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Research article

On generalized $\Im_b$-contractions and related applications

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Abstract: In this manuscript, using the idea of b-simulation functions, certain common fixed point results via generalized $\Im_b$-contractions are investigated in the context of b-metric spaces. These findings generalize and supplement various numbers of established results from the existing literature. Examples and applications are also provided for the authenticity of the presented work. Besides, we ensure the existence of a unique common solution for systems of Volterra-Hammerstein integral and Urysohn integral equations, respectively, by applying the established results.

Keywords: fixed point; b-metric space; b-simulation function; compatible mapping; $\Im$-contraction; generalized $\Im_b$-contraction
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1. Introduction

Many physical phenomenon can be described by integral equations. Non-linear integral equations play a vital role to solve many mathematical problems arising in engineering and applied science. There are various techniques available in the literature for the existence of solutions of these equations. Many researchers used fixed points techniques to the existence of a unique solution to non-linear integral equations. For instance, refer to [1–7]. Especially, Shoaib et al. [2] studied fixed point results and its applications to the systems of non-linear integral and differential equations of arbitrary order, while Rashwan and Saleh [8] established fixed point results to find the existence of a unique common solution to a system of Urysohn integral equations. As opposed to that, Pathak et al. [9] and Rashwan and saleh [8] ensured the existence of a unique common solution to a system of Volterra-Hammerstein non-linear integral equations. Additionally, Baklouti and his co-authors have made significant contributions to related areas, including optimal preventive maintenance policies for solar photovoltaic systems (c.f, [15]) and quadratic Hom-Lie triple systems (c.f, [16, 17]).

In the current article, existence of solutions for the following systems of Volterra-Hammerstein integral and the Urysohn integral equations are investigated, respectively:

$$D(x) = \xi_i(x) + \int_a^b W_i(x, y, D(y))dy,$$

where $x \in (a, b) \subseteq R$; $D, \xi_i \in C((a, b), R^n)$ and $W_i : (a, b) \times (a, b) \times R^n \to R^n$, for $i = 1, 2$;
and

$$D(x) = \tau_i(x) + \lambda \int_0^t m(x, y)g_i(y, D(y))dy + \mu \int_0^\infty n(x, y)h_i(y, D(y))dy,$$

where $x \in (0, \infty), \lambda, \mu \in R, D, T_i, m(x, y), n(x, y), g_i(y, D(y))$ and $h_i(y, D(y))$ for $i = 1, 2$ are measurable functions with real values both in x and y on $(0, \infty)$.

The most elaborated result in this area, known as the Banach fixed point theorem, ensures that a solution exists. The extensive usage of the fixed point theory, particularly in metric spaces [10, 11], has then had an impact on the study of its evolution over the past few decades.

Due to important applications of the Banach contraction principle, many researchers generalized this principle by elaborating the underlying spaces or changing the contractive conditions. See [12–14] for details. In recent decades, scholars concentrated on applying such an aforementioned theorem to various generalized metric spaces, see [18, 19]. Among these generalized spaces, there is the b-metric space, where the coefficient of the triangle inequality is $s \geq 1$. It was introduced by Bakhtin [20]. Moreover, in [26] Czerwik provided the Banach contraction principle on this space. Recently, Salmi and Noorani [21] presented several properties in these spaces and established some common fixed point theorems in ordered cone b-metric spaces. Khojasteh et al. [34] introduced the concept of a simulation function. This concept has been refined in [22] in order to guarantee the presence of a unique coincidence point for two non-linear mappings.

Later, the concept of a b-simulation function was introduced to ensure the existence and uniqueness of a fixed point. In [23], Olgun et al. presented the concept of a generalized $\Im$-contraction. In [24], Jawaher et al. utilized the idea of a b-simulation function and investigated some common fixed points for two contractive mappings. In the same direction, using the concept of a generalized $\Im_b$-contraction with a b-simulation function, Rodjanadid et al. [23] proved some fixed point results in complete b-metric spaces.

Motivated by the above contributions, using b-simulation functions and $\Im_b$-contractions, some fixed point theorems are constructed. As applications of these findings, some examples and existence results for systems of integral equations are also discussed. We note that by using the presented work, some well known results can be deduced from the existence literature.

2. Preliminaries

This section includes all those concepts (definitions, theorems, lemmas, etc.) which will help us to prove the main results of this manuscript. These concepts are taken from different papers, like [25–27] etc. Throughout the manuscript, the following notions and symbols will be utilized. F, Ω, $\beth$ and $\Im$ represent a non-empty set, a metric, a simulation function and a family of simulation functions, respectively. Also, the initials $\mathbb{R}$ and $\mathbb{N}$ in the sequel stand for the sets of all real and natural numbers, respectively.

Definition 2.1. *[25] (Metric space) Let $F \neq \emptyset$. A function $\Omega : F \times F \to [0, \infty)$ is known as a metric on F, if for all $a, b, c \in F$ the following conditions hold:*

m1) $\Omega(a, b) = 0$ *if and only if* $a = b$;
m2) $\Omega(a, b) = \Omega(b, a)$;
m3) $\Omega(a, b) \leq \Omega(a, c) + \Omega(c, b)$.

The pair (F, Ω) is a metric space.

Definition 2.2. *[26] (b-metric space) Let $F \neq \emptyset$ and assume $b \geq 1$. A function $\Omega_b : F \times F \to [0, \infty)$ is called a b-metric on F if for all $a, e, c \in F$, the following requirements are satisfied:*

d1) $\Omega_b(a, e) = 0$ *if and only if* $a = e$;
d2) $\Omega_b(a, e) = \Omega_b(e, a)$;
d3) $\Omega_b(a, e) \leq b[\Omega_b(a, c) + \Omega_b(c, e)]$.

The pair (F, Ω_b) is known as a b-metric space, in short (bMS).

Definition 2.3. *[27] (Convergence, Cauchyness and Completeness) Let $\{f_n\}$ be a sequence in a b-metric space (F, Ω_b, b).*

a) $\{f_n\}$ *is called b-convergent if and only if there is* $f \in F$ *such that* $\Omega_b(f_n, f) \to 0$ *as* $n \to \infty$.
b) $\{f_n\}$ *is a b-Cauchy sequence if and only if* $\Omega_b(f_n, f_m) \to 0$ *as* $n, m \to \infty$.
c) The b-metric space is complete if every b-Cauchy sequence is b-convergent.

Proposition 2.4. *[27] The following assertions hold in a b-metric space (F, Ω_b, b):*
i) The limit of a b-convergent sequence is unique;
ii) Each b-convergent sequence is b-Cauchy;
iii) A b-metric is not continuous generally.

Definition 2.5. *[28] (Simulation function) Let $\beth : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be a function. If $\beth$ satisfies the criteria below:*
($\beth$1) $\beth(0, 0)$=0;
($\beth$2) $\beth(t, s) < s - t$ *for all* $t, s > 0$;
($\beth$3) if $\{t_n\}$,$\{s_n\}$ *are sequences in* $(0, \infty)$ *such that*

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0,$$

then

$$\limsup_{n\to\infty} \beth(t_n, s_n) < 0,$$

then it is referred as a simulation function. The set of all simulation functions is denoted by the symbol $\Im$.

