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Research article

A three-way decision-making technique based on Pythagorean double hierarchy linguistic term sets for selecting logistic service provider and sustainable transportation investments

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Abstract: Finding the best transportation project and logistic service provider is one for the most important aspects of the development of a country. This task becomes more complicated from time to time as different criteria are involved. Hence, this paper proposes an approach to the linguistic three-way decision-making (TWDs) problem for selecting sustainable transportation investments and logistic service providers with unknown criteria and expert weight information. To this end, we first propose a new tool, the Pythagorean double hierarchy linguistic term sets (PyDHLTSs), which is a combination of first hierarchy linguistic term sets and second hierarchy linguistic term sets which can describe uncertainty and fuzziness more flexibly in decision-making (DM) problems. In addition, we propose some aggregation operators and basic operational laws for PyDHLTSs. A new decision-making technique for PyDHLTSs based on decision-theoretic rough sets (DTRSs) is proposed in the three-way decisions. Next, the conditional probability is computed using grey relational analysis in a PyDHLTSs environment, which improves decision-making. The loss function is computed by using the proposed aggregation operator, and the decision's results are determined by the minimum-loss principle. Finally, a real-world case study of a transportation project and logistic service provider is considered to demonstrate the efficiency of the proposed methods.

Keywords: three way decision making; linguistic variable; Pythagorean fuzzy double linguistic term set; Pythagorean fuzzy double linguistic aggregation operators
Mathematics Subject Classification: 03E72, 46S40

1. Introduction

Decision-making (DM) is an important part of human life, referring to the process of listing all the options as assessed by decision experts and then choosing the best option, which is common in our daily life. In the early stages of social development, decision experts used real numbers to provide evaluative information. As DM problems are becoming more complex, experts are unable to provide accurate real numbers to evaluate the alternatives. The ambiguities and imprecision of human judgments exposed the flaws of crisp set theory. Thus, in 1965, Zadeh [1] proposed fuzzy sets (FS) as an effective method for modeling uncertain information by introducing degrees of membership (MD). A few years after its introduction, the FS has gained a lot of attention for representing data that contains uncertainty. Later it was analyzed that the membership degree in an FS did not cover an object's uncertainty. Therefore Attonossov [2] introduced the concept of intuitionistic fuzzy set (IFS) by adding the non-membership degree (NMD) to the FS theory satisfying the condition $MD + NMD \leq 1$. Since its inception, the results of IFS and its application to DM problems have been widely studied [3–5]. But, due to the conditions in some cases, IFS cannot account for the uncertainty of real-world problems. Thus, Yager [6, 7] extended the condition of IFS and developed a new theory called Pythagorean fuzzy sets (PyFS), subject to the constraint $(MD)^2 + (NMD)^2 \leq 1$, which confirms that the theory has good processing potential for solving DM problems. Many researchers have considered PyFS and successfully applied it to DM problems. As these ideas spread, users became more familiar with the data they collected using linguistic variables, such as good, very good, bad, very bad, very very bad, etc. Linguistic term sets (LTS) can handle complex situations efficiently and successfully. Thus Zadeh [8] put forward the framework of computing with words (CWW) and explained its importance, along with various extended forms of LTSs [9–11]. Later, researchers extended the framework of LTSs to many other theories such as FS [13], IFS [12] and PyFS [14]. To model experts, views more deeply, Guo et al., [15, 16] defined a double linguistic term set (DHLTS) to easily convey appropriate data in complex expressions compared to single LTSs. DHLTS is the combination of two sets-namely the first hierarchy linguistic term (FHLT) set and the second hierarchy linguistic term (SHLT) set-allowing more flexibility to describe uncertainty and ambiguity in DM problems. Further, Li et al., [17] introduced the Hamacher aggregation operator and applied it to the DM problem.

In today's world, sustainable mobility and logistics service providers are one of the most controversial concepts in the field of transportation. Sustainable transportation can be defined as any mode of transportation that makes it possible for the movement of goods and people in ways that are socially, economically and environmentally sustainable [18]. Thus, Awasthi et al. [19] categorized common methods for assessing the sustainability of transportation decisions into eight categories, one of which is the multi-criteria decision-making (MCDM) process. It is the preferred technique for solving problems with contradictory objectives and is reliable for sustainable transportation decisions [20]. Therefore, a variety of methods have been presented to handle a sustainable transportation evaluation problem under uncertainty [21–24]. Despite all the efforts to model the uncertainty of a sustainable transportation evaluation problem in MCDM methods, most of them were based on classical FS. There are few studies in the literature concerning sustainable transportation evaluation using an extension of classical fuzzy sets [25–30]. According to the above review, existing methods are helpful in solving DM problems effectively, however, to our knowledge, there is no method to calculate the weights of both experts and criteria under the hybrid study of PyFS and DHLTS. Since sustainable transport assessment problems contain many criteria and are full of uncertainties, it is necessary to study more effective mathematical methods to deal with sustainable transport assessment problems and logistics service providers with high uncertainty.

The main objectives of this work are as follows:
(a) To study the hybrid notion of DHLTSs with the Pythagorean fuzzy set and defined Pythagorean double hierarchy linguistic term sets (PyDHLTS), allowing better application flexibility in real-world scenarios as compared to DHLTSs, and also define a new score and accuracy functions.
(b) To define a series of averaging aggregation operators and basic operational laws for PyDHLTS to aggregate the data from various sources to a single source in the DM process.
(c) To investigate a three-way decision-making technique for solving multi-criteria group decision-making (MCGDM) problems with completely unknown weight information.
(d) To calculate unknown weights for experts and criteria, we propose to use entropy and distance measures.
(e) We further investigate the theoretical and practical interpretation of the proposed tool by solving numerical examples.

