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Research article

A fixed point theorem in strictly convex b-fuzzy metric spaces

Siniša N. Ješić[1], Nataša A. Ćirović[1,*], Rale M. Nikolić[2] and Branislav M. Ranđelović[3]

[1] Department of Applied Mathematics, School of Electrical Engineering, University of Belgrade, Bulevar kralja Aleksandra 73, 11000 Belgrade, Serbia

[2] Department of Natural Sciences and Mathematics, Military Academy, University of Defence, Pavla Jurišića Šturma 33, 11105 Belgrade, Serbia

[3] Department of Mathematics, Faculty of Electronic Engineering, University of Niš, Aleksandra Medvedeva 14, 18000 Niš, Serbia

* **Correspondence:** Email: natasa@etf.bg.ac.rs; Tel: +381113218308; Fax: +381113218308.

Abstract: The main motivation for this paper is to investigate the fixed point property for non-expansive mappings defined on b-fuzzy metric spaces. First, following the idea of S. Ješić's result from 2009, we introduce convex, strictly convex and normal structures for sets in b-fuzzy metric spaces. By using topological methods and these notions, we prove the existence of fixed points for self-mappings defined on b-fuzzy metric spaces satisfying a nonlinear type condition. This result generalizes and improves many previously known results, such as W. Takahashi's result on metric spaces from 1970. A representative example illustrating the main result is provided.

Keywords: b-fuzzy metric spaces; fixed point; convex structure; strictly convex structure; normal structure
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1. Introduction and preliminaries

Schweizer and Sklar provided the axioms of t-norms in 1960 [1].

Definition 1.1. *[1, 2] A t-norm is a binary operation* $T : [0, 1] \times [0, 1] \to [0, 1]$ *such that the following conditions are satisfied for all* $a, b, c \in [0, 1]$*:*

(a) $T(a, b) = T(b, a)$,
(b) $T(a, T(b, c)) = T(T(a, b), c)$,
(c) $T(a, 1) = a$,
(d) $T(a, b) \le T(c, d)$ *whenever* $a \le c$ *and* $b \le d$.

Two basic examples of t-norm are $T_M(a,b) = \min\{a,b\}$ and $T_P(a,b) = ab$. For more examples of t-norm, see [2].

Definition 1.2. *[3] A fuzzy set A in X is a function* $A : X \to [0,1]$.

Fuzzy metric spaces were defined by Kramosil and Michalek in 1975 [4]. George and Veeramani defined different notion of fuzzy metric spaces and discussed several topological properties of these spaces in 1994 [5]. The concept of b-metric spaces was introduced by Bakhtin [6] in 1989, and Czerwik [7, 8] and others [9] expanded the fixed point theory on these spaces. Sedghi and Shobe generalized the concept of fuzzy metric spaces, according to George and Veeramani, by introducing b-fuzzy metric spaces in 2012 [10]. There are several significant fixed point results for functions on b-fuzzy metric spaces, for example Došenović [11]. Recently Badshah-e-Rome et al. introduced μ-extended b-metric spaces in [12] and Mecheraoui et al. introduced E-fuzzy metric spaces in [13]. Fixed point results on these spaces show that the concepts of non-deterministic fuzzy metrics play very important role in several branches of science. Fixed point results are applicable in solving differential or integral equations, for example see Humaira et al. [14].

Definition 1.3. *[10] A 3-tuple (X, M, T) is called a b-fuzzy metric space, if X is an arbitrary set, T is a continuous t-norm and M is a fuzzy set on $X^2 \times (0,+\infty)$ satisfying the following conditions, for all $x, y, z \in X$, $s, t > 0$ and $b \geq 1$ be a given real number:*

(Fb-1) $M(x, y, t) > 0$,

(Fb-2) $M(x, y, t) = 1$ *if and only if* $x = y$,

(Fb-3) $M(x, y, t) = M(y, x, t)$,

(Fb-4) $T\left(M(x, y, \frac{t}{b}), M(y, z, \frac{s}{b})\right) \leq M(x, z, t + s)$,

(Fb-5) $M(x, y, \cdot) : (0,+\infty) \to [0, 1]$ *is continuous.*

Function M is called a b-fuzzy metric on X.

