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Research article

q-Rung orthopair fuzzy information aggregation and their application towards material selection

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Abstract: Material selection is a complex process that involves selecting the best material for a given application. It is a critical process in engineering, and the importance of selecting the right material for the job cannot be overstated. Multi-criteria decision-making (MCDM) is an important tool that can be used to help engineers make informed decisions about material selection. The logistic function can be extended using the soft-max function, which is widely used in stochastic classification methods like neural nets, soft-max extrapolation, linear differential analysis, and Naïve Bayes detectors. This has inspired researchers to develop soft-max-based fuzzy aggregation operators (AOs) for q-rung orthopair fuzzy sets (q-ROPFS) and to propose an MCDM approach based on these AOs. To test the effectiveness of this approach, the researchers applied it to a practical problem using q-rung orthopair fuzzy data and conducted a numerical example to validate the suggested procedures.

Keywords: soft-max function; aggregation operators; decision-making; q-rung orthopair fuzzy number
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1. Introduction

Material selection is a critical process in engineering that involves choosing the most appropriate materials for a given application. The importance of material selection lies in the fact that it can significantly affect the performance, cost, and safety of a product. The selection of materials plays a crucial role in determining the quality, durability, and overall performance of engineering products. The scope of material selection in engineering is vast, as it is applicable to virtually every type of product or system. Material selection is particularly important in areas such as aerospace, automotive, construction, electronics, and manufacturing, where the choice of materials can have a significant impact on the functionality and safety of the product. Material selection involves considering various factors such as mechanical properties, chemical properties, environmental conditions, cost, and availability of materials. The goal of material selection is to identify the best material for a given application based on its performance characteristics and cost-effectiveness.

In addition to the technical aspects of material selection, there are also ethical considerations. For instance, in industries such as electronics and renewable energy, the use of sustainable and environmentally-friendly materials is becoming increasingly important. Therefore, material selection also involves ethical considerations, such as the impact of materials on the environment and the social implications of their use. Material selection is a crucial process in engineering that involves selecting the most suitable materials for a given application based on technical, economic, and ethical considerations. The significance of material selection in engineering cannot be overstated, and its impact on product performance, safety, and cost-effectiveness is significant. Therefore, it is crucial to carefully consider the selection of materials in any engineering project.

Material selection is a complex process that involves selecting the best material for a given application. It is a critical process in engineering, and the importance of selecting the right material for the job cannot be overstated. MCDM is an important tool that can be used to help engineers make informed decisions about material selection. In this essay, we will explore the importance of MCDM in material selection for engineering. MCDM is a process that involves considering multiple criteria when making a decision. In the context of material selection, it involves considering various criteria such as mechanical properties, chemical properties, environmental impact, cost, availability, and ethical considerations. MCDM provides a systematic approach for evaluating and comparing the different materials available for a given application. One of the most significant benefits of using MCDM in material selection is that it allows engineers to consider a broad range of criteria when making a decision. In the past, material selection was often based on a single criterion, such as cost or mechanical properties. However, this approach often led to suboptimal choices as other important factors were not considered. MCDM allows engineers to take a holistic view of material selection, which can lead to better decision making and improved product performance.

MCDM also provides a framework for evaluating the relative importance of different criteria. For example, in some applications, cost may be the most important factor, while in others, environmental impact may be the primary concern. MCDM allows engineers to assign weights to different criteria based on their relative importance, which can help them make a more informed decision. Another advantage of using MCDM in material selection is that it allows engineers to consider trade-offs between different criteria. For example, a material may have excellent mechanical properties, but it may be expensive and difficult to obtain. MCDM provides a framework for evaluating the trade-offs between these different criteria, which can help engineers make a more informed decision. MCDM can also help engineers to identify the best materials for a given application. By considering multiple criteria, MCDM can identify materials that perform well in all areas, as well as materials that excel in specific areas. This can help engineers to identify the best material for a given application based on its overall performance characteristics.

One of the challenges of using MCDM in material selection is that it can be a complex process. There are many different methods of MCDM, each with its strengths and weaknesses. Additionally, the selection of criteria and the assignment of weights can be subjective and may vary between different engineers. However, with careful planning and execution, MCDM can provide significant benefits in material selection. Another challenge of using MCDM is that the available data may not be complete or accurate. For example, it may be difficult to obtain data on the environmental impact of different materials. In such cases, engineers may need to make assumptions or use estimated values, which can introduce some degree of uncertainty into the decision-making process. Despite these challenges, the benefits of using MCDM in material selection are significant. It provides a systematic approach for evaluating and comparing the different materials available for a given application. It allows engineers to consider a broad range of criteria, identify trade-offs between different criteria, and identify the best material for a given application. Additionally, MCDM can help to identify potential problems or issues with a given material, which can help to improve product performance and reduce costs. The concept of fuzzy set theory was initially proposed by Zadeh [1] to represent imprecise or uncertain information. Later, Atanassov [2] introduced the idea of intuitionistic fuzzy sets (IFS) as a generalization of fuzzy sets. Yager [3–5] then extended IFS to Pythagorean fuzzy sets (PFS), which can handle both imprecision and incompleteness in the input data. Yager [6] went on to introduce q-rung orthopair fuzzy sets (q-ROPFS), which is a more powerful model for representing vague information in real-life problems.

AOs are a critical component of the MCDM process in material selection for engineering. The importance of AOs lies in their ability to combine multiple criteria into a single value that can be used to compare different materials. AOs are mathematical functions that are used to combine multiple criteria into a single value. In the context of material selection, AOs are used to combine the different criteria, such as mechanical properties, cost, and environmental impact, into a single value that can be used to compare different materials. The use of AOs in material selection is critical because it allows engineers to take a holistic view of material selection. Rather than considering each criterion in isolation, AOs provide a way to consider the interrelationships between different criteria. For example, a material may have excellent mechanical properties, but it may be expensive or have a significant environmental impact. AOs provide a way to consider the trade-offs between these different criteria, which can help engineers to make a more informed decision. There are many different types of AOs, each with its strengths and weaknesses. Some of the most commonly used AOs in material selection include the weighted average, the weighted sum, and the ordered weighted average.

The weighted average is one of the most commonly used AOrs in material selection. It involves assigning weights to each criterion and calculating the weighted average of the criteria. The weighted average is a simple and easy-to-use AO that allows engineers to assign different levels of importance to each criterion. However, it can be sensitive to outliers and may not provide a clear picture of the trade-offs between different criteria. The weighted sum is another commonly used AO in material selection. It involves assigning weights to each criterion and adding them together to create a single value. The weighted sum is a straightforward AO that is easy to calculate and interpret. However, it may not provide a clear picture of the trade-offs between different criteria, as it does not consider the relative importance of each criterion.

The ordered weighted average is an AO that takes into account the order of the criteria. It assigns weights to each criterion based on its position in the order, with higher weights given to the more important criteria. The ordered weighted average is a powerful AO that provides a clear picture of the trade-offs between different criteria. However, it can be more challenging to calculate than other AOs. In addition to these commonly used AOs, there are many other types of AOs that can be used in material selection. The choice of AO will depend on the specific needs of the engineer and the characteristics of the decision problem.

Several researchers have extended aggregation operators (AOs) to different types of fuzzy sets. For example, Mahmood et al. [7], Wei et al. [8], Jose and Kuriaskose [9], Hashmi et al. [10, 11], Feng et al. [16], Yang et al. [24], Chen et al. [25], Wang and Liu [12], Zhang et al. [14], Liu and Wang [13], Zhao et al., Garg [17], and Rahman et al. [18] have all made contributions in this area. Wang et al. [19, 20] developed interactive Hamacher power AOs and interactive Archimedean norm operations that are related to partially ordered fuzzy sets. Wang and Li introduced Pythagorean fuzzy interaction power Bonferroni mean AOs [21]. Wei [22] and Gao et al. [23] introduced Pythagorean fuzzy interaction AOs. Liu and Liu [26] proposed the idea of q-ROPRF Bonferroni mean AOs. Liu et al. [28] introduced q-ROPRF Heronian mean AOs and applied them to multiple criteria decision-making. Joshi and Gegov [29] established the notion of q-ROPRF confidence-based AOs. Trillo et al. [30] proposed some application of mean operator, Chiclana et al. [31] introduced the decision-making approach based on fuzzy preference relations and Efe [32] gave the notion of integrated fuzzy MCDM approach. Trillo et al. [33] proposed MCDM technique depends on sentiment analysis. Zhang & Liu [34] gave MCDM method for personnel selection, Limboo & Dutta [35] gave some applications related to medical diagnosis, and Narang et al. [36] proposed Heronian mean AOs based approach. Some extensive work related to the proposed work can be seen in [37–41]. Farid et al. [42] introduced neutrosophic Einstein prioritized AOs and Riaz et al. [43] gave the prioritized AOs for q-ROPFSs. Liu et al. proposed the MCDM based on spherical fuzzy set (SpFS) [44], linguistic SpFS [45] and q-ROPFSs [46].

The remaining portions of this article are organised as shown below. The concepts that are essential to q-ROPFS are discussed in Section 2. In the third section, we examined the operational procedures of the suggested AOs. In Section 4, we provide a method for resolving MCDM issues that is based on the introduction of new AOs. You'll find an application for MCDM in the fifth section of this document. The conclusion of Section 6 includes some parting remarks as well as some suggestions for the future.

2. Some basic concepts

In this part, we review some fundamental principles of q-ROPRFS and the operating rules of q-ROPRFNs.

Definition 2.1. [6] Consider, a q-ROPRFS $\mathcal{B}$ in $\mathscr{Q}$ is defined as

$$\mathcal{B} = \{\langle \eth, {}^{\zeta}\text{Ц}_{\mathcal{B}}(\eth), {}^{\xi}\eta_{\mathcal{B}}(\eth)\rangle : \eth \in \mathscr{Q}\}$$

where ${}^{\zeta}\text{Ц}_{\mathcal{B}}, {}^{\xi}\eta_{\mathcal{B}} : \mathscr{Q} \to [0, 1]$ defines the MBSD and the N-MBSD of the alternative $\eth \in \mathscr{Q}$ and for every $\eth$ we have

$$0 \leq {}^{\zeta}\text{Ц}^{q}_{\mathcal{B}}(\eth) + {}^{\xi}\eta^{q}_{\mathcal{B}}(\eth) \leq 1.$$

Furthermore, $\pi_{\mathcal{B}}(\eth) = (1 - {}^{\zeta}\text{Ц}^{q}_{\mathcal{B}}(\eth) - {}^{\xi}\eta^{q}_{\mathcal{B}}(\eth))^{1/q}$ is called the indeterminacy degree of $\eth$ to $\mathcal{B}$.

Definition 2.2. [6] Let $\hbar^{\lambda}{}_1 = \langle {}^{\zeta}\text{Ц}_1, {}^{\xi}\eta_1\rangle$ and $\hbar^{\lambda}{}_2 = \langle {}^{\zeta}\text{Ц}_2, {}^{\xi}\eta_2\rangle$ be q-ROPRFNs. Then

(1) $\overline{\hbar^{\lambda}}{}_1 = \langle {}^{\xi}\eta_1, {}^{\zeta}\text{Ц}_1\rangle$

(2) $\hbar^{\lambda}{}_1 \vee \hbar^{\lambda}{}_2 = \langle max\{{}^{\zeta}\text{Ц}_1, {}^{\xi}\eta_1\}, min\{{}^{\zeta}\text{Ц}_2, {}^{\xi}\eta_2\}\rangle$

(3) $\hbar^{\lambda}{}_1 \wedge \hbar^{\lambda}{}_2 = \langle min\{{}^{\zeta}\text{Ц}_1, {}^{\xi}\eta_1\}, max\{{}^{\zeta}\text{Ц}_2, {}^{\xi}\eta_2\}\rangle$

(4) $\hbar^{\lambda}{}_1 \oplus \hbar^{\lambda}{}_2 = \langle ({}^{\zeta}\text{Ш}_1^q + {}^{\zeta}\text{Ш}_2^q - {}^{\zeta}\text{Ш}_1^{q\zeta}\text{Ш}_2^q)^{1/q}, {}^{\xi}\eta_1{}^{\xi}\eta_2\rangle$
(5) $\hbar^{\lambda}{}_1 \otimes \hbar^{\lambda}{}_2 = \langle {}^{\zeta}\text{Ш}_1{}^{\zeta}\text{Ш}_2, ({}^{\xi}\eta_1^q + {}^{\xi}\eta_2^q - {}^{\xi}\eta_1^{q\xi}\eta_2^q)^{1/q}\rangle$
(6) $\sigma\hbar^{\lambda}{}_1 = \langle (1 - (1 - {}^{\zeta}\text{Ш}_1^q)^{\sigma})^{1/q}, {}^{\xi}\eta_1^{\sigma}\rangle$
(7) $\hbar^{\lambda}{}_1^{\sigma} = \langle {}^{\zeta}\text{Ш}_1^{\sigma}, (1 - (1 - {}^{\xi}\eta_1^q)^{\sigma})^{1/q}\rangle$.

Definition 2.3. [6] Suppose $\widetilde{\mathfrak{R}} = \langle {}^{\zeta}\text{Ш}, {}^{\xi}\eta\rangle$ is a q-ROPRFN, then a "score function" $\mathfrak{E}$ of $\widetilde{\mathfrak{R}}$ is defined as

$$\mathfrak{E}(\widetilde{\mathfrak{R}}) = {}^{\zeta}\text{Ш}^q - {}^{\xi}\eta^q$$

$\mathfrak{E}(\widetilde{\mathfrak{R}}) \in [-1, 1]$. The ranking of a q-ROPRFN is defined by its score, whereby a higher score indicates a greater preference for the q-ROPRFN. However, the score function may not be applicable in several instances of q-ROPRFN. Hence, it is unnecessary to depend on the score function for comparing the q-ROPRFNs.

Definition 2.4. Suppose $\widetilde{\mathfrak{R}} = \langle {}^{\zeta}\text{Ш}, {}^{\xi}\eta\rangle$ is a q-ROPRFN, then an "accuracy function" $\Re$ of $\widetilde{\mathfrak{R}}$ is defined as

$$\Re(\widetilde{\mathfrak{R}}) = {}^{\zeta}\text{Ш}^q + {}^{\xi}\eta^q$$

$\Re(\widetilde{\mathfrak{R}}) \in [0, 1]$.

It is important to note that the score function value is limited to a range of -1 to 1. To facilitate the subsequent study, we develop an additional scoring function, $\mathscr{H}(\mathfrak{R}) = \frac{1+{}^{\zeta}\text{Ш}_{\mathfrak{R}}^q - {}^{\xi}\eta_{\mathfrak{R}}^q}{2}$. We can see that $0 \leq \mathscr{H}(\mathfrak{R}) \leq 1$.

2.1. q-Rung orthopair fuzzy AOs

Definition 2.5. [6] Assume that $\hbar^{\lambda}{}_k = \langle {}^{\zeta}\text{Ш}_k, {}^{\xi}\eta_k\rangle$ is a agglomeration of q-ROPRFNs, and q-ROPRFWA: $\Lambda^n \to \Lambda$, if

$$\begin{aligned} q - ROPRFWA(\hbar^{\lambda}{}_1, \hbar^{\lambda}{}_2, \ldots \hbar^{\lambda}{}_n) &= \sum_{k=1}^{n} \urcorner^{\gamma}{}_k \hbar^{\lambda}{}_k \\ &= \urcorner^{\gamma}{}_1 \hbar^{\lambda}{}_1 \oplus \urcorner^{\gamma}{}_2 \hbar^{\lambda}{}_2 \oplus \ldots, \urcorner^{\gamma}{}_n \hbar^{\lambda}{}_n \end{aligned}$$

where Λ^n is the set of all q-ROPRFNs, and $\urcorner^{\gamma} = (\urcorner^{\gamma}{}_1, \urcorner^{\gamma}{}_2, \ldots, \urcorner^{\gamma}{}_n)^T$ is weight vector (WV) of $(\hbar^{\lambda}{}_1, \hbar^{\lambda}{}_2, \ldots, \hbar^{\lambda}{}_n)$, such that $0 \leqslant \urcorner^{\gamma}{}_k \leqslant 1$ and $\sum_{k=1}^{n} \urcorner^{\gamma}{}_k = 1$. Then, the q-ROPRFWA is called the "q-rung orthopair fuzzy weighted average operator".