We will introduce some background and literature on three-way decision-making in the following sections.

Related work

In literature, various traditional decision-making techniques have been developed which only provide the ranking of the schemes, but do not provide the decision experts with specific recommendations. Three-way decision-making (TWDs) breaks through this limitation because the decision-making method conforms to people's thinking patterns. To this end, Yao [31–33] designed the TWD technique due to their excellent ability to solve DM problems. The three-way decision is a combination of DTRS and Bayesian DM techniques [34, 35], where the general set is divided into three regions, such as a positive region, boundary region and negative region, which have been effectively used to solve many classification problems. This method is applied by many scholars in several fields including work resumption [36], investment decision [37] and medical treatment [38, 39] because they are associated with human decision-making patterns. He et al. [40] consider the TWD technique for solving the hidden property evaluation of judgment debtor under the probabilistic linguistic term set. Later, Wang et al. [41] applied the three-way decision method with a priori probability tolerance dominance relation in fuzzy incomplete information systems. To represent the loss functions (LFs) more accurately, many extended structures of FS are introduced in the TWDs, including FS [42] and IFS [43]. Herbert and Yao [44] studied game theory-related LFs determination methods to construct loss function matrices. Jia et al. [45] presented an optimization problem on the relationship between the threshold value and loss function and then solved the optimization problem to achieve the threshold value. In a DM, Jia et al. [46] provided a new method for computing LFs. However, in practice, the LFs are evaluated by decision experts according to their own historical experience and knowledge and this study adopts the same method. Many scholars have studied the calculation of conditional probability, which is another critical component of TWDs. Ye et al. [42] initially used the entropy weight method to calculate attribute weights and then used weighted aggregation to calculate conditional probability. Later, researchers [48] used the maximizing deviation method to first determine attribute weights and then use a technique called order performance by similarity to ideal solutions (TOPSIS) [47] to achieve conditional probability. Wang et al. [49] calculated the conditional probability using two DM methods based on third-generation prospect theory. Liu and Yang [50] developed a decision theory rough set (DTRS) model under a double hierarchy linguistic term set and applied it to TWDs. According to literature, there is no implementation of TWDs under the hybrid concept of DHLTs and PyFSs. Therefore, the motivation of this work is to investigate the above-mentioned specific goals.

From the above-mentioned goals the main contributions and factors of this work are as follows:
(a) Gou et al. [15, 16] developed DHLTSs by considering only the membership degree, but this idea has some limitations due to the lack of non-membership degrees. Thus we generalized DHLTSs by adding the non-membership degree and defined Pythagorean double hierarchy linguistic term sets (PyDHLTSs). They are adaptable tools that allow decision experts to provide assessments in the form of PyDHLTSs.
(b) Li, Xang et al. [17] defined the LFs by using Hamacher aggregation operators by considering the DHLTSs. We extend this concept to PyDHLTSs for determining the LFs.
(c) Establish the entropy and distance measure for calculating the unknown weights vector of experts and criteria.
(d) Further, we developed the basic operational laws and the Pythagorean double hierarchy linguistic weighted averaging (PyDHLWA), Pythagorean double hierarchy linguistic ordered weighted averaging (PyDHLOWA) and Pythagorean double hierarchy linguistic hybrid averaging (PyDHLHA) operators to aggregate the LFs to an account for various decision attitudes of decision experts, which improves the DM process.
(e) Conditional probabilities are evaluated using the GRA method, taking into account the relationship between relatively positive and negative ideal solutions.
(f) We applied the proposed methodology to TWDs for sustainable transport investment and logistics service provider selection to demonstrate the impact of three-way decision making.

The summary of this article is as follows: Section 2 presents the basic concepts related to IFS, PyFS, and DHLTS. Section 3 introduces the novel notion of PyDHLTSs and score function. Section 4 includes the distance measures and aggregation operators, such as Pythagorean double hierarchy linguistic weighted averaging (PyDHLWA) operators, Pythagorean double hierarchy linguistic order weighted averaging (PyDHLOWA) and Pythagorean double hierarchy linguistic hybrid averaging (PyDHLHA) operators for PyDHLTSs. Section 5 presents the algorithm for determining the conditional probability based on the GRA method and a novel TWD model. Section 6 describes the application of the proposed method by solving a numerical example to illustrate the feasibility of the proposed method. In Section 7, we compare the proposed method with existing techniques to demonstrate the applicability of our proposed method. Section 8 concludes this article.

2. Basic concepts

In this section, we will discuss the concepts of IFS, PyFS, LIFS, LPyFS and DHLTS, as well as their basic operations, which will be used later.

Definition 1. *[2] For a non-empty set $\ddot{U}$, the intuitionistic fuzzy set (IFS) is mathematically defined as*

$$A = \left\{u, \langle \mu(u), \nu(u) \rangle \,| u \in \ddot{U} \right\},$$

where $\mu(u)$ and $\nu(u)$ represents the MD, $NMD \in [0, 1]$, respectively, such that $(\mu(u)) + (\nu(u)) \leq 1$.