Example 2.6. *[28] Let* $\beth : [0,\infty)\times[0,\infty) \to F$ *be defined by* $\beth(t,s) = \lambda s - t$ *for all* $t,s \in [0,\infty)$, *where* $\lambda \in [0,1)$. *Then* $\beth$ *is a simulation function.*

Example 2.7. *[28] Let* $\beth : [0,\infty)\times[0,\infty) \to F$ *be defined by*
$\beth(t,s) = \Psi(s) - \Phi(t)$ *for all* $t,s \in [0,\infty)$, *where* $\Psi, \Phi : [0,\infty) \to [0,\infty)$
are two continuous functions such that $\Psi(t) = \Phi(t) = 0$ *iff* $t = 0$
and $\Psi(t) < t \leq \Phi(t)$ *for all* $t > 0$. *Here,* $\beth$ *is a simulation function.*

Definition 2.8. *[28] ($\Im$-contraction) Let* (F,Ω) *be a metric space,* $T : F \to F$ *be a mapping and* $\beth \in \Im$. T *is called a* $\Im$*-contraction with regard to* $\beth$ *if the following condition holds*

$$\beth(\Omega(Tx,Ty),\Omega(x,y)) \geq 0 \quad for \;\; all \quad x,y \in F.$$

If T *is a* $\Im$*-contraction with respect to* $\beth \in \Im$, *then* $\Omega(Tx,Ty) < \Omega(x,y)$ *for all distinct* $x,y \in F$.

Theorem 2.9. *[28] Suppose* (F,Ω) *is a complete metric space and* $T : F \to F$ *is a* $\Im$*- contraction with respect to* $\beth \in \Im$. *Then* T *has a unique fixed point* $\boldsymbol{u}$ *in* F *and for every* $x_0 \in F$, *the Picard sequence* $\{x_n\}$ *(where* $x_n = Tx_{n-1}$ *for all* $n \in N$*) converges to the fixed point of* T.

Definition 2.10. *[29] (Generalized $\Im$-contraction) Suppose* (F,Ω) *is a metric space,* $T : F \to F$ *is a mapping and* $\beth \in \Im$. *Then* T *is referred to as a generalized* $\Im$*-contraction with regard to* $\beth$ *If the following condition is satisfied:*

$$\beth(\Omega(Tx,Ty),M(x,y)) \geq 0 \quad \forall \quad x,y \in F,$$

where

$$M(x,y) = max\{\Omega(x,y),\Omega(x,Tx),\Omega(y,Ty),\frac{1}{2}(\Omega(x,Ty)+\Omega(y,Tx))\}.$$

Theorem 2.11. *[29] Assume* (F,Ω) *is a complete metric space and* $T : F \to F$ *is a generalized* $\Im$*-contraction with respect to* $\beth \in \Im$, *then* T *has a fixed point in* F. *Moreover, for every* $x_0 \in F$, *the Picard sequence* $\{T^n x_0\}$ *converges to this fixed point.*

Definition 2.12. *[33] (b-simulation function) Let* (F,Ω_b) *be a b-metric space with a constant* $b \geq 1$. *A b-simulation function is a function* $\beth : [0,\infty)\times[0,\infty) \to R$ *satisfying the following conditions:* ($\beth$1) $\beth(t,s) < s - t$ *for all* $t,s > 0$;
($\beth$2) *If* $\{t_n\}$,$\{s_n\}$ *are two sequences in* $(0,\infty)$ *such that*

$$0 < \lim_{n\to\infty} t_n \leq \lim_{n\to\infty} \inf s_n \leq \lim_{n\to\infty} \sup s_n \leq b \lim_{n\to\infty} t_n < \infty,$$

then

$$\lim_{n\to\infty} \sup \beth(bt_n, s_n) < 0.$$

We represent the set of all b-simulation functions by the symbol $\Im_b$.

Some examples of b-simulation functions are as follows.

Example 2.13. *[33] Let $t, s \in [0, \infty)$.*

(1) $\beth(t, s) = \Psi(s) - \Phi(t)$, where $\Phi, \Psi : [0, \infty) \to [0, \infty)$ are two continuous functions such that $\Psi(t) = \Phi(t) = 0 \iff t = 0$ and $\Psi(t) < t \leq \Phi(t)$ for all $t > 0$;

(2) $\beth(t, s) = s\frac{y(t,s)}{z(t,s)}t$, where $y, z : [0, \infty) \times [0, \infty) \to [0, \infty)$ are two functions which are continuous respect to each variable, i.e., $y(t, s) > z(t, s)$ for all $t, s > 0$;

(3) $\beth(t, s)=s - \Phi(s) - t$, where $\Phi : [0, \infty) \to [0, \infty)$ is a lower semi-continuous function such that $\Phi(t) = 0 \Leftrightarrow t = 0$;

(4) $\beth(t, s) = s\Phi(s) - t$, where $\Phi : [0, \infty) \to [0, \infty)$ is such that $\lim_{t \to p^+} \Phi(t) < 1 \ \forall \ p > 0$;

(5) $\beth(t, s) = \lambda s - t$, where $\lambda \in [0, \infty)$.

Definition 2.14. *[31] Let (F, Ω_b, b) be a b-metric and y, z be two self mappings on F. Then the pair $\{y, z\}$ is said to be compatible if*

$$\lim_{n\to\infty} \Omega_b(yzx_n, zyx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in F such that

$$\lim_{n\to\infty} yx_n = \lim_{n\to\infty} zx_n = p \quad \text{for some } p \in F.$$

Lemma 2.15. *[24] Let (F, Ω_b, b) be a b-metric space. If there exist two sequences $\{f_n\}$ and $\{r_n\}$ such that*

$$\lim_{n\to\infty} \Omega_b(f_n, r_n) = 0,$$

whenever $\{f_n\}$ is a sequence in F such that

$$\lim_{n\to\infty} f_n = p \quad \text{for some } p \in F$$

then

$$\lim_{n\to\infty} r_n = p.$$

Theorem 2.16. *[30] Let $T : F \to F$ be a mapping and (F, Ω_b) be a complete b-metric space with a constant $b \geq 1$. Assume there is a b-simulation function $\beth$ such that $\beth(b\Omega_b(Tf, Tr), \Omega_b(f, r)) \geq 0$ for all $f, r \in F$, then T has a unique fixed point.*

In this work, we introduce generalized $\Im_b$-contraction pairs of self-mappings on a b-metric space. We will show that such mappings have a common fixed point. Some examples and applications are presented making effective the new concepts and obtained results. Well known results in literature are investigated and compared.

3. Results via generalized $\Im_b$-contractions

This section includes the main work of this article. We initiate this section with the definition of generalized $\Im_b$-contractions and a related example. Before the proof of the main theorem, we prove some basic lemmas. For the support of the main theorem, some examples are presented. In the last of this section, some remarks are presented.

Definition 3.1. *Consider a b-metric space* (F, Ω_b, b) *with* $b \geq 1$, $f_1, f_2 : F \to F$ *are two mappings and* $\beth \in \Im_b$.