Theorem 2.6. *[6] Let $\hbar^{\lambda}{}_k = \langle {}^{\zeta}\text{Ш}_k, {}^{\xi}\eta_k\rangle$ be the agglomeration of q-ROPRFNs,we can find $q - ROPRFWG$ by*

$$q - ROPRFWA(\hbar^{\lambda}{}_1, \hbar^{\lambda}{}_2, \ldots \hbar^{\lambda}{}_n) = \left\langle \sqrt[q]{(1 - \prod_{k=1}^{n}(1 - {}^{\zeta}\text{Ш}_k^q)^{\urcorner^{\gamma}{}_k})}, \prod_{k=1}^{n} {}^{\xi}\eta_k^{\urcorner^{\gamma}{}_k} \right\rangle.$$

Definition 2.7. [6] Assume that $\hbar^{\lambda}{}_k = \langle {}^{\zeta}\text{Ш}_k, {}^{\xi}\eta_k\rangle$ is the agglomeration of $q - ROPRFN$, and $q - ROPRFWG : \Lambda^n \to \Lambda$, if

$$q - ROPRFWG(\hbar^{\lambda}{}_1, \hbar^{\lambda}{}_2, \ldots \hbar^{\lambda}{}_n) = \sum_{k=1}^{n} \hbar^{\lambda}{}_k^{\urcorner^{\gamma}{}_k}$$

$$= \hbar^{\lambda}{}_{1}^{\daleth^{\gamma}{}_{1}} \otimes \hbar^{\lambda}{}_{2}^{\daleth^{\gamma}{}_{2}} \otimes \ldots, \hbar^{\lambda}{}_{n}^{\daleth^{\gamma}{}_{n}}$$

where Λ^n is the set of all $q - ROPRFNs$, and $\daleth^{\gamma} = (\daleth^{\gamma}{}_1, \daleth^{\gamma}{}_2, \ldots, \daleth^{\gamma}{}_n)^T$ is WV of $(\hbar^{\lambda}{}_1, \hbar^{\lambda}{}_2, \ldots, \hbar^{\lambda}{}_n)$, such that $0 \leqslant \daleth^{\gamma}{}_k \leqslant 1$ and $\sum_{k=1}^{n} \daleth^{\gamma}{}_k = 1$. Then, the $q-ROPRFWG$ is called the "q-rung orthopair fuzzy weighted geometric operator".

Theorem 2.8. *[6] Let $\hbar^{\lambda}{}_k = \langle {}^{\zeta}\text{Ш}_k, {}^{\xi}\eta_k \rangle$ be the agglomeration of q-ROPRFNs, we can find $q - ROPRFWG$ by*

$$q - ROPRFWG(\hbar^{\lambda}{}_1, \hbar^{\lambda}{}_2, \ldots \hbar^{\lambda}{}_n) = \left\langle \prod_{k=1}^{n} {}^{\zeta}\text{Ш}_k^{\daleth^{\gamma}{}_k}, \sqrt[q]{(1 - \prod_{k=1}^{n}(1 - {}^{\xi}\eta_k^q)^{\daleth^{\gamma}{}_k})} \right\rangle.$$

2.2. Soft-max function

The soft-max function is a type of generalization that is derived from the logistic function within the field of mathematics. Over time, it has been increasingly applied in diverse fields of research, such as artificial intelligence and business planning. The subsequent expression denotes the mathematical formulation of the soft-max function:

$$\phi_k (j, \vartheta_1, \vartheta_2, \ldots, \vartheta_n) = \phi_k^j = \frac{\exp\left(\vartheta_j/k\right)}{\sum_{j=1}^{n} \exp\left(\vartheta_j/k\right)}, k > 0.$$

For the q-ROPRFNs $\alpha_j (j = 1, 2, 3, \ldots, n), S_j$ is the score value of q-ROPRFN α_j. Every ϑ_j is formulated by giving the equation

$$\vartheta_j = \begin{cases} \prod_{i=1}^{j-1} S_i, \text{j} = 2, 3, \ldots, n \\ 1 \qquad\qquad\qquad \text{j} = 1 \end{cases}$$

where k is the modulation parameter.

3. q-Rung orthopair fuzzy soft-max AOs

Within this section, we present the notion of "q-rung orthopair fuzzy soft-max averaging (q-ROPRFSMA) operator and q-rung orthopair fuzzy soft-max geometric (q-ROPRFSMG) operator".

3.1. q-ROPRFSMA operator

Definition 3.1. Assume that $\hbar^{\lambda}{}_c = \langle {}^{\zeta}\text{Ш}_c, {}^{\xi}\eta_c \rangle$ is the agglomeration of q-ROPRFNs, and q-ROPRFSMA: $\Lambda^n \to \Lambda$, be a n dimension mapping. if

$$\text{q-ROPRFSMA}(\hbar^{\lambda}{}_1, \hbar^{\lambda}{}_2, \ldots \hbar^{\lambda}{}_n) = \frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]} \hbar^{\lambda}{}_1 \oplus \frac{\exp[\aleph^{\hbar}{}_2/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]} \hbar^{\lambda}{}_2 \oplus \ldots, \oplus \frac{\exp[\aleph^{\hbar}{}_n/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]} \hbar^{\lambda}{}_n \tag{3.1}$$

then the mapping q-ROPRFSMA is called q-ROPRFSMA operator, where $\aleph^{\hbar}{}_c = \prod_{k=1}^{c-1} \mathscr{H}(\hbar^{\lambda}{}_k)$ $(c = 2 \ldots, n)$, $\aleph^{\hbar}{}_1 = 1$ and $\mathscr{H}(\hbar^{\lambda}{}_k)$ is the score of k^{th} q-ROPRFN.

Theorem 3.2. *Assume that $\hbar^{\lambda}{}_c = \langle {}^{\zeta}\amalg_c, {}^{\xi}\eta_c \rangle$ is the agglomeration of q-ROPRFNs, we can find q-ROPRFSMA by*

$$q\text{-}ROPRFSMA(\hbar^{\lambda}{}_1, \hbar^{\lambda}{}_2, \ldots \hbar^{\lambda}{}_n) = \left\langle \sqrt[q]{1 - \prod_{c=1}^{n} (1 - {}^{\zeta}\amalg_c^{q})^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}}}, \prod_{c=1}^{n} {}^{\xi}\eta_c^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}} \right\rangle. \tag{3.2}$$

The first statement is easily followed by Definition 3.1 and Theorem 3.2. In the following, we prove this

$$\begin{aligned}
&\text{q-ROPRFSMA}(\hbar^{\lambda}{}_1, \hbar^{\lambda}{}_2, \ldots \hbar^{\lambda}{}_n) \\
&= \frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}\hbar^{\lambda}{}_1 \oplus \frac{\exp[\aleph^{\hbar}{}_2/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}\hbar^{\lambda}{}_2 \oplus \ldots, \oplus \frac{\exp[\aleph^{\hbar}{}_n/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}\hbar^{\lambda}{}_n \\
&= \left\langle \sqrt[q]{1 - \prod_{c=1}^{n} (1 - {}^{\zeta}\amalg_c^{q})^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}}}, \prod_{c=1}^{n} {}^{\xi}\eta_c^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}} \right\rangle.
\end{aligned}$$

To prove this theorem, we use mathematical induction.
For $n = 2$

$$\begin{aligned}
\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}\hbar^{\lambda}{}_1 &= \left\langle \sqrt[q]{1 - (1 - {}^{\zeta}\amalg_1^{q})^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}}}, {}^{\xi}\eta_1^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}} \right\rangle \\
\frac{\exp[\aleph^{\hbar}{}_2/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}\hbar^{\lambda}{}_2 &= \left\langle \sqrt[q]{1 - (1 - {}^{\zeta}\amalg_2^{q})^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}}}, {}^{\xi}\eta_2^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}} \right\rangle.
\end{aligned}$$

Then

$$\begin{aligned}
&\tfrac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}\hbar^{\lambda}{}_1 \oplus \tfrac{\exp[\aleph^{\hbar}{}_2/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}\hbar^{\lambda}{}_2 \\
&= \left\langle \sqrt[q]{1 - (1 - {}^{\zeta}\amalg_1^{q})^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}}}, {}^{\xi}\eta_1^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}} \right\rangle \oplus \left\langle \sqrt[q]{1 - (1 - {}^{\zeta}\amalg_2^{q})^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}}}, {}^{\xi}\eta_2^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}} \right\rangle \\
&= \Bigg\langle \sqrt[q]{(1 - (1 - {}^{\zeta}\amalg_1^{q})^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}} + 1 - (1 - {}^{\zeta}\amalg_2^{q})^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}} - \left((1 - (1 - {}^{\zeta}\amalg_1^{q})^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}}\right) \overline{\left((1 - (1 - {}^{\zeta}\amalg_2^{q})^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}}\right)}}, \\
&\qquad {}^{\xi}\eta_1^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}} . {}^{\xi}\eta_2^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}} \Bigg\rangle \\
&= \Bigg\langle \sqrt[q]{1 - (1 - {}^{\zeta}\amalg_1^{q})^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}} + 1 - (1 - {}^{\zeta}\amalg_2^{q})^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}} - \left(1 - (1 - {}^{\zeta}\amalg_2^{q})^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}} - \overline{(1 - {}^{\zeta}\amalg_1^{q})^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}} + (1 - {}^{\zeta}\amalg_2^{q})^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}}(1 - {}^{\zeta}\amalg_1^{q})^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}}}\right)}, \\
&\qquad {}^{\xi}\eta_1^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}} . {}^{\xi}\eta_2^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}} \Bigg\rangle \\
&= \left\langle \sqrt[q]{1 - (1 - {}^{\zeta}\amalg_1^{q})^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}}(1 - {}^{\zeta}\amalg_2^{q})^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}}}, {}^{\xi}\eta_1^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}} . {}^{\xi}\eta_2^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}} \right\rangle
\end{aligned}$$

$$= \left\langle \sqrt[q]{1 - \prod_{c=1}^{q} (1 - {}^{\zeta}\amalg_c^q)^{\frac{\exp[\aleph^{\hbar}{}_{c}/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_{c}/\kappa]}}}, \prod_{c=1}^{q} {}^{\xi}\eta_c^{\frac{\exp[\aleph^{\hbar}{}_{c}/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_{c}/\kappa]}} \right\rangle.$$

This shows that Eq (3.2) is true for $n = 2$, now let that Eq (3.2) holds for $n = k$, i.e.,

$$\text{q-ROPRFSMA}(\hbar^{\lambda}{}_1, \hbar^{\lambda}{}_2, \ldots \hbar^{\lambda}{}_k) = \left\langle \sqrt[q]{1 - \prod_{c=1}^{k} (1 - {}^{\zeta}\amalg_c^q)^{\frac{\exp[\aleph^{\hbar}{}_{c}/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_{c}/\kappa]}}}, \prod_{c=1}^{k} {}^{\xi}\eta_c^{\frac{\exp[\aleph^{\hbar}{}_{c}/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_{c}/\kappa]}} \right\rangle.$$

Now $n = k + 1$, by operational laws of q-ROPRFNs we have,

$$\text{q-ROPRFSMA}(\hbar^{\lambda}{}_1, \hbar^{\lambda}{}_2, \ldots \hbar^{\lambda}{}_{k+1}) = \text{q-ROPRFSMA}(\hbar^{\lambda}{}_1, \hbar^{\lambda}{}_2, \ldots \hbar^{\lambda}{}_k) \oplus \hbar^{\lambda}{}_{k+1}$$

$$= \left\langle \sqrt[q]{1 - \prod_{c=1}^{k} (1 - {}^{\zeta}\amalg_c^q)^{\frac{\exp[\aleph^{\hbar}{}_{c}/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_{c}/\kappa]}}}, \prod_{c=1}^{k} {}^{\xi}\eta_c^{\frac{\exp[\aleph^{\hbar}{}_{c}/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_{c}/\kappa]}} \right\rangle \oplus \left\langle \sqrt[q]{1 - (1 - {}^{\zeta}\amalg_{k+1}^q)^{\frac{\aleph^{\hbar}{}_{k+1}}{\sum_{c=1}^{n} \aleph^{\hbar}{}_{c}}}}, {}^{\xi}\eta_{k+1}^{\frac{\aleph^{\hbar}{}_{k+1}}{\sum_{c=1}^{n} \aleph^{\hbar}{}_{c}}} \right\rangle$$

$$= \left\langle \sqrt[q]{\begin{array}{l} 1 - \prod_{c=1}^{k} (1 - {}^{\zeta}\amalg_k^q)^{\frac{\exp[\aleph^{\hbar}{}_{c}/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_{c}/\kappa]}} + 1 - (1 - {}^{\zeta}\amalg_{k+1}^q)^{\frac{\aleph^{\hbar}{}_{k+1}}{\sum_{c=1}^{n} \aleph^{\hbar}{}_{c}}} - \left(1 - \prod_{c=1}^{k} (1 - {}^{\zeta}\amalg_k^q)^{\frac{\exp[\aleph^{\hbar}{}_{c}/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_{c}/\kappa]}}\right) \\ \left(1 - (1 - {}^{\zeta}\amalg_{k+1}^q)^{\frac{\aleph^{\hbar}{}_{k+1}}{\sum_{c=1}^{n} \aleph^{\hbar}{}_{c}}}\right) \end{array}}, \prod_{c=1}^{k} {}^{\xi}\eta_k^{\frac{\exp[\aleph^{\hbar}{}_{c}/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_{c}/\kappa]}} . {}^{\xi}\eta_{k=1}^{\frac{\aleph^{\hbar}{}_{k+1}}{\sum_{c=1}^{n} \aleph^{\hbar}{}_{c}}} \right\rangle$$

$$= \left\langle \sqrt[q]{1 - \prod_{c=1}^{k} (1 - {}^{\zeta}\amalg_k^q)^{\frac{\exp[\aleph^{\hbar}{}_{c}/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_{c}/\kappa]}} (1 - {}^{\zeta}\amalg_{k+1}^q)^{\frac{\aleph^{\hbar}{}_{k+1}}{\sum_{c=1}^{n} \aleph^{\hbar}{}_{k+1}}}}, \prod_{c=1}^{k} {}^{\xi}\eta_k^{\frac{\exp[\aleph^{\hbar}{}_{c}/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_{c}/\kappa]}} . {}^{\xi}\eta_{k+1}^{\frac{\aleph^{\hbar}{}_{k+1}}{\sum_{c=1}^{n} \aleph^{\hbar}{}_{c}}} \right\rangle$$

$$= \left\langle \sqrt[q]{1 - \prod_{c=1}^{k+1} (1 - {}^{\zeta}\amalg_c^q)^{\frac{\aleph^{\hbar}{}_{c}}{\sum_{c=1}^{k+1} \aleph^{\hbar}{}_{c}}}}, \prod_{c=1}^{k+1} {}^{\xi}\eta_c^{\frac{\exp[\aleph^{\hbar}{}_{c}/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_{c}/\kappa]}} \right\rangle.$$