Definition 2. *[6,7] For a non-empty set Ü, the Pythagorean fuzzy set (PyFS) is mathematically defined as*

$$A = \left\{u, \langle \mu(u), \nu(u) \rangle \, | u \in \ddot{U} \right\},$$

where $\mu(u)$ *and* $\nu(u)$ *represents the MD,* $NMD \in [0,1]$ *respectively, such that* $(\mu(u))^2 + (\nu(u))^2 \leq 1$.

Definition 3. *[12] Let Ü be a universal set and* $\underline{S} = \{S_t | S_0 \leq S_t \leq S_\tau, t \in [0,\tau]\}$ *a continuous linguistic term set. Then linguistic intuitionistic fuzzy set (LIFS) in Ü is mathematically defined with the form*

$$A = \left\{u, \left\langle S_\alpha(u), S_\beta(u) \right\rangle | u \in \ddot{U} \right\},$$

where $S_\alpha(u)$ *and* $S_\beta(u)$ *represents the membership and non-membership degree in the form of linguistic terms such that* $\alpha + \beta \leq \tau$ *or* $\frac{\alpha}{\tau} + \frac{\beta}{\tau} \leq 1$. *For simplicity it is denoted by* $A = \left(S_\alpha, S_\beta\right)$.

Definition 4. *[14] Consider Ü to be a universal set and* $\underline{S} = \{S_t | S_0 \leq S_t \leq S_\tau, t \in [0,\tau]\}$ *to be a continuous linguistic term set. Then linguistic Pythagorean fuzzy set (LPyFS) in Ü is mathematically defined as*

$$A = \left\{u, \left\langle S_\alpha(u), S_\beta(u) \right\rangle | u \in \ddot{U} \right\},$$

where $S_\alpha(u)$ *and* $S_\beta(u)$ *represents the membership and non-membership degree in the form of linguistic terms such that* $(\alpha)^2 + (\beta)^2 \leq \tau^2$ *or* $\left(\frac{\alpha}{\tau}\right)^2 + \left(\frac{\beta}{\tau}\right)^2 \leq 1$. *For simplicity it is denoted by* $A = \left(S_\alpha, S_\beta\right)$.

Definition 5. *[14] Let* $A_1 = \left(S_{\alpha_1}, S_{\beta_1}\right), A_2 = \left(S_{\alpha_2}, S_{\beta_2}\right)$ *be two linguistic Pythagorean fuzzy sets. Then algebraic operational laws for linguistic Pythagorean fuzzy set are as follows;*

(1) $A_1 \oplus A_2 = \left(S_{\tau\sqrt{\left(\left(\frac{\alpha_1}{\tau}\right)^2 + \left(\frac{\alpha_2}{\tau}\right)^2 - \left(\frac{\alpha_1}{\tau}\right)^2 . \left(\frac{\alpha_2}{\tau}\right)^2\right)}}, S_{\tau\left(\left(\frac{\beta_1}{\tau}\right).\left(\frac{\beta_2}{\tau}\right)\right)} \right)$,

(2) $A_1 \otimes A_2 = \left(S_{\tau\left(\left(\frac{\alpha_1}{\tau}\right).\left(\frac{\alpha_1}{\tau}\right)\right)}, S_{\tau\sqrt{\left(\left(\frac{\beta_1}{\tau}\right)^2 + \left(\frac{\beta_2}{\tau}\right)^2 - \left(\frac{\beta_1}{\tau}\right)^2 . \left(\frac{\beta_2}{\tau}\right)^2\right)}} \right)$,

(3) $\lambda A_1 = \left(S_{\tau\sqrt{1-\left(1-\left(\frac{\alpha_1}{\tau}\right)^2\right)^\lambda}}, S_{\tau\left(\frac{\beta_1}{\tau}\right)^\lambda} \right)$,

(4) $(A_1)^\lambda = \left(S_{\tau\left(\frac{\alpha_1}{\tau}\right)^\lambda}, S_{\tau\sqrt{1-\left(1-\left(\frac{\beta_1}{\tau}\right)^2\right)^\lambda}} \right)$.

Definition 6. *[15] Let* $\underline{S} = \{S_\alpha | \alpha = -\tau, ..., -1, 0, 1, ..., \tau\}$ *be the first hierarchy linguistic term (FHLT) and* $\underline{O} = \{O_k | k = -\delta, ..., -1, 0, 1, ..., \delta\}$ *be the second hierarchy linguistic term (SHLT) sets, then the structure*

$$\underline{S}_{\underline{o}} = \{S_{\alpha\langle o_k \rangle} | \alpha = -\tau, ..., -1, 0, 1, ..., \tau; k = -\delta, ..., -1, 0, 1, ..., \delta\}$$

is said to be double hierarchy linguistic term sets (DHLTSs), where S_α *is the first hierarchy and* o_k *represents the second hierarchy linguistic terms respectively.*

3. Pythagorean double hierarchy linguistic term sets

In this section, we will develop the hybrid notion of PyFS and DHLTS to obtain the notion of Pythagorean fuzzy double hierarchy linguistic term set (PyDHLTSs), as well as initiate the new score functions and present its basic operations in detail.