Then f_1, f_2 *are called generalized* $\Im_b$*-contractions with respect to* $\beth$ *if the circumstance listed below is true*

$\beth(b\Omega_b(f_1 s, f_1 t), M_b(s,t)) \geq 0$ *for all* $s, t \in F$,

where

$$M_b(s,t) = \max[\Omega_b(f_2 s, f_2 t), \Omega_b(f_2 t, f_1 t), \frac{1}{2b}\Omega_b(f_2 s, f_1 t)].$$

Example 3.2. *Consider the be metric space* (F, Ω_b, b) *with* $F = [1,2]$ *and* $\Omega_b = (s-t)^2$ *for all* $s, t \in F$ *(Here, b=2). Then the self-mappings* $f_1, f_2 : F \to F$ *defined by*

$$f_1(s) = (\frac{s}{8})^2 \quad and$$

$$f_2(s) = \frac{s}{8}$$

are generalized $\Im_b$*-contractions with respect to* $\beth(s,t) = \frac{1}{2}s - t$. *Indeed,*

$$\beth(b\Omega_b(f_1, f_1), M_b(s,t)) \geq 0.$$

Lemma 3.3. *Suppose* (F, Ω_b, b) *is a b-metric space, and* $f_1, f_2 : F \to F$ *are two generalized* $\Im_b$*-contractions. Suppose* $f_1(F) \subseteq f_2(F)$ *and there is a b-simulation function* $\beth$ *such that*

$$\beth[b\Omega_b(f_1 t, f_1 s), M_b(t,s)] \geq 0 \ \ \forall\ t, s \in F, \tag{3.1}$$

then there is a sequence $\{a_n\}$ *in* F *such that*

$$\lim_{n\to\infty} \Omega_b(a_{n-1}, a_n) = 0.$$

Proof. Assume t_0 is an arbitrary point. Since $f_1(F) \subseteq f_2(F)$, we can construct two sequences $\{t_n\}$ and $\{s_n\}$ such that $s_n = f_1(t_n) = f_2(t_{n+1})$ for every $n \in \mathbb{N}$. If there is $n_0 \in N$ such that $t_{n_0} = t_{n_0+1}$, then it follows from the given inequality (3.1) and from ($\beth$1) that for all $n \in \mathbb{N}$,

$$\begin{aligned} 0 &\leq \beth(b\Omega_b(f_1 t_{n_0+1}, f_1 t_{n_0+2}), M_b(f_2 t_{n_0+1}, f_2 t_{n_0+2})) \\ 0 &\leq M_b(f_2 t_{n_0+1}, f_2 t_{n_0+2}) - b\Omega_b(f_1 t_{n_0+1}, f_1 t_{n_0+2}) \\ &= M_b(s_{n_0}, s_{n_0+1}) - b\Omega_b(s_{n_0+1}, s_{n_0+2}). \end{aligned}$$

That is,

$$\begin{aligned} M_b(f_2 t, f_2 s) &= \max[\Omega_b(f_2 t, f_2 s), \Omega_b(f_2 s, f_1 s), \frac{1}{2b}\Omega_b(f_2 t, f_1 s)] \\ M_b(f_2 t_{n_0+1}, f_2 t_{n_0+2}) &= \max[\Omega_b(f_2 t_{n_0+1}, f_2 t_{n_0+2}), \Omega_b(f_2 t_{n_0+2}, f_1 t_{n_0+2}), \frac{1}{2b}\Omega_b(f_2 t_{n_0+1}, f_1 t_{n_0+2})] \\ &= \max[\Omega_b(s_{n_0}, s_{n_0+1}), \Omega_b(s_{n_0+1}, s_{n_0+2}), \frac{1}{2b}\Omega_b(s_{n_0}, s_{n_0+2})]. \end{aligned}$$

Therefore, by triangle inequality,

$$\begin{array}{rcl} M_b(f_2t_{n_0+1}, f_2t_{n_0+2}) & = & \max[\Omega_b(s_{n_0}, s_{n_0+1}), \Omega_b(s_{n_0+1}, s_{n_0+2})] \\ 0 & \leq & \max[\Omega_b(s_{n_0}, s_{n_0+1}), \Omega_b(s_{n_0+1}, s_{n_0+2})] - b\Omega_b(s_{n_0+1}, s_{n_0+2}). \end{array}$$

Since $s_{n_0} = s_{n_0+1}$ implies that $\Omega_b(s_{n_0+1}, s_{n_0}) = 0$, consider

$$\begin{array}{rcl} 0 & \leq & \max[0, \Omega_b(s_{n_0+1}, s_{n_0+2})] - b\Omega_b(s_{n_0+1}, s_{n_0+2}) \\ 0 & < & \Omega_b(s_{n_0+1}, s_{n_0+2}) - b\Omega_b(s_{n_0+1}, s_{n_0+2}) \\ 0 & < & [1-b]\Omega_b(s_{n_0+1}, s_{n_0+2}) \leq 0 \\ \Omega_b(s_{n_0+1}, s_{n_0+2}) & = & 0 \end{array}$$

$$s_{n_0} = s_{n_0+1} = s_{n_0+2} = s_{n_0+3} = \cdots ,$$

which implies that

$$\lim_{n\to\infty}\Omega_b(s_{n-1}, s_n) = 0.$$

Now, suppose that $s_n \neq s_{n+1}$ for every $n \in \mathbb{N}$. Then, it follows from (3.1) and ($\sqsupset$1) that for every $n \in \mathbb{N}$, we have

$$\begin{array}{rcl} 0 & \leq & \sqsupset[b\Omega_b(f_1t_n, f_1t_{n+1}), M_b(f_2t_n, f_2t_{n+1})] \\ & = & \sqsupset[b\Omega_b(s_n, s_{n+1}), M_b(s_{n-1}, s_n)] \\ & < & M_b(s_{n-1}, s_n) - b\Omega_b(s_n, s_{n+1}) \\ & = & \max[\Omega_b(s_{n-1}, s_n), \Omega_b(s_n, s_{n+1})] - b\Omega_b(s_n, s_{n+1}). \end{array}$$

If $\Omega_b(s_n, s_{n+1}) \geq \Omega_b(s_{n-1}, s_n)$, then $0 < \Omega_b(s_n, s_{n+1}) - b\Omega_b(s_n, s_{n+1})$. That is,

$$b\Omega_b(s_n, s_{n+1}) < \Omega_b(s_n, s_{n+1}),$$

which is a contradiction. So we have

$$\begin{array}{rcl} \Omega_b(s_{n-1}, s_n) & \geq & \Omega_b(s_n, s_{n+1}) \\ 0 & < & \Omega_b(s_{n-1}, s_n) - b\Omega_b(s_n, s_{n+1}) \\ b\Omega_b(s_n, s_{n+1}) & = & \Omega_b(s_{n-1}, s_n) \ \forall\ n \in \mathbb{N}. \end{array}$$

This implies that $\{\Omega_b(s_{n-1}, s_n)\}$ is a decreasing sequence of positive real numbers. Thus, there is some $\Gamma \geq 0$, so that

$$\lim_{n\to\infty} \Omega_b(s_{n-1}, s_n) = \Gamma.$$

Assume $\Gamma > 0$, so from the condition ($\sqsupset$2), with $a_n = \Omega_b(s_n, s_{n+1})$ and $b_n = \Omega_b(s_{n-1}, s_n)$, one writes

$$0 \leq \lim_{n\to\infty} \sup \sqsupset[b\Omega_b(s_n, s_{n+1}), M_b(s_{n-1}, s_n)] < 0,$$

which is a contradiction. Hence, we get that $\Gamma = 0$. It ends the proof. □

Remark 3.4. Let (F, Ω_b, b) be a b-metric space and assume $f_1, f_2 : F \to F$ are two generalized $\Im_b$-contractions. Assume that $f_1(F) \subseteq f_2(F)$ and there is a b-simulation function $\beth$ such that

$$0 \leq \beth[b\Omega_b(f_1 t, f_1 s), M_b(f_2 t, f_2 s)] \; \forall\, t, s \in F.$$