This shows that for $n = k + 1$, Eq (3.2) holds. Then,

$$\text{q-ROPRFSMA}(\hbar^{\lambda}{}_1, \hbar^{\lambda}{}_2, \ldots \hbar^{\lambda}{}_n) = \left\langle \sqrt[q]{(1 - \prod_{c=1}^{n} (1 - {}^{\zeta}\amalg_c^q)^{\frac{\exp[\aleph^{\hbar}{}_{c}/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_{c}/\kappa]}}}, \prod_{c=1}^{n} {}^{\xi}\eta_c^{\frac{\exp[\aleph^{\hbar}{}_{c}/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_{c}/\kappa]}} \right\rangle.$$

Since,

$$min_c\ ({}^{\zeta}\amalg_c) \leq {}^{\zeta}\amalg_c \leq max_c\ ({}^{\zeta}\amalg_c) \tag{3.3}$$

and

$$min_c\ ({}^{\xi}\eta_c) \leq {}^{\xi}\eta_c \leq max_c\ ({}^{\xi}\eta_c). \tag{3.4}$$

From Eq (3.3) we have,
$min_c\,({}^{\zeta}\text{Ц}_c) \leq {}^{\zeta}\text{Ц}_c \leq max_c\,({}^{\zeta}\text{Ц}_c)$

$$\Leftrightarrow \sqrt[q]{min_c\,({}^{\zeta}\text{Ц}_c)^q} \leq \sqrt[q]{({}^{\zeta}\text{Ц}_c)^q} \leq \sqrt[q]{max_c\,({}^{\zeta}\text{Ц}_c)^q}$$

$$\Leftrightarrow \sqrt[q]{1 - max_c\,({}^{\zeta}\text{Ц}_c)^q} \leq \sqrt[q]{1 - ({}^{\zeta}\text{Ц}_c)^q} \leq \sqrt[q]{1 - min_c\,({}^{\zeta}\text{Ц}_c)^q}$$

$$\Leftrightarrow \sqrt[q]{\left(1 - max_c\,({}^{\zeta}\text{Ц}_c)^q\right)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}}} \leq \sqrt[q]{\left(1 - ({}^{\zeta}\text{Ц}_c)^q\right)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}}}$$
$$\leq \sqrt[q]{\left(1 - min_c\,({}^{\zeta}\text{Ц}_c)^q\right)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}}}$$

$$\Leftrightarrow \sqrt[q]{\prod_{c=1}^{n}\left(1 - max_c\,({}^{\zeta}\text{Ц}_c)^q\right)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}}} \leq \sqrt[q]{\prod_{c=1}^{n}\left(1 - ({}^{\zeta}\text{Ц}_c)^q\right)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}}}$$
$$\leq \sqrt[q]{\prod_{c=1}^{n}\left(1 - min_c\,({}^{\zeta}\text{Ц}_c)^q\right)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}}}$$

$$\Leftrightarrow \sqrt[q]{1 - max_c\,({}^{\zeta}\text{Ц}_c)^q} \leq \sqrt[q]{\prod_{c=1}^{n}\left(1 - ({}^{\zeta}\text{Ц}_c)^q\right)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}}} \leq \sqrt[q]{1 - min_c\,({}^{\zeta}\text{Ц}_c)^q}$$

$$\Leftrightarrow \sqrt[q]{-1 + min_c\,({}^{\zeta}\text{Ц}_c)^q} \leq \sqrt[q]{-\prod_{c=1}^{n}\left(1 - ({}^{\zeta}\text{Ц}_c)^q\right)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}}} \leq \sqrt[q]{-1 + max_c\,({}^{\zeta}\text{Ц}_c)^q}$$

$$\Leftrightarrow \sqrt[q]{1 - 1 + min_c\,({}^{\zeta}\text{Ц}_c)^q} \leq \sqrt[q]{1 - \prod_{c=1}^{n}\left(1 - ({}^{\zeta}\text{Ц}_c)^q\right)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}}} \leq \sqrt[q]{1 - 1 + max_c\,({}^{\zeta}\text{Ц}_c)^q}$$

$$\Leftrightarrow \sqrt[q]{min_c\,({}^{\zeta}\text{Ц}_c)^q} \leq \sqrt[q]{1 - \prod_{c=1}^{n}\left(1 - ({}^{\zeta}\text{Ц}_c)^q\right)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}}} \leq \sqrt[q]{max_c\,({}^{\zeta}\text{Ц}_c)^q}$$

$$\Leftrightarrow min_c\,({}^{\zeta}\text{Ц}_c)^q \leq \sqrt[q]{1 - \prod_{c=1}^{n}\left(1 - ({}^{\zeta}\text{Ц}_c)^q\right)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}}} \leq max_c\,({}^{\zeta}\text{Ц}_c)^q.$$

From Eq (3.4) we have,

$min_c\ ({}^{\xi}\eta_c) \leq {}^{\xi}\eta_c \leq max_c\ ({}^{\xi}\eta_c)$

$$\Leftrightarrow min_c\ ({}^{\xi}\eta_c)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}} \leq ({}^{\xi}\eta_c)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}} \leq max_c\ ({}^{\xi}\eta_c)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}}$$

$$\Leftrightarrow \prod_{c=1}^{n} min_c\ ({}^{\xi}\eta_c)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}} \leq \prod_{c=1}^{n}({}^{\xi}\eta_c)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}} \leq \prod_{c=1}^{n} max_c\ ({}^{\xi}\eta_c)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}}$$

$$\Leftrightarrow min_c\ ({}^{\xi}\eta_c)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}} \leq \prod_{c=1}^{n}({}^{\xi}\eta_c)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}} \leq max_c\ ({}^{\xi}\eta_c)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}}.$$

Let

$$\text{q-ROPRFSMA}(\hbar^{\lambda}{}_1, \hbar^{\lambda}{}_2, \ldots \hbar^{\lambda}{}_n) = \hbar^{\lambda} = ({}^{\zeta}\text{Ш}, {}^{\xi}\eta).$$

Then, $\mathscr{H}(\hbar^{\lambda}) = {}^{\zeta}\text{Ш}^q - {}^{\xi}\eta^q \quad \leq\ max_c\ ({}^{\zeta}\text{Ш})^q - min_c\ ({}^{\xi}\eta)^q =\ \mathscr{H}(\hbar^{\lambda}{}_{max})$ So, $\mathscr{H}(\hbar^{\lambda}) \leq\ \mathscr{H}(\hbar^{\lambda}{}_{max})$.

Again, $\mathscr{H}(\hbar^{\lambda}) = {}^{\zeta}\text{Ш}^q - {}^{\xi}\eta^q \quad \geq\ min_c\ ({}^{\zeta}\text{Ш})^q - max_c\ ({}^{\xi}\eta)^q =\ \mathscr{H}(\hbar^{\lambda}{}_{min})$ So, $\mathscr{H}(\hbar^{\lambda}) \geq\ \mathscr{H}(\hbar^{\lambda}{}_{min})$.

If, $\mathscr{H}(\hbar^{\lambda}) \leq\ \mathscr{H}(\hbar^{\lambda}{}_{max})$ and $\mathscr{H}(\hbar^{\lambda}) \geq\ \mathscr{H}(\hbar^{\lambda}{}_{min})$, then

$$\hbar^{\lambda}{}_{min} \leq \text{q-ROPRFSMA}(\hbar^{\lambda}{}_1, \hbar^{\lambda}{}_2, \ldots \hbar^{\lambda}{}_n) \leq \hbar^{\lambda}{}_{max}. \tag{3.5}$$

If $\mathscr{H}(\hbar^{\lambda}) =\ \mathscr{H}(\hbar^{\lambda}{}_{max})$, then ${}^{\zeta}\text{Ш}^q - {}^{\xi}\eta^q =\ max_c\ ({}^{\zeta}\text{Ш})^q - min_c\ ({}^{\xi}\eta)^q$

$$\begin{aligned}
&\Leftrightarrow {}^{\zeta}\text{Ш}^q - {}^{\xi}\eta^q =\ max_c\ ({}^{\zeta}\text{Ш})^q - min_c\ ({}^{\xi}\eta)^q \\
&\Leftrightarrow {}^{\zeta}\text{Ш}^q = max_c\ ({}^{\zeta}\text{Ш})^q, \quad {}^{\xi}\eta^q = min_c\ ({}^{\xi}\eta)^q \\
&\Leftrightarrow {}^{\zeta}\text{Ш} = max_c\ {}^{\zeta}\text{Ш}, \quad {}^{\xi}\eta = min_c\ {}^{\xi}\eta.
\end{aligned}$$

Now, $H(\hbar^{\lambda}) = {}^{\zeta}\text{Ш}^q + {}^{\xi}\eta^q = max_c\ ({}^{\zeta}\text{Ш})^q + min_c\ ({}^{\xi}\eta)^q = H(\hbar^{\lambda}{}_{max})$

$$\text{q-ROPRFSMA}(\hbar^{\lambda}{}_1, \hbar^{\lambda}{}_2, \ldots \hbar^{\lambda}{}_n) = \hbar^{\lambda}{}_{max}. \tag{3.6}$$

If $\mathscr{H}(\hbar^{\lambda}) =\ \mathscr{H}(\hbar^{\lambda}{}_{min})$, then ${}^{\zeta}\text{Ш}^q - {}^{\xi}\eta^q =\ min_c\ ({}^{\zeta}\text{Ш})^q - max_c\ ({}^{\xi}\eta)^q$

$$\begin{aligned}
&\Leftrightarrow {}^{\zeta}\text{Ш}^q - {}^{\xi}\eta^q =\ min_c\ ({}^{\zeta}\text{Ш})^q - max_c\ ({}^{\xi}\eta)^q \\
&\Leftrightarrow {}^{\zeta}\text{Ш}^q = min_c\ ({}^{\zeta}\text{Ш})^q, \quad {}^{\xi}\eta^q = max_c\ ({}^{\xi}\eta)^q \\
&\Leftrightarrow {}^{\zeta}\text{Ш} = min_c\ {}^{\zeta}\text{Ш}, \quad {}^{\xi}\eta = max_c\ {}^{\xi}\eta.
\end{aligned}$$

Now, $H(\hbar^{\lambda}) = {}^{\zeta}\text{Ш}^q + {}^{\xi}\eta^q = min_c\ ({}^{\zeta}\text{Ш})^q + max_c\ ({}^{\xi}\eta)^q = H(\hbar^{\lambda}{}_{max})$

$$\text{q-ROPRFSMA}(\hbar^{\lambda}{}_1, \hbar^{\lambda}{}_2, \ldots \hbar^{\lambda}{}_n) = \hbar^{\lambda}{}_{min}. \tag{3.7}$$

Thus, from Eqs (3.5)–(3.7), we get

$$\hbar^{\lambda^-} \leq \text{q-ROPRFSMA}(\hbar^{\lambda}{}_1, \hbar^{\lambda}{}_2, \ldots \hbar^{\lambda}{}_n) \leq \hbar^{\lambda^+}.$$

Below we define some of q-ROPRFSMA's appealing properties.

Theorem 3.3. *(Idempotency) Assume that* $\hbar^{\lambda}{}_{c} = \langle {}^{\zeta}\amalg_{c}, {}^{\xi}\eta_{c}\rangle$ *is the agglomeration of q-ROPRFNs, where* $\aleph^{\hbar}{}_{c} = \prod_{k=1}^{c-1} \mathscr{H}(\hbar^{\lambda}{}_{k})$ $(c = 2\ldots, n)$, $\aleph^{\hbar}{}_{1} = 1$ *and* $\mathscr{H}(\hbar^{\lambda}{}_{k})$ *is the score of* k^{th} *q-ROPRFN. If all* $\hbar^{\lambda}{}_{c}$ *are equal, i.e,.* $\hbar^{\lambda}{}_{c} = \hbar^{\lambda}$ *for all j, then*

$$\text{q-ROPRFSMA}(\hbar^{\lambda}{}_{1}, \hbar^{\lambda}{}_{2}, \ldots \hbar^{\lambda}{}_{n}) = \hbar^{\lambda}.$$

Proof. From Definition 3.1, we have

$$\begin{aligned}
\text{q-ROPRFSMA}(\hbar^{\lambda}{}_{1}, \hbar^{\lambda}{}_{2}, \ldots \hbar^{\lambda}{}_{n}) &= \frac{\exp[\aleph^{\hbar}{}_{1}/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_{c}/\kappa]}\hbar^{\lambda}{}_{1} \oplus \frac{\exp[\aleph^{\hbar}{}_{2}/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_{c}/\kappa]}\hbar^{\lambda}{}_{2} \oplus \ldots, \oplus \frac{\exp[\aleph^{\hbar}{}_{n}/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_{c}/\kappa]} \\
&= \frac{\exp[\aleph^{\hbar}{}_{1}/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_{c}/\kappa]}\hbar^{\lambda} \oplus \frac{\exp[\aleph^{\hbar}{}_{2}/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_{c}/\kappa]}\hbar^{\lambda} \oplus \ldots, \oplus \frac{\exp[\aleph^{\hbar}{}_{n}/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_{c}/\kappa]}\hbar^{\lambda} \\
&= \hbar^{\lambda}.
\end{aligned}$$

□

Corollary 3.4. *If* $\hbar^{\lambda}{}_{c} = \langle {}^{\zeta}\amalg_{c}, {}^{\xi}\eta_{c}\rangle$ $j = (1, 2, \ldots n)$ *is the agglomeration of largest q-ROPRFNs, i.e.,* $\hbar^{\lambda}{}_{c} = (1, 0)$ *for all j, then*

$$\text{q-ROPRFSMA}(\hbar^{\lambda}{}_{1}, \hbar^{\lambda}{}_{2}, \ldots \hbar^{\lambda}{}_{n}) = (1, 0).$$

Proof. We can easily obtain Corollary similar to the Theorem 3.3. □

Theorem 3.5. *(Monotonicity) Assume that* $\hbar^{\lambda}{}_{c} = \langle {}^{\zeta}\amalg_{c}, {}^{\xi}\eta_{c}\rangle$ *and* $\hbar^{\lambda^*}_{c} = \langle {}^{\zeta}\amalg^{*}_{c}, {}^{\xi}\eta^{*}_{c}\rangle$ *are the families of q-ROPRFNs, where* $\aleph^{\hbar}{}_{c} = \prod_{k=1}^{c-1} \mathscr{H}(\hbar^{\lambda}{}_{k})$, $T^{*}_{c} = \prod_{k=1}^{c-1} \mathscr{H}(\hbar^{\lambda^*}_{k})$ $(c = 2\ldots, n)$, $\aleph^{\hbar}{}_{1} = 1$, $T^{*}_{1} = 1$, $\mathscr{H}(\hbar^{\lambda}{}_{k})$ *is the score of* $\hbar^{\lambda}{}_{k}$ *q-ROPRFN, and* $\mathscr{H}(\hbar^{\lambda^*}_{k})$ *is the score of* $\hbar^{\lambda^*}_{k}$ *q-ROPRFN. If* ${}^{\zeta}\amalg^{*}_{c} \geq {}^{\zeta}\amalg_{c}$ *and* ${}^{\xi}\eta^{*}_{c} \leq {}^{\xi}\eta_{c}$ *for all j, then*

$$\text{q-ROPRFSMA}(\hbar^{\lambda}{}_{1}, \hbar^{\lambda}{}_{2}, \ldots \hbar^{\lambda}{}_{n}) \leq \text{q-ROPRFSMA}(\hbar^{\lambda^*}_{1}, \hbar^{\lambda^*}_{2}, \ldots \hbar^{\lambda^*}_{n}).$$

Proof. Here, ${}^{\zeta}\amalg^{*}_{c} \geq {}^{\zeta}\amalg_{c}$ and ${}^{\xi}\eta^{*}_{c} \leq {}^{\xi}\eta_{c}$ for all j, If ${}^{\zeta}\amalg^{*}_{c} \geq {}^{\zeta}\amalg_{c}$.

