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Research article

A new algorithm to compute fuzzy subgroups of a finite group

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Abstract: The enumeration of fuzzy subgroups is a complex problem. Several authors have computed results for special instances of groups. In this paper, we present a novel algorithm that is designed to enumerate the fuzzy subgroups of a finite group. This is achieved through the computation of maximal chains of subgroups. This approach is also useful for writing a program to compute the number of fuzzy subgroups. We provide further elucidation by computing the number of fuzzy subgroups of the groups Q_8 and D_8.

Keywords: fuzzy subgroups; quaternion group; dihedral group; maximal chains
Mathematics Subject Classification: Primary 20N25, 03E72; Secondary 05E15, 20D30, 20D60

1. Introduction

The idea of fuzzy set was introduced by Zadeh [1]. Afterwards, Rosenfeld [2] developed this notion to define fuzzy subgroups. Thus, fuzzy group generalizes the traditional concept of a group. While group theory finds various applications in combinatorics, chemistry, and theoretical physics, fuzzy groups have practical applications in decision-making, pattern recognition, and artificial intelligence [3–5]. It should be noted that subgroups of a group arise as α-cuts of fuzzy subgroups [6]. This fact establishes the connection between fuzzy groups and crisp groups.

There have been several studies on the enumeration of fuzzy subgroups of a group. Sherwood and Anthony [7] discussed the product of fuzzy subgroups in terms of t-norms.

The enumeration of fuzzy subgroups is a complex task, prompting various authors to compute the number of fuzzy subgroups for specific instances only. Further details are provided below. Filep [8] constructed fuzzy subgroups for groups with order up to six. This is an influential paper in the enumeration of fuzzy subgroups.

The case of abelian groups isomorphic to $\mathbb{Z}_{p^n} \oplus \mathbb{Z}_{q^m}$ was solved by Murali and Makamba [9].

Moreover, the interesting case of finite cyclic groups of order $p^n q^m$ was completed by Volf [10].

Tărnăuceanu [11] has done extensive work on the enumeration of fuzzy subgroups, including providing explicit formulas for the counting of fuzzy subgroups of finite abelian groups, as well as for specific cases of dihedral and Hamiltonian groups. He has developed explicit formulas for determining the number of unique fuzzy subgroups in finite p-groups that contains a maximal subgroup and introduced a method for identifying all chains of finite elementary p-groups [12].

Other researchers have studied the cases of finite abelian p-groups, certain dihedral groups, quasi-dihedral groups, quaternion groups, modular p-groups, rectangle groups and finite dicyclic groups [13–15]. They have also developed expansion methods, lattice diagrams and recursive formulas to determine the number of fuzzy subgroups in these groups [16, 17]. Recently, H. Alolaiyan et al. [18] have developed the notion of (α, β)-complex fuzzy subgroups which is a further generalization of a fuzzy subgroup.

We would also like to mention the work of L. Ardekani, [19] in which they have computed the fuzzy normal subgroups of the group U_{6n}.

This paper employs a direct approach by counting the maximal chains in the group G which terminate in G. This method is suitable for the computation of fuzzy subgroups for any finite group.

Fuzzy groups play a significant role in addressing uncertainty and imprecision within the context of rough set theory. Rough set theory and fuzzy set theory are two important frameworks within the field of computational intelligence that have found numerous applications in various domains. Rough set theory focuses on handling uncertainty and imprecision in data by defining lower and upper approximations. Fuzzy set theory deals with ambiguity and vagueness by assigning degrees of membership to elements. In recent years, both rough set theory and fuzzy set theory have been applied to address important issues in healthcare, particularly in the context of lung cancer [22] and COVID-19. Several notable contributions [23] have been made in these areas.

Regarding lung cancer, researchers have utilized rough set theory to analyze medical datasets and extract relevant features for effective diagnosis and classification. By employing rough set-based feature selection techniques, they have achieved improved accuracy and efficiency in lung cancer detection. Moreover, fuzzy set theory has been employed to model uncertainty and imprecision in lung cancer risk assessment, aiding in personalized treatment planning and decision-making.

As for COVID-19, rough set theory has been employed to analyze large-scale datasets, aiding in identifying significant risk factors and predicting disease outcomes. By utilizing rough set-based feature selection and rule induction techniques, researchers have successfully identified key clinical and epidemiological factors associated with disease severity and mortality. Fuzzy set theory has also been applied to model and analyze the linguistic variables associated with COVID-19, enabling better understanding and interpretation of complex and uncertain information [24]. Moreover, the Chikungunya disease can be diagonised using soft rough sets [25]. Extension of fuzzy algebraic structures have been studied in [26–30].

Fuzzy groups can be employed in decision-making processes related to treatment planning and risk assessment. By considering the varying degrees of membership of patients to different groups, healthcare professionals can make more informed and personalized decisions, taking into account the inherent uncertainty and individual variations present in lung cancer and COVID-19 cases. Thus, fuzzy groups provide a flexible framework for handling uncertainty and imprecision within rough set theory and fuzzy set theory applications. They enable a more robust analysis of medical data, enhancing the accuracy and effectiveness of decision-making processes in the context of lung cancer and COVID-19 research.

