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Research article

Legendre-Gauss-Lobatto collocation method for solving multi-dimensional systems of mixed Volterra-Fredholm integral equations

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Abstract: Integral equations play a crucial role in many scientific and engineering problems, though solving them is often challenging. This paper addresses the solution of multi-dimensional systems of mixed Volterra-Fredholm integral equations (SMVF-IEs) by means of a Legendre-Gauss-Lobatto collocation method. The one-dimensional case is addressed first. Afterwards, the method is extended to two-dimensional linear and nonlinear SMVF-IEs. Several numerical examples reveal the effectiveness of the approach and show its superiority in comparison to other alternative techniques for treating SMVF-IEs.

Keywords: system of mixed Volterra-Fredholm integral equations; spectral collocation method; shifted Legendre polynomials; shifted Legendre-Gauss-Lobatto quadrature
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1. Introduction

Integral equations [1–5] can accurately describe many different phenomena in engineering and science. Although there are several powerful techniques to numerically solve integral equations, most reveal limitations with multi-dimensional problems. The systems of mixed Volterra-Fredholm integral equations (SMVF-IEs) [6–12] appear in the scope of parabolic boundary value models, including in physics, mathematics, biology, and other subjects. The solution of SMVF-IEs has been a matter of substantial interest. However, solving SMVF-IEs is a challenging issue, for which effective methods are still lacking. Population dynamics, parabolic boundary value problems, spatio-temporal evolution of epidemics, and other phenomena lead to SMVF-IE-based models.

In reference [13], a variational iteration technique was proposed for solving systems of integro-differential equations. In paper [14], a homotopy perturbation technique was utilized to numerically solve nonlinear SMVF-IEs, while in [15], it was adopted to solve nonlinear two-dimensional SMVF-IEs. In [16], hybrid mixtures were used to solve the three-dimensional L-shaped channel problem and to analyze the properties of heat generation via forced convection. In [17], the Keller Box approach was utilized to simulate nonlinear problems encountered in developing liquid and supplementary algebraic dynamics domains. In [18, 19], numerical and analytical methods were developed to address SMVF-IEs. In [20], the Galerkin finite element method was used to find a closed-form solution for a nonlinear coupled partial differential equation, whereas the author in [21] used a quasi-digital technology called the differential transformation method to address the control of complex PDE devices. In [22], the Caputo–Fabrizio and Atangana–Baleanu techniques were used to solve fractional dimensionless systems, and performed a theoretical analysis via the Chebyshev spectral approach [23].

Spectral methods are effective for solving several types of differential and integral problems [24–27]. Particular kinds of spectral approaches include the Galerkin [28, 29], collocation [30–37] and tau techniques [38–42]. Contrasting with other approaches, namely the finite difference and finite element methods, spectral techniques achieve superior accuracy, even with few nodes, thus involving a smaller computational burden. Indeed, they are characterized by exponential convergence rates. The main idea is to express the solution to the original equation by a finite sum of a certain basis function, and then to choose the functions' coefficients such that the error between the exact and the numerical solutions is minimized. In the spectral collocation variant [43–50], the numerical solution is compelled to closely satisfy the original problem, thus the residuals may approach zero at specific collocation points.

This paper addresses the solution of SMVF-IEs by means of a Legendre-Gauss-Lobatto collocation method. The one-dimensional case is firstly treated. Afterwards, the method is extended to two-dimensional linear and nonlinear SMVF-IEs. Several numerical examples are presented to demonstrate the effectiveness of the approach, and to illustrate its superiority in comparison with alternative techniques for solving SMVF-IEs.

The paper is organized into sections. In section 2, some mathematical preliminaries are outlined. In section 3, one-dimensional SMVF-IEs are solved. In section 4, the novel algorithm is extended for solving two-dimensional SMVF-IEs. In section 5, error analysis of the method is addressed. In section 6, some numerical examples assess and compare the proposed approach with other techniques. In section 7, the main conclusions are summarized.

2. Mathematical preliminaries

The Legendre polynomials $\mathcal{L}_\jmath(\varrho)$ ($\jmath = 0, 1 \ldots$) comply with the Rodrigues' expression [51]:

$$\mathcal{L}_\jmath(\varrho) = \frac{(-1)^\jmath}{2^\jmath \jmath!} D^\jmath((1-\varrho^2)^\jmath), \tag{2.1}$$

where $\jmath$ stands for degree.

Accordingly, the n_1th derivative of $\mathcal{L}_J(\varrho)$, denoted by $\mathcal{L}_J^{n_1}(\varrho)$, is given by the following:

$$\mathcal{L}_J^{n_1}(\varrho) = \sum_{i=0(i+J=even)}^{J-n_1} C_{n_1}(J,i)\mathcal{L}_i(\varrho), \tag{2.2}$$

where

$$C_{\mathcal{L}}(J,i) = \frac{2^{n_1-1}(2i+1)\Gamma(\frac{n_1+J-i}{2})\Gamma(\frac{n_1+J+i+1}{2})}{\Gamma(n_1)\Gamma(\frac{2-n_1+J-i}{2})\Gamma(\frac{3-n_1+J+i}{2})}.$$

Let us denote the norm and inner product by $\|\Lambda\|$ and (Λ,γ), respectively, of space $L^2[-1,1]$. The collection of $\mathcal{L}(\varrho)$ is a whole orthogonal system in $L^2[-1,1]$ [52]

$$\Big(\mathcal{L}_{j_1}(\varrho), \mathcal{L}_J(\varrho)\Big) = \int_{-1}^{1} \mathcal{L}_{j_1}(\varrho)\mathcal{L}_J(\varrho)\, d\varrho = h_J\delta_{j_1J}, \tag{2.3}$$

where $h_i = \frac{2}{2i+1}$, and δ_{j_1J} is the Dirac function. Hence, for each $\gamma \in L^2[-1,1]$,

$$\gamma(\varrho) = \sum_{i=0}^{\infty} a_i\mathcal{L}_i(\varrho), \quad a_i = \frac{1}{h_i}\int_{-1}^{1} \gamma(\varrho)\mathcal{L}_i(\varrho)\, d\varrho. \tag{2.4}$$

Assume that $S_{N_1}[-1,1]$ is a collection of all polynomials of degree at the utmost N_1 ($N_1 \geq 0$). Hence, for each $\varphi \in S_{2N_1-1}[-1,1]$, we have

$$\int_{-1}^{1} \varphi(\varrho)d\varrho = \sum_{i=0}^{N_1} \varpi_{N_1,i}\varphi(\varrho_{N_1,i}), \tag{2.5}$$

where $\varrho_{N_1,J}$ ($0 \leq J \leq N_1$) and $\varpi_{N_1,J}$ ($0 \leq J \leq N_1$) are the nodes and Christoffel numbers of the Legendre-Gauss-Lobatto interpolation on the classical interval $[-1,1]$, respectively. The discrete norm and inner product correspond to

$$\|\Lambda\|_{N_1} = (\Lambda,\gamma)_{N_1}^{\frac{1}{2}}, \qquad (\Lambda,\gamma)_{N_1} = \sum_{j=0}^{N_1} \Lambda(\varrho_{N_1,j_1})\,\gamma(\varrho_{N_1,j_1})\,\varpi_{N_1,j_1}. \tag{2.6}$$

Denote the shifted Legendre polynomials specified on the interval $[0,L]$ by $\mathcal{L}_{L,J}(\varrho)$. These polynomials are obtained by the recurrence [51]

$$(j_1+1)\mathcal{L}_{L,J+1}(\varrho) = (2J+1)(\frac{2\varrho}{L}-1)\mathcal{L}_{L,J}(\varrho) - j_1\mathcal{L}_{L,J-1}(\varrho),\ J=1,2,\cdots. \tag{2.7}$$

Then, we can write

$$\mathcal{L}_{L,j_1}(\varrho) = \sum_{J=0}^{j} (-1)^{j_1+J}\frac{(j_1+J)!}{(j_1-J)!\,(J!)^2\,L^J}\,\varrho^J. \tag{2.8}$$

The integral $I^{\nu}\mathcal{L}_{L,j_1}(\varrho)$ may be obtained from

$$\begin{aligned} I^{\nu}\mathcal{L}_{L,j_1}(\varrho) &= \sum_{\jmath=0}^{j_1} (-1)^{\jmath+j_1} \frac{(\jmath+j_1)!}{(-\jmath+j_1)!\,(\jmath!)^2\,\sigma^{\jmath}}\, I^{\nu}\varrho^{\jmath} \\ &= \sum_{\jmath=0}^{j_1}(-1)^{\jmath+j_1} \frac{(\jmath+j_1)!\,\jmath!}{(-\jmath+j_1)!\,(\jmath!)^2\,\sigma^{\jmath}\,\Gamma(\jmath+\nu+1)}\,\varrho^{\jmath+\nu}, \quad j_1 = 0,1,\cdots,N_1, \end{aligned} \tag{2.9}$$

where $\mathcal{L}_{L,j_1}(0) = (-1)^{j_1}$. The equation of the orthogonality condition is

$$\int_0^L \mathcal{L}_{L,j_1}(\varrho)\mathcal{L}_{L,\jmath}(\varrho)w_L(\varrho)\,d\varrho = h_{\jmath}^L\,\delta_{j_1\jmath}, \tag{2.10}$$

where $w_L(\varrho) = 1$ and $h_{\jmath}^L = \dfrac{L}{2\jmath+1}$.

