},\ \mathbf{\Sigma}_1\,] \right\} \\
&= \min\left\{ k,\ \ r\begin{bmatrix} \mathbf{M}_2 \\ \mathbf{N}_2 \end{bmatrix} - r(\mathbf{X}) - r(\mathbf{X}_0) - r[\,\mathbf{X}_0,\ \mathbf{V}_1\,] \right\} + n - r[\,\mathbf{X},\ \mathbf{\Sigma}_1\,].
\end{aligned} \tag{A.13}$$

Combining (A.12) and (A.13) yields

$$\begin{aligned}
&\max_{\mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}} \min_{\mathbf{P}_{\mathbf{K};\mathbf{J};\mathbf{X};\mathbf{\Sigma}_1}} r(\,\mathbf{P}_{\mathbf{K};\mathbf{J};\mathbf{X};\mathbf{\Sigma}_1} - \mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}\,) \\
&= \min\left\{ k,\ \ r\begin{bmatrix} \mathbf{M}_2 \\ \mathbf{N}_2 \end{bmatrix} - r(\mathbf{X}) - r(\mathbf{X}_0) - r[\,\mathbf{X}_0,\ \mathbf{V}_1\,] \right\} \\
&= \min\left\{ k,\ \ r\begin{bmatrix} \mathbf{M}_2 \\ \mathbf{N}_2 \end{bmatrix} - r(\mathbf{M}_2) + (\,r[\,\mathbf{X},\ \mathbf{X}_0,\ \mathbf{\Sigma}_1,\ \mathbf{V}_1\,] - r[\,\mathbf{X}_0,\ \mathbf{V}_1\,]\,) \right\}\ \ \text{(by (2.7))}.
\end{aligned} \tag{A.14}$$

Setting the right-hand side of (A.14) equal to zero yields $r\begin{bmatrix} \mathbf{M}_2 \\ \mathbf{N}_2 \end{bmatrix} = r(\mathbf{M}_2)$ and $r[\,\mathbf{X},\ \mathbf{X}_0,\ \mathbf{\Sigma}_1,\ \mathbf{V}_1\,] = r[\,\mathbf{X}_0,\ \mathbf{V}_1\,]$, or equivalently, $\mathscr{R}(\mathbf{M}_2') \supseteq \mathscr{R}(\mathbf{N}_2')$ and $\mathscr{R}[\,\mathbf{X},\ \mathbf{\Sigma}_1\,] \subseteq \mathscr{R}[\,\mathbf{X}_0,\ \mathbf{V}_1\,]$ hold by Lemma 2.2(a) and (b). Combining this fact with (2.2) yields the first statement in (b).

By the structural symmetry of $\mathbf{P}_{\mathbf{K};\mathbf{J};\mathbf{X};\mathbf{\Sigma}_1}$ and $\mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}$, we obtain

$$\begin{aligned}
&\max_{\mathbf{P}_{\mathbf{K};\mathbf{J};\mathbf{X};\mathbf{\Sigma}_1}} \min_{\mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}} r(\,\mathbf{P}_{\mathbf{K};\mathbf{J};\mathbf{X};\mathbf{\Sigma}_1} - \mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}\,) \\
&= \min\left\{ k,\ \ r\begin{bmatrix} \mathbf{M}_2 \\ \mathbf{N}_2 \end{bmatrix} - r(\mathbf{M}_2) + (\,r[\,\mathbf{X},\ \mathbf{X}_0,\ \mathbf{\Sigma}_1,\ \mathbf{V}_1\,] - r[\,\mathbf{X},\ \mathbf{\Sigma}_1\,]\,) \right\}.
\end{aligned} \tag{A.15}$$

Setting the right-hand side of (A.15) equal to zero and applying Lemma 2.2(a) and (b) yields the first statement in (c). Combining (b) and (c) yields (d). □

Proof of Theorem 4.3. From Definition 4.1(II) and (4.4), that result (a) is equivalent to

$$(\mathbf{P}_{\mathbf{K};\mathbf{J};\mathbf{X};\mathbf{\Sigma}_1} - \mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1})[\,\mathbf{X},\ \mathbf{\Sigma}_1\mathbf{X}^{\perp}\,] = \mathbf{0}. \tag{A.16}$$

Substituting (3.13) and (3.34) into (A.16) and then simplifying, we obtain

$$\mathbf{U}_0[\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{\perp}[\,\mathbf{X},\ \mathbf{\Sigma}_1\mathbf{X}^{\perp}\,] = \mathbf{G}, \tag{A.17}$$

where $\mathbf{G} = [\,\mathbf{K},\ \mathbf{J}\mathbf{\Sigma}_1\mathbf{X}^{\perp}\,] - [\,\mathbf{K}_0,\ \mathbf{J}_0\mathbf{V}_1\mathbf{X}_0^{\perp}\,][\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{+}[\,\mathbf{X},\ \mathbf{\Sigma}_1\mathbf{X}^{\perp}\,]$ and $\mathbf{U}_0 \in \mathbb{R}^{k\times n}$ is arbitrary. From Lemma 2.5, the matrix equation is solvable for $\mathbf{U}_0$ if and only if

$$r\begin{bmatrix} \mathbf{G} \\ [\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{\perp}[\,\mathbf{X},\ \mathbf{\Sigma}_1\mathbf{X}^{\perp}\,] \end{bmatrix} = r([\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{\perp}[\,\mathbf{X},\ \mathbf{\Sigma}_1\mathbf{X}^{\perp}\,]). \tag{A.18}$$