Definition 7. *Let* $\underline{A}=\left\{\left\langle S_\alpha, S_\beta\right\rangle | \alpha,\beta=0,1,...,\tau\right\}$ *be the first hierarchy linguistic term set and* $\underline{B}=\{\langle O_k, O_l\rangle | k,l=0,1,...,\delta\}$ *be the second hierarchy linguistic term set, then the structure*

$$A_B=\left\{\left\langle S_{\alpha\langle O_k\rangle}, S_{\beta\langle O_l\rangle}\right\rangle | \alpha,\beta=0,1,...,\tau; k,l=0,1,...,\delta\right\}$$

is said to be Pythagorean double hierarchy linguistic term sets (PyDHLTSs) where S_α, S_β *represents the membership and non-membership degree of the first hierarchy linguistic term sets and* O_k, O_l *is the membership and non-membership degree of the second hierarchy linguistic term sets, such that* $\alpha^2+\beta^2\leq\tau^2$ *and* $k^2+l^2\leq\delta^2$ *or* $\left(\frac{\alpha}{\tau}\right)^2+\left(\frac{\beta}{\tau}\right)^2\leq 1$ *and* $\left(\frac{k}{\delta}\right)^2+\left(\frac{l}{\delta}\right)^2\leq 1$. *Simply, it is represented by* $\left(S_{\alpha\langle O_k\rangle}, S_{\beta\langle O_l\rangle}\right)$.

Definition 8. *Let* $A_{B_1}=\left(S_{\alpha_1\langle O_{k_1}\rangle}, S_{\beta_1\langle O_{l_1}\rangle}\right)$, $A_{B_2}=\left(S_{\alpha_2\langle O_{k_2}\rangle}, S_{\beta_2\langle O_{l_2}\rangle}\right)$ *are two Pythagorean fuzzy double hierarchy linguistic sets. Then algebraic operational laws for Pythagorean double hierarchy linguistic set are as follows:*

(1) $$A_{B_1}\oplus A_{B_2}=\left(\begin{array}{c} S_{\tau\sqrt{\left(\left(\frac{\alpha_1}{\tau}\right)^2+\left(\frac{\alpha_2}{\tau}\right)^2-\left(\frac{\alpha_1}{\tau}\right)^2\cdot\left(\frac{\alpha_2}{\tau}\right)^2\right)}\left\langle O_{\delta\sqrt{\left(\left(\frac{k_1}{\delta}\right)^2+\left(\frac{k_2}{\delta}\right)^2-\left(\frac{k_1}{\delta}\right)^2\cdot\left(\frac{k_2}{\delta}\right)^2\right)}}\right\rangle}, \\ S_{\tau\left(\left(\frac{\beta_1}{\tau}\right)\cdot\left(\frac{\beta_2}{\tau}\right)\right)\left\langle O_{\delta\left(\left(\frac{l_1}{\delta}\right)\cdot\left(\frac{l_2}{\delta}\right)\right)}\right\rangle}\end{array}\right);$$

(2) $$A_{B_1}\otimes A_{B_2}=\left(\begin{array}{c} S_{\tau\left(\left(\frac{\alpha_1}{\tau}\right)\cdot\left(\frac{\alpha_2}{\tau}\right)\right)\left\langle O_{\delta\left(\left(\frac{k_1}{\delta}\right)\cdot\left(\frac{k_2}{\delta}\right)\right)}\right\rangle}, \\ S_{\tau\sqrt{\left(\left(\frac{\beta_1}{\tau}\right)^2+\left(\frac{\beta_2}{\tau}\right)^2-\left(\frac{\beta_1}{\tau}\right)^2\cdot\left(\frac{\beta_2}{\tau}\right)^2\right)}\left\langle O_{\delta\sqrt{\left(\left(\frac{l_1}{\delta}\right)^2+\left(\frac{l_2}{\delta}\right)^2-\left(\frac{l_1}{\delta}\right)^2\cdot\left(\frac{l_2}{\delta}\right)^2\right)}}\right\rangle}\end{array}\right);$$

(3) $$\lambda A_{B_1}=\left(S_{\tau\sqrt{1-\left(1-\left(\frac{\alpha_1}{\tau}\right)^2\right)^\lambda}\left\langle O_{\delta\sqrt{1-\left(1-\left(\frac{k_1}{\delta}\right)^2\right)^\lambda}}\right\rangle}, S_{\tau\left(\frac{\beta_1}{\tau}\right)^\lambda\left\langle O_{\delta\left(\frac{l_1}{\delta}\right)^\lambda}\right\rangle}\right);$$

(4) $$(A_{B_1})^\lambda=\left(S_{\tau\left(\frac{\alpha_1}{\tau}\right)^\lambda\left\langle O_{\delta\left(\frac{k_1}{\delta}\right)^\lambda}\right\rangle}, S_{\tau\sqrt{1-\left(1-\left(\frac{\beta_1}{\tau}\right)^2\right)^\lambda}\left\langle O_{\delta\sqrt{1-\left(1-\left(\frac{l_1}{\delta}\right)^2\right)^\lambda}}\right\rangle}\right).$$

Definition 9. *Let* $A_{B_i}=\left(S_{\alpha_i\langle O_{k_i}\rangle}, S_{\beta_i\langle O_{l_i}\rangle}\right)$ *be a Pythagorean fuzzy double hierarchy linguistic set. Then the score* (S_C) *and accuracy* (A_C) *fuctions of PyDHLTS are defined by*

$$\begin{aligned} S_C &= \left(\left(\frac{\alpha_i}{\tau}\right)^2-\left(\frac{\beta_i}{\tau}\right)^2+\left(\frac{k_i}{\delta}\right)^2-\left(\frac{l_i}{\delta}\right)^2\right)/2\in[-1,1] \\ A_C &= \left(\left(\frac{\alpha_i}{\tau}\right)^2+\left(\frac{\beta_i}{\tau}\right)^2+\left(\frac{k_i}{\tau}\right)^2+\left(\frac{l_i}{\tau}\right)^2/2\right)\in[0,1]. \end{aligned}$$