Then there is a sequence $\{s_n\}$ in F such that

$$b\Omega_b(s_m, s_n) \leq M_b(s_{m-1}, s_{n-1}) \; \forall\, m, n \in \mathbb{N}.$$

Proof. By a similar argument of Lemma 3.3 for every $n \in \mathbb{N}$, we have $s_n = f_1(t_n) = f_2(t_{n+1})$. Hence, from (3.1) and ($\beth$1), we have for all $m, n \in \mathbb{N}$,

$$\begin{aligned} 0 &\leq \beth[b\Omega_b(f_1 t_m, f_1 t_n), M_b(f_2 t_m, f_2 t_n)] \\ 0 &\leq \beth[b\Omega_b(s_m, s_n), M_b(s_{m-1}, s_{n-1})] \\ 0 &< M_b(s_{m-1}, s_{n-1}) - b\Omega_b(s_m, s_n) \\ b\Omega_b(s_m, s_n) &< M_b(s_{m-1}, s_{n-1}). \end{aligned}$$

□

Lemma 3.5. *Let (F, Ω_b, b) be a b-metric space and assume $f_1, f_2 : F \to F$ be two generalized $\Im_b$-contractions. Assume that $f_1(F) \subseteq f_2(F)$ and there is a b-simulation function $\beth$ such that the inequality* (3.1) *holds. Then there exists a sequence $\{s_n\}$ in F, such that $\{s_n\}$ is a bounded sequence.*

Proof. By a similar argument of Lemma 3.3, when for some n_0 $s_{n_0} = s_{n_0+1}$ we have $\Omega_b(s_i, s_j) \leq M$ for all $i, j = 0, 1, 2, \cdots$, where

$$M = \max\{\Omega_b(s_i, s_j) : i, j \leq n_0\}.$$

Let us assume that $s_n \neq s_{n+1}$ for each $n \in \mathbb{N}$ and suppose $\{s_n\}$ is a sequence which is not bounded. Then there is a subsequence $\{s\}_{n_k}$ of $\{s_n\}$ such that for $n_1 = 1$ and for each $k \in N$, n_{k+1} is the minimum integer such that $\Omega_b(s_{n_{k+1}}, s_{n_k}) > 1$ and $\Omega_b(s_m, s_{n_k}) \leq 1$ for $n_k \leq m \leq n_{k+1} - 1$. By triangular inequality, we obtain

$$\begin{aligned} 1 &< \Omega_b(s_{n_{k+1}}, s_{n_k}) \\ &\leq b[\Omega_b(s_{n_{k+1}}, s_{n_{k+1}-1}) + \Omega_b(s_{n_{k+1}-1}, s_{n_k})] \\ &\leq b[\Omega_b(s_{n_{k+1}}, s_{n_{k+1}-1}) + 1)] \\ &= b\Omega_b(s_{n_{k+1}}, s_{n_{k+1}-1}) + b). \end{aligned}$$

Letting $k \to \infty$ in the above inequality and using Lemma 3.3, we get

$$1 \leq \lim_{k\to\infty} \inf \Omega_b(s_{n_{k+1}}, s_{n_k}) \leq \lim_{k\to\infty} \sup \Omega_b(s_{n_{k+1}}, s_{n_k}) \leq b.$$

Again from Remark 3.4 we have

$$\begin{aligned} b\Omega_b(s_{n_{k+1}}, s_{n_k}) &\leq M_b(s_{n_{k+1}-1}, s_{n_k-1}) \\ &= \max[\Omega_b(s_{n_{k+1}-1}, s_{n_k-1}), \Omega_b(s_{n_k-1}, s_{n_k}), \frac{1}{2b}\Omega_b(s_{n_{k+1}-1}, s_{n_k})] \end{aligned}$$

$$\begin{aligned}
&\leq \max[b[\Omega_b(s_{n_{k+1}-1}, s_{n_k}) + \Omega_b(s_{n_k}, s_{n_k-1})], \Omega_b(s_{n_k-1}, s_{n_k}), \frac{1}{2b}\Omega_b(s_{n_{k+1}-1}, s_{n_k})] \\
&\leq \max[b[1 + \Omega_b(s_{n_k}, s_{n_k-1})], \Omega_b(s_{n_k-1}, s_{n_k}), \frac{1}{2b}(1)] \\
&\leq \max[b[1 + \Omega_b(s_{n_k}, s_{n_k-1})], \Omega_b(s_{n_k-1}, s_{n_k}), \frac{1}{2b}(b)] \\
&= \max[b(1 + \Omega_b(s_{n_k}, s_{n_k-1}), \Omega_b(s_{n_k-1}, s_{n_k}), \frac{1}{2})].
\end{aligned}$$

Now, as

$$\begin{aligned}
1 &< \Omega_b(s_{n_{k+1}}, s_{n_k}) \\
b &\leq b\Omega_b(s_{n_{k+1}}, s_{n_k}) \\
&< M_b(s_{n_{k+1}-1}, s_{n-k-1}) \\
&\leq \max[b[1 + \Omega_b(s_{n_k}, s_{n_k-1})], \Omega_b(s_{n_k-1}, s_{n_k}), \frac{1}{2}],
\end{aligned}$$

then taking $k \to \infty$, one gets

$$\begin{aligned}
b &\leq \lim_{k\to\infty} M_b(s_{n_{k+1}-1}, s_{n_k}) \\
&\leq \lim_{k\to\infty} \max[b[1 + \Omega_b(s_{n_k}, s_{n_k-1})], \Omega_b(s_{n_k-1}, s_{n_k}), \frac{1}{2}] \\
&= \max[b(1 + 0), 0, \frac{1}{2}] \\
&= \max[b] = b,
\end{aligned}$$

i.e.,

$$\lim_{k\to\infty} \Omega_b(s_{n_k-1}, s_{n_k}) = 0.$$

That is,

$$\lim_{k\to\infty} M_b(s_{n_{k+1}-1}, s_{n_k-1}) = b.$$

Using the inequality (3.1) and ($\sqsupset$2) with $a_k = \Omega_b(s_{n_{k+1}}, s_{n_k})$ and $c_k = M_b(s_{n_{k+1}-1}, s_{n_k-1})$, we have

$$0 \leq \lim_{k\to\infty} \sup \sqsupset[b\Omega_b(s_{n_{k+1}}, s_{n_k}), M_b(s_{n_{k+1}-1}, s_{n_k-1})] < 0,$$

which is a contradiction. Hence, $\{s_n\}$ is a bounded sequence. □

Lemma 3.6. *Suppose (F, Ω_b, b) is a b-metric space and assume $f_1, f_2 : F \to F$ are two generalized $\Im_b$-contractions. Assume that $f_1(F) \subseteq f_2(F)$ and there is a b-simulation function $\sqsupset$ such that the inequality* (3.1) *holds. Then there is a sequence $\{s_n\}$ in F, such that $\{s_n\}$ is a Cauchy sequence.*