If function $\Lambda(\sigma) \in L^2[0,L]$, then it can be expressed by $\mathcal{L}_{L,i}(\sigma)$ as

$$\Lambda(\sigma) = \sum_{i=0}^{\infty} c_i\mathcal{L}_{L,i}(\sigma),$$

with c_i given by

$$c_i = \frac{1}{h_i^L}\int_0^L \Lambda(\sigma)\mathcal{L}_{L,i}(\sigma)d\varrho, \quad i = 0,1,2,\cdots. \tag{2.11}$$

In the approximation, $\Lambda(\varrho)$ can be written as

$$\Lambda_{N_1}(t) \simeq \sum_{i=0}^{N_1} c_i\mathcal{L}_{L,i}(\sigma). \tag{2.12}$$

3. One-dimensional SMVF-IEs

Let us consider the one-dimensional SMVF-IEs

$$\begin{cases} \Lambda(\varrho) = & \Delta_1(\varrho) + \int\limits_0^{\varrho} \jmath_1(\varrho,\sigma)\Lambda(\sigma)d\sigma + \int\limits_0^{L} \jmath_2(\varrho,\sigma)\Lambda(\sigma)d\sigma, \\ \gamma(\varrho) = & \Delta_2(\varrho) + \int\limits_0^{\varrho} \jmath_3(\varrho,\sigma)\gamma(\sigma)d\sigma + \int\limits_0^{L} \jmath_4(\varrho,\sigma)\gamma(\sigma)d\sigma,\ x \in [0,L], \end{cases} \tag{3.1}$$

where $\Delta_1(\varrho)$, $\Delta_2(\varrho)$, $\jmath_1(\varrho,\sigma)$, $\jmath_2(\varrho,\sigma)$, $\jmath_3(\varrho,\sigma)$ and $\jmath_4(\varrho,\sigma)$ are given as real valued functions, while $\Lambda(\varrho)$ and $\gamma(\varrho)$ are unknown functions.

Using $\sigma = \frac{\varrho}{L}\eta$ to write the integrals $\int\limits_0^{\varrho} \jmath_1(\varrho,\sigma)\Lambda(\sigma)d\sigma$, $\int\limits_0^{\varrho} \jmath_3(\varrho,\sigma)\Lambda(\sigma)d\sigma$ in the interval $[0,L]$, for the variable η, we perform the shifted Legendre-Gauss-Lobatto integration

$$\begin{cases} \Lambda(\varrho) = \Delta_1(\varrho) + \frac{\varrho}{L}\int\limits_0^{L} \jmath_1(\varrho,\frac{\varrho}{L}\eta)\Lambda(\frac{\varrho}{L}\eta)d\eta + \int\limits_0^{L} \jmath_2(\varrho,\sigma)\Lambda(\sigma)d\sigma, \\ \gamma(\varrho) = \Delta_2(\varrho) + \frac{\varrho}{L}\int\limits_0^{L} \jmath_3(\varrho,\frac{\varrho}{L}\eta)\gamma(\frac{\varrho}{L}\eta)d\eta + \int\limits_0^{L} \jmath_4(\varrho,\sigma)\gamma(\sigma)d\sigma,\ \varrho \in [0,L]. \end{cases} \tag{3.2}$$

We rely on the shifted Legendre-Gauss-Lobatto collocation technique to turn the previous SMVF-IEs into a system of algebraic equations. Thus, we collocate independent variables at ϱ_{L,N_1,j_1} points, yielding the approximate solution

$$\begin{aligned}\Lambda_{N_1}(\varrho) &= \sum_{j_1=0}^{N_1} a_{j_1}\mathcal{L}_{L,j_1}(\varrho),\\ \gamma_{N_1}(\varrho) &= \sum_{j_1=0}^{N_1} b_{j_1}\mathcal{L}_{L,j_1}(\varrho).\end{aligned} \tag{3.3}$$

Assume $S_{N_1}(0,L)$ is the collection of all polynomials of degree at utmost N_1 for any positive integer N_1. It follows for each $\phi \in S_{2N_1-1}(0,L)$, based on the shifted Legendre-Gauss-Lobatto quadrature,

$$\int_0^L \phi(\varrho)d\varrho = \sum_{i=0}^{N_1} \varpi_{L,N_1,i}\phi(\varrho_{L,N_1,i}), \tag{3.4}$$

where $\varpi_{L,N_1,i}$ are known on the interval [0,L], meaning the Christoffel numbers of the shifted Legendre-Gauss-Lobatto interpolation.

Based on Eqs (3.3) and (3.4), one can write Eq (3.2) as

$$\begin{cases}\sum\limits_{j_1=0}^{N_1} a_{j_1}\mathcal{L}_{L,j_1}(\varrho) = & \Delta_1(\varrho) + \frac{\varrho}{L}\sum\limits_{i=0}^{N_1}\sum\limits_{j_1=0}^{N_1} a_{j_1}\varpi_{L,N_1,i}J_1(\varrho, \frac{\varrho}{L}\varrho_{L,N_1,i})\mathcal{L}_{L,j_1}(\frac{\varrho}{L}\varrho_{L,N_1,i})\\ & +\sum\limits_{i=0}^{N_1}\sum\limits_{j_1=0}^{N_1} a_{j_1}\varpi_{L,N_1,i}J_2(\varrho,\varrho_{L,N_1,i})\mathcal{L}_{L,j_1}(\varrho_{L,N_1,i}),\\ \sum\limits_{j=0}^{N_1} b_{j_1}\mathcal{L}_{L,j_1}(\varrho) = & \Delta_2(\varrho) + \frac{\varrho}{L}\sum\limits_{i=0}^{N_1}\sum\limits_{j_1=0}^{N_1} a_{j_1}\varpi_{L,N_1,i}J_3(\varrho, \frac{\varrho}{L}\varrho_{L,N_1,i})\mathcal{L}_{L,j_1}(\frac{\varrho}{L}\varrho_{L,N_1,i})\\ & +\sum\limits_{i=0}^{N_1}\sum\limits_{j_1=0}^{N_1} a_{j_1}\varpi_{L,N_1,i}J_4(\varrho,\varrho_{L,N_1,i})\mathcal{L}_{L,j_1}(\varrho_{L,N_1,i}).\end{cases} \tag{3.5}$$

In the shifted Legendre-Gauss-Lobatto collocation technique presented herein, the residual of (3.5) is made zero at the $N_1 + 1$ shifted Legendre-Gauss points

$$\begin{cases}\sum\limits_{j_1=0}^{N_1} a_{j_1}\mathcal{L}_{L,j_1}(\varrho_{L,N_1,n_1}) = & \frac{\varrho_{L,N_1,n_1}}{L}\sum\limits_{i=0}^{N_1}\sum\limits_{j_1=0}^{N_1} a_{j_1}\varpi_{L,N_1,i}J_1(\varrho_{L,N_1,n_1}, \frac{\varrho_{L,N_1,n_1}}{L}\varrho_{L,N_1,i})\mathcal{L}_{L,j_1}(\frac{\varrho_{L,N_1,n_1}}{L}\varrho_{L,N_1,i})\\ & +\Delta_1(\varrho_{L,N_1,n_1}) + \sum\limits_{i=0}^{N_1}\sum\limits_{j_1=0}^{N_1} a_{j_1}\varpi_{L,N_1,i}J_2(\varrho_{L,N_1,n_1},\varrho_{L,N_1,i})\mathcal{L}_{L,j_1}(\varrho_{L,N_1,i}),\\ \sum\limits_{j_1=0}^{N_1} b_{j_1}\mathcal{L}_{L,j_1}(\varrho_{L,N_1,n_1}) = & \frac{\varrho_{L,N_1,n_1}}{L}\sum\limits_{i=0}^{N_1}\sum\limits_{j_1=0}^{N_1} a_{j_1}\varpi_{L,N_1,i}J_3(\varrho_{L,N_1,n_1}, \frac{\varrho_{L,N_1,n_1}}{L}\varrho_{L,N_1,i})\mathcal{L}_{L,j_1}(\frac{\varrho_{L,N_1,n_1}}{L}\varrho_{L,N_1,i})\\ & +\Delta_2(\varrho_{L,N_1,n_1}) + \sum\limits_{i=0}^{N_1}\sum\limits_{j_1=0}^{N_1} a_{j_1}\varpi_{L,N_1,i}J_4(\varrho_{L,N_1,n_1},\varrho_{L,N_1,i})\mathcal{L}_{L,j_1}(\varrho_{L,N_1,i}),\\ & \text{where} \qquad n_1 = 0,\dots,N_1.\end{cases} \tag{3.6}$$