2. Methods of construction for fuzzy subgroups

Definition 2.1. [7] Let G be a group then a $\mu \in \mathscr{F}\mathscr{P}(G)$(set of all fuzzy subsets of G) is said to be a fuzzy subgroup of G if

(1) $\mu(xy) \geq \min(\mu(x), \mu(y))$, $\forall x, y \in G$.

(2) $\mu(x^{-1}) \geq \mu(x)$, $\forall x \in G$.

The set of all fuzzy subgroups of G will be denoted by $\mathscr{F}(G)$.

The following Theorem gives a connection between subgroups and fuzzy subgroups of G [6].

Theorem 2.1. *Let G be a group. A $\mu \in \mathscr{F}\mathscr{P}(G)$ is a fuzzy subgroup of G if and only if every α-cut μ_α is a subgroup of G.*

Definition 2.2. Let G be a finite group. A chain of finite subgroups of G is a set of subgroups of G linearly ordered by set inclusion, i.e., it is a finite sequence

$$G_0 \subset G_1 \subset \cdots \subset G_n = G.$$

In the context of fuzzy subgroups of a group G, two fuzzy subgroups μ and ν are considered equivalent, denoted as $\mu \sim \nu$, if the following conditions hold for all x and y in G: $\mu(x)$ is greater than or equal to $\mu(y)$ if and only if $\nu(x)$ is greater than or equal to $\nu(y)$, and $\mu(x)$ is zero if and only if $\nu(x)$ is zero. It can be shown that $\mu \sim \nu$ if and only if there exists a one-to-one correspondence between the equivalence classes of fuzzy subgroups of G and the set of chains of subgroups that end at G. Therefore, to count the distinct fuzzy subgroups of G, we simply need to count all chains of subgroups in G that end at G. The following theorem, proved in [20] gives the number of fuzzy subgroups of a finite group.

Theorem 2.2. *Let G be a finite group, $\mathscr{C}_i$ be a maximal chain of subgroups of G and $\mathscr{F}(G)$ be the set of all fuzzy subgroups of G. If there are r maximal chains in the lattice of subgroups of G then $|\mathscr{F}(G)|$ can be calculated by the following formula*

$$|\mathscr{F}(G)| = \left|\mathscr{F}\left(\bigcup_{i=1}^{r} \mathscr{C}_i\right)\right| + 1.$$

Proof. Let G be a finite group and $\mathscr{S}(G)$ be the lattice of subgroups of G. For each chain of G in $\mathscr{S}(G)$, we add G in the end of each chain. Obviously, $\bigcup_{i=1}^{r} \mathscr{C}_i$ is disjoint union of maximal chains $\mathscr{C}_i$'s and all chains are contained in $\mathscr{C}_i$'s. Furthermore, we have a certain unique chain ($G_i \subset G$) of G in $\mathscr{S}(G)$. So, by counting all those distinct chains of G in $\mathscr{S}(G)$ which terminate in G yields $|\mathscr{F}(G)|$.

Hence without loss generality, we have

$$|\mathscr{F}(G)| = \left|\mathscr{F}\left(\bigcup_{i=1}^{r} \mathscr{C}_i\right)\right| + 1.$$

□

Now we give a detailed working of the general case to compute the number of of chains of subgroups of G that terminate in G. For a particular chain ($G_i \subset G$) of G, which start from G and terminate in G, for a fix $q \in [0, 1]$ there exists a unique trivial fuzzy subgroup of G having order 1.

Theorem 2.3. *Let G be a finite group, then the $|\mathscr{F}(G)|$ (where the order of fuzzy subgroups is greater than one) can be calculated by the following formula:*

$$\left|\mathscr{F}\left(\bigcup_{i=1}^{r}\mathscr{C}_i\right)\right|=\left(\sum_{i=1}^{r}2^{\ell_i}-r\right)-\left(\sum_{i=1}^{r-1}\sum_{2=j>i}^{r}2^{\ell_{ij}}-\frac{r(r-1)}{2}\right)+\left(\sum_{i=1}^{r-2}\sum_{i<j=2}^{r-1}\sum_{i<j<k=3}^{r}2^{\ell_{ijk}}-\frac{1}{2}\sum_{\alpha=2}^{r-1}(r-\alpha+1)(r-\alpha)\right)-\cdots+$$
$$+(-1)^{r-2}\left(\sum_{i_1=1}^{2}\sum_{\zeta_1}2^{\ell_{i_1 i_2 i_3\cdots i_{r-1}}}-r\right)+(-1)^{r-1}\left(\sum_{\zeta_2}2^{\ell_{i_1 i_2 i_3\ldots i_{r-1}\, i_r}}-1\right),$$

where $\zeta_1 : i_1 < i_2, 2 \leq i_2 < i_3 < \ldots < i_{r-1} \leq r$ and $\zeta_2 : 1 = i_1 < i_2 < i_3 < \cdots < i_{r-1} < i_r = r$.