Proof. Using a similar argument as in Lemma 3.3, we have for every $n \in \mathbb{N}$, $s_n = f_1(t_n) = f_2(t_{n+1})$. If there is $n_o \in N$ such that $s_{n_0} = s_{n_0+1}$, then we have $\{s_n\}$ is a cauchy sequence. Let us $s_n \neq s_{n+1}$ for every $n \in \mathbb{N}$ and let

$$C_n = \sup\{\Omega_b(s_i, s_j) : i, j \geq n\}.$$

Now, from Lemma 3.3, $C_n < \infty$ for every $n \in \mathbb{N}$. Since C_n is a positive decreasing sequence, there is some $c \geq 0$ such that

$$\lim_{n\to\infty} C_n = c.$$

Let us consider that $c > 0$. Then by the definition of C_n, for every $k \in N$ there are $n_k, m_k \in N$ such that $m_k > n_k \geq k$ and

$$C_k - \frac{1}{k} < \Omega_b(s_{m_k}, s_{n_k}) \leq C_k.$$

Letting $k \to \infty$ in the inequality above, we have

$$\lim_{k\to\infty} \Omega_b(s_{m_k}, s_{n_k}) = c$$

and

$$\lim_{k\to\infty} \Omega_b(s_{m_k-1}, s_{n_k-1}) = c.$$

By the inequality 3.1 and property ($\beth 1$), we have

$$\begin{aligned}
0 &\leq \beth[b\Omega_b(s_{m_k}, s_{n_k}), M_b(s_{m_k-1}, s_{n_k-1})] \\
&< M_b(s_{m_k-1}, s_{n_k-1}) - b\Omega_b(s_{m_k}, s_{n_k}) \\
b\Omega_b(s_{m_k}, s_{n_k}) &< M_b(s_{m_k-1}, s_{n_k-1}) \\
&= \max[\Omega_b(f_2 t_{m_k}, f_2 t_{n_k}), \Omega_b(f_2 t_{n_k}, f_1 t_{n_k}), \frac{1}{2b}\Omega_b(f_2 t_{m_k}, f_1 t_{n_k})] \\
&= \max[\Omega_b(s_{m_k-1}, s_{n_k-1}), \Omega_b(s_{n_k-1}, s_{n_k}), \frac{1}{2b}\Omega_b(s_{m_k-1}, s_{n_k})] \\
&\leq \max[\Omega_b(s_{m_k-1}, s_{n_k-1}), \Omega_b(s_{n_k-1}, s_{n_k}), \frac{1}{2b}[b(\Omega_b(s_{m_k-1}, s_{m_k}) + \Omega_b(s_{m_k}, s_{n_k}))]] \\
&= \max[\Omega_b(s_{m_k-1}, s_{n_k-1}), \Omega_b(s_{n_k-1}, s_{n_k}), \frac{1}{2}(\Omega_b(s_{m_k-1}, s_{m_k}) + \Omega_b(s_{m_k}, s_{n_k}))].
\end{aligned}$$

Letting $\lim_{k\to\infty}$ in the above inequality using Lemma 3.3,

$$\lim_{k\to\infty} \Omega_b(s_{m_k}, s_{n_k}) = c$$

and

$$\lim_{k\to\infty} \Omega_b(s_{m_k-1}, s_{n_k-1}) = c.$$

We have

$$\begin{aligned}
bc &= \lim_{k\to\infty} b\Omega_b(s_{m_k}, s_{n_k}) \\
&\leq M_b(s_{m_k-1}, s_{n_k-1}) \\
&\leq \lim_{k\to\infty} \max[\Omega_b(s_{m_k-1}, s_{n_k-1}), \Omega_b(s_{n_k-1}, s_{n_k}), \frac{1}{2}(\Omega_b(s_{m_k-1}, s_{m_k}) + \Omega_b(s_{m_k}, s_{n_k}))] \\
&= \max[c, 0, \frac{1}{2}(0 + c)] \\
&= c.
\end{aligned}$$

Then

$$bc \leq \lim_{k\to\infty} \inf M_b(s_{m_k-1}, s_{n_k-1}) \leq \lim_{k\to\infty} \sup M_b(s_{m_k-1}, s_{n_k-1}) \leq c.$$

From the above inequality and since $c > 0$ that is $b = 1$, then by the property ($\beth 2$) with

$$a_k = \Omega_b(s_{m_k}, s_{n_k})$$

and

$$q_k = M_b(s_{m_k-1}, s_{n_k-1})$$

we get

$$0 \leq \lim_{k\to\infty} \sup \beth[b\Omega_b(s_{m_k}, s_{n_k}), M_b(s_{m_k-1}, s_{n_k-1})] < 0$$

which is a contradiction, thus $c = 0$, i.e.,

$$\lim_{n\to\infty} c_n = 0 \quad \forall \quad b \geq 1.$$

This proves that $\{s_n\}$ is a Cauchy sequence. □

We are now going to present our main result.

Theorem 3.7. *Consider* (F, Ω_b, b) *be a complete b-metric space, and* $f_1, f_2 : F \to F$ *be two generalized* $\Im_b$*-contractions with* $f_1(F) \subseteq f_2(F)$ *and the pair* (f_1, f_2) *is compatible. Assume that* $\exists$ *a b-simulation function* $\beth$ *such that 3.1 holds, that is,*

$$\beth[b\Omega_b(f_1 t, f_1 s), M_b(f_2 t, f_2 s)] \geq 0 \quad \forall\, t, s \in F.$$

If f_2 *is continuous, then there is a coincidence point of* f_1 *and* f_2 *, that is, there exists* $t \in F$ *such that* $f_1(t) = f_2(t)$*. Moreover, if* f_2 *is one to one, then* f_1 *and* f_2 *have a unique common fixed point.*

Proof. Consider $x_0 \in F$. Since $f_1(F) \subseteq f_2(F)$, we have for every $n \in \mathbb{N}$, $s_n = f_1(t_n) = f_2(t_{n+1})$. Now, from Lemma 3.6, the sequence $\{s_n\}$ is Cauchy and since (F, Ω_b, b) is a complete b-metric space, there is some $s \in F$ such that

$$\lim_{n\to\infty} s_n = s,$$

that is,

$$s = \lim_{n\to\infty} f_1(t_n) = \lim_{n\to\infty} f_2(t_n).$$

We claim that **s** is a coincidence point of f_1 and f_2. Since f_2 is continuous, we have

$$\lim_{n\to\infty} f_2 f_1(t_n) = f_2 f_2(t_n) = f_2(s).$$

Also, since $\{f_1, f_2\}$ is compatible, we have

$$\lim_{n\to\infty} \Omega_b(f_1 f_2(t_n), f_2 f_1(t_n)) = 0.$$

Hence, by Lemma 2.15 we deduce

$$\lim_{n\to\infty} f_1 f_2(t_n) = f_2(s).$$

Using (3.1), we have

$$0 \leq \beth[b\Omega_b(f_1 s, f_1 f_2(t_n)), M_b(f_2 s, f_2 f_2(t_n))].$$

That is,

$$< M_b(f_2 s, f_2 f_2(t_n)) - b\Omega_b(f_1 s, f_1 f_2(t_n)).$$

Letting $n \to \infty$,

$$0 < \liminf_{n\to\infty} M_b(f_2 s, f_2 f_2(t_n)) - b \limsup_{n\to\infty} \Omega_b(f_1 s, f_1 f_2(t_n)).$$

