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Research article

Product of H-Toeplitz operator and Toeplitz operator on the Bergman space

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Abstract: In this paper, we characterize when the product of two H-Toeplitz operators to be another H-Toeplitz with one general and another quasihomogeneous symbols. Also, we describe the product of H-Toeplitz operator and Toeplitz operator to be another H-Toeplitz with certain harmonic symbols.

Keywords: H-Toeplitz operators; Toeplitz operators; Bergman space; quasihomogeneous function; product problem
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1. Introduction

Let $\mathbb{D}$ denote the unit disc in the complex plane. Let L^2 be the Hilbert space of all Lebesgue square integral functions with respect to the normalized area measure dA on $\mathbb{D}$. The Bergman space L_a^2 is consisting of all holomorphic functions contained in L^2. It is well known that L_a^2 is a closed subspace of L^2 and has an orthogonormal basis $\{e_n\}_{n=0}^{+\infty}$, where $e_n(w) = \sqrt{n+1}w^n$. The Bergman space is a reproducing Hilbert space with the reproducing kernel K_z, which is given explicitly by

$$K_z(w) = \frac{1}{(1-\bar{z}w)^2}, \quad z, w \in \mathbb{D}.$$

Let P be the orthogonal projection from L^2 onto L_a^2, then P is given by

$$(Pf)(w) = \int_{\mathbb{D}} \frac{f(z)}{(1-\bar{z}w)^2} dA(z), \quad f \in L^2, \ \ w \in \mathbb{D}.$$

Denote L^∞ as the set of all bounded measurable functions on $\mathbb{D}$. For $f \in L^\infty$, the Toeplitz operator T_f with symbol f is defined by

$$T_f g = P(fg), \quad g \in L_a^2.$$

It is easy to see that T_f is a bounded operator on the Bergman space.

Let L_h^2 be the harmonic Bergman space which is the collection of all harmonic functions in L^2. Define a unitary operator $K : L_a^2 \to L_h^2$ by $K(e_{2n}) = e_n$ and $K(e_{2n+1}) = \overline{e_{n+1}}$, $n = 0, 1, 2, \cdots$. The H-Toeplitz operator B_f with symbol $f \in L^\infty$ is defined by

$$B_f g = P(fKg), \quad g \in L_a^2.$$

Obviously B_f is a bounded operator on the Bergman space.

Let $\mathcal{R}$ be the space of square integrable functions on $[0, 1]$ with respect to the measure rdr. It is clear that the functions in $\mathcal{R}$ are radial functions on $\mathbb{D}$. Since trigonometric polynomials are dense in L^2 and $e^{ik_1\theta}\mathcal{R}$ is orthogonal to $e^{ik_2\theta}\mathcal{R}$ for $k_1 \neq k_2$, one can see that

$$L^2 = \bigoplus_{k\in\mathbb{Z}} e^{ik\theta}\mathcal{R}.$$

So, for each $f \in L^2$, it can be written as (see [4])

$$f(re^{i\theta}) = \sum_{k\in\mathbb{Z}} e^{ik\theta}\varphi_k(r),$$

where each $\varphi_k \in \mathcal{R}$ is bounded radial function when $f \in L^\infty$. Each function in $e^{ik\theta}\mathcal{R}$ is called a quasihomogeneous function with degree k.

In 1964, Brown and Halmos [1] showed that for Toeplitz operators on the Hardy space, $T_f T_g = T_h$ holds if and only if either $\bar{f}$ or g is analytic and $h = fg$. In 1989, Zheng [2] showed that if f, g are bounded harmonic functions such that $T_f T_g = T_h$ on the Bergman space, then either $\bar{f}$ or g is analytic. The product problem on Toeplitz operators with general symbols turns out to be much more complicated. In [5] Louhichi and Zakariasy showed that if the product of two Toeplitz operators with the quasihomogeneous symbols on the Bergman space with the degree p and s respectively to be another Toeplitz operator, then the symbol functions must be quasihomogeneous with the degree $p + s$. In [6] Louhichi, Strouse and Zakariasy showd the relationship between the radial part of the quasihomogeneous symbols.

In 2007, Arora and Paliwal [7] started to study the H-Toeplitz on the Hardy space. Gupta and Singh expand this definition for Slant H-Toeplitz operators on the Hardy space [8] and for H-Toeplitz operator on the Bergman space [9]. In 2022, Liang et al. characterized the commuting of H-Toeplitz operators with quasihomogeneous symbols on the Bergman space, see [10].

Motivated by the mentioned works, in Section 3 of this paper we will characterize when the product of two H-Toeplitz operators to be another H-Toeplitz with one general and another quasihomogeneous symbols, see Theorems 3.1 and 3.3. Also, in Section 4 we will consider the product of Toeplitz operator and H-Toeplitz operator to be another H-Toeplitz with certain harmonic symbols, that is, when $T_f B_g = B_h$ or $B_f T_g = B_h$ holds for certain harmonic symbols f, g, h, see Theorems 4.1 and 4.2 respectively. With non-harmonic symbols, we consider a simple case which tells the answer of when $B_f T_g = B_h$ is not trivial, see Theorem 4.4.