After the coefficients a_j, b_j are specified, the approximate solution $\Lambda_{N_1}(\varrho)$, $\gamma_{N_1}(\varrho)$ at any value of $\varrho \in [0,L]$ in the specific domain can be easily computed from the equations

$$\Lambda_{N_1}(\varrho) = \sum_{j_1=0}^{N_1} a_{j_1} \mathcal{L}_{L,j_1}(\varrho),$$
$$\gamma_{N_1}(\varrho) = \sum_{j_1=0}^{N_1} b_{j_1} \mathcal{L}_{L,j_1}(\varrho). \tag{3.7}$$

4. Two-dimensional SMVF-IEs

The preceding numerical algorithm is extended to solve linear and nonlinear two-dimensional SMVF-IEs. The collocation points are chosen at the shifted Legendre-Gauss-Lobatto interpolation nodes. The idea is to discretize the SMVF-IEs and to construct a system of algebraic equations.

4.1. Linear SMVF-IEs

Let us consider the two dimensional SMVF-IEs

$$\begin{cases} \Lambda(\varrho,\sigma) = \Delta_1(\varrho,\sigma) + \int\limits_0^{\sigma}\int\limits_0^{L} \jmath_1(\varrho,\sigma,\chi,\psi)\Lambda(\chi,\psi)d\chi d\psi, \\ \gamma(\varrho,\sigma) = \Delta_2(\varrho,\sigma) + \int\limits_0^{\sigma}\int\limits_0^{L} \jmath_2(\varrho,\sigma,\chi,\psi)\gamma(\chi,\psi)d\chi d\psi, (\varrho,\sigma) \in [0,L]\times[0,\tau], \end{cases} \tag{4.1}$$

where $\Lambda(\varrho,\sigma)$ and $\gamma(\varrho,\sigma)$ are unknown functions, whilst $\Delta_1(\varrho,\sigma)$, $\Delta_2(\varrho,\sigma)$, $\jmath_1(\varrho,\sigma)$ and $\jmath_2(\varrho,\sigma,\chi,\psi)$, are given as real valued functions.

Using the change of variable $\psi = \frac{\sigma}{\tau}\eta$, we can transform the integrals $\int\limits_0^{\sigma}\int\limits_0^{L} \jmath_1(\varrho,\sigma,\chi,\psi)\Lambda(\chi,\psi)d\chi d\psi$, $\int\limits_0^{\sigma}\int\limits_0^{L} \jmath_2(\varrho,\sigma,\chi,\psi)\gamma(\chi,\psi)d\chi d\psi$, into the interval, $[0,\tau]$, for the variable η, to immediately execute the shifted Legendre-Gauss-Lobatto integration,

$$\begin{cases} \Lambda(\varrho,\sigma) = \Delta_1(\varrho,\sigma) + \frac{\sigma}{\tau}\int\limits_0^{\tau}\int\limits_0^{L} \jmath_1(\varrho,\sigma,\chi,\frac{\sigma}{\tau}\eta)\Lambda(\chi,\frac{\sigma}{\tau}\eta)d\chi d\eta, \\ \gamma(\varrho,\sigma) = \Delta_2(\varrho,\sigma) + \frac{\sigma}{\tau}\int\limits_0^{\tau}\int\limits_0^{L} \jmath_2(\varrho,\sigma,\chi,\frac{\sigma}{\tau}\eta)\gamma(\chi,\frac{\sigma}{\tau}\eta)d\chi d\eta, (\varrho,\sigma) \in [0,L]\times[0,\tau]. \end{cases} \tag{4.2}$$

We extend the dependent variable by the model

$$\Lambda_{N_1,N_2}(\varrho,\sigma) = \sum_{i=0}^{N_2}\sum_{j_1=0}^{N_1} a_{ij_1}\mathcal{L}_{\tau,i}(\sigma)\mathcal{L}_{L,j_1}(\varrho),$$
$$\gamma_{N_1,N_2}(\varrho,\sigma) = \sum_{i=0}^{N_2}\sum_{j_1=0}^{N_1} b_{ij_1}\mathcal{L}_{\tau,i}(\sigma)\mathcal{L}_{L,j_1}(\varrho). \tag{4.3}$$

In virtue of the Eqs (4.3) and (3.4), we can rewrite Eq (4.2) as

$$\begin{cases}\sum_{i=0}^{N_2}\sum_{j_1=0}^{N_1} a_{ij_1}\mathcal{L}_{\tau,i}(\sigma)\mathcal{L}_{L,j_1}(\varrho) = & \sum_{i=0}^{N_2}\sum_{j_1=0}^{N_1} a_{ij_1}\chi_{i,j_1}(\varrho,\sigma) + \Delta_1(\varrho,\sigma),\\ \sum_{i=0}^{N_2}\sum_{j_1=0}^{N_1} b_{ij_1}\mathcal{L}_{\tau,i}(\sigma)\mathcal{L}_{L,j_1}(\varrho) = & \sum_{i=0}^{N_2}\sum_{j_1=0}^{N_1} b_{ij_1}\psi_{i,j_1}(\varrho,\sigma) + \Delta_2(\varrho,\sigma),\ (\varrho,\sigma)\in[0,L]\times[0,\tau],\end{cases} \tag{4.4}$$

where

$$\chi_{i,j_1}(\varrho,\sigma) = \frac{\sigma}{\tau}\sum_{r=0}^{N_1}\sum_{s=0}^{N_2}\varpi_{L,N_1,r}\varpi_{\tau,N_2,s}J_1(\varrho,\sigma,\chi_{L,N_1,r},\frac{\sigma}{\tau}\eta_{\tau,N_2,s})\mathcal{L}_{\tau,i}(\frac{\sigma}{\tau}\eta_{\tau,N_2,s})\mathcal{L}_{L,j_1}(\chi_{L,N_1,r}),$$

$$\psi_{i,j_1}(\varrho,\sigma) = \frac{\sigma}{\tau}\sum_{r=0}^{N_1}\sum_{s=0}^{N_2}\varpi_{L,N_1,r}\varpi_{\tau,N_2,s}J_2(\varrho,\sigma,\chi_{L,N_1,r},\frac{\sigma}{\tau}\eta_{\tau,N_2,s})\mathcal{L}_{\tau,i}(\frac{\sigma}{\tau}\eta_{\tau,N_2,s})\mathcal{L}_{L,j_1}(\chi_{L,N_1,r}).$$

The residual of (4.4) is set to be zero in the suggested shifted Legendre-Gauss-Lobatto collocation technique at $(N_1 + 1)\times(N_2 + 1)$ of shifted Legendre-Gauss-Lobatto points

$$\begin{aligned}\sum_{i=0}^{N_2}\sum_{j_1=0}^{N_1} a_{ij_1}\mathcal{L}_{\tau,i}(\sigma_{\tau,N_2,n_2})\mathcal{L}_{L,j_1}(\varrho_{L,N_1,n_1}) &= \sum_{i=0}^{N_2}\sum_{j_1=0}^{N_1} a_{ij_1}\chi_{i,j_1}(\varrho_{L,N_1,n_1},\sigma_{\tau,N_2,n_2})\\ &\quad + \Delta_1(\varrho_{L,N_1,n_1},\sigma_{\tau,N_2,n_2}),\\ \sum_{i=0}^{N_2}\sum_{j_1=0}^{N_1} b_{ij_1}\mathcal{L}_{\tau,i}(\sigma_{\tau,N_2,n_2})\mathcal{L}_{L,j_1}(\varrho_{L,N_1,n_1}) &= \sum_{i=0}^{N_2}\sum_{j_1=0}^{N_1} b_{ij_1}\psi_{i,j_1}(\varrho_{L,N_1,n_1},\sigma_{\tau,N_2,n_2})\\ &\quad + \Delta_2(\varrho_{L,N_1,n_1},\sigma_{\tau,N_2,n_2}),\end{aligned} \tag{4.5}$$

where $n_1 = 0,\dots,N_1$ and $n_2 = 0,\dots,N_2$.