But

$$\begin{aligned}
M_b(f_2 s, f_2 f_2(t_n)) &= \max[\Omega_b(f_2 s, f_2 f_2(t_n)), \Omega_b(f_2 s, f_1 f_2(t_n)), \frac{1}{2b}\Omega_b(f_2 s, f_1 f_2(t_n))] \\
\liminf_{n\to\infty} M_b(f_2 s, f_2 f_2(t_n)) &= \liminf_{n\to\infty} \max[\Omega_b(f_2 s, f_2 f_2(t_n)), \Omega_b(f_2 s, f_1 f_2(t_n)), \frac{1}{2b}\Omega_b(f_2 s, f_1 f_2(t_n))] \\
&= \max[\Omega_b(f_2 s, f_2 s), \Omega_b(f_2 s, f_2 s), \frac{1}{2b}\Omega_b(f_2 s, f_2 s)] \\
&= 0
\end{aligned}$$

which implies that

$$\begin{aligned}
0 &< M_b(f_2 s, f_2 f_2(t_n)) - b\Omega_b(f_1 s, f_1 f_2(t_n)) \\
0 &< \liminf_{n\to\infty} M_b(f_2 s, f_2 f_2(t_n)) - b \limsup_{n\to\infty} \Omega_b(f_1 s, f_1 f_2(t_n)) \\
&= -b \limsup_{n\to\infty} \Omega_b(f_1 s, f_1 f_2(t_n)) \leq 0.
\end{aligned}$$

Thus,

$$\limsup_{n\to\infty} \Omega_b(f_1 s, f_1 f_2(t_n)) = 0$$

that is

$$\lim_{n\to\infty} f_1 f_2(t_n) = f_1(s)$$

therefore $f_1(s) = f_2(s)$.

Now, assume there is $p \in F$ such that $f_1(p) = f_2(p)$ then the inequality (3.1) and ($\beth$2) imply that

$$0 \leq \beth[b\Omega_b(f_1 s, f_1 p), M_b(f_2 s, f_2 p)]$$

$$= M_b(f_2 s, f_2 p) - b\Omega_b(f_1 s, f_1 p),$$

where

$$\begin{aligned}
M_b(f_2 s, f_2 p) &= \max[\Omega_b(f_2 s, f_2 p), \Omega_b(f_2 p, f_1 p), \frac{1}{2b}\Omega_b(f_2 s, f_1 p)] \\
&= \max[\Omega_b(f_2 s, f_1 p), \frac{1}{2b}\Omega_b(f_2 s, f_1 p)] \\
M_b(f_2 s, f_2 p) &= \Omega_b(f_2 s, f_1 p) \\
0 &< \Omega_b(f_2 s, f_1 p) - b\Omega_b(f_1 s, f_1 p)
\end{aligned}$$

$$
\begin{aligned}
&= \Omega_b(f_2 s, f_1 p) - b\Omega_b(f_2 s, f_1 p) \\
&= [1-b]\Omega_b(f_2 s, f_1 p) \\
&\le 0 \\
\Omega_b(f_2 s, f_1 p) &= 0.
\end{aligned}
$$

Hence,

$$b\Omega_b(f_1 s, f_1 p) \le \Omega_b(f_1 s, f_1 p).$$

If $b > 1$ then $f_1(s) = f_1(p)$. If $b = 1$, by the condition ($\beth 2$) with

$$a_k = \Omega_b(f_1 s, f_1 p)$$

and

$$v_k = M_b(f_2 s, f_2 p)$$

we get

$$0 \le \limsup_{k\to\infty} \beth[b\Omega_b(f_1 s, f_1 p), M_b(f_2 s, f_2 p)] < 0$$

which is a contradiction. Therefore,

$$f_1(p) = f_1(s) = f_2(p) = f_2(s).$$

Now, suppose that f_2 is one to one. If s, p are two coincidence points of f_1 and f_2, In this case, by the above argument we have

$$f_1(s) = f_2(s) = f_1(p) = f_2(p).$$

Since f_2 is one to one, it follows that $p = s$. Also, since $f_2(s) = f_1(s)$ and the pair $\{f_1, f_2\}$ is compatible we have

$$f_1 f_2(s) = f_2 f_1(s).$$

Therefore,

$$f_2 f_1(s) = f_1 f_2(s) = f_1 f_1(s).$$

That is, $f_1(s)$ is a coincidence point of f_1 and f_2. Therefore, $f_1(s) = s$ and hence

$$f_1(s) = f_2(s) = s.$$

That is, f_1 and f_2 have a unique common fixed point $s \in F$. □

Corollary 3.8. *Let (F, Ω_b, b) be a complete b-metric space and $f_1, f_2 : F \to F$ be two generalized $\Im_b$-contractions with $f_1(F) \subseteq f_2(F)$ and the pair (f_1, f_2) is compatible. Suppose that there is $\lambda \in (0, 1)$ such that*

$$b\Omega_b(f_1 s, f_1 t) \le \lambda M_b(s, t) \quad \forall\ s, t \in F.$$

If f_2 is continuous, then there is a coincidence point of f_1 and f_2, that is, there is $t \in F$ such that $f_1(t) = f_2(t)$. Moreover, if f_2 is one to one, then f_1 and f_2 have a unique common fixed point.

Proof. By taking the b-simulation function

$$\beth(x,y) = \lambda y - x \quad \forall\ x, y \geq 0.$$

The result follows from Theorem 3.7. □

Example 3.9. *Let* $X = [0,\infty)$ *and* $d : X \times X \to X$ *be defined by*

$$d(s,t) = \begin{cases} 0 & \text{if } s=t; \\ 8 & \text{if } s,t \in [0,1); \\ 3 + \frac{1}{s+t} & \text{if } s,t \in [1,\infty); \\ \frac{33}{25} & \text{otherwise.} \end{cases}$$

Then, clearly d *is a b-metric on* X *with* $b = \frac{5}{4}$.

Here, we observe that when $s = \frac{3}{2}$ *and* $u = 2$ *(both belong to* $[1,\infty)$ *and* $t \in [0,\infty)$*), we have*

$$d(s,u) = 3 + \frac{1}{\frac{3}{2}+2} = \frac{23}{7}$$

and

$$d(s,t) + d(t,u) = \frac{33}{25} + \frac{33}{25} = \frac{66}{25}.$$

Hence, $d(s,u) \neq d(s,t) + d(t,u)$. *Hence,* d *is a b-metric with* $b = \frac{5}{4}(> 1)$, *but it is not a metric. We now define* $f, g : X \to X$ *by*

$$f(s) = \begin{cases} \frac{s}{4} + 2 & \text{if } s \in [0,1); \\ 3s - 2 & \text{if } s \in [1,\infty) \end{cases}$$

and

$$g(s) = \begin{cases} s & \text{if } s \in [0,1); \\ \frac{1}{s} & \text{if } s \in [1,\infty). \end{cases}$$

Clearly, f *and* g *are b-continuous functions. Now, we define* $\beth : [0,\infty) \times [0,\infty) \to [0,\infty)$ *by* $\beth(x,y) = \frac{4}{5}y - x$. *We have the following possible cases:*

Case 1: $s,t \in [0,1)$.

In this case , $d(fs, ft) = 3 + \frac{1}{s+t}$ *and* $M_b(s,t) = 8$

$$\beth(bd(fs,ft), M_b(s,t)) = \frac{4}{5}(8) - \frac{5}{4}(3 + \frac{1}{s+t}) > 0.$$

Case 2: $s,t \in [1,\infty)$.

Here, $d(fs, ft) = 3 + \frac{1}{s+t}$ *and* $M_b = 8$. *We have*

$$\beth(bd(fs,ft), M_b(s,t)) = \frac{4}{5}(8) - \frac{5}{4}(3 + \frac{1}{s+t}) > 0.$$

Case 3: $s \in [0,1)$ *and* $t \in [1,\infty)$.