2. Preliminaries

In this section, we will present some lemmas which will be used frequently.

The Mellin transform $\widehat{\varphi}$ of a function $\varphi \in L^1([0,1], rdr)$ which plays an important role is defined by

$$\widehat{\varphi}(w) = \int_0^1 \varphi(r) r^{w-1} dr.$$

It is clear that $\widehat{\varphi}$ is analytic on $\{w : \operatorname{Re} w > 2\}$. The following two lemmas have been proved in [10] which will be used often in the paper.

Lemma 2.1. *Let $\varphi \in L^1([0,1], rdr)$. If there exist a sequence of positive integers $\{n_k\}$ satisfying that $\sum\limits_{k=1}^{\infty} \frac{1}{n_k} = \infty$ and $\widehat{\varphi}(n_k) = 0$ for all k, then $\varphi = 0$.*

Lemma 2.2. *Let ϕ be a bounded radial function and p an integer. Then for any nonnegative integer n,*

$$B_{e^{ip\theta}\phi}(w^{2n}) = \begin{cases} 2\sqrt{\dfrac{n+1}{2n+1}}(n+p+1)\widehat{\phi}(2n+p+2)w^{n+p}, & n+p \geq 0, \\ 0 & , \ n+p < 0, \end{cases}$$

$$B_{e^{ip\theta}\phi}(w^{2n+1}) = \begin{cases} 2\sqrt{\dfrac{n+2}{2n+2}}(p-n)\widehat{\phi}(p+2)w^{p-n-1}, & n+1 \leq p, \\ 0 & , \ n+1 > p. \end{cases}$$

By Lemma 2.2, we obtain the following two lemmas immediately.

Lemma 2.3. *Let p be a nonnegative integer. Then for each nonnegative integer n,*

$$B_{\overline{w}^p}(w^{2n+1}) = 0, \quad B_{w^p}(w^{2n}) = \sqrt{\frac{n+1}{2n+1}} w^{n+p},$$

$$B_{\overline{w}^p}(w^{2n}) = \begin{cases} \sqrt{\dfrac{n+1}{2n+1}} \dfrac{n-p+1}{n+1} w^{n-p}, & n \geq p, \\ 0 & , \ n<p, \end{cases}$$

$$B_{w^p}(w^{2n+1}) = \begin{cases} \sqrt{\dfrac{n+2}{2n+2}} \dfrac{p-n}{p+1} w^{p-n-1}, & n \leq p-1, \\ 0 & , \ n>p-1. \end{cases}$$

Lemma 2.4. *Suppose $f = \sum_{k\in\mathbb{Z}} e^{ik\theta}\varphi_k(r)$, $h = \sum_{s=-M}^{\infty} e^{is\theta}\psi_s(r) \in L^\infty$, where M is a nonnegative integer. Then for nonnegative integer n,*

$$B_f(w^{2n}) = 2\sum_{k=-n}^{\infty} \sqrt{\frac{n+1}{2n+1}}(n+k+1)\widehat{\varphi}_k(k+2n+2)w^{n+k},$$

and for $n \geq M$,

$$B_h(w^{2n}) = 2\sum_{s=-M}^{\infty} \sqrt{\frac{n+1}{2n+1}}(n+s+1)\widehat{\psi}_s(s+2n+2)w^{n+s}.$$

In [9], it is showed that the map $f \to B_f$ is one to one, then the following lemma holds.

Lemma 2.5. *Suppose $f \in L^\infty$, then $B_f = 0$ if and only if $f = 0$.*

3. Product of H-Toeplitz operators

In this section, we focus on the product of two H-Toeplitz operators. Our aim here is to provide a sufficient and necessary condition for the product of two H-Toeplitz operators to be another H-Toeplitz operator with more general symbols.

Theorem 3.1. *Let p be an integer and M a nonnegative integer. Suppose ϕ is a bounded radial function on $\mathbb{D}$ and $f, h \in L^\infty$ with $h = \sum_{s=-M}^{\infty} e^{is\theta}\psi_s(r)$. Then the following statements are equivalent:*

(1) $B_f B_{e^{ip\theta}\phi} = B_h$,
(2) $B_{e^{ip\theta}\phi} B_f = B_h$,
(3) $f = h = 0$ *or* $\phi = h = 0$.