4. Pythagorean double hierarchy linguistic term averaging operators

This section develops a list of averaging aggregation operators such as PyDHLWA, PyDHLOWA and PyDHLHA for Pythagorean double hierarchy linguistic term set (PyDHLTSs) and also describes its basic properties as follows:

Definition 10. *Let $A_{B_i} = \left(S_{\alpha_i\langle O_{k_i}\rangle}, S_{\beta_i\langle O_{l_i}\rangle}\right)(i \in \mathbb{N})$ be the collection of PyFDHLVs and $\varpi = (\varpi_1, \varpi_2, ..., \varpi_n)^T$ represent the weight vectors of a given collection restricted to $\varpi_i > 0, \sum_{i=1}^{n} \varpi_i = 1$. Then, according to definition 8, of operational laws (1) and (3), the Pythagorean double hierarchy linguistic weighted averaging (PyDHLWA) operator is defined as:*

$$PyDHLWA(A_{B_1}, A_{B_2}, ..., A_{B_n}) = \varpi_1 A_{B_1} \oplus \varpi_2 A_{B_2}, ..., \oplus \varpi_n A_{B_n}$$

$$= \left[\begin{array}{c} S_{\tau\sqrt{1-\prod_{i=1}^{n}\left(1-\left(\frac{\alpha_i}{\tau}\right)^2\right)^{\varpi_i}}\left\langle O_{\delta\sqrt{1-\prod_{i=1}^{n}\left(1-\left(\frac{k_i}{\delta}\right)^2\right)^{\varpi_i}}}\right\rangle}, \\ S_{\tau\prod_{i=1}^{n}\left(\frac{\beta_i}{\tau}\right)^{\varpi_i}\left\langle O_{\delta\prod_{i=1}^{n}\left(\frac{l_i}{\delta}\right)^{\varpi_i}}\right\rangle} \end{array}\right].$$

Theorem 1. Let $A_{B_i} = \left(S_{\alpha_i\langle O_{k_i}\rangle}, S_{\beta_i\langle O_{l_i}\rangle}\right)(i \in \mathbb{N})$ be the collection of PyFDHLVs and $\varpi = (\varpi_1, \varpi_2, ..., \varpi_n)^T$ represent the weight vectors of a given collection restricted to $\varpi_i > 0, \sum_{i=1}^{n} \varpi_i = 1$. Then, the basic properties of *PyDHLWA* are as follows:

(1) (Idempotency): Suppose $A_{B_i} = \left(S_{\alpha_i\langle O_{k_i}\rangle}, S_{\beta_i\langle O_{l_i}\rangle}\right) = \left(S_{\alpha\langle O_k\rangle}, S_{\beta\langle O_l\rangle}\right), \vee i \in \mathbb{N}$ then

$$PyDHLWA(A_{B_1}, A_{B_2}, ..., A_{B_n}) = \left(S_{\alpha\langle O_k\rangle}, S_{\beta\langle O_l\rangle}\right).$$

(2) (Monotonicity): Suppose $C_{D_i} = \left(S^*_{\alpha_i\left\langle O^*_{k_i}\right\rangle}, S^*_{\beta_i\left\langle O^*_{l_i}\right\rangle}\right)$ $(i \in \mathbb{N})$ be another collection of PyFDHLVs such that $S^*_{\alpha_i} \geq S_{\alpha_i}$ $S^*_{\beta_i} \leq S_{\beta_i}$ and $O^*_{k_i} \geq O_{k_i}, O^*_{l_i} \leq O_{l_i}$, then

$$PyDHLWA(A_{B_1}, A_{B_2}, ..., A_{B_n}) \leq PyDHLWA(C_{D_1}, C_{D_2}, ..., C_{D_n}).$$

(3) (Boundedness): Suppose $A^-_{B^-_i} = \left(\min_i S_{\alpha_i\langle \min_i O_{k_i}\rangle}, \max_i S_{\beta_i\langle \max_i O_{l_i}\rangle}\right)$ and $A^+_{B^+_i} = \left((\max_i S_{\alpha_i\langle \max_i O_{k_i}\rangle}, \min_i S_{\beta_i\langle \min_i O_{l_i}\rangle}\right)$ are two collection PyFDHLVs, then

$$A^-_{B^-_i} \leq PyDHLWA(A_{B_1}, A_{B_2}, ..., A_{B_n}) \leq A^+_{B^+_i}.$$

Proof. Straight forward. □

Definition 11. *Let $A_{B_i} = \left(S_{\alpha_i\langle O_{k_i}\rangle}, S_{\beta_i\langle O_{l_i}\rangle}\right)(i \in \mathbb{N})$ be the collection of PyFDHLVs and $\varpi = (\varpi_1, \varpi_2, ..., \varpi_n)^T$ represent the weight vectors of a given collection restricted to $\varpi_i > 0, \sum_{i=1}^{n} \varpi_i = 1$.*