Proof. From Theorem 2.2 it follows that,

$$\left|\mathscr{F}(\bigcup_{i=1}^{r}\mathscr{C}_i)\right| = \sum_{i=1}^{r}|\mathscr{F}(\mathscr{C}_i)| - \sum_{1\leq i<j\leq r}\left|\mathscr{F}(\mathscr{C}_i\cap\mathscr{C}_j)\right| + \sum_{1\leq i<j<k\leq r}\left|\mathscr{F}(\mathscr{C}_i\cap\mathscr{C}_j\cap\mathscr{C}_k)\right| + \cdots + (-1)^{r-1}\,|\mathscr{F}(\mathscr{C}_1\cap\mathscr{C}_2\cap\ldots\cap\mathscr{C}_r)|.$$

We denote $|\mathscr{F}(\mathscr{C}_i)|$, $\left|\mathscr{F}(\mathscr{C}_i\cap\mathscr{C}_j)\right|$, $\left|\mathscr{F}(\mathscr{C}_i\cap\mathscr{C}_j\cap\mathscr{C}_k)\right|$ and $|\mathscr{F}(\mathscr{C}_1\cap\mathscr{C}_2\cap\cdots\cap\mathscr{C}_r)|$ by $\mathscr{S}_i$, $(i = 1, \ldots, r)$, $\mathscr{S}_{ij}$ $(1 \leq i < j \leq r)$, $\mathscr{S}_{ijk}$, $(1 \leq i < j < k \leq r)$ and $\mathscr{S}_{12\ldots r}$ respectively.

Let $\mathscr{S}_i$ represents the $|\mathscr{F}(G)|$ except of first order, formed by corresponding $\mathscr{C}_i$ of G and $\mathscr{S}_{ij}$, $\mathscr{S}_{ijk}$, $\ldots$, $\mathscr{S}_{12\cdots r}$ represents the cardinality of all distinct possible fuzzy subgroups of G excluding first order, formed by corresponding $\left(\mathscr{C}_i\cap\mathscr{C}_j\right)$, $\left(\mathscr{C}_i\cap\mathscr{C}_j\cap\mathscr{C}_k\right)$, $\ldots$,$(\mathscr{C}_1\cap\mathscr{C}_2\cap\cdots\cap\mathscr{C}_r)$ respectively of G. For ease of computations, the above equation can be written as

$$\left|\mathscr{F}\left(\bigcup_{i=1}^{r}\mathscr{C}_i\right)\right| = \sum_{i=1}^{r}\mathscr{S}_i - \sum_{1\leq i<j\leq r}\mathscr{S}_{ij} + \sum_{1\leq i<j<k\leq r}\mathscr{S}_{ijk} - \cdots + (-1)^{r-2}\sum_{\zeta_1}\mathscr{S}_{i_1 i_2 i_3\ldots i_{r-1}} + (-1)^{r-1}\sum_{\zeta_2}\mathscr{S}_{i_1 i_2 i_3\ldots i_{r-1} i_r}.$$

Where $\zeta_1 : 1 \leq i_1 < i_2 < i_3 < \cdots < i_{r-1} \leq r$ and $\zeta_2 : 1 = i_1 < i_2 < i_3 < \ldots < i_{r-1} < i_r = r$.

We define lengths of chains as under

$$1\leq\ \ell_i=|\mathscr{C}_i|-1\ \ (1\leq i\leq r),\ 1\leq\ \ell_{ij}=\left|\mathscr{C}_i\cap\mathscr{C}_j\right|-1\ \ (1\leq i<j\leq r),$$

$$1\leq\ \ell_{ijk}=\left|\mathscr{C}_i\cap\mathscr{C}_j\cap\mathscr{C}_k\right|-1\ \ (1\leq i<j<k\leq r),\ldots,\ 1\leq\ \ell_{i_1 i_2 i_3\ldots i_{r-1}}=\left|\mathscr{C}_{i_1}\cap\mathscr{C}_{i_2}\cap\mathscr{C}_{i_3}\cap\cdots\cap\mathscr{C}_{i_{r-1}}\right|-1\ \ \ (1\leq i_1<i_2<i_3<\cdots<i_{r-1}\leq r),$$

$$1\leq\ \ell_{i_1 i_2 i_3\ldots i_{r-1} i_r}=\left|\mathscr{C}_{i_1}\cap\mathscr{C}_{i_2}\cap\mathscr{C}_{i_3}\cap\cdots\cap\mathscr{C}_{i_{r-1}}\cap\mathscr{C}_{i_r}\right|-1\ \ \ \ (1=i_1<i_2<i_3<\cdots<i_{r-1}<i_r=r).$$

We compute the lengths of chains in the following steps.