Proof. If (3) holds, then (1) and (2) hold clearly. Conversely, suppose one of (1) and (2) holds. If $\phi = 0$, then by Lemma 2.5 we can obtain $\phi = h = 0$ immediately. So in the following we assume $\phi \neq 0$ and show that $f = h = 0$. For this end, we first write

$$f = \sum_{k\in\mathbb{Z}} e^{ik\theta}\varphi_k(r),$$

where each φ_k is bounded radial function. Choose n satisfying $2n \geq M$. By Lemma 2.4,

$$B_h(w^{4n}) = 2\sum_{s=-M}^{\infty} \sqrt{\frac{2n+1}{4n+1}}(2n+s+1)\widehat{\phi_s}(4n+s+2)w^{2n+s}, \tag{3.1}$$

$$B_h(w^{4n+2}) = 2\sum_{s=-M}^{\infty} \sqrt{\frac{2n+2}{4n+3}}(2n+s+2)\widehat{\phi_s}(4n+s+4)w^{2n+s+1}. \tag{3.2}$$

"(1) $\Rightarrow$ (3)". Suppose $B_f B_{e^{ip\theta}\phi} = B_h$ and $\phi \neq 0$. We show the result in the following two cases.
Case 1. p is even. Let $n > \max\{0, \frac{p}{2} + M + 1\}$, by Lemmas 2.2 and 2.4, direct computations give that

$$\begin{aligned}
&B_f B_{e^{ip\theta}\phi}(w^{4n})\\
&= 2\sqrt{\frac{2n+1}{4n+1}}(2n+p+1)\widehat{\phi}(4n+p+2)B_f(w^{2n+p})\\
&= 4\sqrt{\frac{2n+1}{4n+1}}(2n+p+1)\widehat{\phi}(4n+p+2)\\
&\quad\times\sum_{k=-n-\frac{p}{2}}^{\infty}\sqrt{\frac{n+\frac{p}{2}+1}{2n+p+1}}\Big(n+\frac{p}{2}+k+1\Big)\widehat{\varphi_k}(k+2n+p+2)w^{k+n+\frac{p}{2}}.
\end{aligned} \tag{3.3}$$

Since (3.1) equals to (3.3), we obtain that

$$\widehat{\phi}(4n+p+2)\widehat{\varphi_k}(k+2n+p+2) = 0$$

for any $k = -n-\frac{p}{2}, \cdots, n-\frac{p}{2}-M-1$. In other words, the above holds when $n > N_k = \max\{0, \frac{p}{2} + M + 1, \frac{p}{2} + M + 1 + k, -\frac{p}{2} - k\}$ for each integer k. Set

$$E_k = \Big\{n > N_k : \quad \widehat{\phi}(4n+p+2) \neq 0\Big\}.$$

By Lemma 2.1 and $\phi \neq 0$, we have $\sum_{n\in E_k}\frac{1}{n}=\infty$. For each fixed k, choose $n\in E_k$, then $\widehat{\varphi_k}(k+2n+p+2)=0$ with $\sum_{n\in E_k}\frac{1}{k+2n+p+2}=\infty$. By Lemma 2.1 we get $\varphi_k=0$ for each integer k. So we obtain $f=0$ and hence $h=0$.

Case 2. p is odd. Let $n>\max\{0, M+\frac{p+1}{2}\}$, by Lemmas 2.2 and 2.4 again, we have

$$\begin{aligned}
&B_f B_{e^{ip\theta}\phi}(w^{4n+2})\\
&=2\sqrt{\frac{2n+2}{4n+3}}(2n+p+2)\widehat{\phi}(4n+p+4)B_f(w^{2n+p+1})\\
&=4\sqrt{\frac{2n+2}{4n+3}}(2n+p+2)\widehat{\phi}(4n+p+4)\\
&\quad\times\sum_{k=-n-\frac{p+1}{2}}^{\infty}\sqrt{\frac{n+\frac{p+1}{2}+1}{2n+p+2}}\Big(n+\frac{p+1}{2}+k+1\Big)\widehat{\varphi_k}(k+2n+p+3)w^{k+n+\frac{p+1}{2}}.
\end{aligned}\tag{3.4}$$

Because (3.2) equals to (3.4), it follows that

$$\widehat{\phi}(4n+p+4)\widehat{\varphi_k}(k+2n+p+3)=0,$$

where $k=-n-\frac{p+1}{2},\cdots,n-M-\frac{p+1}{2}$. With the similar arguments as done in Case 1, we can obtain $f=0$ and then $h=0$. Therefore, (3) holds.

"(2) ⇒ (3)". Suppose $B_{e^{ip\theta}\phi}B_f=B_h$ and $\phi\neq 0$. Let the integer $n>|p|+M+1$, we deduce (3) by the following two cases.