Then, according to Definition 8, of operational laws (1) and (3), the Pythagorean double hierarchy linguistic order weighted averaging (PyDHLOWA) operator is defined as:

$$PyDHLOWA(A_{B_1}, A_{B_2}, ..., A_{B_n}) = \varpi_1 A_{\delta B_1} \oplus \varpi_2 A_{\delta B_2}, ..., \oplus \varpi_n A_{\delta B_n}$$

$$= \begin{bmatrix} S_{\tau\sqrt{1-\prod_{i=1}^{n}\left(1-\left(\frac{\alpha_{\delta_i}}{\tau}\right)^2\right)^{\varpi_i}}\left\langle O_{\delta\sqrt{1-\prod_{i=1}^{n}\left(1-\left(\frac{k_{\delta_i}}{\delta}\right)^2\right)^{\varpi_i}}}\right\rangle}, \\ S_{\tau\prod_{i=1}^{n}\left(\frac{\beta_{\delta_i}}{\tau}\right)^{\varpi_i}\left\langle O_{\delta\prod_{i=1}^{n}\left(\frac{l_{\delta_i}}{\delta}\right)^{\varpi_i}}\right\rangle} \end{bmatrix},$$

where $A_{\delta B_i}$ is the largest permuation from the given collecton.

Theorem 2. Let $A_{B_i} = \left(S_{\alpha_i\langle O_{k_i}\rangle}, S_{\beta_i\langle O_{l_i}\rangle}\right)(i \in \mathbb{N})$ be the collection of PyFDHLVs and $\varpi = (\varpi_1, \varpi_2, ..., \varpi_n)^T$ represent the weight vectors of a given collection restricted to $\varpi_i > 0, \sum_{i=1}^{n} \varpi_i = 1$. Then, the basic properties of *PyDHLOWA* are as follows:

(1) (Idempotency): Suppose $A_{B_i} = \left(S_{\alpha_i\langle O_{k_i}\rangle}, S_{\beta_i\langle O_{l_i}\rangle}\right) = \left(S_{\alpha\langle O_k\rangle}, S_{\beta\langle O_l\rangle}\right), \vee i \in \mathbb{N}$ then

$$PyDHLOWA(A_{B_1}, A_{B_2}, ..., A_{B_n}) = \left(S_{\alpha\langle O_k\rangle}, S_{\beta\langle O_l\rangle}\right).$$

(2) (Monotonicity): Suppose $C_{D_i} = \left(S^*_{\alpha_i\left\langle O^*_{k_i}\right\rangle}, S^*_{\beta_i\left\langle O^*_{l_i}\right\rangle}\right)$ $(i \in \mathbb{N})$ be another collection of PyFDHLVs such that $S^*_{\alpha_i} \geq S_{\alpha_i}$ $S^*_{\beta_i} \leq S_{\beta_i}$ and $O^*_{k_i} \geq O_{k_i}, O^*_{l_i} \leq O_{l_i}$, then

$$PyDHLOWA(A_{B_1}, A_{B_2}, ..., A_{B_n}) \leq PyDHLOWA(C_{D_1}, C_{D_2}, ..., C_{D_n}).$$

(3) (Boundedness): Suppose $A^-_{B^-_i} = \left(\min_i S_{\alpha_i\langle \min_i O_{k_i}\rangle}, \max_i S_{\beta_i\langle \max_i O_{l_i}\rangle}\right)$ and

$A^+_{B^+_i} = \left((\max_i S_{\alpha_i\langle \max_i O_{k_i}\rangle}, \min_i S_{\beta_i\langle \min_i O_{l_i}\rangle}\right)$ are two collection PyFDHLVs,

then

$$A^-_{B^-_i} \leq PyDHLOWA(A_{B_1}, A_{B_2}, ..., A_{B_n}) \leq A^+_{B^+_i}.$$

Proof. Straight forward. □

Definition 12. *Let* $A_{B_i} = \left(S_{\alpha_i\langle O_{k_i}\rangle}, S_{\beta_i\langle O_{l_i}\rangle}\right)(i \in \mathbb{N})$ *be the collection of PyFDHLVs with weight vectors* $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ *such that* $\omega_i > 0, \sum_{i=1}^{n} \omega_i = 1$, *and let* $\varpi = (\varpi_1, \varpi_2, ..., \varpi_n)^T$ *represent the associated weight vectors of a given collection restricted to* $\varpi_i > 0, \sum_{i=1}^{n} \varpi_i = 1$. *Then, according to Definition 8, of operational laws (1) and (3), the Pythagorean double hierarchy linguistic hybrid averaging (PyDHLHA) operator is defined as:*

$$PFDHLHA(A_{B_1}, A_{B_2}, ..., A_{B_n}) = \varpi_1 \ddot{A}_{\delta B_1} \oplus \varpi_2 \ddot{A}_{\delta B_2}, ..., \oplus \varpi_n \ddot{A}_{\delta B_n}$$

$$= \begin{bmatrix} S_{\tau\sqrt{1-\prod_{i=1}^{n}\left(1-\left(\frac{\ddot{\alpha}_{\delta_i}}{\tau}\right)^2\right)^{\varpi_i}}\left\langle O_{\delta\sqrt{1-\prod_{i=1}^{n}\left(1-\left(\frac{\ddot{k}_{\delta_i}}{\delta}\right)^2\right)^{\varpi_i}}}\right\rangle}, \\ S_{\tau\prod_{i=1}^{n}\left(\frac{\ddot{\beta}_{\delta i}}{\tau}\right)^{\varpi_i}\left\langle O_{\delta\prod_{i=1}^{n}\left(\frac{\ddot{l}_{\delta_i}}{\delta}\right)^{\varpi_i}}\right\rangle} \end{bmatrix}.$$

where $\ddot{A}_{\delta B_i} = n\varpi_i A_{B_1}$ represents the largest value of permutation from the given collection and n denotes the balancing coefficient.