Here, $d(fs, ft) = 3 + \frac{1}{s+t}$ *and* $M_b = 8$. *Also,*

$$\beth(bd(fs,ft),M_b(s,t)) = \frac{4}{5}(8) - \frac{5}{4}(3+\frac{1}{s+t}) > 0.$$

Case 4: $s \in [1,\infty)$ and $t \in [0,1)$.
In this case $d(fs,ft) = 3 + \frac{1}{s+t}$ and $M_b = 8$. Also,

$$\beth(bd(fs,ft),M_b(s,t)) = \frac{4}{5}(8) - \frac{5}{4}(3+\frac{1}{s+t}) > 0.$$

So the pair $\{f,g\}$ is a generalized $\Im_b$-contraction. It satisfies all the conditions of Theorem 3.7. Hence, f and g have a common unique fixed point.

Example 3.10. *Let $\mathsf{F} = [0,1]$ be endowed with the b-metric $\Omega_b(s,t) = (s-t)^2$, where $b = 2$. Define f_1 and f_2 on F by*

$$f_1(s) = (\frac{s}{4})^2$$

and

$$f_2(s) = (\frac{s}{4}).$$

Obviously $f_1(\mathsf{F}) \subseteq f_2(\mathsf{F})$ and furthermore the pair $\{f_1,f_2\}$ is compatible. Consider the b-simulation function given as

$$\beth(r,q) = \frac{1}{2}q - r \ \ \forall \ \ r,q \geq 0.$$

For all $s,t \in \mathsf{F}$ we have

$$\begin{aligned} 0 &\leq \beth[2\Omega_b(f_1s,f_1t),M_b(s,t)] \\ 0 &< \frac{1}{2}M_b(s,t) - 2\Omega_b(f_1s,f_1t) \\ \Omega_b(f_1s,f_1t) &< \frac{1}{4}M_b(s,t). \end{aligned}$$

Now,

$$\begin{aligned} \Omega_b(f_1s,f_1t) &= (f_1s - f_1t)^2 \\ &= ((\frac{s}{4})^2 - (\frac{t}{4})^2)^2 \\ &= (\frac{s}{4}+\frac{t}{4})^2(\frac{s}{4}-\frac{t}{4})^2 \\ &\leq (\frac{1}{4}+\frac{1}{4})^2(\frac{s}{4}-\frac{t}{4})^2 \\ &= (\frac{2}{4})^2\Omega_b(f_2s,f_2t) \\ &\leq \frac{1}{4}M_b(s,t) \\ &= \frac{1}{4}\max[\Omega_b(f_2s,f_2t),\Omega_b(f_2t,f_1t),\frac{1}{2b}\Omega_b(f_2s,f_1t)]. \end{aligned}$$

As all the requirements of Theorem 3.7 are satisfied, so f_1 and f_2 have a unique common fixed point, which is 0.

Example 3.11. *Take $F = [0,1]$. Define $\Omega_b : F\times F \to R$ by $\Omega_b(s,t) = (s-t)^2$. Clearly, (F,Ω_b) is a complete b-metric with $b = 2$.*

Now, we define the functions $f_1, f_2 : [0,1] \to [0,1]$ by

$$f_1 s = \frac{as}{1+s} \quad \forall \quad s \in F, a \in (0, \frac{1}{\sqrt{2}}]$$

and

$$f_2 t = \frac{t}{1+t} \quad \forall \quad t \in F.$$

Clearly,

$$f_1(s) \subseteq f_2(t)$$

and furthermore the pair is $[f_1, f_2]$ is compatible. Now, consider the b-simulation function $\beth : [0,\infty)\times[0,\infty) \to R$ defined by

$$\beth(x,y) = \frac{f_1}{f_1+1} - x.$$

We have

$$\beth(2\Omega_b(f_1 s, f_1 t), M_b(s,t)) = \frac{M_b(s,t)}{M_b(s,t)+1} - 2\Omega_b(f_1 s, f_1 t)$$

$$\begin{aligned}
&\geq \frac{\Omega_b(s,t)}{\Omega_b(s,t)+1} - 2\Omega_b(f_1 s, f_1 t)\\
&= \frac{(s-t)^2}{(s-t)^2+1} - 2[\frac{as}{1+s} - \frac{at}{1+t}]^2\\
&= \frac{(s-t)^2}{(s-t)^2+1} - 2\frac{a^2(s-t)^2}{[(1+s)(1+t)]^2}\\
&\geq \frac{(s-t)^2}{(s-t)^2+1} - 2\frac{a^2(s-t)^2}{(s-t)^2+1}\\
&= \frac{(s-t)^2 - 2a^2(s-t)^2}{(s-t)^2+1}\\
&= \frac{(1-2a^2)(s-t)^2}{(s-t)^2+1}\\
&\geq 0 \quad \forall\ s,t \in F.
\end{aligned}$$

Thus, all the assumption are satisfied of Theorem 3.7, and hence f_1 and f_2 have a unique common fixed point, which is 0.

Remark 3.12. If in Lemma 3.3, $M_b = \Omega_b(f_2 t, f_2 s)$ in inequality (3.1), then we will get Lemma 3.1 of [24].

Remark 3.13. If in Remark 3.4, $M_b = \Omega_b(f_2 t, f_2 s)$ in inequality (3.1), then we will get Remark 3.2 of [24].

Remark 3.14. If in Lemma 3.5, $M_b = \Omega_b(f_2 t, f_2 s)$ in inequality (3.1), then we will get Lemma 3.3 of [24].

Remark 3.15. If in Lemma 3.6, $M_b = \Omega_b(f_2t, f_2s)$ in inequality (3.1), then we will get Lemma 3.4 of [24].

Remark 3.16. If in Theorem 3.7, $M_b = \Omega_b(f_2t, f_2s)$ in inequality (3.1), then we will get Theorem 3.5 of [24].

4. A system of non linear Urysohn integral equations

In this section, we present an application of our result to integral equations. Namely, we study the existence of the unique common solution of a system of non linear Urysohn integral equations.

Let us consider the integral equations

$$f(x) = r_1(x) + \int_a^b k_1(x, t, f(t))dt \tag{4.1}$$

and

$$g(x) = r_2(x) + \int_a^b k_2(x, t, g(t))dt \tag{4.2}$$

where (i) f, r_1, r_2 and g are unknown functions for each $x \in [a, b]$.

(ii) k_1 and k_2 are kernels defined for $x, t \in [a, b]$.

Let us denote

$$\vartheta_1 f(x) = \int_a^b k_1(x, t, f(t))dt$$

and

$$\vartheta_2 g(x) = \int_a^b k_2(x, t, g(t))dt.$$

Assume that

- (A_1) $\vartheta_1 f(x) + r_1(x) + r_2(x) - \vartheta_2(\vartheta_1 f(x) + r_1(x)) + r_2(x) = 0$
- (A_2) $r_1(x) - r_2(x) + \vartheta_1 f(x) - \vartheta_1 g(x) = 0.$

We will ensure the existence of a unique common solution of (4.1) and (4.2) that belong to $G = (C[a, b], R^n)$ (the set of continuous mappings defined on $[a, b]$). For this, define the continuous mappings $T_1, T_2 : G \to G$ by

$$T_1 f(x) = r_1(x) + \vartheta_1 f(x)$$

and

$$T_2 g(x) = 2f(x) - \vartheta_2 f(x) - r_2(x)$$

where $f, g, r_1, r_2 \in G$. We claim $T_1 \subseteq T_2$.