Case 1. $p\le 0$. By Lemmas 2.2 and 2.4, we may obtain that

$$\begin{aligned}
&B_{e^{ip\theta}\phi}B_f(w^{4n})\\
&=4\sum_{k=-n-p}^{\infty}\sqrt{\frac{2n+1}{4n+1}}(2n+2k+1)\widehat{\varphi_{2k}}(4n+2k+2)\\
&\quad\times\sqrt{\frac{n+k+1}{2n+2k+1}}(n+k+p+1)\widehat{\phi}(2n+2k+p+2)w^{n+k+p}.
\end{aligned}\tag{3.5}$$

Since (3.1) equals to (3.5), then we have $\widehat{\varphi_{2k}}(4n+2k+2)\widehat{\phi}(2n+2k+p+2)=0$ for $k=-n-p,\dots,n-p-M-1$, where $n>|p|+M+1$. As done in Case 1 of "(1) ⇒ (3)", one may obtain $\varphi_{2k}=0$ for any integer k. Also, by Lemmas 2.2 and 2.4 again, we get

$$\begin{aligned}
&B_{e^{ip\theta}\phi}B_f(w^{4n+2})\\
&=4\sum_{k=-n-p-1}^{\infty}\sqrt{\frac{2n+2}{4n+3}}(2n+2k+3)\widehat{\varphi_{2k+1}}(4n+2k+5)\\
&\quad\times\sqrt{\frac{n+k+2}{2n+2k+3}}(n+k+p+2)\widehat{\phi}(2n+2k+p+4)w^{n+k+p+1}.
\end{aligned}\tag{3.6}$$

Because (3.2) equals to (3.6), we have $\widehat{\varphi_{2k+1}}(4n+2k+5)\widehat{\phi}(2n+2k+p+4)=0$ for $k=-n-p-1,\dots,n-p-M-1$, where $n>|p|+M+1$. As done before, we then obtain $\varphi_{2k+1}=0$ for each integer k. Thus we get $f=0$, and hence $h=0$. So (3) holds.

Case 2. $p > 0$. By Lemma 2.2 and (2.4), we have

$$\begin{aligned} B_{e^{ip\theta}\phi}B_f(w^{4n}) = 4\sum_{k=-n}^{\infty} & \sqrt{\frac{2n+1}{4n+1}}(2n+2k+1)\widehat{\varphi}_{2k}(4n+2k+2) \\ & \times \sqrt{\frac{n+k+1}{2n+2k+1}}(n+k+p+1)\widehat{\phi}(2n+2k+p+2)w^{n+k+p} \\ + 4\sum_{k=-n}^{p-n-1} & \sqrt{\frac{2n+1}{4n+1}}(2n+2k+2)\widehat{\varphi}_{2k+1}(4n+2k+3) \\ & \times \sqrt{\frac{n+k+2}{2n+2k+2}}(p-n-k)\widehat{\phi}(p+2)w^{p-n-k-1}, \end{aligned} \tag{3.7}$$

Comparing (3.1) with (3.7), it gives that $\widehat{\varphi}_{2k}(4n+2k+2)\widehat{\phi}(2n+2k+p+2) = 0$ for $k = -n, \dots, n-p-M-1$, where $n > p+M+1$. By using the same arguments as done in Case 1, we have $\widehat{\varphi}_{2k} = 0$ for any integer k. Also, by Lemma 2.2 and (2.4),

$$\begin{aligned} B_{e^{ip\theta}\phi}B_f(w^{4n+2}) = 4\sum_{k=-n}^{p-n-1} & \sqrt{\frac{2n+2}{4n+3}}(2n+2k+2)\widehat{\varphi}_{2k}(4n+2k+4) \\ & \times \sqrt{\frac{n+k+2}{2n+2k+2}}(p-n-k)\widehat{\phi}(p+2)w^{p-n-k-1} \\ + \sum_{k=-n-1}^{\infty} & \sqrt{\frac{2n+2}{4n+3}}(2n+2k+3)\widehat{\varphi}_{2k+1}(4n+2k+5) \\ & \times \sqrt{\frac{n+k+2}{2n+2k+3}}(n+k+p+2)\widehat{\phi}(2n+2k+p+3)w^{n+k+p+1}. \end{aligned} \tag{3.8}$$

By comparing (3.2) with (3.8), it follows that $\widehat{\varphi}_{2k+1}(4n+2k+5)\widehat{\phi}(2n+2k+p+4) = 0$ for $k = -n-1, \dots, n-p-M-1$, where $n > p+M+1$. Similarly we have $\widehat{\varphi}_{2k+1} = 0$ for any integer k. Above all, $f = 0$. Hence $h = 0$, so (3) holds. □

The following zero product problem holds immediately.

Corollary 3.2. *Suppose $f \in L^\infty$ and ϕ is a bounded radial function. Let p be an integer. Then the following statements are equivalent:*

(1) $B_f B_{e^{ip\theta}\phi} = 0$,
(2) $B_{e^{ip\theta}\phi}B_f = 0$,
(3) $f = 0$ or $\phi = 0$.

Now we are ready to characterize the product of two H-Toeplitz operators to be another H-Toeplitz operator with harmonic and radial symbols.