(1) We calculate,

$$\sum_{i=1}^{r}\mathscr{S}_i, \qquad 1 \leq i \leq r.$$

Since, ℓ_i is the length for the maximal chain $\mathscr{C}_i$ and $\ell_i = |\mathscr{C}_i| - 1$.
We have

$$\mathscr{S}_i = 2^{\ell_i} - 1,$$
$$\sum_{i=1}^{r}\mathscr{S}_i = (2^{\ell_1} - 1) + (2^{\ell_2} - 1) + \cdots + (2^{\ell_r} - 1) = \sum_{i=1}^{r}2^{\ell_i} - r.$$

(2) In this step, we calculate $\sum\limits_{1=i<j\leq r} \mathscr{S}_{ij}, \ (1 \leq i < j \leq r)$.

Let ℓ_{ij} be the length for the overlapping of maximal chains $\mathscr{C}_i$ and $\mathscr{C}_j$ which is $\ell_{ij} = \left|\mathscr{C}_i \cap \mathscr{C}_j\right| - 1$.
Fix $i = 1 \ \ for \ \ 2 \leq j \leq r$,

$$\mathscr{S}_{1r} = 2^{\ell_{1r}} - 1,$$

Now,

$$\sum_{j=2}^{r} \mathscr{S}_{1j} = \sum_{j=2}^{r} \left(2^{\ell_{1j}} - 1^j\right) = \sum_{j=2}^{r} 2^{\ell_{1j}} - (r-1).$$

Fix $i = 2, \ for \ 3 \leq j \leq r$

$$\mathscr{S}_{2r} = 2^{\ell_{2r}} - 1.$$

We have

$$\sum_{j=3}^{r} \mathscr{S}_{2j} = \sum_{j=3}^{r} \left(2^{\ell_{2j}} - 1^j\right) = \sum_{j=3}^{r} 2^{\ell_{2j}} - (r-2).$$

Now fix $i = r-2, \ for \ r-1 \leq j \leq r$,

$$\mathscr{S}_{r-2\ r-1} = 2^{\ell_{r-2\ r-1}} - 1,$$

$$\mathscr{S}_{r-2\ r} = 2^{\ell_{r-2\ r}} - 1,$$

$$\sum_{j=r-1}^{r} \mathscr{S}_{r-2\ j} = \sum_{j=r-1}^{r} \left(2^{\ell_{r-2\ j}} - 1^j\right) = \sum_{j=r-1}^{r} 2^{\ell_{r-2\ j}} - 2.$$

Fixing $i = r-1, \ j = r$,

$$\mathscr{S}_{r-1\ r} = 2^{\ell_{r-1\ r}} - 1.$$

$$\sum_{j=r} \mathscr{S}_{r-1\ j} = \sum_{j=r} \left(2^{\ell_{r-1\ j}} - 1^j\right) = \sum_{j=r} 2^{l_{r-1\ j}} \ - 1.$$

$$\sum_{1\leq i<j\leq r} \mathscr{S}_{ij} = \sum_{j=2}^{r} \mathscr{S}_{1j} + \sum_{j=3}^{r} \mathscr{S}_{2j} + \ldots + \sum_{j=r-1}^{r} \mathscr{S}_{r-2\ j} + \sum_{j=r} \mathscr{S}_{r-1\ j},$$

hence,

$$\sum_{1\leq i<j\leq r} \mathscr{S}_{ij} = \sum_{i=1}^{r-1} \sum_{2=j>i}^{r} 2^{\ell_{ij}} - \frac{r(r-1)}{2}.$$

(3) In this step, we calculate $\sum\limits_{1=i<j<k\leq r} \mathscr{S}_{ijk}, \ (1 \leq i < j < k \leq r)$.

Let ℓ_{ijk} be the length for overlapping of maximal chains $\mathscr{C}_i$, $\mathscr{C}_j$ and $\mathscr{C}_k$ which is $\ell_{ijk} = \left|\mathscr{C}_i \cap \mathscr{C}_j \cap \mathscr{C}_k\right| - 1$.
We $fix\ i = 1, \ \ for \ \ 2 \leq j < k \leq m$.
When $j = 2$ and $3 \leq k \leq r$,

$$\mathscr{S}_{123} = 2^{\ell_{123}} - 1,$$

$$\cdots\cdots\cdots$$

$$\mathscr{S}_{12r} = 2^{\ell_{12r}} - 1,$$

$$\sum_{k=3}^{r}\mathscr{S}_{12k}=\sum_{k=3}^{r}\left(2^{\ell_{12k}}-1^{k}\right)=\sum_{k=3}^{r}2^{\ell_{12k}}-(r-2).$$