Finally, Eq (4.5) is enforced to exactly satisfy (4.1) at the shifted Legendre-Gauss-Lobatto interpolation nodes ϱ_{L,N_1,n_1}, σ_{L,N_1,n_2}. This provides $2(N_1 + 1)(N_2 + 1)$ equations for a_{i,j_1}, b_{ij_1}; $i = 0,\cdots,N_1,\ j_1 = 0,\cdots,N_2$. Consequently, the approximate solution (4.3) can be evaluated as

$$\begin{aligned}\Lambda_{N_1,N_2}(\varrho,\sigma) &= \sum_{i=0}^{N_2}\sum_{j_1=0}^{N_1} a_{ij_1}\mathcal{L}_{\tau,i}(\sigma)\mathcal{L}_{L,j_1}(\varrho),\\ \gamma_{N_1,N_2}(\varrho,\sigma) &= \sum_{i=0}^{N_2}\sum_{j_1=0}^{N_1} b_{ij_1}\mathcal{L}_{\tau,i}(\sigma)\mathcal{L}_{L,j_1}(\varrho).\end{aligned} \tag{4.6}$$

4.2. Nonlinear SMVF-IEs

We expand the technique for numerically handling nonlinear SMVF-IEs

$$\begin{cases}\Lambda(\varrho,\sigma) = \Delta_1(\varrho,\sigma) + \int_0^{\sigma}\int_0^{L} J_1(\varrho,\sigma,\chi,\psi,\Lambda(\chi,\psi),\gamma(\chi,\psi))d\chi d\psi,\\ \gamma(\varrho,\sigma) = \Delta_2(\varrho,\sigma) + \int_0^{\sigma}\int_0^{L} J_2(\varrho,\sigma,\chi,\psi,\Lambda(\chi,\psi),\gamma(\chi,\psi))d\chi d\psi, (\varrho,\sigma)\in[0,L]\times[0,\tau],\end{cases} \tag{4.7}$$

where $\Lambda(\varrho,\sigma)$ and $\gamma(\varrho,\sigma)$ are unknown functions, whilst $f(\varrho,\sigma)$, $k(\varrho,\sigma,\chi,\psi,(\Lambda(\chi,\psi)))$, $J_1(\varrho,\sigma,\chi,\psi,\Lambda(\chi,\psi),(\gamma(\chi,\psi)))$ and $J_2(\varrho,\sigma,\chi,\psi,\Lambda(\chi,\psi),\gamma(\chi,\psi))$ are given as functions.

Using the change of variable $\psi=\frac{\sigma}{\tau}\eta$, we can transform the integrals $\int_0^\sigma\int_0^L J_1(\varrho,\sigma,\chi,\psi)\Lambda(\chi,\psi)d\chi d\psi$, $\int_0^\sigma\int_0^L J_2(\varrho,\sigma,\chi,\psi)\gamma(\chi,\psi)d\chi d\psi$, into the interval, $[0,\tau]$, for the variable η, and apply the shifted Legendre-Gauss-Lobatto integration

$$\begin{cases}\Lambda(\varrho,\sigma)=\Delta_1(\varrho,\sigma)+\frac{\sigma}{\tau}\int_0^\tau\int_0^L J_1(\varrho,\sigma,\chi,\frac{\sigma}{\tau}\eta,\Lambda(\chi,\frac{\sigma}{\tau}\eta),\gamma(\chi,\frac{\sigma}{\tau}\eta))d\chi d\eta,\\ \gamma(\varrho,\sigma)=\Delta_2(\varrho,\sigma)+\frac{\sigma}{\tau}\int_0^\tau\int_0^L J_2(\varrho,\sigma,\chi,\frac{\sigma}{\tau}\eta,\Lambda(\psi\chi,\frac{\sigma}{\tau}\eta),\gamma(\chi,\frac{\sigma}{\tau}\eta))d\chi d\eta,(\varrho,\sigma)\in[0,L]\times[0,\tau].\end{cases}\tag{4.8}$$

We select the approximate solution from the model

$$\begin{aligned}\Lambda_{N_1,N_2}(\varrho,\sigma)&=\sum_{i=0}^{N_2}\sum_{j_1=0}^{N_1}a_{ij_1}\mathcal{L}_{\tau,i}(\sigma)\mathcal{L}_{L,j_1}(\varrho),\\ \gamma_{N_1,N_2}(\varrho,\sigma)&=\sum_{i=0}^{N_2}\sum_{j_1=0}^{N_1}b_{ij_1}\mathcal{L}_{\tau,i}(\sigma)\mathcal{L}_{L,j_1}(\varrho).\end{aligned}\tag{4.9}$$

Proceeding as in the previous subsection, we can rewrite the problem in the form

$$\begin{cases}\sum_{i=0}^{N_2}\sum_{j_1=0}^{N_1}a_{ij_1}\mathcal{L}_{\tau,i}(\sigma)\mathcal{L}_{L,j_1}(\varrho)= & \frac{\sigma}{\tau}\sum_{r=0}^{N_1}\sum_{s=0}^{N_2}\varpi_{L,N_1,r}\varpi_{\tau,N_2,s}J_1\Big(\varrho,\sigma,\chi_{L,N_1,r},\frac{\sigma}{\tau}\eta_{\tau,N_2,s},\delta_{r,s}(\sigma),\lambda_{r,s}(\sigma)\Big)\\ & +\Delta_1(\varrho,\sigma),\\ \sum_{i=0}^{N_2}\sum_{j_1=0}^{N_1}b_{ij_1}\mathcal{L}_{\tau,i}(\sigma)\mathcal{L}_{L,j_1}(\varrho)= & \frac{\sigma}{\tau}\sum_{r=0}^{N_1}\sum_{s=0}^{N_2}\varpi_{L,N_1,r}\varpi_{\tau,N_2,s}J_2\Big(\varrho,\sigma,\chi_{L,N_1,r},\frac{\sigma}{\tau}\eta_{\tau,N_2,s},\delta_{r,s}(\sigma),\lambda_{r,s}(\sigma)\Big)\\ & +\Delta_2(\varrho,\sigma),\\ & (\varrho,\sigma)\in[0,L]\times[0,\tau],\end{cases}\tag{4.10}$$

where

$$\begin{aligned}\delta_{r,s}(\sigma)&=\sum_{i=0}^{N_2}\sum_{j_1=0}^{N_1}a_{ij_1}\mathcal{L}_{\tau,i}(\frac{\sigma}{\tau}\eta_{\tau,N_2,s})\mathcal{L}_{L,j_1}(\chi_{L,N_1,r}),\\ \lambda_{r,s}(\sigma)&=\sum_{i=0}^{N_2}\sum_{j_1=0}^{N_1}b_{ij_1}\mathcal{L}_{\tau,i}(\frac{\sigma}{\tau}\eta_{\tau,N_2,s})\mathcal{L}_{L,j_1}(\chi_{L,N_1,r}).\end{aligned}$$