Every fuzzy metric space is a b-fuzzy metric space for $b = 1$. In general, the converse is not true. For examples of b-fuzzy metric spaces that are not fuzzy metric spaces, see [10, 15].

Definition 1.4. *[10] A function $f : \mathbb{R} \to \mathbb{R}$ is b-nondecreasing if the following implication holds, for all $x, y \in \mathbb{R}$,*

$$x > by \text{ implies } f(x) \geq f(y).$$

Lemma 1.1. *[10] Let (X, M, T) be a b-fuzzy metric space. Then, b-fuzzy metric $M(x, y, t)$ is b-nondecreasing with respect to t, for all $x, y \in X$. Additionally,*

$$M(x, y, b^n t) \geq M(x, y, t),$$

for every $n \in \mathbb{N}$.

2. Compact sets in the topology induced by a b-fuzzy metric

The topology induced by the b-fuzzy metric spaces was introduced by Sedghi and Shobe in [10]. In this section, we introduce the concept of compact sets in b-fuzzy metric spaces. Additionally, we show that every compact subset of b-fuzzy metric space is bF-bounded.

Definition 2.1. *[10] Let (X, M, T) be a b-fuzzy metric space. The set $B(x, r, t) = \{y \in X \mid M(x, y, t) > 1 - r\}$ is called an open ball $B(x, r, t)$ with centre $x \in X$ and radius $r \in (0, 1)$, with respect to $t > 0$.*

Definition 2.2. *Let (X, M, T) be a b-fuzzy metric space and $A \subset X$. The set A is called an open set if for every $x \in A$ there exist $t > 0$ and $r \in (0, 1)$ such that $B(x, r, t) \subset A$.*

Theorem 2.1. *Every open ball $B(x, r, t)$ in a b-fuzzy metric space (X, M, T) is an open set.*

Proof. For arbitrary point $y \in B(x, r, t)$, it holds that $M(x, y, t) > 1 - r$. There exists $t_0 > 0$ such that $\frac{t_0}{b} \in (0, t)$, for every $b \geq 1$, and $M(x, y, \frac{t_0}{b}) > 1 - r$. If we denote $r_0 = M(x, y, \frac{t_0}{b})$, then there exists $s \in (0, 1)$, such that $r_0 > 1 - s > 1 - r$. Hence, for given r_0 and s satisfying $r_0 > 1 - s$, there exists $r_1 \in (0, 1)$ such that $T(r_0, r_1) > 1 - s$. We will prove that $B(y, 1 - r_1, \frac{t-t_0}{b}) \subset B(x, r, t)$. Let $z \in B(y, 1 - r_1, \frac{t-t_0}{b})$ be arbitrary. Then, $M(y, z, \frac{t-t_0}{b}) > r_1$ and it follows that

$$M(x, z, t) \geq T\left(M\left(x, y, \frac{t_0}{b}\right), M\left(y, z, \frac{t - t_0}{b}\right)\right) \geq T(r_0, r_1) \geq 1 - s > 1 - r.$$

Hence, we obtain that $z \in B(x, r, t)$ (i.e., $B(y, 1 - r_1, \frac{t-t_0}{b}) \subset B(x, r, t)$). □

Topology τ in b-fuzzy metric space was defined in [10]:

$$\tau = \{A \subseteq X \mid \text{for every } x \in A \text{ there exist } t > 0 \text{ and } r \in (0, 1) \text{ such that } B(x, r, t) \subset A\}.$$

Theorem 2.2. *Every b-fuzzy metric space is Hausdorff.*

Proof. Let (X, M, T) be a b-fuzzy metric space. Let $x, y \in X$ such that $x \neq y$ be arbitrary. Then, $r = M(x, y, t) \in (0, 1)$, and for every $r_0 \in (r, 1)$, there exist r_1 such that $T(r_1, r_1) \geq r_0$. Let us consider the open balls $B(x, 1 - r_1, \frac{t}{2b})$ and $B(y, 1 - r_1, \frac{t}{2b})$. We will prove that these balls have an empty intersection. Indeed, if we assume the contrary (i.e. that there exists $z \in X$ such that $z \in B(x, 1 - r_1, \frac{t}{2b}) \cap B(y, 1 - r_1, \frac{t}{2b})$), then we obtain

$$r = M(x, y, t) \geq T\left(M\left(x, z, \frac{t}{2b}\right), M\left(y, z, \frac{t}{2b}\right)\right) \geq T(r_1, r_1) \geq r_0 > r,$$

which is a contradiction, implying that (X, M, T) is a Hausdorff space. □

A sequence $\{x_n\}$ in b-fuzzy metric space converges to $x \in X$ if $\lim_{n \to +\infty} M(x_n, x, t) = 1$ for every $t > 0$. It is called a Cauchy sequence if for arbitrary $\varepsilon \in (0, 1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ holds for all $n, m \geq n_0$. A b-fuzzy metric space is complete if every Cauchy sequence in X converges to a point in X [10].