Theorem 3.3. *Suppose f and h are bounded harmonic functions on $\mathbb{D}$, ϕ is a bounded radial function. Then $B_f B_\phi = B_h$ if and only if $f = h = 0$ or $\phi = h = 0$.*

Proof. The sufficiency is obvious, now we prove the necessity. First we write $f = f_+ + \overline{f_-}$ and $h = h_+ + \overline{h_-}$, where $f_+ = \sum_{j=0}^{\infty} a_j w^j$, $f_- = \sum_{s=1}^{\infty} b_s w^s$, $h_+ = \sum_{t=0}^{\infty} c_t w^t$ and $h_- = \sum_{m=1}^{\infty} d_m w^m$.

If $\phi = 0$, then the necessity holds. In the following we assume $\phi \neq 0$. By Lemma 2.3,

$$B_h(w) = \sum_{t=1}^{\infty} \frac{t}{t+1} c_t w^{t-1}, \tag{3.9}$$

and by Lemma 2.2,

$$B_f B_\phi(w) = 0. \tag{3.10}$$

Since (3.9) equals to (3.10), we have $c_t = 0$, $t \geq 1$. Hence $h = c_0 + \overline{h_-}$. For the nonnegative integer n, direct calculations show that

$$\begin{aligned} B_f B_\phi(w^{4n}) = 2\sqrt{\frac{2n+1}{4n+1}}(2n+2)\widehat{\phi}(4n+2)\sqrt{\frac{n+1}{2n+1}} \\ \times \Big(\sum_{j=0}^{\infty} a_j w^{j+n} + \sum_{s=0}^{n} \overline{b_s} \frac{n-s+1}{n+1} w^{n-s} \Big) \end{aligned} \tag{3.11}$$

and

$$B_h(w^{4n}) = \sqrt{\frac{2n+1}{4n+1}} \sum_{m=0}^{2n} \overline{d_m} \frac{2n-m+1}{2n+1} w^{2n-m}, \tag{3.12}$$

where $d_0 = c_0$. By comparing (3.11) with (3.12), we then get $\widehat{\phi}(4n+2)a_j = 0$ for $j \geq 2n$. As we assume $\phi \neq 0$, there must be a positive integer n_0 such that $\widehat{\phi}(4n_0+2) \neq 0$. Thus $a_j = 0$ for any integer $j \geq 2n_0$. Then (3.11) becomes

$$\begin{aligned} B_f B_\phi(w^{4n}) = 2\sqrt{\frac{2n+1}{4n+1}}(2n+1)\widehat{\phi}(4n+2)\sqrt{\frac{n+1}{2n+1}} \\ \times \Big(\sum_{j=0}^{2n_0-1} a_j w^{j+n} + \sum_{s=0}^{n} \overline{b_s} \frac{n-s+1}{n+1} w^{n-s} \Big). \end{aligned} \tag{3.13}$$

Let $n \geq 2n_0$. Observe that the biggest degree of w is $2n-1$ in (3.12), and $n+2n_0-1$ in (3.13), so we may obtain that $d_m = 0$ for $m = n-2n_0, \cdots, n+1$. Note that n is any nonnegative integer with $n \geq 2n_0$, hence $d_m = 0$ for any integer $m \geq 0$. It follows that $h = 0$. By Corollary 3.2, we then obtain $f = 0$. □

4. Product of Toeplitz operator and H-Toeplitz operator

In this section, we focus on the product of Toeplitz operator and H-Toeplitz operator to be another H-Toeplitz operator. First, we discuss the case of $T_f B_g = B_h$ with bounded harmonic symbols f, g, h. For this case we can apply the known result used for the product of two Toeplitz operators case on the Bergman space (see [3]).

Theorem 4.1. *Suppose f, g and h are bounded harmonic functions. Then $T_f B_g = B_h$ if and only if one of the following statements holds:*

(1) f is a constant and $fg = h$.
(2) f and g are co-analytic and $fg = h$.

Proof. We notice a fact: for any nonnegative integer n, it has that

$$T_f B_g(w^{2n}) = B_h(w^{2n}) \Longleftrightarrow T_f T_g(w^n) = T_h(w^n). \tag{4.1}$$

So by Corollary 1 in [3], we see that the above holds if and only if

$$fg = h \tag{4.2}$$

with f, g are both analytic or f, g are both co-analytic or one of f and g is constant.

We first show the sufficiency. If (1) holds, then it is clear that $T_f B_g = B_h$. If (2) holds, then h is also co-analytic, and so by Lemma 2.1, we have $T_f B_g(w^{2n+1}) = 0 = B_h(w^{2n+1})$ for any integer $n \geq 0$; on the other hand, by (4.1), we see that $T_f B_g(w^{2n}) = B_h(w^{2n})$ holds for each nonnegative integer n. Thus $T_f B_g = B_h$.