The residual of (4.10) is set to be zero at $(N_1+1)\times(N_2+1)$ for the shifted Legendre-Gauss-Lobatto points

$$\begin{aligned}\sum_{i=0}^{N_2}\sum_{j_1=0}^{N_1}a_{ij_1}\mathcal{L}_{\tau,i}(\sigma_{\tau,N_2,n_2})\mathcal{L}_{L,j_1}(\varrho_{L,N_1,n_1})&=\xi_{n_2,n_1}+\Delta_1(\varrho_{L,N_1,n_1},\sigma_{\tau,N_2,n_2}),\\ \sum_{i=0}^{N_2}\sum_{j_1=0}^{N_1}b_{ij_1}\mathcal{L}_{\tau,i}(\sigma_{\tau,N_2,n_2})\mathcal{L}_{L,j_1}(\varrho_{L,N_1,n_1})&=\zeta_{n_2,n_1}+\Delta_2(\varrho_{L,N_1,n_1},\sigma_{\tau,N_2,n_2}),\end{aligned}\tag{4.11}$$

where

$$\xi_{n_2,n_1} = \frac{\sigma_{\tau,N_2,n_2}}{\tau}\sum_{r=0}^{N_1}\sum_{s=0}^{N_2}\varpi_{L,N_1,r}\varpi_{\tau,N_2,s}J1 \times\Big(\varrho_{L,N_1,n_1},\sigma_{\tau,N_2,n_2},\chi_{L,N_1,r},\frac{\sigma_{\tau,N_2,n_2}}{\tau}\eta_{\tau,N_2,s},\delta_{r,s}(\sigma_{\tau,N_2,n_2}),\lambda_{r,s}(\sigma_{\tau,N_2,n_2})\Big), \tag{4.12}$$

$$\zeta_{n_2,n_1} = \frac{\sigma_{\tau,N_2,n_2}}{\tau}\sum_{r=0}^{N_1}\sum_{s=0}^{N_2}\varpi_{L,N_1,r}\varpi_{\tau,N_2,s}J2 \times\Big(\varrho_{L,N_1,n_1},\sigma_{\tau,N_2,n_2},\chi_{L,N_1,r},\frac{\sigma_{\tau,N_2,n_2}}{\tau}\eta_{\tau,N_2,s},\delta_{r,s}(\sigma_{\tau,N_2,n_2}),\lambda_{r,s}(\sigma_{\tau,N_2,n_2})\Big), \tag{4.13}$$

where $n_1 = 0,\ldots,N_1$ and $n_2 = 0,\ldots,N_2$.

By utilizing Newton's iterative technique, we can solve the preceding nonlinear system of algebraic equations. As a result, the coefficients a_{ij}, b_{ij} are specified, and the approximate solution can be accounted for $\Lambda_{N_1,N_2}(\varrho,\sigma)$, $v_{N_1,N_2}(\varrho,\sigma)$ at any value of (ϱ,σ) in the known domain, by means of the next equation

$$\begin{aligned}\Lambda_{N_1,N_2}(\varrho,\sigma) &= \sum_{i=0}^{N_2}\sum_{j_1=0}^{N_1} a_{ij_1}\mathcal{L}_{\tau,i}(\sigma)\mathcal{L}_{L,j_1}(\varrho),\\ \gamma_{N_1,N_2}(\varrho,\sigma) &= \sum_{i=0}^{N_2}\sum_{j=0}^{N_1} b_{ij_1}\mathcal{L}_{\tau,i}(\sigma)\mathcal{L}_{L,j_1}(\varrho).\end{aligned} \tag{4.14}$$

5. Error analysis

In this section, we investigate an error analysis of the present method.

Definition 5.1. For a nonnegative integer ρ, we have [51, 53]

$$H^{\rho}(-1,1) = \{\Lambda : \partial_z^{\iota}\Lambda \in L^2(-1,1), 0 \le i \le \rho\},$$

whereas $\partial_z^{\iota}\Lambda(z) = \frac{\partial^{\iota}\Lambda(z)}{\partial z^{\iota}}$, and

$$\|\Lambda\|_{\rho} = (\sum_{\iota=0}^{\rho}\|\partial_z^{\iota}\Lambda\|^2)^{\frac{1}{2}}.$$

$$|\Lambda|_{\rho} = \|\partial_z^{\rho}\Lambda\|.$$

Lemma 5.2. *For $\Lambda \in B^q(\omega^d)$ with $d \le q \le N+1$ [54]*

$$\left\|\boldsymbol{I}_{m,x}\Lambda - \Lambda\right\| \le c\sqrt{\frac{(N-q1)!}{N!}}(N+q)^{-(q+1)/2}\|\Lambda\|_{B^q(\omega^d)}. \tag{5.1}$$

Theorem 5.3. *Let $\boldsymbol{I}_N\boldsymbol{\Lambda}(\varrho)$ and $\boldsymbol{\Lambda}(\varrho)$ be the spectral approximation and the exact solution of the Volterra-Fredholm system. Thus, we have*

$$\|E_N\|_{L^2(I)} \le C\sqrt{\frac{(N-\rho+1)!}{N!}}(N+\rho)^{-(\rho+1)/2}\Big[|F(\boldsymbol{\Lambda}(\cdot))|_{H^1(I)} + |\boldsymbol{\Lambda}|_{H^1(I)}\Big] + LM\|E_N\|. \tag{5.2}$$

Proof. We can write the system in Eq (3.1) as a multivariate system

$$\mathbf{\Lambda}(\varrho) = \boldsymbol{I}_{\varrho,N}\mathbf{\Delta}(\varrho) + \boldsymbol{I}_{\varrho,N}\int_0^{\varrho} \boldsymbol{J}(\varrho,\sigma)\mathbf{\Lambda}(\varrho)d\sigma + \boldsymbol{I}_{\varrho,N}\int_0^{1} \boldsymbol{J}(\varrho,\sigma)\mathbf{\Lambda}(\varrho)d\sigma. \tag{5.3}$$

When utilizing the approximate solution we have

$$\mathbf{\Lambda}_N(\varrho) = \boldsymbol{I}_{\varrho,N}\mathbf{\Delta}(\varrho) + \boldsymbol{I}_{\varrho,N}\boldsymbol{I}_{\sigma,N}\int_0^{\varrho} \boldsymbol{J}(\varrho,\sigma)\mathbf{\Lambda}_N(\varrho)d\sigma + \boldsymbol{I}_{\varrho,N}\boldsymbol{I}_{\sigma,N}\int_0^{1} \boldsymbol{J}(\varrho,\sigma)\mathbf{\Lambda}_N(\varrho)d\sigma. \tag{5.4}$$

Subtracting (5.4) from (5.3) yields

$$||e|| \leq \sum_{\ell=1}^{4} ||B_\ell||, \tag{5.5}$$

where

$$\begin{aligned}
B_1 &= \boldsymbol{I}_{\varrho,N}\int_0^{\varrho}(\boldsymbol{I} - \boldsymbol{I}_{\sigma,N})\Big[\boldsymbol{J}(\varrho,\sigma)\mathbf{\Lambda}(\varrho)d\sigma\Big],\\
B_2 &= \boldsymbol{I}_{\varrho,N}\int_0^{\varrho}\boldsymbol{I}_{\sigma,N}\Big[\boldsymbol{J}(\varrho,\sigma)\mathbf{\Lambda}(\varrho) - \boldsymbol{J}(\varrho,\sigma)\mathbf{\Lambda}_N(\varrho)\Big]d\sigma,\\
B_3 &= \boldsymbol{I}_{\varrho,N}\int_0^{1}(\boldsymbol{I} - \boldsymbol{I}_{\sigma,N})\Big[\boldsymbol{J}(\varrho,\sigma)\mathbf{\Lambda}_N(\varrho)d\sigma\Big],\\
B_4 &= \boldsymbol{I}_{\varrho,N}\int_0^{1}\boldsymbol{I}_{\sigma,N}\Big[\boldsymbol{J}(\varrho,\sigma)\mathbf{\Lambda}(\varrho) - \boldsymbol{J}(\varrho,\sigma)\mathbf{\Lambda}_N(\varrho)\Big]d\sigma.
\end{aligned}$$

We can write (5.5) by using Gronwall inequality

$$\parallel e(x) \parallel_{L_2} \leq \parallel B_1 \parallel_{L_2} + \parallel B_2 \parallel_{L_2} + \parallel B_3 \parallel_{L_2} + \parallel B_4 \parallel_{L_2}. \tag{5.6}$$

Then, the term $\parallel B_1 \parallel$ is estimated as the following:

$$\begin{aligned}
\parallel B_1 \parallel &= \left\|\boldsymbol{I}_{\varrho,N}\int_0^{\varrho}(\boldsymbol{I} - \boldsymbol{I}_{\sigma,N})\Big[\boldsymbol{J}(\varrho,\sigma)\mathbf{\Lambda}(\varrho)\Big]d\sigma\right\|\\
&= \Big[\sum_{|\imath|_\infty \leq N}\varpi_\imath\Big(\int_0^{\varrho}(\boldsymbol{I} - \boldsymbol{I}_{\sigma,N})\Big[\boldsymbol{J}(\varrho,\sigma)\mathbf{\Lambda}(\varrho)\Big]d\sigma\Big)^2\Big]^{\frac{1}{2}}.
\end{aligned} \tag{5.7}$$