Definition 2.3. *[16] Let (X, M, T) be a b-fuzzy metric space. Define the mapping $\delta_A(t) : (0, +\infty) \to [0, 1]$ by*

$$\delta_A(t) = \inf_{x,y \in A} \sup_{\varepsilon < t} M(x, y, \varepsilon).$$

The constant $\delta_A = \sup_{t>0} \delta_A(t)$ is a b-fuzzy diameter of set A.

Sedghi and Shobe [10] introduced the notion of bF-bounded set.

Definition 2.4. *[10] Let (X, M, T) be a b-fuzzy metric space. A set A is bF-bounded if there exist $t > 0$ and $r \in (0, 1)$ such that $M(x, y, t) > 1 - r$ for all $x, y \in A$.*

Next two definition were introduced by Ranđelović et al. [16].

Definition 2.5. *[16] Let (X, M, T) be a b-fuzzy metric space and $A \subseteq X$. Closure $\bar{A}$ of the set A is the smallest closed set containing A.*

Definition 2.6. *[16] Let (X, M, T) be a b-fuzzy metric space. The set $B[x, r, t] = \{y \in X \,|\, M(x, y, t) \geq 1 - r\}$ is called a closed ball $B[x, r, t]$ with centre $x \in X$ and radius $r \in (0, 1)$, with respect to $t > 0$.*

Now, we will introduce the concept of compact set in b-fuzzy metric space.

Definition 2.7. *A subset K of a b-fuzzy metric space is called compact if the following statement holds:*

$$K \subseteq \bigcup_{\alpha \in \Lambda} U_\alpha \quad \text{implies} \quad K \subseteq \bigcup_{i=1}^{n} U_{\alpha_i} \quad \text{for some} \quad \alpha_1, \ldots, \alpha_n \in \Lambda,$$

for every family $\{U_\alpha : \alpha \in \Lambda\}$ of open sets $U_\alpha \subset X$.

By applying the De-Morgan's laws to the previous definition, we get the next lemma.

Lemma 2.1. *Let (X, M, T) be a b-fuzzy metric space. Then, the set $K \subseteq X$ is compact if and only if for every family of closed sets $\{F_\alpha\}_{\alpha \in \Lambda}$ such that $F_\alpha \subseteq K$, the following holds:*

$$\bigcap_{\alpha \in \Lambda} F_\alpha = \emptyset \quad \text{implies} \quad \bigcap_{i=1}^{n} F_{\alpha_i} = \emptyset \quad \text{for some} \quad \alpha_1, \ldots, \alpha_n \in \Lambda.$$

Theorem 2.3. *Every compact subset of a b-fuzzy metric space is bF-bounded.*

Proof. Let K be a compact subset of a b-fuzzy metric space (X, M, T). Let $t > 0$ and $r \in (0, 1)$ be arbitrary and consider the open cover $\{B(x, r, t) \,|\, x \in K\}$ of set K. From the compactness of K, it follows that there exist $x_1, x_2, \ldots x_n \in K$, such that $K \subseteq \bigcup_{i=1}^{n} B(x_i, r, \frac{t}{4b^2})$.