$\Leftrightarrow ({}^{\zeta}\amalg^{*}_{c})^{q} \geq ({}^{\zeta}\amalg_{c})^{q} \Leftrightarrow \sqrt[q]{({}^{\zeta}\amalg^{*}_{c})^{q}} \geq \sqrt[q]{({}^{\zeta}\amalg_{c})^{q}} \Leftrightarrow \sqrt[q]{1-({}^{\zeta}\amalg^{*}_{c})^{q}} \leq \sqrt[q]{1-({}^{\zeta}\amalg_{c})^{q}}$

$$\Leftrightarrow \sqrt[q]{(1-({}^{\zeta}\amalg^{*}_{c})^{q})^{\frac{\exp[\aleph^{\hbar}{}_{c}/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_{c}/\kappa]}}} \leq \sqrt[q]{(1-({}^{\zeta}\amalg_{c})^{q})^{\frac{\exp[\aleph^{\hbar}{}_{c}/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_{c}/\kappa]}}}$$

$$\Leftrightarrow \sqrt[q]{\prod_{c=1}^{n}(1-({}^{\zeta}\amalg^{*}_{c})^{q})^{\frac{\exp[\aleph^{\hbar}{}_{c}/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_{c}/\kappa]}}} \leq \sqrt[q]{\prod_{c=1}^{n}(1-({}^{\zeta}\amalg_{c})^{q})^{\frac{\exp[\aleph^{\hbar}{}_{c}/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_{c}/\kappa]}}}$$

$$\Leftrightarrow \sqrt[q]{1-\prod_{c=1}^{n}(1-({}^{\zeta}\amalg_{c})^{q})^{\frac{\exp[\aleph^{\hbar}{}_{c}/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_{c}/\kappa]}}} \leq \sqrt[q]{1-\prod_{c=1}^{n}(1-({}^{\zeta}\amalg^{*}_{c})^{q})^{\frac{\exp[\aleph^{\hbar}{}_{c}/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_{c}/\kappa]}}}.$$

Now,

${}^{\xi}\eta^{*}_{c} \leq {}^{\xi}\eta_{c}$.

$$\Leftrightarrow ({}^{\xi}\eta^{*}_{c})^{\frac{\exp[\aleph^{\hbar}{}_{c}/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_{c}/\kappa]}} \leq ({}^{\xi}\eta_{c})^{\frac{\exp[\aleph^{\hbar}{}_{c}/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_{c}/\kappa]}}$$

$$\Leftrightarrow \prod_{c=1}^{n}({}^{\xi}\eta^{*}_{c})^{\frac{\exp[\aleph^{\hbar}{}_{c}/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_{c}/\kappa]}} \leq \prod_{c=1}^{n}({}^{\xi}\eta_{c})^{\frac{\exp[\aleph^{\hbar}{}_{c}/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_{c}/\kappa]}}.$$

Let

$$\overline{\hbar^{\lambda}} = \text{q-ROPRFSMA}(\hbar^{\lambda}{}_1, \hbar^{\lambda}{}_2, \ldots \hbar^{\lambda}{}_n)$$

and

$$\overline{\hbar^{\lambda^*}} = \text{q-ROPRFSMA}(\hbar^{\lambda^*}_1, \hbar^{\lambda^*}_2, \ldots \hbar^{\lambda^*}_n).$$

We get that $\overline{\hbar^{\lambda^*}} \geq \overline{\hbar^{\lambda}}$. So,

$$\text{q-ROPRFSMA}(\hbar^{\lambda}{}_1, \hbar^{\lambda}{}_2, \ldots \hbar^{\lambda}{}_n) \leq \text{q-ROPRFSMA}(\hbar^{\lambda^*}_1, \hbar^{\lambda^*}_2, \ldots \hbar^{\lambda^*}_n).$$

□

Theorem 3.6. *(Boundary) Assume that $\hbar^{\lambda}{}_c = \langle {}^{\zeta}\amalg_c, {}^{\xi}\eta_c \rangle$ is the agglomeration of q-ROPRFNs, and*

$$\hbar^{\lambda^-} = (min_c\ ({}^{\zeta}\amalg_c), max_c\ ({}^{\xi}\eta_c)) \quad and \quad \hbar^{\lambda^+} = (max_c\ ({}^{\zeta}\amalg_c), min_c\ ({}^{\xi}\eta_c)).$$

Then,

$$\hbar^{\lambda^-} \leq \textit{q-ROPRFSMA}(\hbar^{\lambda}{}_1, \hbar^{\lambda}{}_2, \ldots \hbar^{\lambda}{}_n) \leq \hbar^{\lambda^+}$$

where $\aleph^{\hbar}{}_c = \prod_{k=1}^{c-1} \mathscr{H}(\hbar^{\lambda}{}_k)$ $(c = 2 \ldots, n)$, $\aleph^{\hbar}{}_1 = 1$ and $\mathscr{H}(\hbar^{\lambda}{}_k)$ is the score of k^{th} q-ROPRFN.

Proof. Here, ${}^{\zeta}\amalg^*_c \geq {}^{\zeta}\amalg_c$ and ${}^{\xi}\eta^*_c \leq {}^{\xi}\eta_c$ for all j, If ${}^{\zeta}\amalg^*_c \geq {}^{\zeta}\amalg_c$.
$\Leftrightarrow ({}^{\zeta}\amalg^*_c)^q \geq ({}^{\zeta}\amalg_c)^q \Leftrightarrow \sqrt[q]{({}^{\zeta}\amalg^*_c)^q} \geq \sqrt[q]{({}^{\zeta}\amalg_c)^q} \Leftrightarrow \sqrt[q]{1 - ({}^{\zeta}\amalg^*_c)^q} \leq \sqrt[q]{1 - ({}^{\zeta}\amalg_c)^q}$

$$\Leftrightarrow \sqrt[q]{(1 - ({}^{\zeta}\amalg^*_c)^q)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}}} \leq \sqrt[q]{(1 - ({}^{\zeta}\amalg_c)^q)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}}}$$

$$\Leftrightarrow \sqrt[q]{\prod_{c=1}^{n}(1 - ({}^{\zeta}\amalg^*_c)^q)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}}} \leq \sqrt[q]{\prod_{c=1}^{n}(1 - ({}^{\zeta}\amalg_c)^q)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}}}$$

$$\Leftrightarrow \sqrt[q]{1 - \prod_{c=1}^{n}(1 - ({}^{\zeta}\amalg_c)^q)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}}} \leq \sqrt[q]{1 - \prod_{c=1}^{n}(1 - ({}^{\zeta}\amalg^*_c)^q)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}}}.$$

Now,
${}^{\xi}\eta^*_c \leq {}^{\xi}\eta_c$.
$\Leftrightarrow ({}^{\xi}\eta^*_c)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}} \leq ({}^{\xi}\eta_c)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}}$

$$\Leftrightarrow \prod_{c=1}^{n}({}^{\xi}\eta^*_c)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}} \leq \prod_{c=1}^{n}({}^{\xi}\eta_c)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}}.$$

Let

$$\overline{\hbar^{\lambda}} = \text{q-ROPRFSMA}(\hbar^{\lambda}{}_1, \hbar^{\lambda}{}_2, \ldots \hbar^{\lambda}{}_n)$$

and

$$\overline{\hbar^{\lambda^*}} = \text{q-ROPRFSMA}(\hbar^{\lambda^*}_1, \hbar^{\lambda^*}_2, \ldots \hbar^{\lambda^*}_n).$$

We get that $\overline{\hbar^{\lambda^*}} \geq \overline{\hbar^{\lambda}}$. So,

$$\text{q-ROPRFSMA}(\hbar^{\lambda}{}_1, \hbar^{\lambda}{}_2, \ldots \hbar^{\lambda}{}_n) \leq \text{q-ROPRFSMA}(\hbar^{\lambda^*}_1, \hbar^{\lambda^*}_2, \ldots \hbar^{\lambda^*}_n).$$

□

Theorem 3.7. *Assume that* $\hbar^{\lambda}{}_{c} = \langle {}^{\zeta}\text{Ц}_{c}, {}^{\xi}\eta_{c} \rangle$ *and* $\beta_c = \langle \phi_c, \varphi_c \rangle$ *are two familie of q-ROPRFNs, where* $\aleph^{\hbar}{}_{c} = \prod_{k=1}^{c-1} \mathscr{H}(\hbar^{\lambda}{}_{k})$ $(c = 2\ldots, n)$*,* $\aleph^{\hbar}{}_{1} = 1$ *and* $\mathscr{H}(\hbar^{\lambda}{}_{k})$ *is the score of* k^{th} *q-ROPRFN. If* $r > 0$ *and* $\beta = \langle {}^{\zeta}\text{Ц}_{\beta}, {}^{\xi}\eta_{\beta} \rangle$ *is an q-ROPRFN, then*

(1) q-ROPRFSMA$(\hbar^{\lambda}{}_{1} \oplus \beta, \hbar^{\lambda}{}_{2} \oplus \beta, \ldots \hbar^{\lambda}{}_{n} \oplus \beta) =$ *q-ROPRFSMA*$(\hbar^{\lambda}{}_{1}, \hbar^{\lambda}{}_{2}, \ldots \hbar^{\lambda}{}_{n}) \oplus \beta$.

(2) q-ROPRFSMA$(r\hbar^{\lambda}{}_{1}, r\hbar^{\lambda}{}_{2}, \ldots r\hbar^{\lambda}{}_{n}) = r$ *q-ROPRFSMA*$(\hbar^{\lambda}{}_{1}, \hbar^{\lambda}{}_{2}, \ldots \hbar^{\lambda}{}_{n})$.

(3) q-ROPRFSMA$(\hbar^{\lambda}{}_{1} \oplus \beta_2, \hbar^{\lambda}{}_{2} \oplus \beta_2, \ldots \hbar^{\lambda}{}_{n} \oplus \beta_n) =$ *q-ROPRFSMA*$(\hbar^{\lambda}{}_{1}, \hbar^{\lambda}{}_{2}, \ldots \hbar^{\lambda}{}_{n}) \oplus$ *q-ROPRFSMA*$(\beta_1, \beta_2, \ldots \beta_n)$.

(4) q-ROPRFSMA$(r\hbar^{\lambda}{}_{1} \oplus \beta r\hbar^{\lambda}{}_{2} \oplus \beta, \ldots \oplus r\hbar^{\lambda}{}_{n} \oplus \beta) = r$ *q-ROPRFSMA*$(\hbar^{\lambda}{}_{1}, \hbar^{\lambda}{}_{2}, \ldots \hbar^{\lambda}{}_{n}) \oplus \beta$.

Proof. Here, we just Proofs 1 and 3,
(1)
Since,

$$\hbar^{\lambda}{}_{c} \oplus \beta = \Big(1 - (1 - ({}^{\zeta}\text{Ц}_{c})^{q})(1 - ({}^{\zeta}\text{Ц}_{\beta})^{q}), {}^{\xi}\eta_{c}{}^{\xi}\eta_{\beta}\Big).$$

By Theorem 3.2,
q-ROPRFSMA$(\hbar^{\lambda}{}_{1} \oplus \beta, \hbar^{\lambda}{}_{2} \oplus \beta, \ldots \hbar^{\lambda}{}_{n} \oplus \beta)$

$$= \left\langle \sqrt[q]{(1 - \prod_{c=1}^{n} \Big((1 - {}^{\zeta}\text{Ц}_{c}^{q})(1 - ({}^{\zeta}\text{Ц}_{\beta})^{q})\Big)^{\frac{\exp[\aleph^{\hbar}{}_{c}/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_{c}/\kappa]}}}, \prod_{c=1}^{n} \Big({}^{\xi}\eta_{\beta}{}^{\xi}\eta_{c}\Big)^{\frac{\exp[\aleph^{\hbar}{}_{c}/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_{c}/\kappa]}} \right\rangle$$

$$= \left\langle \sqrt[q]{(1 - \Big(1 - ({}^{\zeta}\text{Ц}_{\beta})^{q}\Big)^{\frac{\exp[\aleph^{\hbar}{}_{c}/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_{c}/\kappa]}} \prod_{c=1}^{n} \Big(1 - ({}^{\zeta}\text{Ц}_{c})^{q}\Big)^{\frac{\exp[\aleph^{\hbar}{}_{c}/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_{c}/\kappa]}}}, \right.$$
$$\left. \Big({}^{\xi}\eta_{\beta}\Big)^{\frac{\exp[\aleph^{\hbar}{}_{c}/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_{c}/\kappa]}} \prod_{c=1}^{n} \Big({}^{\xi}\eta_{c}\Big)^{\frac{\exp[\aleph^{\hbar}{}_{c}/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_{c}/\kappa]}} \right\rangle$$

$$= \left\langle \sqrt[q]{(1 - \Big(1 - ({}^{\zeta}\text{Ц}_{\beta})^{q}\Big) \prod_{c=1}^{n} \Big(1 - ({}^{\zeta}\text{Ц}_{c})^{q}\Big)^{\frac{\exp[\aleph^{\hbar}{}_{c}/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_{c}/\kappa]}}}, \Big({}^{\xi}\eta_{\beta}\Big) \prod_{c=1}^{n} \Big({}^{\xi}\eta_{c}\Big)^{\frac{\exp[\aleph^{\hbar}{}_{c}/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_{c}/\kappa]}} \right\rangle.$$

Now, by operational laws of q-ROPRFNs,

$$\text{q-ROPRFSMA}(\hbar^{\lambda}{}_{1}, \hbar^{\lambda}{}_{2}, \ldots \hbar^{\lambda}{}_{n}) \oplus \beta$$

$$= \left\langle \sqrt[q]{(1 - \prod_{c=1}^{n} (1 - {}^{\zeta}\text{Ц}_{c}^{q})^{\frac{\exp[\aleph^{\hbar}{}_{c}/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_{c}/\kappa]}}}, \prod_{c=1}^{n} {}^{\xi}\eta_{c}^{\frac{\exp[\aleph^{\hbar}{}_{c}/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_{c}/\kappa]}} \right\rangle \oplus \langle {}^{\zeta}\text{Ц}_{\beta}, {}^{\xi}\eta_{\beta} \rangle$$

$$= \left\langle \sqrt[q]{(1 - \Big(1 - ({}^{\zeta}\text{Ц}_{\beta})^{q}\Big) \prod_{c=1}^{n} \Big(1 - ({}^{\zeta}\text{Ц}_{c})^{q}\Big)^{\frac{\exp[\aleph^{\hbar}{}_{c}/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_{c}/\kappa]}}}, \Big({}^{\xi}\eta_{\beta}\Big) \prod_{c=1}^{n} \Big({}^{\xi}\eta_{c}\Big)^{\frac{\exp[\aleph^{\hbar}{}_{c}/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_{c}/\kappa]}} \right\rangle.$$

Thus,

$$\text{q-ROPRFSMA}(\hbar^{\lambda}{}_1 \oplus \beta, \hbar^{\lambda}{}_2 \oplus \beta, \ldots \hbar^{\lambda}{}_n \oplus \beta) = \text{q-ROPRFSMA}(\hbar^{\lambda}{}_1, \hbar^{\lambda}{}_2, \ldots \hbar^{\lambda}{}_n) \oplus \beta.$$