By using the Cauchy inequality, we can get,

$$\begin{aligned}
\parallel B_1 \parallel &\leq \Big[\sum_{|\imath|_\infty \leq N}\varpi_\imath\int_0^{\varrho}\Big|(\boldsymbol{I} - \boldsymbol{I}_{\sigma,N})(\boldsymbol{J}(\varrho,\sigma)\mathbf{\Lambda}(\varrho))\Big|^2 d\sigma\Big]^{\frac{1}{2}}\\
&\leq \Big(\sum_{|\imath|_\infty \leq N}\varpi_\imath\Big)^{\frac{1}{2}}\max_{|\imath|_\infty \leq N}\Big(\int_0^{\sigma}\Big|(\boldsymbol{I} - \boldsymbol{I}_{\sigma,N})(\boldsymbol{J}(\varrho,\sigma)\mathbf{\Lambda}(\varrho))\Big|^2 d\sigma\Big)^{\frac{1}{2}}.
\end{aligned} \tag{5.8}$$

Hence,

$$\parallel B_1 \parallel \leq c\sqrt{\frac{(N-\rho+1)!}{N!}}(N+\rho)^{-(\rho+1)/2}|F(\mathbf{\Lambda}(\cdot))|. \tag{5.9}$$

Now we estimate the term $\|B_2\|$. We use the Legendre-Gauss integration formula (2.3) to obtain

$$\begin{aligned}\| B_2 \| &= \left\| \boldsymbol{I}_{\varrho,N} \int_0^{\varrho} \boldsymbol{I}_{\sigma,N}\Big[\boldsymbol{J}(\varrho,\sigma)\boldsymbol{\Lambda}(\varrho) - \boldsymbol{J}(\varrho,\sigma)\boldsymbol{\Lambda}_N(\varrho)\Big]d\sigma\right\| \\ &= \Big[\sum_{|\iota|_\infty \le N} \varpi_\iota \Big(\int_0^{\varrho} \boldsymbol{I}_{\sigma,N}\Big[\boldsymbol{J}(\varrho,\sigma)\boldsymbol{\Lambda}(\varrho) - \boldsymbol{J}(\varrho,\sigma)\boldsymbol{\Lambda}_N(\varrho)\Big]d\sigma\Big)^2\Big]^{\frac{1}{2}}.\end{aligned} \tag{5.10}$$

We obtain it by using the Cauchy-Schwarz inequality

$$\begin{aligned}\| B_2 \| &\le \Big[\sum_{|\iota|_\infty \le N} \varpi_\iota \int_0^{\varrho} \boldsymbol{I}_{\sigma,N}|\boldsymbol{J}(\varrho,\sigma)\boldsymbol{\Lambda}(\varrho) - \boldsymbol{J}(\varrho,\sigma)\boldsymbol{\Lambda}_N(\varrho)|^2 d\sigma\Big]^{\frac{1}{2}} \\ &\le \Big[\sum_{|\iota|_\infty \le N} \varpi_\iota \sum_{|\ell|_\infty \le N} \varpi_\ell|\boldsymbol{J}(\varrho,\sigma)\boldsymbol{\Lambda}(\varrho) - \boldsymbol{J}(\varrho,\sigma)\boldsymbol{\Lambda}_N(\varrho)|^2\Big]^{\frac{1}{2}}.\end{aligned} \tag{5.11}$$

By using the Lipschitz condition, we can write

$$\begin{aligned}\| B_2 \| &\le LM\Big[\sum_{|\iota|_\infty \le N} \varpi_\iota \sum_{|\ell|_\infty \le N} |\boldsymbol{\Lambda}(\varrho) - \boldsymbol{\Lambda}_N(\varrho)|^2 \varpi_\ell\Big]^{\frac{1}{2}} \\ &\le LM\Big[\int_0^1 |\boldsymbol{I}_{\sigma,N}(\boldsymbol{\Lambda}(\varrho) - \boldsymbol{\Lambda}_N(\varrho)|^2 d\eta\Big]^{\frac{1}{2}}.\end{aligned} \tag{5.12}$$

By using the triangle inequality, we derive that

$$\| B_2 \| \le LM\Big[\Big(\int_0^1 |\boldsymbol{I}_{\sigma,N}(\boldsymbol{\Lambda}(\varrho) - \boldsymbol{\Lambda}_N(\varrho)|^2 d\sigma\Big)^{\frac{1}{2}} + \Big(\int_0^1 |\boldsymbol{\Lambda}(\varrho) - \boldsymbol{\Lambda}_N(\varrho)|^2 d\eta\Big)^{\frac{1}{2}}\Big]. \tag{5.13}$$

Furthermore, from Lemma 5.2, we can deduce that

$$\| B_2 \| \le c\sqrt{\frac{(N-\rho+1)!}{N!}}(N+\rho)^{-(\rho+1)/2}|\boldsymbol{\Lambda}| + LM \| E_N \| . \tag{5.14}$$

Then, the term $\| B_3 \|$ is estimated as the following:

$$\begin{aligned}\| B_3 \| &= \left\| \boldsymbol{I}_{\varrho,N} \int_0^1 (\boldsymbol{I} - \boldsymbol{I}_{\sigma,N})\Big[\boldsymbol{J}(\varrho,\sigma)\boldsymbol{\Lambda}(\varrho)\Big]d\sigma\right\| \\ &= \Big[\sum_{|\iota|_\infty \le N} \varpi_\iota \Big(\int_0^1 (\boldsymbol{I} - \boldsymbol{I}_{\sigma,N})\Big[\boldsymbol{J}(\varrho,\sigma)\boldsymbol{\Lambda}(\varrho)\Big]d\sigma\Big)^2\Big]^{\frac{1}{2}}.\end{aligned} \tag{5.15}$$

By using the Cauchy inequality, we can get,

$$\begin{aligned}\| B_3 \| &\le \Big[\sum_{|\iota|_\infty \le N} \varpi_\iota \int_0^1 \Big|(\boldsymbol{I} - \boldsymbol{I}_{\sigma,N})(\boldsymbol{J}(\varrho,\sigma)\boldsymbol{\Lambda}(\varrho))\Big|^2 d\sigma\Big]^{\frac{1}{2}} \\ &\le \Big(\sum_{|\iota|_\infty \le N} \varpi_\iota\Big)^{\frac{1}{2}} \max_{|\iota|_\infty \le N}\Big(\int_0^1 \Big|(\boldsymbol{I} - \boldsymbol{I}_{\sigma,N})(\boldsymbol{J}(\varrho,\sigma)\boldsymbol{\Lambda}(\varrho))\Big|^2 d\sigma\Big)^{\frac{1}{2}}.\end{aligned} \tag{5.16}$$

Hence,

$$\| B_3 \| \leq c\sqrt{\frac{(N-\rho+1)!}{N!}}(N+\rho)^{-(\rho+1)/2}|F(\mathbf{\Lambda}(\cdot))|. \tag{5.17}$$

Now, we estimate $\|B_4\|$. We use Eq (2.3) to obtain

$$\begin{aligned} \| B_4 \| &= \left\| \boldsymbol{I}_{\varrho,N} \int_0^1 \boldsymbol{I}_{\sigma,N}\Big[\boldsymbol{J}(\varrho,\sigma)\mathbf{\Lambda}(\varrho) - \boldsymbol{J}(\varrho,\sigma)\mathbf{\Lambda}_N(\varrho)\Big]d\sigma \right\| \\ &= \Big[\sum_{|\iota|_\infty \leq N} \varpi_\iota \Big(\int_0^1 \boldsymbol{I}_{\sigma,N}\Big[\boldsymbol{J}(\varrho,\sigma)\mathbf{\Lambda}(\varrho) - \boldsymbol{J}(\varrho,\sigma)\mathbf{\Lambda}_N(\varrho)\Big]d\sigma\Big)^2\Big]^{\frac{1}{2}}. \end{aligned} \tag{5.18}$$

We obtain it by using the Cauchy-Schwarz inequality

$$\begin{aligned} \| B_4 \| &\leq \Big[\sum_{|\iota|_\infty \leq N} \varpi_\iota \int_0^1 \boldsymbol{I}_{\sigma,N}|\boldsymbol{J}(\varrho,\sigma)\mathbf{\Lambda}(\varrho) - \boldsymbol{J}(\varrho,\sigma)\mathbf{\Lambda}_N(\varrho)|^2 d\sigma\Big]^{\frac{1}{2}} \\ &\leq \Big[\sum_{|\iota|_\infty \leq N} \varpi_\iota \sum_{|\ell|_\infty \leq N} \varpi_\ell |\boldsymbol{J}(\varrho,\sigma)\mathbf{\Lambda}(\varrho) - \boldsymbol{J}(\varrho,\sigma)\mathbf{\Lambda}_N(\varrho)|^2\Big]^{\frac{1}{2}}. \end{aligned} \tag{5.19}$$