Let $x, y \in K$. Then, $x \in B(x_i, r, \frac{t}{4b^2})$ and $y \in B(x_j, r, \frac{t}{4b^2})$ for some $i, j \in \{1, \ldots, n\}$. Thus, we have that $M(x, x_i, \frac{t}{4b^2}) > 1 - r$ and $M(y, x_j, \frac{t}{4b^2}) > 1 - r$. Additionally, let $\alpha = \min\{M\left(x_i, x_j, \frac{t}{4b^2}\right) \,|\, 1 \leq i, j \leq n\}$. Obviously, $\alpha > 0$. Then, it follows that

$$\begin{aligned} M(x, y, t) &\geq T\left(M\left(x, x_i, \frac{t}{2b}\right), M\left(x_i, y, \frac{t}{2b}\right)\right) \\ &\geq T\left(M\left(x, x_i, \frac{t}{2b}\right), T\left(M\left(x_i, x_j, \frac{t}{4b^2}\right), M\left(x_j, y, \frac{t}{4b^2}\right)\right)\right) \\ &\geq T\left(M\left(x, x_i, \frac{t}{2b}\right), T(\alpha, 1 - r)\right), \end{aligned}$$

for all $x, y \in K$. Since $M(x, y, .)$ is b-nondecreasing function and $\frac{t}{2b} > b \cdot \frac{t}{4b^2}$, it follows that

$$M\left(x, x_i, \frac{t}{2b}\right) \geq M\left(x, x_i, \frac{t}{4b^2}\right) > 1 - r,$$

i.e.,

$$M(x, y, t) \geq T(1 - r, T(\alpha, 1 - r)) > 1 - s,$$

for all $x, y \in K$ and some $s \in (0, 1)$. Hence, we obtain that K is bF-bounded set. □

Remark 2.1. *From Theorems 2.2 and 2.3, it follows that every compact set in a b-fuzzy metric space is closed. Additionally, every closed subset of compact set in b-fuzzy metric space is compact.*

3. Convex structure, normal structure and strictly convex structure on b-fuzzy metric spaces

Brodskii and Milman introduced the concept of normal structure in 1948 [17], and Takahashi [18] defined convex and normal structures for sets in metric spaces in 1970. Hadžić [19] generalized the Takahashi's concept of convex structure on Menger probabilistic metric spaces. Ješić [20] defined the convex structure on intuitionistic fuzzy metric spaces and was the first to introduce the concept of strictly convex structure and normal structure in spaces with non-deterministic distances. This notion has attracted the attention of other researchers, for example, Gabeleh, Ekici and De La Sen use the notion of strictly convex fuzzy metric spaces in [21]. Here, we introduce the notion of convex, strictly convex and normal structure in b-fuzzy metric spaces.

Definition 3.1. *Let (X, M, T) be a b-fuzzy metric space. A mapping $S : X \times X \times [0, 1] \to X$ is a convex structure on X if $S(x, y, 0) = y$, $S(x, y, 1) = x$ and*

$$M(S(x, y, \theta), z, 2t) \geq T\left(M\left(x, z, \frac{t}{b\theta}\right), M\left(y, z, \frac{t}{b(1-\theta)}\right)\right) \tag{3.1}$$

hold for all $x, y, z \in X, \theta \in (0, 1)$ and $t > 0$.

A b-fuzzy metric space (X, M, T) with a convex structure is called a convex b-fuzzy metric space.

Definition 3.2. *Let (X, M, T) be a b-fuzzy metric space with convex structure $S(x, y, \theta)$. A subset $A \subseteq X$ is called a convex set if for all $x, y \in A$ and $\theta \in [0, 1]$, it follows that $S(x, y, \theta) \in A$.*

The following lemma holds (for proof see [20]).

Lemma 3.1. *Let (X, M, T) be a b-fuzzy metric space. Let $\{K_\alpha\}_{\alpha\in\Delta}$ be a family of convex subsets of X. Then, the intersection $K = \cap_{\alpha\in\Delta} K_\alpha$ is a convex set.*

Definition 3.3. *A convex b-fuzzy metric space (X, M, T) with a convex structure $S(x, y, \theta)$ is strictly convex if for arbitrary $x, y \in X$ and $\theta \in (0, 1)$ the element $z = S(x, y, \theta)$ is the unique element such that*

$$M\left(x, y, \frac{t}{\theta}\right) = M(y, z, t), \quad M\left(x, y, \frac{t}{1-\theta}\right) = M(y, z, t), \tag{3.2}$$

holds for all $t > 0$.

Lemma 3.2. *Let (X, M, T) be a b-fuzzy metric space with a convex structure $S(x, y, \theta)$. Suppose that*

$$M(S(x, y, \theta), z, t) > \min\{M(z, x, t), M(z, y, t)\} \tag{3.3}$$

holds for every $\theta \in (0, 1)$, $t > 0$ and $x, y, z \in X$.