When $j=3$ and $4\le k\le r$,

$$\mathscr{S}_{134}=2^{\ell_{134}}-1,$$

$$\cdots\cdots\cdots$$

$$\mathscr{S}_{13r}=2^{\ell_{13r}}-1,$$

$$\sum_{k=4}^{r}\mathscr{S}_{13k}=\sum_{k=4}^{r}\left(2^{\ell_{13k}}-1^{k}\right)=\sum_{k=4}^{r}2^{\ell_{13k}}-(r-3),$$

$$\cdots\cdots\cdots$$

When $j=r-2$ and $r-1\le k\le r$,

$$\mathscr{S}_{1\ r-2\ r-1}=2^{\ell_{1\ r-2\ r-1}}-1.$$

$$\mathscr{S}_{1\ r-2\ r}=2^{\ell_{1\ r-2\ r}}-1.$$

$$\sum_{k=r-1}^{r}\mathscr{S}_{1\ r-2\ k}=\sum_{k=r-1}^{r}\left(2^{\ell_{1\ r-2\ k}}-1^{k}\right)=\sum_{k=r-1}^{r}2^{\ell_{1\ r-2\ k}}-2.$$

When $j=r-1$ and $k=r$,

$$\mathscr{S}_{1\ r-1\ r}=2^{\ell_{1\ r-1\ r}}-1,$$

$$\sum_{k=r}\mathscr{S}_{1\ r-1\ k}=\sum_{k=r}\left(2^{\ell_{1\ r-1\ k}}-1^{k}\right)=\sum_{k=r}2^{\ell_{1\ r-1\ k}}-1.$$

$$\sum_{2\le j<k\le r}\mathscr{S}_{1jk}=\sum_{k=3}^{r}\mathscr{S}_{12k}+\sum_{k=4}^{r}\mathscr{S}_{13k}+\ldots+\sum_{k=r-1}^{r}\mathscr{S}_{1\ r-2\ k}+\sum_{k=r}\mathscr{S}_{1\ r-1\ k},$$

Thus,

$$\sum_{2\le j<k\le r}\mathscr{S}_{1jk}=\sum_{j=2}^{r-1}\sum_{j<k=3}^{r}2^{\ell_{12k}}-\frac{(r-1)(r-2)}{2}.$$

Similarly, fix $i=2$, for $3\le j<k\le r$,

$$\sum_{3\le j<k\le r}\mathscr{S}_{2jk}=\sum_{j=3}^{r-1}\sum_{j<k=4}^{r}2^{\ell_{2jk}}-\frac{(r-2)(r-3)}{2}.$$

$$\cdots\cdots\cdots$$

Fix $i=r-3$, for $r-2\le j<k\le r$,

$$\sum_{r-2\le j<k\le r}\mathscr{S}_{r-3\ j\ k}=\sum_{j=r-2}^{r-1}\sum_{j<k=r-1}^{r}2^{\ell_{r-3\ j\ k}}-3.$$

Fix $i = r-2$, for $r-1 \le j < k \le r$

$$\sum_{r-1\le j<k\le r} \mathscr{S}_{r-2\ j\ k} = \sum_{k=r} \mathscr{S}_{r-2\ r-1\ k} = \sum_{k=r} 2^{l_{r-2\ r-1\ k}} - 1$$

$$\sum_{1\le i<j<k\le r} \mathscr{S}_{ijk} = \sum_{2\le j<k\le r} \mathscr{S}_{1jk} + \sum_{3\le j<k\le r} \mathscr{S}_{2jk} + \ldots + \sum_{r-2\le j<k\le r} \mathscr{S}_{r-3\ j\ k} + \sum_{r-1\le j<k\le r} \mathscr{S}_{r-2\ j\ k}$$

$$\sum_{1\le i<j<k\le r} \mathscr{S}_{ijk} = \sum_{j=2}^{r-1} \sum_{j<k=3}^{r} 2^{\ell_{1jk}} - \frac{(r-1)(r-2)}{2} + \sum_{j=3}^{r-1} \sum_{j<k=4}^{r} 2^{\ell_{2jk}} - \frac{(r-2)(r-3)}{2} + \ldots + \sum_{j=r-2}^{r-1} \sum_{j<k=r-1}^{r} 2^{\ell_{r-3\ j\ k}} - 3 + \sum_{k=r} 2^{\ell_{r-2\ r-1\ k}} - 1,$$

$$\sum_{1\le i<j<k\le r} \mathscr{S}_{ijk} = \sum_{i=1}^{r-2} \sum_{i<j=2}^{r-1} \sum_{i<j<k=3}^{r} 2^{\ell_{ijk}} - \frac{1}{2} \sum_{\alpha=2}^{r-1} (r+1-\alpha)(r-\alpha).$$

....

$(r-1)^{th}$ Step:
We find, $\sum_{\zeta} \mathscr{S}_{i_1 i_2 i_3 \ldots i_{r-1}}$ *where* $i_1 < i_2 < i_3 < \ldots < i_{r-1}$,

$$\sum_{\zeta} \mathscr{S}_{i_1 i_2 i_3 \ldots i_{r-1}} = \sum_{i_1=1}^{2} \sum_{2\le i_2<i_3<\ldots<i_{r-1}\le r} 2^{\ell_{i_1 i_2 i_3 \cdots i_{r-1}}} - r.$$