Applying (2.3) and (2.4) to both sides and then simplifying, we obtain

$$\begin{aligned}
& r\begin{bmatrix} \mathbf{G} \\ [\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{\perp}[\,\mathbf{X},\ \mathbf{\Sigma}_1\mathbf{X}^{\perp}\,] \end{bmatrix} \\
=& r\begin{bmatrix} [\,\mathbf{K},\ \mathbf{J}\mathbf{\Sigma}_1\mathbf{X}^{\perp}\,] - [\,\mathbf{K}_0,\ \mathbf{J}_0\mathbf{V}_1\mathbf{X}_0^{\perp}\,][\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{+}[\,\mathbf{X},\ \mathbf{\Sigma}_1\mathbf{X}^{\perp}\,] & \mathbf{0} \\ [\,\mathbf{X},\ \mathbf{\Sigma}_1\mathbf{X}^{\perp}\,] & [\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,] \end{bmatrix} - r[\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,] \\
=& r\begin{bmatrix} [\,\mathbf{K},\ \mathbf{J}\mathbf{\Sigma}_1\mathbf{X}^{\perp}\,] & [\,\mathbf{K}_0,\ \mathbf{J}_0\mathbf{V}_1\mathbf{X}_0^{\perp}\,] \\ [\,\mathbf{X},\ \mathbf{\Sigma}_1\mathbf{X}^{\perp}\,] & [\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,] \end{bmatrix} - r[\,\mathbf{X}_0,\ \mathbf{V}_1\,] \\
=& r\begin{bmatrix} \mathbf{X} & \mathbf{X}_0 & \mathbf{\Sigma}_1 & \mathbf{V}_1 \\ \mathbf{0} & \mathbf{0} & \mathbf{X}' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X}_0' \\ \mathbf{K} & \mathbf{K}_0 & \mathbf{J}\mathbf{\Sigma}_1 & \mathbf{J}_0\mathbf{V}_1 \end{bmatrix} - r(\mathbf{X}) - r(\mathbf{X}_0) - r[\,\mathbf{X}_0,\ \mathbf{V}_1\,] \\
=& r\begin{bmatrix} \mathbf{M}_2 \\ \mathbf{N}_2 \end{bmatrix} - r(\mathbf{X}) - r(\mathbf{X}_0) - r[\,\mathbf{X}_0,\ \mathbf{V}_1\,],
\end{aligned} \tag{A.19}$$

and

$$\begin{aligned}
r([\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{\perp}[\,\mathbf{X},\ \mathbf{\Sigma}_1\mathbf{X}^{\perp}\,]) &= r[\,\mathbf{X},\ \mathbf{\Sigma}_1,\ \mathbf{X}_0,\ \mathbf{V}_1\,] - r[\,\mathbf{X}_0,\ \mathbf{V}_1\,] \\
&= r(\mathbf{M}_2) - r(\mathbf{X}) - r(\mathbf{X}_0) - r[\,\mathbf{X}_0,\ \mathbf{V}_1\,].
\end{aligned} \tag{A.20}$$

Substitution of (A.19) and (A.20) into (A.18) leads to $r\begin{bmatrix} \mathbf{M}_2 \\ \mathbf{N}_2 \end{bmatrix} = r(\mathbf{M}_2)$, or equivalently, $\mathscr{R}(\mathbf{M}_2') \supseteq \mathscr{R}(\mathbf{N}_2')$ by Lemma 2.2(b), thus establishing the equivalence of (a) and (c).

It follows from Lemma 4.1(b) that the statement in (b) holds if and only if

$$\max_{\mathbf{P}_{\mathbf{K};\mathbf{J};\mathbf{X};\mathbf{\Sigma}_1}} \min_{\mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}} r((\,\mathbf{P}_{\mathbf{K};\mathbf{J};\mathbf{X};\mathbf{\Sigma}_1} - \mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}\,)[\,\mathbf{X},\ \mathbf{\Sigma}_1\mathbf{X}^{\perp}\,]) = \mathbf{0}. \tag{A.21}$$

By (2.9), (3.11), (A.19) and (A.20),

$$\max_{\mathbf{P}_{\mathbf{K};\mathbf{J};\mathbf{X};\mathbf{\Sigma}_1}} \min_{\mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}} r((\,\mathbf{P}_{\mathbf{K};\mathbf{J};\mathbf{X};\mathbf{\Sigma}_1} - \mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}\,)[\,\mathbf{X},\ \mathbf{\Sigma}_1\mathbf{X}^{\perp}\,])$$

$$= \min_{\mathbf{U}_0} r(\mathbf{G} - \mathbf{U}_0[\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{\perp}[\,\mathbf{X},\ \mathbf{\Sigma}_1\mathbf{X}^{\perp}\,])$$

$$= r\begin{bmatrix} \mathbf{G} \\ [\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{\perp}[\,\mathbf{X},\ \mathbf{\Sigma}_1\mathbf{X}^{\perp}\,] \end{bmatrix} - r([\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{\perp}[\,\mathbf{X},\ \mathbf{\Sigma}_1\mathbf{X}^{\perp}\,])$$

$$= r\begin{bmatrix} \mathbf{M}_2 \\ \mathbf{N}_2 \end{bmatrix} - r(\mathbf{M}_2). \tag{A.22}$$