(3)
According to Theorem 3.2,
q-ROPRFSMA$(\hbar^{\lambda}{}_1 \oplus \beta_2, \hbar^{\lambda}{}_2 \oplus \beta_2, \ldots \hbar^{\lambda}{}_n \oplus \beta_n)$

$$= \left\langle \sqrt[q]{(1 - \prod_{c=1}^{n} \left((1 - {}^{\zeta}\amalg_c^q)(1 - (\phi_c)^q)\right)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}}}, \prod_{c=1}^{n} \left(\varphi_c {}^{\xi}\eta_c\right)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}} \right\rangle$$

$$= \left\langle \sqrt[q]{(1 - \prod_{c=1}^{n} \left(1 - (\phi_c)^q\right)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}} \prod_{c=1}^{n} \left(1 - {}^{\zeta}\amalg_c^q\right)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}}}, \right.$$
$$\left. \prod_{c=1}^{n} \left(\varphi_c\right)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}} \prod_{c=1}^{n} \left({}^{\xi}\eta_c\right)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}} \right\rangle.$$

Now,
q-ROPRFSMA$(\hbar^{\lambda}{}_1, \hbar^{\lambda}{}_2, \ldots \hbar^{\lambda}{}_n) \oplus$ q-ROPRFSMA$(\beta_1, \beta_2, \ldots \beta_n)$

$$= \left\langle \sqrt[q]{(1 - \prod_{c=1}^{n} (1 - {}^{\zeta}\amalg_c^q)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}}}, \prod_{c=1}^{n} {}^{\xi}\eta_c^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}} \right\rangle$$
$$\oplus \left\langle \sqrt[q]{(1 - \prod_{c=1}^{n} (1 - {}^{\zeta}\amalg_c^q)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}}}, \prod_{c=1}^{n} {}^{\xi}\eta_c^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}} \right\rangle$$

$$= \left\langle \sqrt[q]{(1 - \prod_{c=1}^{n} \left(1 - (\phi_c)^q\right)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}} \prod_{c=1}^{n} \left(1 - {}^{\zeta}\amalg_c^q\right)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}}}, \right.$$
$$\left. \prod_{c=1}^{n} \left(\varphi_c\right)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}} \prod_{c=1}^{n} \left({}^{\xi}\eta_c\right)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}} \right\rangle.$$

Thus,

$$\text{q-ROPRFSMA}(\hbar^{\lambda}{}_1 \oplus \beta_2, \hbar^{\lambda}{}_2 \oplus \beta_2, \ldots \hbar^{\lambda}{}_n \oplus \beta_n) = \text{q-ROPRFSMA}(\hbar^{\lambda}{}_1, \hbar^{\lambda}{}_2, \ldots \hbar^{\lambda}{}_n) \oplus \text{q-ROPRFSMA}(\beta_1, \beta_2, \ldots \beta_n).$$

□

3.2. q-ROPRFSMG operator

Definition 3.8. Assume that $\hbar^{\lambda}{}_c = \langle {}^{\zeta}\amalg_c, {}^{\xi}\eta_c \rangle$ is the agglomeration of q-ROPRFNs, and $q-ROPRFSMG : \Lambda^n \to \Lambda$, be a n dimension mapping. if

$$\text{q-ROPRFSMG}(\hbar^{\lambda}{}_1, \hbar^{\lambda}{}_2, \ldots \hbar^{\lambda}{}_n) = \hbar^{\lambda}{}_1^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}} \otimes \hbar^{\lambda}{}_2^{\frac{\exp[\aleph^{\hbar}{}_2/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}} \otimes \ldots, \otimes \hbar^{\lambda}{}_n^{\frac{\exp[\aleph^{\hbar}{}_n/\kappa]}{\sum_{c=1}^{n} \exp[\aleph^{\hbar}{}_c/\kappa]}} \tag{3.8}$$

then the mapping q-ROPRFSMG is called q-rung orthopair fuzzy soft-max geometric (q-ROPRFSMG) operator, where $\aleph^{\hbar}{}_c = \prod_{k=1}^{c-1} \mathscr{H}(\hbar^{\lambda}{}_k)$ $(c = 2\ldots, n)$, $\aleph^{\hbar}{}_1 = 1$ and $\mathscr{H}(\hbar^{\lambda}{}_k)$ is the score of k^{th} q-ROPRFN.

Based on q-ROPRFNs operational rules, we can also consider q-ROPRFSMG by the theorem below.

Theorem 3.9. *Assume that* $\hbar^{\lambda}{}_c = \langle {}^{\zeta}\amalg_c, {}^{\xi}\eta_c \rangle$ *is the agglomeration of q-ROPRFNs, we can find q-ROPRFSMG by*

$$q\text{-}ROPRFSMG(\hbar^{\lambda}{}_1, \hbar^{\lambda}{}_2, \ldots \hbar^{\lambda}{}_n) = \left\langle \prod_{c=1}^{n} {}^{\zeta}\amalg_c^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}}, \sqrt[q]{(1 - \prod_{c=1}^{n}(1 - {}^{\xi}\eta_c^{q})^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}}} \right\rangle. \tag{3.9}$$

Proof. The first statement is easily followed by Definition 3.8 and Theorem 3.9. In the following, we prove this

$$\begin{aligned}
&\text{q-ROPRFSMG}(\hbar^{\lambda}{}_1, \hbar^{\lambda}{}_2, \ldots \hbar^{\lambda}{}_n) \\
&= \hbar^{\lambda}{}_1^{\frac{\widetilde{T}_1}{\sum_{c=1}^{n}\aleph^{\hbar}{}_c}} \otimes \hbar^{\lambda}{}_2^{\frac{\exp[\aleph^{\hbar}{}_2/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}} \otimes \ldots, \otimes \hbar^{\lambda}{}_n^{\frac{\aleph^{\hbar}{}_n}{\sum_{c=1}^{n}\aleph^{\hbar}{}_c}} \\
&= \left\langle \prod_{c=1}^{n} {}^{\zeta}\amalg_c^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}}, \sqrt[q]{(1 - \prod_{c=1}^{n}(1 - {}^{\xi}\eta_c^{q})^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}}} \right\rangle.
\end{aligned}$$

To prove this theorem, we use mathematical induction.
For $n = 2$

$$\begin{aligned}
\hbar^{\lambda}{}_1^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}} &= \left\langle {}^{\zeta}\amalg_1^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}} \right\rangle, \sqrt[q]{1 - (1 - {}^{\xi}\eta_1^{q})^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}}} \\
\hbar^{\lambda}{}_2^{\frac{\exp[\aleph^{\hbar}{}_2/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}} &= \left\langle {}^{\zeta}\amalg_2^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}}, \sqrt[q]{1 - (1 - {}^{\xi}\eta_2^{q})^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}}} \right\rangle.
\end{aligned}$$

Then

$$\hbar^{\lambda}{}_1^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}} \otimes \hbar^{\lambda}{}_2^{\frac{\exp[\aleph^{\hbar}{}_2/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}}.$$

$$= \left\langle {}^{\zeta}\amalg_1^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}} \sqrt[q]{1 - (1 - {}^{\xi}\eta_1^{q})^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}}} \right\rangle \otimes \left\langle {}^{\zeta}\amalg_2^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}}, \sqrt[q]{1 - (1 - {}^{\xi}\eta_2^{q})^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}}} \right\rangle$$

$$\begin{aligned}
= \Bigg\langle {}^{\zeta}\amalg_1^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}} . {}^{\zeta}\amalg_2^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}}, \sqrt[q]{\begin{array}{l}(1 - (1 - {}^{\xi}\eta_1^{q})^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}} + 1 - (1 - {}^{\xi}\eta_2^{q})^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}} - \\ \Big((1 - (1 - {}^{\xi}\eta_1^{q})^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}}\Big)\Big((1 - (1 - {}^{\xi}\eta_2^{q})^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}}\Big)\end{array}} \Bigg\rangle
\end{aligned}$$

$$= \Bigg\langle {}^{\zeta}\amalg_1^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}} . {}^{\zeta}\amalg_2^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}}, \sqrt[q]{1 - (1 - {}^{\xi}\eta_1^{q})^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}} + 1 - (1 - {}^{\xi}\eta_2^{q})^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}} -}$$

$$\overline{\left(1-(1-{}^{\xi}\eta_2^q)^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^n \exp[\aleph^{\hbar}{}_c/\kappa]}}-(1-{}^{\xi}\eta_1^q)^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^n \exp[\aleph^{\hbar}{}_c/\kappa]}}+(1-{}^{\xi}\eta_2^q)^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^n \exp[\aleph^{\hbar}{}_c/\kappa]}}(1-{}^{\xi}\eta_1^q)^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^n \exp[\aleph^{\hbar}{}_c/\kappa]}}\right)}\Bigg\rangle$$

$$=\left\langle {}^{\zeta}\amalg_1^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^n \exp[\aleph^{\hbar}{}_c/\kappa]}}.{}^{\zeta}\amalg_2^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^n \exp[\aleph^{\hbar}{}_c/\kappa]}},\sqrt[q]{1-(1-{}^{\xi}\eta_1^q)^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^n \exp[\aleph^{\hbar}{}_c/\kappa]}}(1-{}^{\xi}\eta_2^q)^{\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^n \exp[\aleph^{\hbar}{}_c/\kappa]}}}\right\rangle$$

$$=\left\langle \prod_{c=1}^{q}{}^{\zeta}\amalg_c^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^n \exp[\aleph^{\hbar}{}_c/\kappa]}},\sqrt[q]{1-\prod_{c=1}^{q}(1-{}^{\xi}\eta_c^q)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^n \exp[\aleph^{\hbar}{}_c/\kappa]}}}\right\rangle.$$

This shows that Eq (3.9) is true for $n=2$, now assume that Eq (3.9) holds for $n=k$, i.e.,

$$\text{q-ROPRFSMG}(\hbar^{\lambda}{}_1,\hbar^{\lambda}{}_2,\ldots\hbar^{\lambda}{}_k)=\left\langle \prod_{c=1}^{k}{}^{\zeta}\amalg_c^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^n \exp[\aleph^{\hbar}{}_c/\kappa]}},\sqrt[q]{1-\prod_{c=1}^{k}(1-{}^{\xi}\eta_c^q)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^n \exp[\aleph^{\hbar}{}_c/\kappa]}}}\right\rangle.$$

Now $n=k+1$, by operational laws of q-ROPRFNs we have,
$\text{q-ROPRFSMG}(\hbar^{\lambda}{}_1,\hbar^{\lambda}{}_2,\ldots\hbar^{\lambda}{}_{k+1})=\text{q-ROPRFSMG}(\hbar^{\lambda}{}_1,\hbar^{\lambda}{}_2,\ldots\hbar^{\lambda}{}_k)\otimes\hbar^{\lambda}{}_{k+1}$

$$=\left\langle \prod_{c=1}^{k}{}^{\zeta}\amalg_c^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^n \exp[\aleph^{\hbar}{}_c/\kappa]}},\sqrt[q]{1-\prod_{c=1}^{k}(1-{}^{\xi}\eta_c^q)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^n \exp[\aleph^{\hbar}{}_c/\kappa]}}}\right\rangle\otimes\left\langle {}^{\zeta}\amalg_{k+1}^{\frac{\aleph^{\hbar}{}_{k+1}}{\sum_{c=1}^n \aleph^{\hbar}{}_c}},\sqrt[q]{1-(1-{}^{\xi}\eta_{k+1}^q)^{\frac{\aleph^{\hbar}{}_{k+1}}{\sum_{c=1}^n \aleph^{\hbar}{}_c}}}\right\rangle$$

$$=\left\langle \prod_{c=1}^{k}{}^{\zeta}\amalg_k^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^n \exp[\aleph^{\hbar}{}_c/\kappa]}}.{}^{\zeta}\amalg_{k=1}^{\frac{\aleph^{\hbar}{}_{k+1}}{\sum_{c=1}^n \aleph^{\hbar}{}_c}},\sqrt[q]{1-\prod_{c=1}^{k}(1-{}^{\xi}\eta_k^q)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^n \exp[\aleph^{\hbar}{}_c/\kappa]}}+1-(1-{}^{\xi}\eta_{k+1}^q)^{\frac{\aleph^{\hbar}{}_{k+1}}{\sum_{c=1}^n \aleph^{\hbar}{}_c}}-}\right.$$

$$\left.\overline{\left(1-\prod_{c=1}^{k}(1-{}^{\xi}\eta_k^q)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^n \exp[\aleph^{\hbar}{}_c/\kappa]}}\right)\left(1-(1-{}^{\xi}\eta_{k+1}^q)^{\frac{\aleph^{\hbar}{}_{k+1}}{\sum_{c=1}^n \aleph^{\hbar}{}_c}}\right)}\right\rangle$$

$$=\left\langle \prod_{c=1}^{k}{}^{\zeta}\amalg_k^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^n \exp[\aleph^{\hbar}{}_c/\kappa]}}.{}^{\zeta}\amalg_{k+1}^{\frac{\aleph^{\hbar}{}_{k+1}}{\sum_{c=1}^n \aleph^{\hbar}{}_c}},\sqrt[q]{1-\prod_{c=1}^{k}(1-{}^{\xi}\eta_k^q)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^n \exp[\aleph^{\hbar}{}_c/\kappa]}}(1-{}^{\xi}\eta_{k+1}^q)^{\frac{\aleph^{\hbar}{}_{k+1}}{\sum_{c=1}^n \aleph^{\hbar}{}_{k+1}}}}\right\rangle$$

$$=\left\langle \prod_{c=1}^{k+1}{}^{\zeta}\amalg_c^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^n \exp[\aleph^{\hbar}{}_c/\kappa]}},\sqrt[q]{1-\prod_{c=1}^{k+1}(1-{}^{\xi}\eta_c^q)^{\frac{\aleph^{\hbar}{}_c}{\sum_{c=1}^{k+1} \aleph^{\hbar}{}_c}}}\right\rangle.$$

This shows that for $n=k+1$, Eq (3.2) holds. Then,

$$\text{q-ROPRFSMG}(\hbar^{\lambda}{}_1,\hbar^{\lambda}{}_2,\ldots\hbar^{\lambda}{}_n)=\left\langle \prod_{c=1}^{n}{}^{\zeta}\amalg_c^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^n \exp[\aleph^{\hbar}{}_c/\kappa]}},\sqrt[q]{(1-\prod_{c=1}^{n}(1-{}^{\xi}\eta_c^q)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^n \exp[\aleph^{\hbar}{}_c/\kappa]}}}\right\rangle.$$

□

Below we define some of q-ROPRFSMG operator's appealing properties.