Proof. If we show that $T_2(T_1 f(x) + r_2(x)) = T_1 f(x)$, then it is conformed that $T_1 \subseteq T_2$. Hence,

$$\begin{aligned} T_2(T_1 f(x) + r_2(x)) &= 2[T_1 f(x) + r_2(x)] - \vartheta_2[T_1 f(x) + r_2(x)] - r_2(x) \\ &= 2T_1 f(x) + 2r_2(x) - \vartheta_2[T_1 f(x) + r_2(x)] - r_2(x) \\ &= T_1 f(x) + r_1(x) + r_2(x) + \vartheta_1 f(x) - \vartheta_2[T_1 f(x) + r_2(x)] \\ &= T_1 f(x) + [r_1(x) + r_2(x) + \vartheta_1 f(x) - \vartheta_2[T_1 f(x) + r_2(x)]]. \end{aligned}$$

Using (A_1),

$$T_2[T_1f(x) + r_2(x)] = T_1f(x).$$

It shows that

$$T_1f(x) \subseteq T_2f(x).$$

We endow on G the b-metric (with $b = 2$) given as $\Omega_b(x, y) = |x - y|^2$. Here, (G, Ω_b) is complete. Further, let us suppose that $k_1, k_2 : [a, b] \times [a, b] \times R^n \to R^n$ are continuous functions satisfying

$$|k_1(x, t, f(t)) - k_2(x, t, f(t))| \leq \frac{\sqrt{M_b(f, g)}}{\sqrt{2}(b - a)}, \tag{4.3}$$

where

$$M_b(f, g) = \max\{\Omega_b(T_2(f), T_2(g)), \Omega_b(T_2(g), T_1(g)), \frac{1}{4}\Omega_b(T_2(f), T_1(g))\}.$$

□

Theorem 4.1. *Under the conditions* (A_1) , (A_2) *and* (4.3)*, the Eqs* (4.1) *and* (4.2) *have a unique common solution.*

Proof. For $f, g \in (G, R^n)$ and $x \in [a, b]$, we define the continuous mappings $T_1, T_2 : G \to G$ by

$$T_1f(x) = r_1(x) + \vartheta_1 f(x)$$

and

$$T_2f(x) = 2f(x) - \vartheta_2 f(x) - r_2(x).$$

Then we have

$$\begin{aligned}
2\Omega_b(T_1(f), T_1(g)) &= 2|T_1(f) - T_2(g)|^2 \\
&\leq 2|T_2f(x) - T_1g(x)|^2 \\
&= 2|2f(x) - \vartheta_2 f(x) - r_2(x) - r_1(x) - \vartheta_1 g(x)|^2 \\
&= 2|[r_1(x) - r_2(x) + \vartheta_1 f(x) - \vartheta_1 g(x)] + \vartheta_1 f(x) - \vartheta_2 f(x)|^2 \\
&= 2|\vartheta_1 f(x) - \vartheta_2 f(x)|^2 \\
&\leq 2(\int_a^b |k_1 - k_2|)^2 dt \leq 2(\int_a^b \frac{\sqrt{M_b(f, g)}}{\sqrt{2}(b - a)} dt) \\
&= M_b(f, g).
\end{aligned}$$

This shows that all the requirements of our main theorem are satisfied, i.e., $2\Omega_b(T_1(f), T_1(g)) \leq M_b(f, g)$. Therefore, the integral equations (4.1) and (4.2) have a unique common solution.

□

5. A system of non linear Volterra-Hammerstein integral equations

In this section, we give a second application and we ensure the existence of the unique common solution of a system of non linear Volterra-Hammerstein integral equations.

Let us take $F = (L(0,\infty), R)$ the space of real-valued measurable functions on $(0,\infty)$. Consider

$$D(x) = \tau_1(x) + \lambda \int_0^t m(x,y)g_1(y, D(y))dy + \mu \int_0^\infty n(x,y)h_1(y, D(y))dy \tag{5.1}$$

and

$$D(x) = \tau_2(x) + \lambda \int_0^t m(x,y)g_2(y, D(y))dy + \mu \int_0^\infty n(x,y)h_2(y, D(y))dy \tag{5.2}$$

for all $x, y \in (0,\infty)$, where $\lambda, \mu \in \mathbb{R}$, and $D, \tau_1, \tau_2, m(x,y), n(x,y), g_1, g_2, h_1, and\ h_2$ are measurable functions with real values in S and r on $(0,\infty)$,

$$\varpi_i = \int_0^t m(x,y)g_i(y, D(y))dy$$

$$\psi_i = \int_0^\infty n(x,y)h_i(y, D(y))dy$$

for $i = 1, 2$

$$f_1D(x) = \varpi_1 D(x) + \psi_1 D(x) + \tau_1(x)$$
$$f_2D(x) = 2D(x) - \varpi_2 D(x) - \psi_2 D(x) - \tau_2(x)$$

with

$$f_1(D(x)) \quad \subseteq \quad f_2(v(x)). \tag{5.3}$$

Assume that

$$(c_1): \quad \varpi_1 D(x) + \psi_1 D(x) + \tau_1(x) + \tau_2(x) - \varpi_2(\varpi_1 D(x) + \psi_1 D(x) + \tau_1(x) + \tau_2(x))$$
$$-\psi_2(\varpi_1 D(x) + \psi_1 D(x) + \tau_1(x) + \tau_2(x)) = 0.$$

We consider the b-metric space $\Omega_b(x,y) = |x-y|^2$.

Theorem 5.1. *Under the assumption* (c_1) *and the condition* (5.3)*, the system of non linear Volterra-Hammerstein integral equations has a unique common solution.*

Proof. Note that the system of non linear Volterra-Hammerstein integral equations (5.1) and (5.2) has a unique common solution if and only if the system of operator f_1 and f_2 has a unique common fixed point.

Now,

$$\Omega_b(f_1D(x), f_1v(x)) = |(f_1D(x) - f_1v(x))|^2$$
$$\leq |(f_2v(x) - f_1v(x))|^2$$

$$= \Omega_b(f_2(v(x)), f_1(v(x)))$$

$$\leq M_b(D(x), v(x)) = \max \Omega_b(f_2 D(x), f_2 v(x)), \Omega_b(f_2 v(x), f_1 v(x)), \frac{1}{4}\Omega_b(f_2 D(x), f_1 v(x))$$

$$\Omega_b(f_1 D(x), f_1 v(x)) \leq M_b(D(x), v(x)).$$

This shows that all the requirements of our main theorem are satisfied, and therefore the integral equations (5.1) and (5.2) have a unique common solution. □

6. Conclusions

Rodjanadid et al. [23] used the idea of generalized $\Im_b$-contractions with b-simulation functions and proved some fixed point results in complete b-metric spaces. Jawaher et al. [24] utilized the idea of b-simulation functions and investigated some common fixed points for two contractive mappings. In this manuscript, we combined these two ideas and proved some common fixed points for two contractive mappings using the idea of generalized $\Im_b$-contractions with b-simulation functions in b-metric spaces. Different examples and applications are given to demonstrate the validity of the concept and the degree of applicability of our findings. Many applied problems can be described by systems of Fredholm and Volterra integral equations. The presented results can be utilized to study the existence of unique common solutions of these systems.

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Conflict of interest

The authors declare that they have no conflict of interest.

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