Now we show the necessity. As discussed before, when $T_f B_g = B_h$, then (4.2) holds and f and g are analytic, or f and g are co-analytic, or one of f and g is constant.

Case 1. Suppose f, g are analytic. Then h is also analytic by (4.2). We write $f = \sum_{j=0}^{\infty} a_j w^j$, $g = \sum_{s=0}^{\infty} b_s w^s$ and $h = \sum_{t=0}^{\infty} c_t w^t$, then by Lemma 2.3, $T_f B_g(w^{2n+1}) = B_h(w^{2n+1})$ gives that

$$\sum_{j=0}^{\infty} \sum_{s=n+1}^{\infty} \frac{s-n}{s+1} a_j b_s w^{j+s-n-1} = \sum_{t=n+1}^{\infty} \frac{t-n}{t+1} c_t w^{t-n-1}. \tag{4.3}$$

On comparing the coefficient of w^0 of both sides of (4.3), we get $c_{n+1} = a_0 b_{n+1}$ for any nonnegative integer n. Therefore,

$$f(0)(g - g(0)) = h - h(0). \tag{4.4}$$

If $f(0) \neq 0$, then puting the above into (4.2) to get $h(0) = f(0)g(0)$, so $f(0)g = h$. By (4.2) again we obtain that f is constant. If $f(0) = 0$, then (4.4) gives that h is a constant. By (4.2) we see that f and g both are constants. Hence (1) holds.

Case 2. f and g are co-analytic and $fg = h$, this is (2).

Case 3. If g is constant, then for any nonnegative integer n, $0 = T_f B_g(w^{2n+1}) = B_h(w^{2n+1})$. It follows from the right side of (4.3) that $h = h(0)$. Thus by (4.2), we see that f is constant. This is a special case of (1).

Case 4. If f is constant, then it is easy to see that (1) holds. □

Now we discuss the case of $B_f T_g = B_h$ with bounded harmonic symbols f, g, h. Although we only prove the case when $g = w^p$, the obtained result tells us that it may hold only in the trivial case.

Theorem 4.2. *Suppose f and h are bounded harmonic functions on $\mathbb{D}$. Let p be a nonnegative integer. Then $B_f T_{w^p} = B_h$ if and only if one of the following statements holds:*

(1) $p = 0$, $f = h$.
(2) $p \neq 0$, $f = h = 0$.

Proof. The sufficiency is obvious, now we prove the necessity. Suppose $B_f T_{w^p} = B_h$. If $p = 0$, we obtain $f = h$ immediately. In the following we suppose $p \neq 0$.

Write f and h as $f_+ + \overline{f_-}$ and $h_+ + \overline{h_-}$ respectively, where $f_+ = \sum_{j=0}^{\infty} a_j w^j$, $f_- = \sum_{s=1}^{\infty} b_s w^s$, $h_+ = \sum_{t=0}^{\infty} c_t w^t$ and $h_- = \sum_{m=1}^{\infty} d_m w^m$. We show the result by two cases.

Case 1. p is even. For any nonnegative integer n, by Lemma 2.3, we have

$$B_f T_{w^p}(w^{2n}) = \sqrt{\frac{n+\frac{p}{2}+1}{2n+p+1}}\Big(f_+ \cdot w^{n+\frac{p}{2}} + \sum_{s=1}^{n+\frac{p}{2}} \overline{b_s}\frac{n+\frac{p}{2}-s+1}{n+\frac{p}{2}+1} w^{n+\frac{p}{2}-s}\Big) \tag{4.5}$$

and

$$B_h(w^{2n}) = \sqrt{\frac{n+1}{2n+1}}\Big(h_+ \cdot w^n + \sum_{m=1}^{n} \overline{d_m}\frac{n-m+1}{n+1} w^{n-m}\Big). \tag{4.6}$$

Write $h_+^1 = \sum_{t=\frac{p}{2}}^{\infty} c_t w^t$ and $h_+^2 = \sum_{t=0}^{\frac{p}{2}-1} c_t w^t$, then $h_+ = h_+^1 + h_+^2$. Because (4.5) equals to (4.6), so for each nonnegative integer n, we have

$$\sqrt{\frac{n+\frac{p}{2}+1}{2n+p+1}} f_+ \cdot w^{n+\frac{p}{2}} = \sqrt{\frac{n+1}{2n+1}} h_+^1 \cdot w^n,$$

that is,

$$f_+ = \sqrt{\frac{(2n+p+1)(n+1)}{(n+\frac{p}{2}+1)(2n+1)}} \cdot \frac{h_+^1}{w^{p/2}}, \quad n \geq 0.$$

Hence $f_+ = h_+^1 = 0$. Also we have

$$B_f T_{w^p}(w) = B_f(w^{p+1}) = 0, \tag{4.7}$$

$$B_h(w) = \sqrt{\frac{n+1}{2n+1}} \sum_{t=1}^{\frac{p}{2}-1} c_t \frac{t}{t+1} w^t. \tag{4.8}$$