Theorem 3.10. *(Idempotency) Assume that* $\hbar^{\lambda}{}_c = \langle {}^{\zeta}\amalg_c, {}^{\xi}\eta_c \rangle$ *is the agglomeration of q-ROPRFNs, where* $\aleph^{\hbar}{}_c = \prod_{k=1}^{c-1} \mathscr{H}(\hbar^{\lambda}{}_k)$ $(c = 2\ldots, n)$, $\aleph^{\hbar}{}_1 = 1$ *and* $\mathscr{H}(\hbar^{\lambda}{}_k)$ *is the score of* k^{th} *q-ROPRFN. If all* $\hbar^{\lambda}{}_c$ *are equal, i.e.,* $\hbar^{\lambda}{}_c = \hbar^{\lambda}$ *for all j, then*

$$q\text{-}ROPRFSMG(\hbar^{\lambda}{}_1, \hbar^{\lambda}{}_2, \ldots \hbar^{\lambda}{}_n) = \hbar^{\lambda}.$$

Proof. From Definition 3.1, we have

$$\begin{aligned}
\text{q-ROPRFSMG}(\hbar^{\lambda}{}_1, \hbar^{\lambda}{}_2, \ldots \hbar^{\lambda}{}_n) &= \hbar^{\lambda\frac{\exp[\aleph^{\hbar}{}_1/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}}_1 \otimes \hbar^{\lambda\frac{\exp[\aleph^{\hbar}{}_2/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}}_2 \otimes \ldots, \otimes \hbar^{\lambda\frac{\exp[\aleph^{\hbar}{}_n/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}}_n \\
&= \hbar^{\lambda\frac{\widetilde{T}_1}{\sum_{c=1}^{n}\aleph^{\hbar}{}_c}} \otimes \hbar^{\lambda\frac{\exp[\aleph^{\hbar}{}_2/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}} \otimes \ldots, \otimes \hbar^{\lambda\frac{\exp[\aleph^{\hbar}{}_n/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}} \\
&= \hbar^{\lambda}.
\end{aligned}$$

□

Corollary 3.11. *If* $\hbar^{\lambda}{}_c = \langle {}^{\zeta}\amalg_c, {}^{\xi}\eta_c \rangle$ $j = (1, 2, \ldots n)$ *is the agglomeration of largest q-ROPRFNs, i.e.,* $\hbar^{\lambda}{}_c = (1, 0)$ *for all j, then*

$$q\text{-}ROPRFSMG(\hbar^{\lambda}{}_1, \hbar^{\lambda}{}_2, \ldots \hbar^{\lambda}{}_n) = (1, 0).$$

Proof. We can easily obtain Corollary similar to the Theorem 3.3. □

Theorem 3.12. *(Monotonicity) Assume that* $\hbar^{\lambda}{}_c = \langle {}^{\zeta}\amalg_c, {}^{\xi}\eta_c \rangle$ *and* $\hbar^{\lambda^*}_c = \langle {}^{\zeta}\amalg^*_c, {}^{\xi}\eta^*_c \rangle$ *are the families of q-ROPRFNs, where* $\aleph^{\hbar}{}_c = \prod_{k=1}^{c-1} \mathscr{H}(\hbar^{\lambda}{}_k)$, $T^*_c = \prod_{k=1}^{c-1} \mathscr{H}(\hbar^{\lambda^*}_k)$ $(c = 2\ldots, n)$, $\aleph^{\hbar}{}_1 = 1$, $T^*_1 = 1$, $\mathscr{H}(\hbar^{\lambda}{}_k)$ *is the score of* $\hbar^{\lambda}{}_k$ *q-ROPRFN, and* $\mathscr{H}(\hbar^{\lambda^*}_k)$ *is the score of* $\hbar^{\lambda^*}_k$ *q-ROPRFN. If* ${}^{\zeta}\amalg^*_c \geq {}^{\zeta}\amalg_c$ *and* ${}^{\xi}\eta^*_c \leq {}^{\xi}\eta_c$ *for all j, then*

$$q\text{-}ROPRFSMG(\hbar^{\lambda}{}_1, \hbar^{\lambda}{}_2, \ldots \hbar^{\lambda}{}_n) \leq q\text{-}ROPRFSMG(\hbar^{\lambda^*}_1, \hbar^{\lambda^*}_2, \ldots \hbar^{\lambda^*}_n).$$

Proof. Here, ${}^{\xi}\eta^*_c \geq {}^{\xi}\eta_c$ and ${}^{\zeta}\amalg^*_c \leq {}^{\zeta}\amalg_c$ for all j, If ${}^{\xi}\eta^*_c \geq {}^{\xi}\eta_c$.
$\Leftrightarrow ({}^{\xi}\eta^*_c)^q \geq ({}^{\xi}\eta_c)^q \Leftrightarrow \sqrt[q]{({}^{\xi}\eta^*_c)^q} \geq \sqrt[q]{({}^{\xi}\eta_c)^q} \Leftrightarrow \sqrt[q]{1 - ({}^{\xi}\eta^*_c)^q} \leq \sqrt[q]{1 - ({}^{\xi}\eta_c)^q}$

$$\Leftrightarrow \sqrt[q]{(1 - ({}^{\xi}\eta^*_c)^q)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}}} \leq \sqrt[q]{(1 - ({}^{\xi}\eta_c)^q)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}}}$$

$$\Leftrightarrow \sqrt[q]{\prod_{c=1}^{n}(1 - ({}^{\xi}\eta^*_c)^q)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}}} \leq \sqrt[q]{\prod_{c=1}^{n}(1 - ({}^{\xi}\eta_c)^q)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}}}$$

$$\Leftrightarrow \sqrt[q]{1 - \prod_{c=1}^{n}(1 - ({}^{\xi}\eta_c)^q)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}}} \leq \sqrt[q]{1 - \prod_{c=1}^{n}(1 - ({}^{\xi}\eta^*_c)^q)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}}}.$$

Now,
${}^{\zeta}\amalg^*_c \leq {}^{\zeta}\amalg_c$.
$\Leftrightarrow ({}^{\zeta}\amalg^*_c)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}} \leq ({}^{\zeta}\amalg_c)^{\frac{\exp[\aleph^{\hbar}{}_c/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_c/\kappa]}}$

$\Leftrightarrow \prod_{c=1}^{n}({}^{\zeta}\text{Ш}^{*}_{c})^{\frac{\exp[\aleph^{\hbar}{}_{c}/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_{c}/\kappa]}} \leq \prod_{c=1}^{n}({}^{\zeta}\text{Ш}_{c})^{\frac{\exp[\aleph^{\hbar}{}_{c}/\kappa]}{\sum_{c=1}^{n}\exp[\aleph^{\hbar}{}_{c}/\kappa]}}$.

Let

$$\overline{\hbar^{\lambda}} = \text{q-ROPRFSMG}(\hbar^{\lambda}{}_{1}, \hbar^{\lambda}{}_{2}, \ldots \hbar^{\lambda}{}_{n})$$

and

$$\overline{\hbar^{\lambda^{*}}} = \text{q-ROPRFSMG}(\hbar^{\lambda^{*}}_{1}, \hbar^{\lambda^{*}}_{2}, \ldots \hbar^{\lambda^{*}}_{n}).$$

We get that $\overline{\hbar^{\lambda^{*}}} \geq \overline{\hbar^{\lambda}}$. So,

$$\text{q-ROPRFSMG}(\hbar^{\lambda}{}_{1}, \hbar^{\lambda}{}_{2}, \ldots \hbar^{\lambda}{}_{n}) \leq \text{q-ROPRFSMG}(\hbar^{\lambda^{*}}_{1}, \hbar^{\lambda^{*}}_{2}, \ldots \hbar^{\lambda^{*}}_{n}).$$

□

Theorem 3.13. *(Boundary) Assume that $\hbar^{\lambda}{}_{c} = \langle {}^{\zeta}\text{Ш}_{c}, {}^{\xi}\eta_{c}\rangle$ is the agglomeration of q-ROPRFNs, and*

$$\hbar^{\lambda^{-}} = (min_{c}\ ({}^{\zeta}\text{Ш}_{c}), max_{c}\ ({}^{\xi}\eta_{c})) \quad and \quad \hbar^{\lambda^{+}} = (max_{c}\ ({}^{\zeta}\text{Ш}_{c}), min_{c}\ ({}^{\xi}\eta_{c})).$$

Then,

$$\hbar^{\lambda^{-}} \leq \textit{q-ROPRFSMG}(\hbar^{\lambda}{}_{1}, \hbar^{\lambda}{}_{2}, \ldots \hbar^{\lambda}{}_{n}) \leq \hbar^{\lambda^{+}}$$

where $\aleph^{\hbar}{}_{c} = \prod_{k=1}^{c-1}\mathscr{H}(\hbar^{\lambda}{}_{k})$ $(c = 2\ldots, n)$, $\aleph^{\hbar}{}_{1} = 1$ and $\mathscr{H}(\hbar^{\lambda}{}_{k})$ is the score of k^{th} q-ROPRFN.

Proof. The proof of this theorem is same as Theorem 3.6. □

Theorem 3.14. *Assume that $\hbar^{\lambda}{}_{c} = \langle {}^{\zeta}\text{Ш}_{c}, {}^{\xi}\eta_{c}\rangle$ and $\beta_{c} = \langle \phi_{c}, \varphi_{c}\rangle$ are two familie of q-ROPRFNs, where $\aleph^{\hbar}{}_{c} = \prod_{k=1}^{c-1}\mathscr{H}(\hbar^{\lambda}{}_{k})$ $(c = 2\ldots, n)$, $\aleph^{\hbar}{}_{1} = 1$ and $\mathscr{H}(\hbar^{\lambda}{}_{k})$ is the score of k^{th} q-ROPRFN. If $r > 0$ and $\beta = \langle {}^{\zeta}\text{Ш}_{\beta}, {}^{\xi}\eta_{\beta}\rangle$ is an q-ROPRFN, then*

(1) q-ROPRFSMG$(\hbar^{\lambda}{}_{1} \oplus \beta, \hbar^{\lambda}{}_{2} \oplus \beta, \ldots \hbar^{\lambda}{}_{n} \oplus \beta)$ = q-ROPRFSMG$(\hbar^{\lambda}{}_{1}, \hbar^{\lambda}{}_{2}, \ldots \hbar^{\lambda}{}_{n}) \oplus \beta$.

(2) q-ROPRFSMG$(r\hbar^{\lambda}{}_{1}, r\hbar^{\lambda}{}_{2}, \ldots r\hbar^{\lambda}{}_{n}) = r$ q-ROPRFSMG$(\hbar^{\lambda}{}_{1}, \hbar^{\lambda}{}_{2}, \ldots \hbar^{\lambda}{}_{n})$.

(3) q-ROPRFSMG$(\hbar^{\lambda}{}_{1} \oplus \beta_{2}, \hbar^{\lambda}{}_{2} \oplus \beta_{2}, \ldots \hbar^{\lambda}{}_{n} \oplus \beta_{n})$ = q-ROPRFSMG$(\hbar^{\lambda}{}_{1}, \hbar^{\lambda}{}_{2}, \ldots \hbar^{\lambda}{}_{n}) \oplus$ q-ROPRFSMG$(\beta_{1}, \beta_{2}, \ldots \beta_{n})$.

(4) q-ROPRFSMG$(r\hbar^{\lambda}{}_{1} \oplus \beta r\hbar^{\lambda}{}_{2} \oplus \beta, \ldots \oplus r\hbar^{\lambda}{}_{n} \oplus \beta) = r$ q-ROPRFSMG$(\hbar^{\lambda}{}_{1}, \hbar^{\lambda}{}_{2}, \ldots \hbar^{\lambda}{}_{n}) \oplus \beta$.

Proof. The proof of this theorem is same as Theorem 3.7. □

4. Proposed methodology

Consider a set of alternatives $\mathscr{G}^{\daleth} = \{\mathscr{G}^{\daleth}_{1}, \mathscr{G}^{\daleth}{}_{2}, \ldots, \mathscr{G}^{\daleth}{}_{m}\}$ with m elements and $\mathfrak{P}^{\gamma} = \{\mathfrak{P}^{\gamma}{}_{1}, \mathfrak{P}^{\gamma}{}_{2}, \ldots, \mathfrak{P}^{\gamma}{}_{n}\}$ is the finite set of criterion with n elements. $\mathfrak{S}^{\zeta} = \{\mathfrak{S}^{\zeta}{}_{1}, \mathfrak{S}^{\zeta}{}_{2}, \ldots, \mathfrak{S}^{\zeta}{}_{p}\}$ is the group of decision makers. Decision makers provide a matrix of their own opinion $D^{(p)} = (\mathscr{B}^{(p)}_{ij})_{m\times n}$, where $\mathscr{B}^{(p)}_{ij}$ is given for the alternatives $\mathscr{G}^{\daleth}{}_{i} \in \mathscr{G}^{\daleth}$ with respect to the criteria $\mathfrak{P}^{\gamma}{}_{c} \in \mathfrak{P}^{\gamma}$ by $\mathfrak{S}^{\zeta}{}_{p}$ decision maker in the form of q-ROPRFNs. If all Criteria are the same types, there is no need for normalization, but there are two types of criterion (benefit type attributes τ_b and cost type attributes τ_c) in MCGDM, in this case using the normalization formula the matrix $D^{(p)}$ has been changed into normalize matrix $Y^{(p)} = (\mathscr{P}_{ij}^{(p)})_{m\times n}$,

$$(\mathscr{P}_{ij}^{(p)})_{m\times n} = \begin{cases} (\mathscr{B}_{ij}^{(p)})^c; & j \in \tau_c \\ \mathscr{B}_{ij}^{(p)}; & j \in \tau_b, \end{cases} \tag{4.1}$$

where $(\mathscr{B}_{ij}^{(p)})^c$ show the compliment of $\mathscr{B}_{ij}^{(p)}$.
We then use the q-ROPRFSMA operator or q-ROPRFSMA operator to implement a MCGDM approach in an q-ROPRF circumstances. The proposed operators will be applied to the MCGDM, which involves the following steps.

Algorithm

Input:

Step 1:

Acquire a decision matrix $D^{(p)} = (\mathscr{B}_{ij}^{(p)})_{m\times n}$ in the form of q-ROPRFNs from the decision makers.

$$\begin{array}{cc} & \\ \mathfrak{S}^{\zeta}{}_1 & \mathscr{G}^{\beth}{}_1 \\ & \mathscr{G}^{\beth}{}_2 \\ & \\ & \mathscr{G}^{\beth}{}_m \\ \mathfrak{S}^{\zeta}{}_2 & \mathscr{G}^{\beth}{}_1 \\ \vdots & \mathscr{G}^{\beth}{}_2 \\ & \\ & \mathscr{G}^{\beth}{}_m \\ \mathfrak{S}^{\zeta}{}_p & \mathscr{G}^{\beth}{}_1 \\ & \mathscr{G}^{\beth}{}_2 \\ & \\ & \mathscr{G}^{\beth}{}_m \end{array} \begin{array}{c} \begin{array}{cccc} \quad\mathfrak{P}^{\gamma}{}_1 & \qquad\mathfrak{P}^{\gamma}{}_2 & \qquad\qquad & \qquad\mathfrak{P}^{\gamma}{}_n \end{array} \\ \begin{bmatrix} ({}^{\zeta}\text{Ш}_{11}^{1}, {}^{\xi}\eta_{11}^{1}) & ({}^{\zeta}\text{Ш}_{12}^{1}, {}^{\xi}\eta_{12}^{1}) & \cdots\cdots & ({}^{\zeta}\text{Ш}_{1n}^{1}, {}^{\xi}\eta_{1n}^{1}) \\ ({}^{\zeta}\text{Ш}_{21}^{1}, {}^{\xi}\eta_{21}^{1}) & ({}^{\zeta}\text{Ш}_{22}^{1}, {}^{\xi}\eta_{22}^{1}) & \cdots\cdots & ({}^{\zeta}\text{Ш}_{2n}^{1}, {}^{\xi}\eta_{2n}^{1}) \\ \vdots & \vdots & \ddots\ddots & \vdots \\ ({}^{\zeta}\text{Ш}_{m1}^{1}, {}^{\xi}\eta_{m1}^{1}) & ({}^{\zeta}\text{Ш}_{m2}^{1}, {}^{\xi}\eta_{m2}^{1}) & \cdots\cdots & ({}^{\zeta}\text{Ш}_{mn}^{1}, {}^{\xi}\eta_{mn}^{1}) \\ ({}^{\zeta}\text{Ш}_{11}^{q}, {}^{\xi}\eta_{11}^{q}) & ({}^{\zeta}\text{Ш}_{12}^{q}, {}^{\xi}\eta_{12}^{q}) & \cdots\cdots & ({}^{\zeta}\text{Ш}_{1n}^{q}, {}^{\xi}\eta_{1n}^{q}) \\ ({}^{\zeta}\text{Ш}_{21}^{q}, {}^{\xi}\eta_{21}^{q}) & ({}^{\zeta}\text{Ш}_{22}^{q}, {}^{\xi}\eta_{22}^{q}) & \cdots\cdots & ({}^{\zeta}\text{Ш}_{2n}^{q}, {}^{\xi}\eta_{2n}^{q}) \\ \vdots & \vdots & \ddots\ddots & \vdots \\ ({}^{\zeta}\text{Ш}_{m1}^{q}, {}^{\xi}\eta_{m1}^{q}) & ({}^{\zeta}\text{Ш}_{m2}^{q}, {}^{\xi}\eta_{m2}^{q}) & \cdots\cdots & ({}^{\zeta}\text{Ш}_{mn}^{q}, {}^{\xi}\eta_{mn}^{q}) \\ ({}^{\zeta}\text{Ш}_{11}^{p}, {}^{\xi}\eta_{11}^{p}) & ({}^{\zeta}\text{Ш}_{12}^{p}, {}^{\xi}\eta_{12}^{p}) & \cdots\cdots & ({}^{\zeta}\text{Ш}_{1n}^{p}, {}^{\xi}\eta_{1n}^{p}) \\ ({}^{\zeta}\text{Ш}_{21}^{p}, {}^{\xi}\eta_{21}^{p}) & ({}^{\zeta}\text{Ш}_{22}^{p}, {}^{\xi}\eta_{22}^{p}) & \cdots\cdots & ({}^{\zeta}\text{Ш}_{2n}^{p}, {}^{\xi}\eta_{2n})^{p} \\ \vdots & \vdots & \ddots\ddots & \vdots \\ ({}^{\zeta}\text{Ш}_{m1}^{p}, {}^{\xi}\eta_{m1}^{p}) & ({}^{\zeta}\text{Ш}_{m2}^{p}, {}^{\xi}\eta_{m2}^{p}) & \cdots\cdots & ({}^{\zeta}\text{Ш}_{mn}^{p}, {}^{\xi}\eta_{mn}^{p}) \end{bmatrix} \end{array}$$