Where $\zeta : 1 \le i_1 < i_2 < i_3 < \ldots < i_{r-1} \le r$.
r^{th} Step:
We find $\sum_{\zeta} \mathscr{S}_{i_1 i_2 i_3 \ldots i_{r-1} i_r}$ where $i_1 < i_2 < i_3 < \ldots < i_{r-1} < i_r$,

$$\sum_{\zeta} \mathscr{S}_{i_1 i_2 i_3 \ldots i_{r-1} i_r} = \sum_{\zeta} 2^{\ell_{i_1 i_2 i_3 \cdots i_{r-1} i_r}} - 1,$$

and $\zeta : 1 = i_1 < i_2 < i_3 < \ldots < i_{r-1} < i_r = r$. Summing together all previous steps,

$$\left|\mathscr{F}\left(\bigcup_{i=1}^{r} B_i\right)\right| = \sum_{i=1}^{r} \mathscr{S}_i - \sum_{1\le i<j\le r} \mathscr{S}_{ij} + \sum_{1\le i<j<k\le r} \mathscr{S}_{ijk} - \cdots + (-1)^{r-2} \sum_{1\le i_1<i_2<i_3<\cdots<i_{r-1}\le r} \mathscr{S}_{i_1 i_2 i_3 \ldots i_{r-1}} + (-1)^{r-1} \sum_{1=i_1<i_2<i_3<\ldots<i_{r-1}<i_r=r} \mathscr{S}_{i_1 i_2 i_3 \ldots i_{r-1} i_r}.$$

Finally, we have:

$$\left|\mathscr{F}\left(\bigcup_{i=1}^{r} B_i\right)\right| = \left(\sum_{i=1}^{r} 2^{\ell_i} - r\right) - \left(\sum_{i=1}^{r-1} \sum_{2=j>i}^{r} 2^{\ell_{ij}} - \frac{r(r-1)}{2}\right) + \left(\sum_{i=1}^{r-2} \sum_{i<j=2}^{r-1} \sum_{i<j<k=3}^{r} 2^{\ell_{ijk}} - \frac{1}{2} \sum_{\alpha=2}^{r-1} (r-\alpha+1)(r-\alpha)\right) - \cdots + (-1)^{r-2} \left(\sum_{i_1=1}^{2} \sum_{i_1<i_2, 2\le i_2<i_3<\ldots<i_{r-1}\le r} 2^{\ell_{i_1 i_2 i_3 \cdots i_{r-1}}} - r\right) + (-1)^{r-1} \left(\sum_{1=i_1<i_2<i_3<\cdots<i_{r-1}<i_r=r} 2^{\ell_{i_1 i_2 i_3 \ldots i_{r-1} i_r}} - 1\right).$$

□

The following algorithm is the consequence of the Theorem 2.3.

Algorithm 1: Enumerating fuzzy subgroups of a finite group

Input: A finite group G and $\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_r$, the maximal chains of subgroups of G.

Output: Number of all distinct possible fuzzy subgroups (having the order greater than one) of G.

Data: Lattice of subgroups of the group G.

Step 1. Determine all distinct fuzzy subgroups corresponding to every maximal chain, excluding the fuzzy subgroup of order 1. $|\mathscr{F}(\mathscr{C}_i)|$ is the number of fuzzy subgroups corresponding to the i^{th} maximal chain, where $1 \leq i \leq r$.

Step 2: Find the sum $\sum_{i=1}^{r} |\mathscr{F}(\mathscr{C}_i)|$.

Step 3. Find intersections of l maximal chains, where $2 \leq \ell \leq r$.

Step 4: Find number of all distinct fuzzy subgroups (except of first order) corresponding to the intersection of every pair of maximal chains $\mathscr{C}_i \cap \mathscr{C}_j$, $\left|\mathscr{F}(\mathscr{C}_i \cap \mathscr{C}_j)\right|$ is the number of fuzzy subgroups corresponding to the intersection of every pair of maximal chains $\mathscr{C}_i \cap \mathscr{C}_j$ maximal chain, where $1 \leq i < j \leq r$.

Step 5: Find the sum $\sum_{1\leq i<j\leq r} \left|\mathscr{F}(\mathscr{C}_i \cap \mathscr{C}_j)\right|$.

Step 6. Find number of all distinct fuzzy subgroups (except of first order) corresponding to the intersection of every l maximal chains, where $3 \leq l \leq r$. Thus, we find $\left|\mathscr{F}(\mathscr{C}_i \cap \mathscr{C}_j \cap \mathscr{C}_k)\right|$ where $(1 \leq i < j < k \leq r), \ldots, \left|\mathscr{F}(\mathscr{C}_{i_1} \cap \mathscr{C}_{i_2} \cap \mathscr{C}_{i_3} \cap \cdots \cap \mathscr{C}_{i_{r-1}})\right|$ where $(1 \leq i_1 < i_2 < i_3 < \cdots < i_{r-1} \leq r)$ and $|\mathscr{F}(\mathscr{C}_1 \cap \mathscr{C}_2 \cap \ldots \cap \mathscr{C}_r)|$.