Theorem 3. Let $A_{B_i} = \left(S_{\alpha_i\langle O_{k_i}\rangle}, S_{\beta_i\langle O_{l_i}\rangle}\right)(i \in \mathbb{N})$ be the collection of PyFDHLVs with weight vectors $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ such that $\omega_i > 0, \sum_{i=1}^{n} \omega_i = 1$, and let $\varpi = (\varpi_1, \varpi_2, ..., \varpi_n)^T$ represent the associated weight vectors of a given collection restricted to $\varpi_i > 0, \sum_{i=1}^{n} \varpi_i = 1$. Then, the basic properties of *PyDHLHA* are as follows:

(1) (Idempotency): Suppose $A_{B_i} = \left(S_{\alpha_i\langle O_{k_i}\rangle}, S_{\beta_i\langle O_{l_i}\rangle}\right) = \left(S_{\alpha\langle O_k\rangle}, S_{\beta\langle O_l\rangle}\right), \vee i \in \mathbb{N}$ then

$$PyDHLHA(A_{B_1}, A_{B_2}, ..., A_{B_n}) = \left(S_{\alpha\langle O_k\rangle}, S_{\beta\langle O_l\rangle}\right).$$

(2) (Monotonicity): Suppose $C_{D_i} = \left(S^*_{\alpha_i\left\langle O^*_{k_i}\right\rangle}, S^*_{\beta_i\left\langle O^*_{l_i}\right\rangle}\right)(i \in \mathbb{N})$ be another collection of PyFDHLVs such that $S^*_{\alpha_i} \geq S_{\alpha_i}$ and $S^*_{\beta_i} \leq S_{\beta_i}$, $O^*_{k_i} \geq O_{k_i}$, $O^*_{l_i} \leq O_{l_i}$, then

$$PyDHLHA(A_{B_1}, A_{B_2}, ..., A_{B_n}) \leq PyDHLHA(C_{D_1}, C_{D_2}, ..., C_{D_n}).$$

(3) (Boundedness): Suppose $A^-_{B^-_i} = \left(\min_i S_{\alpha_i\left\langle \min_i O_{k_i}\right\rangle}, \max_i S_{\beta_i\left\langle \max_i O_{l_i}\right\rangle}\right)$ and
$A^+_{B^+_i} = \left((\max_i S_{\alpha_i\left\langle \max_i O_{k_i}\right\rangle}, \min_i S_{\beta_i\left\langle \min_i O_{l_i}\right\rangle}\right)$ are two collection PyFDHLVs, then

$$A^-_{B^-_i} \leq PyDHLHA(A_{B_1}, A_{B_2}, ..., A_{B_n}) \leq A^+_{B^+_i}.$$

Proof. Straight forward. □

Definition 13. *Suppose* $A_{B_1} = \left(S_{\alpha_1\langle O_{k_1}\rangle}, S_{\beta_1\langle O_{l_1}\rangle}\right)$, $A_{B_2} = \left(S_{\alpha_2\langle O_{k_2}\rangle}, S_{\beta_2\langle O_{l_2}\rangle}\right)$ *are two Pythagorean double hierarchy linguistic sets. Then, the Hamming distance between any two PyDHLTSs* A_{B_1} *and* A_{B_2} *for any* $\Delta > 0 \,(\in \mathbb{R})$ *is defined as follows:*

$$d(A_{B_1}, A_{B_2}) = \left(\frac{1}{4n}\left[\begin{array}{c} \left|S_{\left(\frac{\alpha_1}{\tau}\right)^2} - S_{\left(\frac{\alpha_2}{\tau}\right)^2}\right|^\Delta + \left|O_{\left(\frac{k_1}{\delta}\right)^2} - O_{\left(\frac{k_2}{\delta}\right)^2}\right|^\Delta + \\ \left|S_{\left(\frac{\beta_1}{\tau}\right)^2} - S_{\left(\frac{\beta_2}{\tau}\right)^2}\right|^\Delta + \left|O_{\left(\frac{l_1}{\delta}\right)^2} - O_{\left(\frac{l_2}{\delta}\right)^2}\right|^\Delta \end{array}\right]\right)^{\frac{1}{\Delta}}. \tag{4.1}$$

5. Conditional probablity based on GRA method

The TWD approach depends on two main things, namely loss function (LF) and conditional probability. To find the conditional probability first we defined PyDHLTSs and let $U = \{u_1, u_2, ..., u_m\}$ be a set of alternatives and $C = \{\Psi_1, \Psi_2, ..., \Psi_n\}$ be a criteria in the form of PyDHLTSs with unknown weights vectors $\omega_j = (\omega_1, \omega_2, ..., \omega_n)$ subject to ω_i, $\sum_{j=1}^{n} \omega_i = 1$. To provide an evaluation report for each alternative based on criteria there exists an e number of decision experts represented by a set $E_p = \{E_1, E_2, ..., E_e\}$ with unknown weight vector $\varpi_p = (\varpi_1, \varpi_2, ..., \varpi_e)$ subject to ϖ_p, $\sum_{p=1}^{e} \varpi_p = 1$.

Then the expert evaluation matrix in the form of PyDHLTS is represented by $M^p = [A^p_{B_{ij}}]_{m\times n}$. The DM process is categorized into the following four phases.