Equation (A.21) thereby is equivalent to $r\big[\mathbf{M}_2', \mathbf{N}_2'\big] = r(\mathbf{M}_2)$, that is, $\mathscr{R}(\mathbf{M}_2') \supseteq \mathscr{R}(\mathbf{N}_2')$ holds by Lemma 2.2(b). Combining the fact with (2.2) leads to the equivalence of (b) and (c). □

Proof of Theorem 4.4. It follows from Lemma 4.1(b) that the statement in (a) holds if and only if

$$\max_{\mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}} \min_{\mathbf{P}_{\mathbf{K};\mathbf{J};\mathbf{X};\mathbf{\Sigma}_1}} r\,((\,\mathbf{P}_{\mathbf{K};\mathbf{J};\mathbf{X};\mathbf{\Sigma}_1} - \mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}\,)[\,\mathbf{X},\ \mathbf{\Sigma}_1\mathbf{X}^{\perp}\,]) = \mathbf{0}. \tag{A.23}$$

From (2.7), (2.8), (3.11), (3.13), (3.34) and (A.19), we obtain

$$\max_{\mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}} \min_{\mathbf{P}_{\mathbf{K};\mathbf{J};\mathbf{X};\mathbf{\Sigma}_1}} r((\,\mathbf{P}_{\mathbf{K};\mathbf{J};\mathbf{X};\mathbf{\Sigma}_1} - \mathbf{P}_{\mathbf{K}_0;\mathbf{J}_0;\mathbf{X}_0;\mathbf{V}_1}\,)[\,\mathbf{X},\ \mathbf{\Sigma}_1\mathbf{X}^{\perp}\,])$$

$$= \max_{\mathbf{U}_0} r\,(\mathbf{G} - \mathbf{U}_0[\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{\perp}[\,\mathbf{X},\ \mathbf{\Sigma}_1\mathbf{X}^{\perp}\,])$$

$$= \min\left\{ r\begin{bmatrix} \mathbf{G} \\ [\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{\perp}[\,\mathbf{X},\ \mathbf{\Sigma}_1\mathbf{X}^{\perp}\,] \end{bmatrix},\ \ k \right\}$$

$$= \min\left\{ r\begin{bmatrix} \mathbf{M}_2 \\ \mathbf{N}_2 \end{bmatrix} - r(\mathbf{X}) - r(\mathbf{X}_0) - r[\,\mathbf{X}_0,\ \mathbf{V}_1\,],\ \ k \right\}$$

$$= \min\left\{ r\begin{bmatrix} \mathbf{M}_2 \\ \mathbf{N}_2 \end{bmatrix} - r(\mathbf{M}_2) + (\,r[\,\mathbf{X},\ \mathbf{X}_0,\ \mathbf{\Sigma}_1,\ \mathbf{V}_1\,] - r[\,\mathbf{X}_0,\ \mathbf{V}_1\,]),\ \ k \right\}, \tag{A.24}$$

where $\mathbf{G} = [\,\mathbf{K},\ \mathbf{J}\mathbf{\Sigma}_1\mathbf{X}^{\perp}\,] - [\,\mathbf{K}_0,\ \mathbf{J}_0\mathbf{V}_1\mathbf{X}_0^{\perp}\,][\,\mathbf{X}_0,\ \mathbf{V}_1\mathbf{X}_0^{\perp}\,]^{+}[\,\mathbf{X},\ \mathbf{\Sigma}_1\mathbf{X}^{\perp}\,]$, and $\mathbf{U}_0 \in \mathbb{R}^{k\times n}$ is arbitrary.

Setting the right-hand side of (A.24) equal to zero yields $r\begin{bmatrix} \mathbf{M}_2 \\ \mathbf{N}_2 \end{bmatrix} = r(\mathbf{M}_2)$ and $r[\,\mathbf{X},\ \mathbf{X}_0,\ \mathbf{\Sigma}_1,\ \mathbf{V}_1\,] = r[\,\mathbf{X}_0,\ \mathbf{V}_1\,]$, or equivalently, $\mathscr{R}(\mathbf{M}_2') \supseteq \mathscr{R}(\mathbf{N}_2')$ and $\mathscr{R}[\,\mathbf{X},\ \mathbf{\Sigma}_1\,] \subseteq \mathscr{R}[\,\mathbf{X}_0,\ \mathbf{V}_1\,]$ hold by Lemma 2.2(a) and (b). Combining this fact with (2.2) leads to the equivalence of (a) and (c). Combining these facts with Theorem 4.3 yields the equivalence of (b) and (c). □