Step 7: Find the following sums $\sum_{1\leq i<j<k\leq r} \left|\mathscr{F}(\mathscr{C}_i \cap \mathscr{C}_j \cap \mathscr{C}_k)\right|, \ldots, \sum_{\zeta_1} \left|\mathscr{F}(\mathscr{C}_{i_1} \cap \mathscr{C}_{i_2} \cap \mathscr{C}_{i_3} \cap \cdots \cap \mathscr{C}_{i_{r-1}})\right|$ and $\sum_{\zeta_2} \left|\mathscr{F}(\mathscr{C}_{i_1} \cap \mathscr{C}_{i_2} \cap \mathscr{C}_{i_3} \cap \cdots \cap \mathscr{C}_{i_{r-1}} \cap \mathscr{C}_{i_r})\right|$, where, $\zeta_1 : 1 \leq i_1 < i_2 < i_3 < \cdots < i_{r-1} \leq r$ and $\zeta_2 : 1 = i_1 < i_2 < i_3 < \cdots < i_{r-1} < i_r = r$.

Step 8. By using combinatorial tools on maximal chains, find the overall sum of both the sum of all distinct fuzzy subgroups (excluding first order) corresponding to every i^{th} maximal chain, where $1 \leq i \leq r$ and the sum of all distinct fuzzy subgroups (except of first order) corresponding to the intersection of every ℓ maximal chains, where $2 \leq \ell \leq r$. Thus, we have computed $\left|\mathscr{F}\left(\bigcup_{i=1}^{r} \mathscr{C}_i\right)\right|$.

We enumerate the fuzzy subgroups of the quaternion group of Q_8 as an illustration of the above algorithm.

Example 2.3. The quaternion group Q_8 is generated by $\langle i, j\rangle$, where $i^2 = -1 = j^2 = k^2 = ijk$. Let B_1, B_2, B_3 be the possible maximal chains of Q_8. The subgroups of Q_8 are:
$H_1 = \langle 1\rangle$, $H_2 = \langle -1\rangle$, $H_3 = \langle i\rangle$, $H_4 = \langle j\rangle$, $H_5 = \langle k\rangle$ and $H_6 = Q_8$.

The maximal chains of Q_8 are:

$$B_1 \; : \; H_1{}^{q_0} \subset H_2{}^{q_1} \subset H_3{}^{q_2} \subset H_6{}^{q_3}$$

$$B_2 \ : \ H_1{}^{q_0} \subset H_2{}^{q_1} \subset H_4{}^{q_2} \subset H_6{}^{q_3}$$

$$B_3 \ : \ H_1{}^{q_0} \subset H_2{}^{q_1} \subset H_5{}^{q_2} \subset H_6{}^{q_3}$$

$$\text{where } 1 \geq q_0 > q_1 > q_2 > q_3 \geq 0.$$

Step 1:
Here B_1 defines the following distinct possible fuzzy subgroups (except of first order) Q_8;
$H_1{}^{q_0} \subset H_2{}^{q_1} \subset H_3{}^{q_2} \subset H_6{}^{q_3}$

$$\lambda_1(g) = \begin{cases} q_0, & for \ \ g = e, \\ q_1, & for \ \ g \in H_2 \setminus H_1, \\ q_2, & for \ \ g \in H_3 \setminus H_2, \\ q_3, & for \ \ g \in H_6 \setminus H_3. \end{cases}$$

$H_1{}^{q_0} \subset H_2{}^{q_1} \subset H_6{}^{q_2}$

$$\lambda_2(g) = \begin{cases} q_0, & for \ \ g = e, \\ q_1, & for \ \ g \in H_2 \setminus H_1, \\ q_2, & for \ \ g \in H_6 \setminus H_2. \end{cases}$$

$H_1{}^{q_0} \subset H_3{}^{q_1} \subset H_6{}^{q_2}$

$$\lambda_3(g) = \begin{cases} q_0, & for \ \ g \in H_1, \\ q_1, & for \ \ g \in H_3 \setminus H_1, \\ q_2, & for \ \ g \in H_6 \setminus H_3. \end{cases}$$

$H_2{}^{q_0} \subset H_3{}^{q_1} \subset H_6{}^{q_2}$

$$\lambda_4(g) = \begin{cases} q_0, & for \ \ g \in H_2, \\ q_1, & for \ \ g \in H_3 \setminus H_2, \\ q_2, & for \ \ g \in H_6 \setminus H_3. \end{cases}$$

$H_1{}^{q_0} \subset H_6{}^{q_1}$

$$\lambda_5(g) = \begin{cases} q_0, & for \ \ g \in H_1, \\ q_1, & for \ \ g \in H_6 \setminus H_1. \end{cases}$$

$H_2{}^{q_0} \subset H_6{}^{q_1}$

$$\lambda_6(g) = \begin{cases} q_0, & for \ \ g \in H_2, \\ q_1, & for \ \ g \in H_6 \setminus H_2. \end{cases}$$

$H_3{}^{q_0} \subset H_6{}^{q_1}$

$$\lambda_7(g) = \begin{cases} q_0, & for \ \ g \in H_3, \\ q_1, & for \ \ g \in H_6 \setminus H_3. \end{cases}$$