Since (4.7) equals to (4.8), we get $c_t = 0$ for $t = 1, 2, \cdots, \frac{p}{2}-1$. Now $h = c_0 + \overline{h_-}$. Putting $f_+ = 0$ and $h = c_0 + \overline{h_-}$ into (4.5) and (4.6) respectively, we then get

$$\begin{aligned}\sqrt{\frac{n+\frac{p}{2}+1}{2n+p+1}} &\sum_{s=1}^{n+\frac{p}{2}} \overline{b_s}\frac{n+\frac{p}{2}-s+1}{n+\frac{p}{2}+1} w^{n+\frac{p}{2}-s} \\ &= \sqrt{\frac{n+1}{2n+1}}\Big(c_0 w^n + \sum_{m=1}^{n} \overline{d_m}\frac{n-m+1}{n+1} w^{n-m}\Big),\end{aligned} \tag{4.9}$$

which shows that $b_s = 0$ for $s = 1, 2, \cdots, \frac{p}{2}-1$. For any nonnegative integer n, the coefficients of w^n in the above equation gives that

$$\sqrt{\frac{n+1}{2n+1}} c_0 = \sqrt{\frac{n+\frac{p}{2}+1}{2n+p+1}} \frac{n+1}{n+\frac{p}{2}+1} \overline{b_{\frac{p}{2}}}.$$

Hence $c_0 = b_{\frac{p}{2}} = 0$. Now $h_+ = 0$ and (4.9) can be rewritten as

$$\sqrt{\frac{n+\frac{p}{2}+1}{2n+p+1}} \sum_{s=\frac{p}{2}+1}^{n+\frac{p}{2}} \overline{b_s} \frac{n+\frac{p}{2}-s+1}{n+\frac{p}{2}+1} w^{n+\frac{p}{2}-s} = \sqrt{\frac{n+1}{2n+1}} \sum_{m=1}^{n} \overline{d_m} \frac{n-m+1}{n+1} w^{n-m}$$

for any integer $n \geq 1$. Now for fixed integer $m : 1 \leq m \leq n$, comparing the coefficients of w^{n-m}, we get

$$\sqrt{\frac{n+\frac{p}{2}+1}{2n+p+1}} \frac{\overline{b_{\frac{p}{2}+m}}}{n+\frac{p}{2}+1} = \sqrt{\frac{n+1}{2n+1}} \frac{\overline{d_m}}{n+1}, \quad \forall n \geq m.$$

Similarly we may get $d_m = b_{\frac{p}{2}+m} = 0$ for $1 \leq m \leq n$. Since n is any nonnegative integer, so we obtain that $d_m = b_{\frac{p}{2}+m} = 0$ for any integer $m \geq 1$. Therefore $h_- = f_- = 0$ and then it follows that $f = h = 0$.
Case 2. p is odd. By Lemma 2.3, for any nonnegative integer n, $B_f T_{w^p}(w^{2n+1}) = B_h(w^{2n+1})$ gives that

$$\begin{aligned} &\sqrt{\frac{n+\frac{p+1}{2}+1}{2n+p+2}} \Big(f_+ \cdot w^{n+\frac{p+1}{2}} + \sum_{s=1}^{n+\frac{p+1}{2}} \overline{b_s} \frac{n+\frac{p+1}{2}-s+1}{n+\frac{p+1}{2}+1} w^{n+\frac{p+1}{2}-s} \Big) \\ &= \sqrt{\frac{n+2}{2n+2}} \sum_{t=n+1}^{\infty} c_t \frac{t-n}{t+1} w^{t-n-1}. \end{aligned} \tag{4.10}$$

For fixed $s : 1 \leq s \leq n + \frac{p+1}{2}$, comparing the coefficients of $w^{n+\frac{p+1}{2}-s}$ in the above induces

$$\sqrt{\frac{n+\frac{p+1}{2}+1}{2n+p+2}} \cdot \frac{\overline{b_s}}{n+\frac{p+1}{2}+1} = \sqrt{\frac{n+2}{2n+2}} \cdot \frac{c_{2n+\frac{p+1}{2}-s+1}}{2n+\frac{p+1}{2}+1}.$$

Since $\lim_{n\to\infty} c_{2n+\frac{p+1}{2}-s+1} = 0$, then $b_s = 0$. Because s is any term of $1, 2, \cdots, n + \frac{p+1}{2}$ and n is any nonnegative integer, it implies that $b_s = 0$ for any $s \geq 1$. Hence, $f_- = 0$. By (4.10), one can get that

$$\sqrt{\frac{n+\frac{p+1}{2}+1}{2n+p+2}} f_+ \cdot w^{n+\frac{p+1}{2}} = \sqrt{\frac{n+2}{2n+2}} \sum_{t=n+1}^{\infty} c_t \frac{t-n}{t+1} w^{t-n-1}. \tag{4.11}$$