If there exists $z \in X$ such that

$$M(S(x, y, \theta), z, t) = \min\{M(z, x, t), M(z, y, t)\} \tag{3.4}$$

holds for all $t > 0$, then $S(x, y, \theta) \in \{x, y\}$.

Proof. Assuming that the condition (3.4) holds for some $z \in X$ and for all $t > 0$; from the definition of the convex structure S for $\theta = 0$ an $\theta = 1$, it follows the statement of the lemma. □

Lemma 3.3. *Let (X, M, T) be a strictly convex b-fuzzy metric space with a convex structure $S(x, y, \theta)$. Let $x, y \in X, x \neq y$ be arbitrary. Then, there exists $\theta \in (0, 1)$ such that $S(x, y, \theta) \notin \{x, y\}$.*

The proof follows from the condition (3.2) in the definition of strictly convex structure, see [20].

Definition 3.4. *A point $x \in A$ is diametral if*

$$\sup_{\varepsilon < t} \inf_{y \in A} M(x, y, \varepsilon) = \delta_A(t)$$

holds for all $t > 0$.

Definition 3.5. *A b-fuzzy metric space (X, M, T) possesses a normal structure if, for every closed, bF-bounded and convex set $Y \subset X$, which consists of at least two different points, there exists a point $x \in Y$ which is non-diametral, i.e., there exists $t_0 > 0$ such that*

$$\sup_{\varepsilon < t_0} \inf_{y \in Y} M(x, y, \varepsilon) > \delta_Y(t_0)$$

holds.

It is easy to see that compact and convex sets in convex metric space possess a normal structure (see [18]).

Definition 3.6. *Let (X, M, T) be a convex b-fuzzy metric space and $Y \subseteq X$. The intersection of all closed, convex sets in X that contain Y is called the closed convex shell of Y, denoted by $conv(Y)$.*

Keeping in mind that X belongs to the family of closed, convex sets that contain Y, it is clear that the set $conv(Y)$ exists. The closed convex shell is closed set as an intersection of closed sets, and from the Lemma 3.1 it follows that it is convex set, too.

4. Main results

There are several fixed point and common fixed point results for mappings defined on b-fuzzy metric spaces that observe linear (generalized) contractive type condition. On the other hand, there is no fixed point result for mappings which do not increase distances on b-fuzzy metric spaces. We prove a fixed point result for a wide class of mappings, satisfying non-linear condition, that include mappings which do not increase distances.

Lemma 4.1. *Let (X, M, T) be a strictly convex b-fuzzy metric space with a convex structure $S(x, y, \theta)$ satisfying* (3.3). *Then, arbitrary nonempty, convex and compact set $K \subseteq X$ possesses a normal structure.*

Proof. Suppose that K does not possess a normal structure. It follows that there exists a closed, convex bF-bounded subset $Y \subset K$ that contains at least two different points and does not contain a non-diametral point, i.e.,

$$\sup_{\varepsilon < t} \inf_{y \in Y} M(x, y, \varepsilon) = \delta_Y(t),$$

holds for every $x \in Y$.

Since X is strictly convex and condition (3.3) holds, then the statements of Lemmas 3.2 and 3.3 also hold. Let x_1 and x_2 be arbitrary points in Y. From Lemma 3.3, there exists $\theta_0 \in (0, 1)$ such that $S(x_1, x_2, \theta_0) \notin \{x_1, x_2\}$. Since Y is a convex set, it follows that $S(x_1, x_2, \theta_0) \in Y$.

Set Y is a closed subset of the compact set K, so Y is also compact. Since $\delta_Y(t) = \sup_{\varepsilon<t} \inf_{y\in Y} M(y, S(x_1, x_2, \theta), \varepsilon)$ is a continuous function for arbitrary $t > 0$, there exists $x_3 \in Y$ such that $\sup_{\varepsilon<t} M(x_3, S(x_1, x_2, \theta_0), \varepsilon) = \delta_Y(t)$. From Lemma 3.2 and the fact that $M(x, y, \cdot)$ is a b-nondecreasing function with respect to t, it follows that

$$\begin{aligned}\delta_Y(t) &= \sup_{\varepsilon<t} M(x_3, S(x_1, x_2, \theta_0), \varepsilon) = M(x_3, S(x_1, x_2, \theta_0), t)\\ &> \min\{M(x_3, x_1, t), M(x_3, x_2, t)\}\\ &= \min\left\{\sup_{\varepsilon<t} M(x_3, x_1, \varepsilon), \sup_{\varepsilon<t} M(x_3, x_2, \varepsilon)\right\} \geq \delta_Y(t).\end{aligned} \tag{4.1}$$