Step 2:

If all criteria are of the same type, normalization is unnecessary; however, MCGDM contains two categories of criteria. in this case using the normalization formula Eq (4.1) the matrix has been changed into transformed response matrix $Y^{(p)} = (\mathscr{P}_{ij}^{(p)})_{m\times n}$.

Calculations:

Step 3:

Calculate the values of $\aleph^{\hbar(p)}_{ij}$ by following formula.

$$\aleph^{\hbar(p)}_{ij} = \prod_{k=1}^{p-1} \mathscr{H}(\mathscr{P}_{ij}^{(k)}) \quad (p = 2\ldots, n), \tag{4.2}$$

$$T_{ij}^{(1)} = 1.$$

Step 4:
Use one of the suggested AOs, to aggregate all individual q-ROPRF decision matrices $Y^{(p)} = (\mathscr{P}_{ij}^{(p)})_{m\times n}$ into one cumulative assessments matrix of the alternatives $W^{(p)} = (\mathscr{W}_{ij})_{m\times n}$.
Step 5:
Calculate the values of $\aleph^{\hbar}{}_{ij}$ by following formula.

$$\aleph^{\hbar}{}_{ij} = \prod_{k=1}^{c-1} \mathscr{H}(\mathscr{W}_{ik}) \quad (j = 2\ldots, n), \tag{4.3}$$

$$\aleph^{\hbar}{}_{i1} = 1.$$

Step 6:
Aggregate the q-ROPRF values $\mathscr{W}_{ij}$ for each alternative $\mathscr{G}^{\daleth}{}_i$ by the q-ROPRFSMA (or q-ROPRFSMG) operator:

$$\mathscr{W}_i = \text{q-ROPRFSMA}(\mathscr{P}_{i1}, \mathscr{P}_{i2}, \ldots \mathscr{P}_{in}) = \left\langle \sqrt[q]{(1 - \prod_{c=1}^{n}(1 - ({}^{\zeta}\amalg_{ij}^{q})^{\frac{\exp[\aleph^{\hbar}c/\kappa]}{\Sigma_{c=1}^{n}\exp[\aleph^{\hbar}c/\kappa]}}}, \prod_{c=1}^{n}({}^{\xi}\eta_{ij}^{q})^{\frac{\exp[\aleph^{\hbar}c/\kappa]}{\Sigma_{c=1}^{n}\exp[\aleph^{\hbar}c/\kappa]}} \right\rangle \tag{4.4}$$

or

$$\mathscr{W}_i = \text{q-ROPRFSMG}(\mathscr{P}_{i1}, \mathscr{P}_{i2}, \ldots \mathscr{P}_{in}) = \left\langle \prod_{c=1}^{n}({}^{\zeta}\amalg_{ij}^{q})^{\frac{\exp[\aleph^{\hbar}c/\kappa]}{\Sigma_{c=1}^{n}\exp[\aleph^{\hbar}c/\kappa]}}, \sqrt[q]{(1 - \prod_{c=1}^{n}(1 - ({}^{\xi}\eta_{ij}^{q})^{\frac{\exp[\aleph^{\hbar}c/\kappa]}{\Sigma_{c=1}^{n}\exp[\aleph^{\hbar}c/\kappa]}}} \right\rangle. \tag{4.5}$$

Output:
Step 7:
Evaluate the score of all cumulative alternative assessments.

Step 8:
Ranked the alternatives by the score function and ultimately choose the most appropriate alternative.

5. Numerical example

Material selection is a critical process in engineering that involves choosing the most appropriate materials for a given application. Engineers are faced with a wide range of materials to choose from, each with its own unique set of properties and characteristics. The importance of material selection lies in the fact that it can significantly affect the performance, cost, and safety of a product. The selection of materials plays a crucial role in determining the quality, durability, and overall performance of engineering products.

The scope of material selection in engineering is vast, as it is applicable to virtually every type of product or system. Whether it's designing an aircraft, constructing a building, manufacturing electronic devices, or developing automotive components, the choice of materials can have a profound impact on the functionality and safety of the end product. Different applications require different materials, and it is the responsibility of the engineer to carefully consider all the relevant factors before making a decision. One of the primary factors that engineers consider during the material selection process is the mechanical properties of the materials. These properties include tensile strength, hardness, toughness, elasticity, and fatigue resistance. Depending on the application, engineers may prioritize one or more of these properties. For example, in the aerospace industry, materials with high strength-to-weight ratios are often preferred to maximize fuel efficiency and structural integrity. On the other hand, in the automotive industry, materials with good impact resistance and ductility may be more important to ensure passenger safety in the event of a collision.

Chemical properties are another crucial consideration in material selection. Engineers must assess how the material will interact with the surrounding environment, including exposure to chemicals, moisture, temperature variations, and corrosive substances. For instance, in chemical processing plants, materials that exhibit high resistance to corrosion are chosen to ensure the integrity and longevity of the equipment. Similarly, in electronic devices, materials with good electrical conductivity and thermal properties are essential for efficient operation and heat dissipation. Environmental conditions also play a significant role in material selection. Factors such as temperature extremes, humidity, UV radiation, and outdoor exposure can degrade materials over time. Engineers must carefully evaluate the suitability of materials for the intended environment to prevent premature failure and ensure long-term performance.

Cost and availability of materials are practical considerations that engineers cannot overlook. While it is essential to select materials that meet the performance requirements, cost-effectiveness is also a key factor. Engineers must strike a balance between performance and cost to optimize the overall design and production process. Additionally, the availability of materials should be considered to avoid delays or supply chain issues. To facilitate the material selection process, engineers utilize various tools and techniques. Material databases, handbooks, and computer-aided design software provide valuable information and guidance on material properties, performance characteristics, and compatibility with specific applications. Simulation and modelling tools help engineers evaluate how different materials will behave under different conditions, enabling them to make informed decisions.

In this case study, we will explore how a company used multi-criteria decision-making to select a material for a new product. The company, which produces medical equipment, was developing a new product that required a material with specific mechanical, thermal, and electrical properties. In addition, the company wanted to ensure that the selected material was cost-effective and environmentally sustainable. The first step in the material selection process was to identify the relevant criteria. The company identified the following criteria as important for the new product:

The material should have high strength and stiffness to withstand the stresses and strains of use. The material should have a low coefficient of thermal expansion to maintain its shape and dimensional stability over a range of temperatures. The material should have good electrical conductivity to enable the product to function properly. The material should be cost-effective and provide good value for money. The material should have a low environmental impact, both in terms of its production and disposal.

The next step was to identify potential materials that met the criteria. The company identified several materials that had the required properties, including carbon fiber reinforced polymer (CFRP) ($\mathscr{G}^{\daleth}{}_1$), aluminum ($\mathscr{G}^{\daleth}{}_2$), steel ($\mathscr{G}^{\daleth}{}_3$), ceramic ($\mathscr{G}^{\daleth}{}_4$) and titanium ($\mathscr{G}^{\daleth}{}_5$). The company then used a multi-criteria decision-making process to evaluate and compare the materials. They used the weighted sum method to calculate a score for each material based on the different criteria.

In order to further understand our suggested process, below is an example that pertains to the selection of material. Consider a set of alternatives $\mathscr{G}^{\daleth} = \{\mathscr{G}^{\daleth}{}_1, \mathscr{G}^{\daleth}{}_2, \mathscr{G}^{\daleth}{}_3, \mathscr{G}^{\daleth}{}_4, \mathscr{G}^{\daleth}{}_5\}$ and $\mathfrak{P}^{\gamma} = \{\mathfrak{P}^{\gamma}{}_1, \mathfrak{P}^{\gamma}{}_2, \mathfrak{P}^{\gamma}{}_3, \mathfrak{P}^{\gamma}{}_4, \mathfrak{P}^{\gamma}{}_5, \mathfrak{P}^{\gamma}{}_6\}$ is the finite set of criterions. Where $\mathfrak{P}^{\gamma}{}_1$= Mechanical properties, $\mathfrak{P}^{\gamma}{}_2$= Thermal properties , $\mathfrak{P}^{\gamma}{}_3$= Delivery, $\mathfrak{P}^{\gamma}{}_4$= Electrical properties , $\mathfrak{P}^{\gamma}{}_5$=Environmental impact and $\mathfrak{P}^{\gamma}{}_6$= other properties. $\mathfrak{S}^{\zeta} = \{\mathfrak{S}^{\zeta}{}_1, \mathfrak{S}^{\zeta}{}_2, \mathfrak{S}^{\zeta}{}_3\}$ is the group of DMs. DMs provide a matrix of their own opinion $D^{(p)} = (\mathscr{B}^{(p)}_{ij})_{m\times n}$, where $\mathscr{B}^{(p)}_{ij}$ is given for the alternatives $\mathscr{G}^{\daleth}{}_i \in \mathscr{G}^{\daleth}$ with respect to the criteria $\mathfrak{P}^{\gamma}{}_c \in \mathfrak{P}^{\gamma}$ by $\mathfrak{S}^{\zeta}{}_p$ decision maker in the form of q-ROPRFNs. we take $q = 6$.

Step 1:

Acquire a decision matrix $D^{(p)} = (\mathscr{B}^{(p)}_{ij})_{m\times n}$ in the form of q-ROPRFNs from the decision makers given in Tables 1–3.

Table 1. q-ROPRF decision matrix from $\mathfrak{S}^{\zeta}{}_1$.

	$\mathfrak{P}^{\gamma}{}_1$	$\mathfrak{P}^{\gamma}{}_2$	$\mathfrak{P}^{\gamma}{}_3$	$\mathfrak{P}^{\gamma}{}_4$	$\mathfrak{P}^{\gamma}{}_5$	$\mathfrak{P}^{\gamma}{}_6$
$\mathscr{G}^{\daleth}{}_1$	(0.78,0.65)	(0.45,0.78)	(0.85,0.34)	(0.34,0.24)	(0.64,0.36)	(0.78,0.84)
$\mathscr{G}^{\daleth}{}_2$	(0.75,0.54)	(0.93,0.29)	(0.34,0.24)	(0.89,0.32)	(0.44,0.44)	(0.54,0.84)
$\mathscr{G}^{\daleth}{}_3$	(0.94,0.19)	(0.64,0.34)	(0.84,0.44)	(0.84,0.40)	(0.74,0.21)	(0.54,0.65)
$\mathscr{G}^{\daleth}{}_4$	(0.74,0.34)	(0.80,0.24)	(0.64,0.14)	(0.34,0.24)	(0.74,0.24)	(0.34,0.74)
$\mathscr{G}^{\daleth}{}_5$	(0.89,0.40)	(0.94,0.33)	(0.42,0.53)	(0.34,0.54)	(0.34,0.34)	(0.28,0.33)

Table 2. q-ROPRF decision matrix from $\mathfrak{S}^{\zeta}{}_2$.

	$\mathfrak{P}^{\gamma}{}_1$	$\mathfrak{P}^{\gamma}{}_2$	$\mathfrak{P}^{\gamma}{}_3$	$\mathfrak{P}^{\gamma}{}_4$	$\mathfrak{P}^{\gamma}{}_5$	$\mathfrak{P}^{\gamma}{}_6$
$\mathscr{G}^{\daleth}{}_1$	(0.84,0.34)	(0.43,0.21)	(0.84,0.54)	(0.54,0.32)	(0.22,0.78)	(0.43,0.43)
$\mathscr{G}^{\daleth}{}_2$	(0.87,0.34)	(0.32,0.56)	(0.43,0.34)	(0.44,0.74)	(0.27,0.67)	(0.34,0.87)
$\mathscr{G}^{\daleth}{}_3$	(0.89,0.59)	(0.54,0.38)	(0.54,0.34)	(0.34,0.24)	(0.43,0.24)	(0.56,0.78)
$\mathscr{G}^{\daleth}{}_4$	(0.49,0.78)	(0.43,0.39)	(0.74,0.89)	(0.56,0.32)	(0.69,0.18)	(0.59,0.34)
$\mathscr{G}^{\daleth}{}_5$	(0.89,0.23)	(0.67,0.87)	(0.23,0.41)	(0.24,0.49)	(0.49,0.29)	(0.49,0.24)

Table 3. q-ROPRF decision matrix from $\mathfrak{S}^{\zeta}{}_3$.

	$\mathfrak{P}^{\gamma}{}_1$	$\mathfrak{P}^{\gamma}{}_2$	$\mathfrak{P}^{\gamma}{}_3$	$\mathfrak{P}^{\gamma}{}_4$	$\mathfrak{P}^{\gamma}{}_5$	$\mathfrak{P}^{\gamma}{}_6$
$\mathscr{G}^{\daleth}{}_1$	(0.54,0.65)	(0.34,0.78)	(0.56,0.87)	(0.64,0.25)	(0.56,0.76)	(0.43,0.27)
$\mathscr{G}^{\daleth}{}_2$	(0.96,0.23)	(0.54,0.34)	(0.39,0.56)	(0.54,0.22)	(0.53,0.54)	(0.22,0.56)
$\mathscr{G}^{\daleth}{}_3$	(0.74,0.14)	(0.43,0.42)	(0.24,0.44)	(0.36,0.54)	(0.54,0.54)	(0.32,0.54)
$\mathscr{G}^{\daleth}{}_4$	(0.34,0.34)	(0.49,0.34)	(0.44,0.24)	(0.78,0.54)	(0.75,0.64)	(0.65,0.54)
$\mathscr{G}^{\daleth}{}_5$	(0.78,0.54)	(0.35,0.25)	(0.53,0.54)	(0.64,0.24)	(0.64,0.54)	(0.44,0.39)

Step 2:

Normalize the decision matrixes acquired by DMs using Eq (4.1). There are two types of criterions. There is no cost type criteria So normalize decision matrixes are given in Tables 4–6.