So, $c_{n+1} = \cdots = c_{2n+\frac{p+1}{2}+1} = 0$. Because n is any nonnegative integer, thus $c_t = 0$ for $t \geq 1$. Now we obtain that the left side of (4.11) is also zero. Therefore $f_+ = 0$. Above all, $f = 0$. By Lemma 2.5, we have $h = 0$. □

For the case of $B_f T_g = B_h$ with non-harmonic symbols, it becomes much complicated. So we only focus on the simple case with f and g both are radial functions and h is a general one. Even for such simple case, the obtained relation of f, g and h is not explicit, but it still tells that $B_f T_g = B_h$ holds with nontrivial case which is different from the previous result.

We need the following lemma which is proved in [5].

Lemma 4.3. *Let p be an integer and ψ a bounded radial function on $\mathbb{D}$. Then for any nonnegative integer n,*

$$T_{e^{ip\theta}\psi}(w^n) = \begin{cases} 2(n+p+1)\widehat{\psi}(2n+p+2)w^{n+p}, & n+p \geq 0, \\ 0 \qquad\qquad\qquad\qquad\quad, & n+p < 0. \end{cases}$$

Theorem 4.4. *Suppose ϕ and ψ are bounded radial functions on $\mathbb{D}$, $h \in L^\infty$. Then $B_\phi T_\psi = B_h$ if and only if h is a radial function and a solution of the equation*

$$2w\widehat{\psi}(2w)\widehat{\phi}(w+1) = \widehat{h}(w+1), \quad \operatorname{Re} w > 1. \tag{4.12}$$

Proof. We first show the necessity. Write h as $h = \sum_{k\in\mathbb{Z}} e^{ik\theta}\varphi_k(r)$, where each φ_k is bounded radial function. For any nonnegative integer n, by Lemmas 2.2 and 4.3, it follows from $B_\phi T_\psi(w^{2n}) = B_h(w^{2n})$ that

$$\begin{aligned} &4(2n+1)\widehat{\psi}(4n+2)\sqrt{\frac{n+1}{2n+1}}(n+1)\widehat{\phi}(2n+2)w^n \\ &= 2\sum_{k=-n}^{\infty}\sqrt{\frac{n+1}{2n+1}}(n+k+1)\widehat{\varphi_k}(k+2n+2)w^{n+k}. \end{aligned} \tag{4.13}$$

Hence for $n \geq 0$, we have $\widehat{\varphi_k}(k+2n+2) = 0$, $k \neq 0$. Note that $\sum_{n=0}^{\infty}\frac{1}{k+2n+2} = \infty$, so by Lemma 2.1, we get $\varphi_k = 0$ for all $k \neq 0$, which means that h is a radial function. Furthermore, we see that $B_\phi T_\psi(w^{2n+1}) = 0 = B_h(w^{2n+1})$, so (4.13) becomes

$$2(2n+1)\widehat{\psi}(4n+2)\widehat{\phi}(2n+2) = \widehat{h}(2n+2).$$

It implies that h is a solution of the equation

$$2w\widehat{\psi}(2w)\widehat{\phi}(w+1) = \widehat{h}(w+1), \quad \operatorname{Re} w > 1.$$

The sufficiency is obvious by the above arguments. □

We note that the Eq (4.12) has a nontrivial solution

$$\psi = ar^2 + c,\ \phi = r,\ h = (2a+c)r - a,$$

where a and c are any constants.

5. Conclusions

In this research, it obtains the following characterizations for the product of H-Toeplitz operators and Toeplitz operators with certain symbols on the Bergman space.

(1) Let p be an integer and M a nonnegative integer. Suppose ϕ is a bounded radial function on $\mathbb{D}$ and $f, h \in L^\infty$ with $h = \sum_{s=-M}^{\infty} e^{is\theta}\psi_s(r)$. Then $B_f B_{e^{ip\theta}\phi} = B_h$ if and only if $B_{e^{ip\theta}\phi} B_f = B_h$, and if and only if $f = h = 0$ or $\phi = h = 0$.

(2) Suppose f and h are bounded harmonic functions on $\mathbb{D}$, ϕ is a bounded radial function. Then $B_f B_\phi = B_h$ if and only if $f = h = 0$ or $\phi = h = 0$.

(3) Suppose f, g and h are bounded harmonic functions. Then $T_f B_g = B_h$ if and only if f is a constant and $fg = h$, or, f and g are co-analytic and $fg = h$.

(4) Suppose f and h are bounded harmonic functions on $\mathbb{D}$. Let p be a nonnegative integer. Then $B_f T_{w^p} = B_h$ if and only if $p = 0$ and $f = h$, or, $p \neq 0$ and $f = h = 0$.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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