By using the Lipschitz condition, we can write

$$\begin{aligned} \| B_4 \| &\leq LM\Big[\sum_{|\iota|_\infty \leq N} \varpi_\iota \sum_{|\ell|_\infty \leq N} |\mathbf{\Lambda}(\varrho) - \mathbf{\Lambda}_N(\varrho)|^2 \varpi_\ell\Big]^{\frac{1}{2}} \\ &\leq LM\Big[\int_0^1 |\boldsymbol{I}_{\sigma,N}(\mathbf{\Lambda}(\varrho) - \mathbf{\Lambda}_N(\varrho)|^2 d\eta\Big]^{\frac{1}{2}}. \end{aligned} \tag{5.20}$$

By using the triangle inequality, we have that

$$\| B_4 \| \leq LM\Big[\Big(\int_0^1 |\boldsymbol{I}_{\sigma,N}(\mathbf{\Lambda}(\varrho) - \mathbf{\Lambda}_N(\varrho)|^2 d\sigma\Big)^{\frac{1}{2}} + \Big(\int_0^1 |\mathbf{\Lambda}(\varrho) - \mathbf{\Lambda}_N(\varrho)|^2 d\eta\Big)^{\frac{1}{2}}\Big]. \tag{5.21}$$

Additionally, from Lemma 5.2, we can deduce that

$$\| B_4 \| \leq c\sqrt{\frac{(N-\rho+1)!}{N!}}(N+\rho)^{-(\rho+1)/2}|\mathbf{\Lambda}| + LM \| E_N \| . \tag{5.22}$$

6. Numerical results

To illustrate the performance of the proposed scheme and the thoroughness of the results, we present some numerical examples. The results with the new method are compared with those yielded by others [11, 15, 18, 19]. For assessing accuracy, the difference between the exact, $\Lambda(\varrho)$, and the approximate, $\Lambda_{N,M}(\varrho)$, solutions at the point ϱ, meaning the absolute error E is adopted, that is,

$$E(\varrho,\sigma) = | \Lambda(\varrho,\sigma) - \Lambda_{N_1,N_2}(\varrho,\sigma) | . \tag{6.1}$$

Additionally, the maximum absolute error (M_E) is specified by

$$M_E = \text{Max}\{E(\varrho,\sigma) : (\varrho,\sigma) \in [0,L] \times [0,\tau]\}. \tag{6.2}$$

Example 1. Let us consider the one-dimensional linear SMVF-IEs [18]:

$$\begin{cases} 5\Lambda_1(\varrho) &= \Delta_1(\varrho) - \int\limits_0^1 \sin(\varrho-\chi)\cos(\Lambda_1(\chi))\cos(\gamma_2(\chi))d\chi - \int\limits_0^\varrho (2\varrho\Lambda_1(\chi)+\varrho\chi\gamma_2(\chi))d\chi, \\ 5\gamma_2(\varrho) &= \Delta_2(\varrho) - \int\limits_0^1 (\varrho\chi^2\cos(\Lambda_1(\chi))+\chi\cos(\Lambda_1(\chi)))d\chi - \int\limits_0^\varrho (\varrho^2 sin(\Lambda_1(\chi))+\chi^2\gamma_2(\chi))d\chi, \\ & \qquad \varrho\in[0,1], \end{cases} \tag{6.3}$$

where $\Delta_1(\varrho)$ and $\Delta_2(\varrho)$ are given functions, founded by the exact solution $\Lambda_1(\varrho) = 1-\varrho$, $\gamma_2(\varrho) = \varrho$.

In order to illustrate the convergence rate of our technique, we list the M_E for several options of N_1 in Table 1, while comparing our method with the fixed point collocation approach (FPCA) [18]. We verify that the new technique leads to a better numerical solution with far fewer nodes, and that our numerical solutions are very close to the exact ones. In Figure 1, we illustrate the M_E (i.e., $\log_{10} M_E$) obtained with the new technique for diverse values of N, where E_1 and E_2 correspond to $\Lambda_1(\varrho)$ and $\gamma_2(\varrho)$, respectively, in logarithmic graphs being computed as in Eq (6.1).

Table 1. The M_E concerning the problem (1).

	FPCA [18] at	New method at		
	N=6	$N=4$	$N=6$	$N=8$
Λ_1	9.53×10^{-6}	1.04×10^{-7}	2.68×10^{-12}	3.73×10^{-16}
γ_2	4.93×10^{-5}	8.24×10^{-9}	4.55×10^{-14}	3.26×10^{-16}

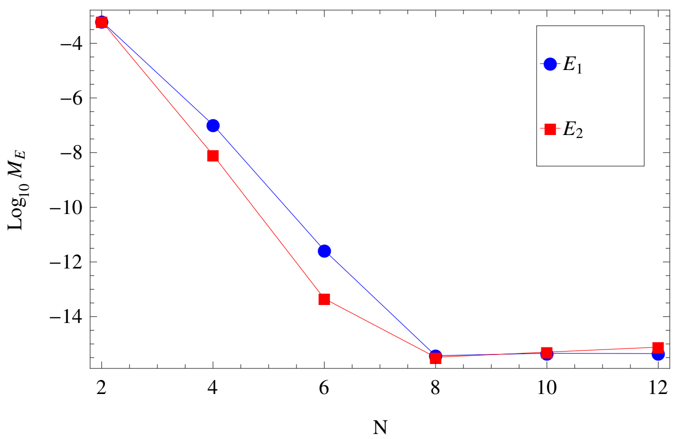

Figure 1. M_E convergence of problem 1.

Example 2. We solve the two-dimensional linear SMVF-IEs [19]:

$$\begin{cases} \Lambda_1(\varrho,\sigma) = \sigma\sin(\varrho) - \frac{1}{2} + \frac{1}{2}\cos(\varrho) - \frac{1}{2}\sin(\varrho) + \int\limits_0^\varrho\int\limits_0^1 (\Lambda_1(\chi,\psi)+\gamma_2(\chi,\psi))d\psi d\chi, \\ \gamma_2(\varrho,\sigma) = \sigma\cos(\varrho) - \frac{1}{2} + \frac{1}{2}\sin(\varrho) - \frac{1}{2}\cos(\varrho) + \int\limits_0^\varrho\int\limits_0^1 (\Lambda_1(\chi,\psi)-\gamma_2(\chi,\psi))d\psi d\chi, \\ \qquad (\varrho,\sigma)\in[0,1]\times[0,1], \end{cases} \tag{6.4}$$

that has the exact solution $\Lambda_1(\varrho,\sigma) = \sigma\sin(\varrho)$, $\gamma_2(\varrho,\sigma) = \sigma\cos(\varrho)$.

For several choices of N, we verify that the new technique is more accurate than the homotopy method (HAM) [19]. Table 2 summarizes the values of the absolute errors E of problem 2.

Figures 2 and 3, illustrate the evolution of the absolute errors along each dimension, E_1 and E_2, for $N = 12$. To emphasize the high thoroughness and convergence rate of the new scheme, we depict the maximum absolute errors in log scale in Figure 4. Based on the results, we may conclude that the proposed technique yields excellent approximations and exponential convergence rates.

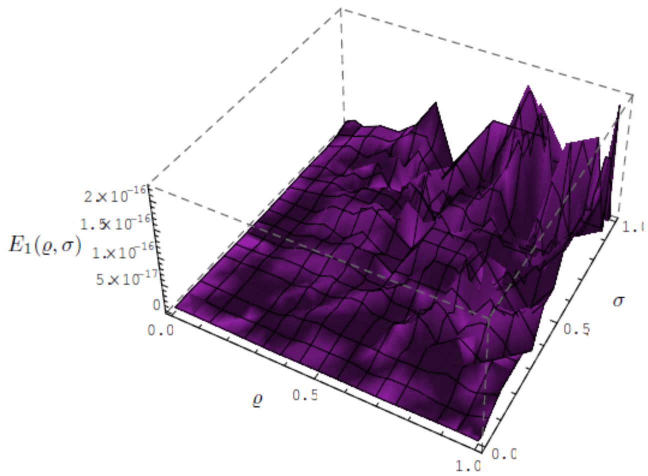

Figure 2. The E_1 of Λ_1 of problem 2, for $N = 12$.