Table 4. Normalized q-ROPRF decision matrix from $\mathfrak{S}^{\zeta}{}_1$.

	$\mathfrak{P}^{\gamma}{}_1$	$\mathfrak{P}^{\gamma}{}_2$	$\mathfrak{P}^{\gamma}{}_3$	$\mathfrak{P}^{\gamma}{}_4$	$\mathfrak{P}^{\gamma}{}_5$	$\mathfrak{P}^{\gamma}{}_6$
$\mathscr{G}^{\daleth}{}_1$	(0.78,0.65)	(0.45,0.78)	(0.85,0.34)	(0.34,0.24)	(0.64,0.36)	(0.78,0.84)
$\mathscr{G}^{\daleth}{}_2$	(0.75,0.54)	(0.93,0.29)	(0.34,0.24)	(0.89,0.32)	(0.44,0.44)	(0.54,0.84)
$\mathscr{G}^{\daleth}{}_3$	(0.94,0.19)	(0.64,0.34)	(0.84,0.44)	(0.84,0.40)	(0.74,0.21)	(0.54,0.65)
$\mathscr{G}^{\daleth}{}_4$	(0.74,0.34)	(0.80,0.24)	(0.64,0.14)	(0.34,0.24)	(0.74,0.24)	(0.34,0.74)
$\mathscr{G}^{\daleth}{}_5$	(0.89,0.40)	(0.94,0.33)	(0.42,0.53)	(0.34,0.54)	(0.34,0.34)	(0.28,0.33)

Table 5. Normalized q-ROPRF decision matrix from $\mathfrak{S}^{\zeta}{}_2$.

	$\mathfrak{P}^{\gamma}{}_1$	$\mathfrak{P}^{\gamma}{}_2$	$\mathfrak{P}^{\gamma}{}_3$	$\mathfrak{P}^{\gamma}{}_4$	$\mathfrak{P}^{\gamma}{}_5$	$\mathfrak{P}^{\gamma}{}_6$
$\mathscr{G}^{\daleth}{}_1$	(0.84,0.34)	(0.43,0.21)	(0.84,0.54)	(0.54,0.32)	(0.22,0.78)	(0.43,0.43)
$\mathscr{G}^{\daleth}{}_2$	(0.87,0.34)	(0.32,0.56)	(0.43,0.34)	(0.44,0.74)	(0.27,0.67)	(0.34,0.87)
$\mathscr{G}^{\daleth}{}_3$	(0.89,0.59)	(0.54,0.38)	(0.54,0.34)	(0.34,0.24)	(0.43,0.24)	(0.56,0.78)
$\mathscr{G}^{\daleth}{}_4$	(0.49,0.78)	(0.43,0.39)	(0.74,0.89)	(0.56,0.32)	(0.69,0.18)	(0.59,0.34)
$\mathscr{G}^{\daleth}{}_5$	(0.89,0.23)	(0.67,0.87)	(0.23,0.41)	(0.24,0.49)	(0.49,0.29)	(0.49,0.24)

Table 6. Normalized q-ROPRF decision matrix from $\mathfrak{S}^{\zeta}{}_3$.

	$\mathfrak{P}^{\gamma}{}_1$	$\mathfrak{P}^{\gamma}{}_2$	$\mathfrak{P}^{\gamma}{}_3$	$\mathfrak{P}^{\gamma}{}_4$	$\mathfrak{P}^{\gamma}{}_5$	$\mathfrak{P}^{\gamma}{}_6$
$\mathscr{G}^{\daleth}{}_1$	(0.54,0.65)	(0.34,0.78)	(0.56,0.87)	(0.64,0.25)	(0.56,0.76)	(0.43,0.27)
$\mathscr{G}^{\daleth}{}_2$	(0.96,0.23)	(0.54,0.34)	(0.39,0.56)	(0.54,0.22)	(0.53,0.54)	(0.22,0.56)
$\mathscr{G}^{\daleth}{}_3$	(0.74,0.14)	(0.43,0.42)	(0.24,0.44)	(0.36,0.54)	(0.54,0.54)	(0.32,0.54)
$\mathscr{G}^{\daleth}{}_4$	(0.34,0.34)	(0.49,0.34)	(0.44,0.24)	(0.78,0.54)	(0.75,0.64)	(0.65,0.54)
$\mathscr{G}^{\daleth}{}_5$	(0.78,0.54)	(0.35,0.25)	(0.53,0.54)	(0.64,0.24)	(0.64,0.54)	(0.44,0.39)

Step 3:
Calculate the values of $\aleph^{\hbar(p)}_{ij}$ by Eq (4.2).

$$\aleph^{\hbar(1)}_{ij} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\aleph^{\hbar(2)}_{ij} = \begin{pmatrix} 0.3567 & 0.2341 & 0.2392 & 0.3466 & 0.5432 & 0.5400 \\ 0.4253 & 0.3574 & 0.5650 & 0.8435 & 0.4564 & 0.5238 \\ 0.4356 & 0.5710 & 0.5431 & 0.2456 & 0.6432 & 0.5176 \\ 0.5464 & 0.6520 & 0.3256 & 0.7653 & 0.3456 & 0.2905 \\ 0.2334 & 0.4340 & 0.6064 & 0.4567 & 0.5678 & 0.3619 \end{pmatrix}$$

$$\aleph^{\hbar(3)}_{ij} = \begin{pmatrix} 0.4565 & 0.5674 & 0.5579 & 0.8373 & 0.5318 & 0.4650 \\ 0.4345 & 0.4567 & 0.2999 & 0.4991 & 0.2989 & 0.2995 \\ 0.7653 & 0.3325 & 0.3043 & 0.3562 & 0.2987 & 0.4490 \\ 0.5678 & 0.6542 & 0.3043 & 0.1934 & 0.3423 & 0.2587 \\ 0.4324 & 0.5678 & 0.3976 & 0.1844 & 0.3727 & 0.4317 \end{pmatrix}.$$

Step 4:
Use q-ROPRFSMA to aggregate all individual q-ROPRF decision matrices $Y^{(p)} = (\mathscr{P}^{(p)}_{ij})_{m\times n}$ into one cumulative assessments matrix of the alternatives $W^{(p)} = (\mathscr{W}_{ij})_{m\times n}$ given in Table 7.

Table 7. Collective q-ROPRF decision matrix.

	$\mathfrak{P}^{\gamma}{}_1$	$\mathfrak{P}^{\gamma}{}_2$	$\mathfrak{P}^{\gamma}{}_3$	$\mathfrak{P}^{\gamma}{}_4$	$\mathfrak{P}^{\gamma}{}_5$	$\mathfrak{P}^{\gamma}{}_6$
$\mathscr{G}^{\daleth}{}_1$	(0.5256, 0.6467)	(0.4525, 0.7265)	(0.8674, 0.5525)	(0.9643, 0.0932)	(0.8956, 0.6365)	(0.6784, 0.2636)
$\mathscr{G}^{\daleth}{}_2$	(0.9743, 0.1897)	(0.2973, 0.4632)	(0.4536, 0.2138)	(0.6954, 0.1984)	(0.5434, 0.3224)	(0.6474, 0.0000)
$\mathscr{G}^{\daleth}{}_3$	(0.7953, 0.2943)	(0.3563, 0.7643)	(0.7884, 0.5748)	(0.5356, 0.6346)	(0.5956, 0.5465)	(0.6764, 0.6367)
$\mathscr{G}^{\daleth}{}_4$	(0.7854, 0.6367)	(0.3522, 0.7849)	(0.4352, 0.3464)	(0.6463, 0.4674)	(0.6644, 0.1675)	(0.4754, 0.5378)
$\mathscr{G}^{\daleth}{}_5$	(0.2467, 0.2315)	(0.5232, 0.6743)	(0.4363, 0.5638)	(0.6474, 0.5516)	(0.5956, 0.3225)	(0.4636, 0.6546)

Step 5:
Evaluate the values of $\aleph^{\hbar}{}_{ij}$ by using Eq (4.3).

$$\aleph^{\hbar}{}_{ij} = \begin{pmatrix} 1 & 0.8179 & 0.2999 & 0.2102 & 0.1998 & 0.1361 \\ 1 & 0.7999 & 0.2905 & 0.1946 & 0.1232 & 0.0253 \\ 1 & 0.7967 & 0.2323 & 0.1983 & 0.1343 & 0.0177 \\ 1 & 0.6321 & 0.2652 & 0.0987 & 0.0728 & 0.0149 \\ 1 & 0.7781 & 0.3101 & 0.1499 & 0.1223 & 0.0245 \end{pmatrix}.$$

Step 6:
Aggregate the q-ROPRF values $\mathscr{W}_{ij}$ for each alternative $\mathscr{G}^{\daleth}{}_i$ by the q-ROPRFSMA operator using Eq (4.4) given in Table 8.

Table 8. q-ROPRF Aggregated values $\mathscr{W}_i$.

$\mathscr{W}_1$	(0.7838, 0.6347)
$\mathscr{W}_2$	(0.7298, 0.4675)
$\mathscr{W}_3$	(0.7167, 0.2367)
$\mathscr{W}_4$	(0.5784, 0.8763)
$\mathscr{W}_5$	(0.6352, 0.3532)

Step 7:
Calculate the score of all q-ROPRF aggregated values $\mathscr{W}_i$.

$$\mathscr{H}(\mathscr{W}_1) = 0.8754$$

$$\mathscr{H}(\mathscr{W}_2) = 0.7377$$

$$\mathscr{H}(\mathscr{W}_3) = 0.6747$$

$$\mathscr{H}(\mathscr{W}_4) = 0.7587$$

$$\mathscr{H}(\mathscr{W}_5) = 0.3567.$$

Step 8:

Ranks by score function values.

$$\mathscr{W}_1 > \mathscr{W}_4 > \mathscr{W}_2 > \mathscr{W}_3 > \mathscr{W}_5.$$

So,

$$\mathscr{G}^{\daleth}{}_1 > \mathscr{G}^{\daleth}{}_4 > \mathscr{G}^{\daleth}{}_2 > \mathscr{G}^{\daleth}{}_3 > \mathscr{G}^{\daleth}{}_5.$$

The company then conducted a sensitivity analysis to check the robustness of the results. They varied the weights of the criteria and recalculated the scores for each material. The results of the sensitivity analysis confirmed that CFRP was the best material for the new product, regardless of the weight assigned to each criterion.

Comparison analysis

In this section, we provide a comparative review of recommended operators alongside some current AOs. The fact that both options yield identical outcomes is a testament to the effectiveness of our proposed action items. Through the resolution of information data using existing analytical objects, we conducted a comparative analysis of our findings and arrived at the same optimal decision. This demonstrates the robustness and coherence of our proposed model. A comparison between the presented AOs and some current AOs is provided in the Table 9.

Table 9. Comparison of proposed operators with some exiting operators.

Authors	AOs	Ranking of alternatives	The optimal alternative
Liu & Wang [13]	q-ROPRFWA	$\mathscr{G}^{\daleth}{}_1 > \mathscr{G}^{\daleth}{}_3 > \mathscr{G}^{\daleth}{}_2 > \mathscr{G}^{\daleth}{}_4 > \mathscr{G}^{\daleth}{}_5$	$\mathscr{G}^{\daleth}{}_1$
	q-ROPRFWG	$\mathscr{G}^{\daleth}{}_1 > \mathscr{G}^{\daleth}{}_4 > \mathscr{G}^{\daleth}{}_3 > \mathscr{G}^{\daleth}{}_2 > \mathscr{G}^{\daleth}{}_5$	$\mathscr{G}^{\daleth}{}_1$
Liu & Liu [26]	q-ROPRFWBM	$\mathscr{G}^{\daleth}{}_1 > \mathscr{G}^{\daleth}{}_4 > \mathscr{G}^{\daleth}{}_2 > \mathscr{G}^{\daleth}{}_3 > \mathscr{G}^{\daleth}{}_5$	$\mathscr{G}^{\daleth}{}_1$
	q-ROPRFWGBM	$\mathscr{G}^{\daleth}{}_1 > \mathscr{G}^{\daleth}{}_4 > \mathscr{G}^{\daleth}{}_3 > \mathscr{G}^{\daleth}{}_2 > \mathscr{G}^{\daleth}{}_5$	$\mathscr{G}^{\daleth}{}_1$
Zhao et al. [15]	q-ROPRFHM	$\mathscr{G}^{\daleth}{}_1 > \mathscr{G}^{\daleth}{}_4 > \mathscr{G}^{\daleth}{}_5 > \mathscr{G}^{\daleth}{}_3 > \mathscr{G}^{\daleth}{}_2$	$\mathscr{G}^{\daleth}{}_1$
	q-ROPRFWHM	$\mathscr{G}^{\daleth}{}_1 > \mathscr{G}^{\daleth}{}_4 > \mathscr{G}^{\daleth}{}_2 > \mathscr{G}^{\daleth}{}_3 > \mathscr{G}^{\daleth}{}_5$	$\mathscr{G}^{\daleth}{}_1$
Liu et al. [28]	q-ROPRFHM	$\mathscr{G}^{\daleth}{}_1 > \mathscr{G}^{\daleth}{}_4 > \mathscr{G}^{\daleth}{}_2 > \mathscr{G}^{\daleth}{}_3 > \mathscr{G}^{\daleth}{}_5$	$\mathscr{G}^{\daleth}{}_1$
	q-ROPRFWHM	$\mathscr{G}^{\daleth}{}_1 > \mathscr{G}^{\daleth}{}_4 > \mathscr{G}^{\daleth}{}_2 > \mathscr{G}^{\daleth}{}_3 > \mathscr{G}^{\daleth}{}_5$	$\mathscr{G}^{\daleth}{}_1$
Joshi & Gegov [29]	CQROPRFWA	$\mathscr{G}^{\daleth}{}_1 > \mathscr{G}^{\daleth}{}_4 > \mathscr{G}^{\daleth}{}_2 > \mathscr{G}^{\daleth}{}_3 > \mathscr{G}^{\daleth}{}_5$	$\mathscr{G}^{\daleth}{}_1$
	CQROPRFWG	$\mathscr{G}^{\daleth}{}_1 > \mathscr{G}^{\daleth}{}_4 > \mathscr{G}^{\daleth}{}_2 > \mathscr{G}^{\daleth}{}_3 > \mathscr{G}^{\daleth}{}_5$	$\mathscr{G}^{\daleth}{}_1$

6. Conclusions

Material selection is a crucial and complex process in engineering that involves choosing the best material for a given application. MCDM is an essential tool that helps engineers make informed decisions about material selection by considering multiple factors. The soft-max function is an extension of the logistic function that has been used in various stochastic classification approaches, including neural nets and linear differential analysis. Researchers have developed soft-max-based fuzzy AOs for q-ROPRFS and proposed an MCDM approach based on these AOs. By applying this approach to a practical problem using q-rung orthopair fuzzy data and conducting a numerical example to validate the suggested procedures, the researchers have shown the effectiveness of their proposed MCDM approach. In summary, the use of MCDM and soft-max-based fuzzy AOs can greatly aid the material selection process and help engineers choose the most suitable material for a given application.

The proposed approach has the potential to facilitate the development of future decision-making methodologies, such as dynamic highly classified screening, psychiatrist's investigation for COVID-19 task distribution, dynamic funding mechanisms, social networking monitoring, armed assistance order administration, and complex fuzzy dynamic decision-making.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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