From the previous statement, it follows that $\delta_Y(t) > \delta_Y(t)$, which is a contradiction. □

Lemma 4.2. *Let (X, M, T) be a convex b-fuzzy metric space with a convex structure $S(x, y, \theta)$ satisfying* (3.3)*. Then, closed balls in X are convex sets.*

Proof. Let $y_1, y_2 \in B[x, r, t]$ be arbitrary points. This implies that $M(x, y_1, t) \geq 1 - r$ and $M(x, y_2, t) \geq 1 - r$, for every $t > 0$. We will prove that $M(S(y_1, y_2, \theta), x, t) \geq 1 - r$, for every $t > 0$, i.e., $S(y_1, y_2, \theta) \in B[x, r, t]$. Indeed, for arbitrary $\theta \in (0, 1)$, from (3.3) it follows that

$$M(S(y_1, y_2, \theta), x, t) > \min\{M(y_1, x, t), M(y_2, x, t)\} \geq \min\{1 - r, 1 - r\} = 1 - r.$$

Taking either $\theta = 0$ or $\theta = 1$, we get that $S(y_1, y_2, 0) = y_2, S(y_1, y_2, 1) = y_1$, in $B[x, r, t]$. □

Lemma 4.3. *(Zorn's lemma) Let X be a nonempty partially ordered set in which every chain has a lower (upper) bound. Then X has a minimal (maximal) element.*

Next we shall give the main result of this paper.

Theorem 4.1. *Let (X, M, T) be a strictly convex b-fuzzy metric space with a convex structure $S(x, y, \theta)$ satisfying (3.3) and $K \subseteq X$ a nonempty, convex and compact subset of X. Let f be a self mapping on K, satisfying the condition*

$$M(f(x), f(y), \varphi(t)) \geq M(x, y, t), \tag{4.2}$$

for all $x, y \in K$, and for some continuous function $\varphi : (0, +\infty) \mapsto (0, +\infty)$ satisfying $\varphi(t) \leq \frac{t}{b}$. Then, f has at least one fixed point on K.

Proof. Let Γ be a family of nonempty, closed, convex sets $K_\gamma \subseteq K$ such that $f(K_\gamma) \subseteq K_\gamma$. This family is nonempty because $K \subseteq \Gamma$. Indeed, set K is closed as a compact set in Hausdorff's space and $f(K) \subseteq K$. By ordering this family with an inclusion, we get a partially ordered set $(\Gamma, \subseteq)$. Let $\{K_\gamma \,|\, \gamma \in \Delta\}$ be an arbitrary chain of this family. Then, the set $\cap_{\gamma\in\Delta} K_\gamma$ is a nonempty, closed, convex subset of K, and it is a lower bound of this chain. We will show that this set is nonempty. Let us assume the contrary, that $\cap_{\gamma\in\Delta} K_\gamma = \emptyset$. Then, from Lemma 2.1, it follows that there exists a finite sub-family $K_{\gamma_1} \supseteq \ldots \supseteq K_{\gamma_n}$ of the chain $\{K_\gamma \,|\, \gamma \in \Delta\}$ that has an empty intersection, which is a contradiction, since the intersection is the nonempty set K_{γ_n}. From Zorn's lemma, it follows that there exists a minimal element K_0 of the family Γ such that $f(K_0) \subseteq K_0$. We will prove that K_0 contains exactly one point, and since $f : K_0 \to K_0$ this will mean that the mapping f has a fixed point.