$\Rightarrow \qquad |\mathscr{F}(B_1)| = 7.$

On the same pattern, $|\mathscr{F}(B_2)| = 7$ and $|\mathscr{F}(B_3)| = 7$.

$$|\mathscr{F}(B_1)| = |\mathscr{F}(B_2)| = |\mathscr{F}(B_3)| = 7.$$

Step 2:

$$|\mathscr{F}(B_1 \cap B_2)| = |\mathscr{F}(B_1 \cap B_3)| = |\mathscr{F}(B_2 \cap B_3)| = 3.$$

Step 3:

$$|\mathscr{F}(B_1 \cap B_2 \cap B_3)| = 3,$$

$$\text{since} \quad |\mathscr{F}(Q_8)| = \left|\mathscr{F}(\bigcup_{i=1}^{3} B_i)\right| + 1.$$

Thus , we have,

$$\begin{aligned} \left|\mathscr{F}\left(\bigcup_{i=1}^{3} B_i\right)\right| &= |\mathscr{F}(B_1)| + |\mathscr{F}(B_2)| + |\mathscr{F}(B_3)| \\ &- |\mathscr{F}(B_1 \cap B_2)| - |\mathscr{F}(B_1 \cap B_3)| - |\mathscr{F}(B_2 \cap B_3)| \\ &+ |\mathscr{F}(B_1 \cap B_2 \cap B_3)| \end{aligned}$$

$$\left|\mathscr{F}\left(\bigcup_{i=1}^{3} B_i\right)\right| = (7 + 7 + 7) - (3 + 3 + 3) + 3 = 15.$$

Hence

$$|\mathscr{F}\,(Q_8)| = \left|\mathscr{F}\left(\bigcup_{i=1}^{3} B_i\right)\right| + 1 = 16,$$

which is in agreement with the above Theorem 2.3.

We give another example of computing fuzzy subgroups.

Example 2.4. The Dihedral group D_8 has elements $\{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$. The subgroups of D_8 are:
$H_1 = \{e\}$, $H_2 = \{e, a, a^2, a^3\}$, $H_3 = \{e, b\}$, $H_4 = \{e, a^2\}$, $H_5 = \{e, ab\}$, $H_6 = \{e, a^2b\}$, $H_7 = \{e, a^3b\}$, $H_8 = \{e, a^2, ab, a^3b\}$, $H_9 = \{e, b, a^2, a^2b\}$ and $H_{10} = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$.
The maximal chains of D_8 are:

$$B_1 : H_1^{q_0} \subset H_5^{q_1} \subset H_9^{q_2} \subset H_{10}^{q_3}.$$
$$B_2 : H_1^{q_0} \subset H_3^{q_1} \subset H_9^{q_2} \subset H_{10}^{q_3}.$$
$$B_3 : H_1^{q_0} \subset H_6^{q_1} \subset H_9^{q_2} \subset H_{10}^{q_3}.$$
$$B_4 : H_1^{q_0} \subset H_6^{q_1} \subset H_8^{q_2} \subset H_{10}^{q_3}.$$
$$B_5 : H_1^{q_0} \subset H_6^{q_1} \subset H_7^{q_2} \subset H_{10}^{q_3}.$$
$$B_6 : H_1^{q_0} \subset H_2^{q_1} \subset H_7^{q_2} \subset H_{10}^{q_3},$$

where $1 \geq q_0 \geq q_1 \geq q_2 \geq q_3 \geq 0$.

We find that $|F(D_8)| = \left|\bigcup_{i=1}^{i=7} B_i\right| - 5 = 32$. Thus, there are 32 fuzzy subgroups of D_8.

3. Conclusions

The number of fuzzy subgroups of a finite group G depend on its maximal chains because every maximal chain and intersection of maximal chains define fuzzy subgroups of a finite group G. We can get exactly one distinct fuzzy subgroup (trivial) of G of order 1, corresponding to a unique chain ($G_i \subset G$) of G. We can find the number of all distinct possible fuzzy subgroups (of order greater than one) of a finite group G by an algorithm which is consequence of above Theorem 2.3 as

$$\left|\mathscr{F}(\bigcup_{i=1}^{m} B_i)\right| = \sum_{i=1}^{m} |\mathscr{F}(B_i)| - \sum_{1\le i<j\le m} \left|\mathscr{F}(B_i \cap B_j)\right| + \sum_{1\le i<j<k\le m} \left|\mathscr{F}(B_i \cap B_j \cap B_k)\right| + \cdots + (-1)^{m-1} |\mathscr{F}(B_1 \cap B_2 \cap \ldots \cap B_m)|.$$

This formula is only valid and useful for all chains in the lattice of G of i^{th} length and terminate in G, where $i \geq 1$. Due to the comlexity of the algorithm used, this method is feasible for groups of reasonably small orders. Moreover, the computer algebra system GAP [21] can be used to find the number of fuzzy subgroups of a group using the approach given in this paper.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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