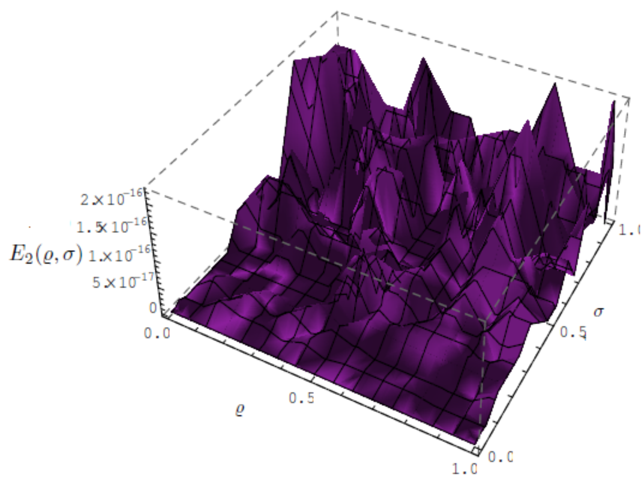

Figure 3. The E_2 of γ_2 of problem 2, for $N = 12$.

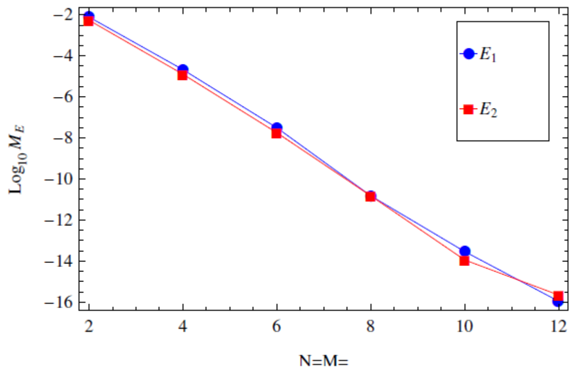

Figure 4. M_E convergence for problem 2.

Table 2. The absolute errors E of problem 2.

	HAM [19] at		New method at			
	$N = 12$		$N = 8$		$N = 12$	
(ϱ, σ)	Λ_1	γ_2	Λ_1	γ_2	Λ_1	γ_2
(0.0, 0.0)	0	0	2.07×10^{-24}	7.44×10^{-24}	2.9×10^{-18}	1.06×10^{-27}
(0.2, 0.2)	1.05×10^{-6}	9.99×10^{-6}	3.60×10^{-12}	1.94×10^{-12}	6.94×10^{-18}	2.78×10^{-17}
(0.4, 0.4)	3.70×10^{-7}	3.95×10^{-6}	1.06×10^{-11}	5.59×10^{-12}	8.33×10^{-17}	0
(0.6, 0.6)	2.91×10^{-6}	4.07×10^{-6}	1.58×10^{-11}	8.95×10^{-12}	5.55×10^{-17}	0
(0.8, 0.8)	1.01×10^{-5}	1.87×10^{-6}	1.43×10^{-11}	8.28×10^{-12}	0	1.11×10^{-16}
(1.0, 1.0)	1.04×10^{-9}	0	2.22×10^{-16}	1.11×10^{-16}	1.11×10^{-16}	1.11×10^{-16}

Example 3. We now consider the nonlinear two-dimensional SMVF-IEs [15]:

$$\begin{cases} \Lambda_1(\varrho,\sigma) = \varrho + \sigma - \frac{2}{9}\varrho^2\chi^3 - \frac{1}{4}\varrho^2\chi^4 + \int\limits_0^\sigma \int\limits_0^1 \varrho^2\psi^2((\Lambda_1(\chi,\psi))^2 + \gamma_2(\chi,\psi))d\psi d\chi, \\ \gamma_2(\varrho,\sigma) = \varrho^2 - \chi^2 + \frac{1}{5}\varrho\chi^6 - \frac{2}{9}\varrho\chi^4 - \frac{1}{2}\varrho\chi^3 - \frac{3}{10}\varrho\chi^2 + \int\limits_0^\sigma \int\limits_0^1 \varrho\sigma(\Lambda_1(\chi,\psi) - (\gamma_2(\chi,\psi))^2)d\psi d\chi, \end{cases} \tag{6.5}$$

where $(\varrho, \sigma) \in [0, 1] \times [0, 1]$, and with the exact solution $\Lambda_1(\varrho, \sigma) = \varrho + \sigma$, $\gamma_2(\varrho, \sigma) = \varrho^2 - \sigma^2$.

Table 3 shows the absolute errors by utilizing the new algorithm and those with the homotopy perturbation method (HPM) [15]. It is observed that the proposed technique is more precise than the HPM.

Table 3. The absolute errors E of problem 3.

	HPM [15] at $N = 11$		New method at $N = 4$	
(ϱ, σ)	Λ_1	γ_2	Λ_1	γ_2
(0.0,0.0)	0	0	2.6×10^{-16}	4.7×10^{-17}
(0.1,0.1)	5.5×10^{-21}	4.0×10^{-18}	3.4×10^{-16}	6.7×10^{-16}
(0.2,0.2)	1.2×10^{-16}	5.2×10^{-14}	2.7×10^{-16}	5.5×10^{-16}
(0.3,0.3)	4.4×10^{-15}	1.5×10^{-11}	3.8×10^{-16}	3.8×10^{-16}
(0.4,0.4)	6.0×10^{-12}	8.1×1.10^{-10}	4.7×10^{-16}	5.5×10^{-16}
(0.5,0.5)	4.5×10^{-10}	1.7×1.10^{-8}	2.3×10^{-16}	4.3×10^{-16}
(0.6,0.6)	1.3×10^{-8}	1.7×1.10^{-7}	1.6×10^{-16}	6.3×10^{-16}
(0.7,0.7)	2.1×10^{-7}	7.4×1.10^{-7}	6.7×10^{-16}	1.9×10^{-17}
(0.8,0.8)	2.5×10^{-6}	2.1×1.10^{-6}	3.5×10^{-16}	3.9×10^{-16}

Example 4. We consider the nonlinear two-dimensional SMVF-IEs [19]:

$$\begin{cases} \Lambda_1(\varrho,\sigma) = \Delta_1(\varrho,\sigma) + \int_0^{\varrho}\int_0^1 (\chi - \psi^2)((\Lambda_1(\chi,\psi))^2 + \gamma_2(\chi,\psi))d\psi d\chi, \\ \gamma_2(\varrho,\sigma) = \Delta_2(\varrho,\sigma) + \int_0^{\sigma}\int_0^1 (2\Lambda_1(\chi,\psi) - 3\varrho\gamma_2(\chi,\psi)d\psi d\chi, (\varrho,\sigma) \in [0,1]\times[0,1], \end{cases} \tag{6.6}$$

where $\Delta_1(\varrho,\sigma)$ and $\Delta_2(\varrho,\sigma)$ are the given real valued functions, and the exact solution is $\Lambda_1(\varrho,\sigma) = -2\varrho + 2\varrho\sigma$, $\gamma_2(\varrho,\sigma) = 1 + 2\varrho\sin(\sigma)$.

Table 4 compares the absolute error E resulting from the application of the new proposed method with that by the approach in [19] for several values of N and M. The numerical results show that the solutions are extremely precise, even with small values of N and M.

Table 4. The absolute errors E of problem 4.

	HAM [19] at $N = 10$		New Method at $N = 4$	
(ϱ, σ)	Λ_1	γ_2	Λ_1	γ_2
(0.0, 0.0)	0	0	0	1.83×10^{-17}
(0.2, 0.2)	4.72×10^{-7}	5.37×10^{-6}	9.38×10^{-10}	2.45×10^{-6}
(0.4, 0.4)	1.16×10^{-7}	1.44×10^{-8}	3.72×10^{-9}	1.36×10^{-5}
(0.6, 0.6)	4.42×10^{-7}	4.11×10^{-6}	8.10×10^{-9}	2.01×10^{-5}
(0.8, 0.8)	4.50×10^{-6}	5.29×10^{-6}	1.33×10^{-8}	9.26×10^{-6}
(1.0, 1.0)	1.53×10^{-6}	1.77×10^{-5}	1.83×10^{-8}	6.41×10^{-9}

7. Conclusions

In this paper, a shifted Legendre-Gauss-Lobatto collocation scheme was proposed for numerically solving SMVF-IEs. First, the one-dimensional case was solved, and then the method was extended to address two-dimensional linear and nonlinear SMVF-IEs. Different numerical examples revealed the superiority of the proposed approach when compared with alternative techniques for treating SMVF-

IEs. The SMVF-IEs play a crucial role in many scientific and engineering problems and, thus, methods for solving then accurately are crucial.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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