Assume that K_0 contains at least two different points. From Theorem 2.3, it follows that K_0 is a bF-bounded set. From Lemma 4.1, it follows that K possesses a normal structure. Since K_0 is a closed and convex set, there exists some non-diametral point $x_0 \in K_0$, i.e., there exists $t_0 > 0$ such that the following inequality holds:

$$\sup_{\varepsilon< t_0} \inf_{y\in K_0} M(x_0, y, \varepsilon) > \delta_{K_0}(t_0). \tag{4.3}$$

Denote $1 - \mu := \sup_{\varepsilon< t_0} \inf_{y\in K_0} M(x_0, y, \varepsilon)$ and $K_1 = conv(f(K_0))$. Since $f(K_0) \subseteq K_0$, it holds that

$$K_1 = conv(f(K_0)) = \overline{conv(f(K_0))} \subseteq \overline{conv(K_0)} = \overline{K_0} = K_0,$$

i.e., $K_1 \subseteq K_0$. From this, it follows that

$$f(K_1) \subseteq f(K_0) \subseteq \overline{(conv(f(K_0))} = K_1,$$

i.e., $f(K_1) \subseteq K_1$. This means that $K_1 \in \Gamma$, and since K_0 is the minimal element of Γ and $K_1 \subseteq K_0$, we have that $K_1 = K_0$.

Supposing that (4.3) holds (i.e., $1 - \mu > \delta_{K_0}(t_0)$), let us define sets

$$C := \left(\bigcap_{y\in K_0} B[y, \xi, t_0]\right)\bigcap K_0 \quad \text{and} \quad C_1 := \left(\bigcap_{y\in f(K_0)} B[y, \xi, t_0]\right)\bigcap K_0.$$

The set C is nonempty since $x_0 \in C$. Indeed, from inequality (4.3) it follows that $M(x_0, y, t_0) \geq 1 - \mu$. From the previous it follows that x_0 belongs to $B[y, \mu, t_0]$ for all $y \in K_0$. Consequently, x_0 belongs to C.

We will show that $C = C_1$. Since $f(K_0) \subseteq K_0$, it follows that $C_1 \supseteq C$.

Let $z \in C_1$. Then, for arbitrary $y \in f(K_0)$, it holds that $M(y, z, t_0) \geq 1 - \mu$ (i.e., $y \in B[z, \mu, t_0]$). Since y is arbitrary point from $f(K_0)$, it follows that $f(K_0) \subseteq B[z, \mu, t_0]$. Because $B[z, \mu, t_0]$ is a closed and convex set which contains $f(K_0)$, we conclude that

$$K_1 = \overline{conv\,(f(K_0))} \subseteq B[z, \mu, t_0].$$

Since $K_0 = K_1$, it follows that $K_0 \subseteq B[z, \mu, t_0]$. From this, we have that for every $y \in K_0$ it holds that $z \in B[y, \mu, t_0]$ (i.e., $C_1 \subseteq C$). This shows that $C = C_1$.

Let us show that $C \in \Gamma$. The set C is closed and convex set as an intersection of closed and convex sets. We will prove that $f(C) \subseteq C$. Let $z \in C$ and $y \in f(K_0)$. Then, there exists $x \in K_0$ such that $y = f(x)$. Because $M(x, y, \cdot)$ is b-nondecreasing function, and if we apply inequality (4.2) for $t = t_0$, since $\varphi(t_0) \leq \frac{t_0}{b}$, from Lemma 1.1 we obtain

$$M(f(z), y, t_0) = M(f(z), f(x), t_0) \geq M(f(z), f(x), b\varphi(t_0)) \geq M(f(z), f(x), \varphi(t_0)) \geq M(z, x, t_0) \geq 1 - \mu.$$

This means that $f(z) \in C_1$. Since z is arbitrary point from $z \in C$, we obtain $f(C) \subseteq C_1$, and because $C_1 = C$, we have that $f(C) \subseteq C$.

Since $C \subseteq K_0$ and K_0 is the minimal element of collection Γ, it follows that $C = K_0$. Now, we have that $\delta_C(t_0) \geq 1-\mu > \delta_{K_0}(t_0)$, which is a contradiction with $C = K_0$ (i.e., the assumption that K_0 contains at least two different points is wrong). This means that K_0 contains only one point which is a fixed point of the mapping f. $\square$

Example 4.1. Let (X, M, T) be a complete b-fuzzy metric space, $(b = 2)$ with $X = [0, +\infty) \subset \mathbb{R}$ and $M(x, y, t) = e^{-\frac{|x-y|^2}{t}}$ (see [10]). We shall prove that convex structure in X can be defined by $S(x, y, \lambda) = \lambda x + (1 - \lambda)y, \lambda \in [0, 1]$. Finally, we give one example for previous theorem.

Let us show that the convex structure $S(x, y, \lambda)$ satisfies the condition (3.3). Let $x, y, z \in X$, $t > 0$ and $\lambda \in (0, 1)$ be arbitrary. Without loss of generality, we can assume that $M(x, z, t) = \min\{M(x, z, t), M(y, z, t)\}$ (i.e., $M(x, z, t) < M(y, z, t)$, for $x \neq y$). From this inequality, since e^{-s} is a decreasing function, it follows that $|x - z| > |y - z|$, for $x \neq y$. Then, the following holds:

$$|\lambda x + (1 - \lambda)y - z| = |\lambda(x - z) + (1 - \lambda)(y - z)| < |\lambda(x - z) + (1 - \lambda)(x - z)| = |x - z|.$$

Since the mapping s^2 is increasing on $X = [0, +\infty)$, it follows that

$$|\lambda x + (1 - \lambda)y - z|^2 < |x - z|^2,$$

and since e^{-s} is a decreasing function, we have that

$$M(S(x, y, \lambda), z, t) > M(x, z, t) = \min\{M(x, z, t), M(y, z, t)\},$$

i.e., the condition (3.3) holds.

Let

$$K = [0, 1], \qquad f(x) = \frac{x^2}{4}, \qquad \varphi(t) = \frac{t}{4}\ (t > 0).$$

It is easy to see that $f(K) \subseteq K$ and

$$\varphi(t) = \frac{t}{4} \leq \frac{t}{2} = \frac{t}{b},$$

for all $t > 0$.

The condition (4.2) is satisfied. Indeed, since for $x, y \in [0, 1] = K$, it holds that $|x + y| \leq 2$, and we obtain

$$\frac{|f(x) - f(y)|^2}{\varphi(t)} = \frac{4\left|x^2 - y^2\right|^2}{16t} = \frac{|x - y|^2|x + y|^2}{4t} \leq \frac{4|x - y|^2}{4t} = \frac{|x - y|^2}{t}.$$

Since e^{-s} is decreasing, it follows that

$$M(f(x), f(y), \varphi(t)) \geq M(x, y, t).$$

Since all the conditions of the Theorem 4.1 are satisfied, we get that $f(x)$ has at least one fixed point on K. It is easy to see that this fixed point is $x = 0$.

5. Connection with previously known results

First, we introduce the notion of nonexpansive mappings on b-fuzzy metric spaces.

Definition 5.1. *Let (X, M, T) be a b-fuzzy metric space and let f be a self-mapping on X. The mapping f is called nonexpansive if*

$$M(fx, fy, t) \geq M(x, y, t) \tag{5.1}$$

holds for all $x, y \in X$ and every $t > 0$.

The main result of our paper is a generalization of known results for nonexpansive selfmappings on metric spaces obtained by Takahashi [18], in 1970, as well as Kirk [22] and Browder [23]. Additionally, in [20], Ješić proved a fixed point theorem for nonexpansive mappings on strictly convex intuitionistic fuzzy metric spaces, and a version of this result is also a consequence of our result.

Let us state the corresponding result on b-fuzzy metric spaces for nonexpansive mappings, which is a direct corollary of Theorem 4.1, by taking that $b = 1$ and $\varphi(t) = t$.

Corollary 5.1. *Let (X, M, T) be a strictly convex b-fuzzy metric space with a convex structure $S(x, y, \theta)$ satisfying (3.3) and $K \subseteq X$ be a nonempty, convex and compact subset of X. Let f be a nonexpansive self-mapping on K. Then, f has at least one fixed point on K.*

6. Conclusions

In this paper, we introduced the notions of convex, strictly convex and normal structure in b-fuzzy metric spaces. Using these notions and topological methods, we proved existence of fixed point for self mappings defined on b-fuzzy metric spaces. This result is significant in the fact that we observe a wide class of mappings that includes non-expansive mappings. There are several possibilities for further research: proving fixed point results for class of non-expansive mappings defined on star-shaped sets using topological methods, analysing existence of common fixed points for several non-expansive mappings and solving some classes of differential and integral equations. Finally, presented topological method is useful for mathematical description of the stability of dynamical systems.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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