Phase I.

In this phase, first construct the decision experts matrix in the form of PyDHLTSs with unknown weights for each decision expert. Thus, when the weights of decision experts are unknown it is very difficult to make an accurate decision. Hence it is important to evaluate the weights of each decision expert. For this, we first construct the ideal opinion matrix, right ideal ideal opinion matrix and left ideal opinion matrix, represented by IO, RIO and LIO respectively. Then we determine the distance measure denoted by $dIO_i^{(p)}, dRIO_i^{(p)}$ and $dLIO_i^{(p)}$ from decision experts matrix M^p to IO, RIO and LIO respectively. Further, we find the closeness index and at last calculate the weights of each decision expert.

The stepwise details are as follows:

(a) Construct decision experts matrix in the form of PyDHLTSs as follows.

$$M^p = [A_{B_{ij}}]_{m\times n} = \begin{matrix} u_1 \\ \\ u_2 \\ \vdots \\ u_m \end{matrix} \begin{pmatrix} \Psi_1 & \Psi_2 & ... & \Psi_n \\ A^p_{B_{11}} & A^p_{B_{12}} & ... & A^p_{B_{1n}} \\ A^p_{B_{21}} & A^p_{B_{22}} & ... & A^p_{B_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ A^p_{B_{m1}} & A^p_{B_{m2}} & ... & A^p_{B_{mn}} \end{pmatrix}$$

where $A^p_{B_{ij}} = \left(S^p_{\alpha_{ij}\left\langle O^p_{k_{ij}} \right\rangle}, S^p_{\beta_{ij}\left\langle O^p_{l_{ij}} \right\rangle} \right), (i = 1, 2, ...m)\,(j = 1, 2, ...n)\,(p = 1, 2, ...e)$.

(b) Construct the ideal opinion matrix IO that is closer to each decision expert matrix.

$$IO = \begin{pmatrix} IO_{12} & IO_{12} & ... & IO_{1n} \\ IO_{21} & IO_{22} & ... & IO_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ IO_{m1} & IO_{m2} & ... & IO_{mn} \end{pmatrix}$$

where

$$\begin{aligned} IO_{ij} &= \sum_{p=1}^{e} 1/e \left(S^p_{\alpha_{ij}\left\langle O^p_{k_{ij}} \right\rangle}, S^p_{\beta_{ij}\left\langle O^p_{l_{ij}} \right\rangle} \right) \\ &= \left\{ \begin{array}{c} S_{\tau\left(\sqrt{1-\prod_{p=1}^{e}\left(1-\left(\frac{\alpha_{ij}}{\tau}\right)^2\right)^{1/e}}\right)\left\langle O_{\delta\sqrt{1-\prod_{p=1}^{e}\left(1-\left(\frac{k_{ij}}{\delta}\right)^2\right)^{1/e}}} \right\rangle}, \\ S_{\tau\prod_{\iota=1}^{e}\left(\frac{\beta_{ij}}{\tau}\right)^{1/e}\left\langle O_{\delta\prod_{\iota=1}^{e}\left(\frac{l_{ij}}{\delta}\right)^{1/e}} \right\rangle} \end{array} \right\}. \end{aligned} \tag{5.1}$$

(c) Evaluate the right ideal RIO and left ideal LIO opinion matrices by using Eqs (5.2) and (5.3) as follows:

$$RIO = \begin{pmatrix} RIO_{12} & RIO_{12} & ... & RIO_{1n} \\ RIO_{21} & RIO_{22} & ... & RIO_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ RIO_{m1} & RIO_{m2} & ... & RIO_{mn} \end{pmatrix}$$

and

$$LIO=\begin{pmatrix} LIO_{12} & \overline{LO}_{12} & ... & LIO_{1n} \\ LIO_{21} & \overline{LO}_{22} & ... & LIO_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ LIO_{m1} & LIO_{m2} & ... & LIO_{mn} \end{pmatrix}$$

whereas

$$RIO_{ij}=\left\{\max_{p} Sc\left(S^{p}_{\alpha_{ij}\left\langle O^{p}_{k_{ij}}\right\rangle}, S^{p}_{\beta_{ij}\left\langle O^{p}_{l_{ij}}\right\rangle}\right)\right\}, \tag{5.2}$$

and

$$LIO_{ij}=\left\{\min_{p} Sc\left(S^{p}_{\alpha_{ij}\left\langle O^{p}_{k_{ij}}\right\rangle}, S^{p}_{\beta_{ij}\left\langle O^{p}_{l_{ij}}\right\rangle}\right)\right\}. \tag{5.3}$$

(d) Calculate the distance measure $dIO_i^{(p)}, dRIO_i^{(p)}$ and $dLIO_i^{(p)}$ from decision experts matrix M^p from IO, RIO and LIO by Eq (4.1).

(e) Find the closeness index CI_p by using the [59].

$$CI_p=\frac{\sum_{i=1}^{m} dRIO_i^{(p)}+\sum_{i=1}^{m} dLIO_i^{(p)}}{\sum_{i=1}^{m} dIO_i^{(p)}+\sum_{i=1}^{m} dRIO_i^{(p)}+\sum_{i=1}^{m} dLIO_i^{(p)}}. \tag{5.4}$$

(f) Evaluate the experts' weight as follows:

$$\varpi_p=\frac{CI_p}{\sum_{p=1}^{e} CI_p} \tag{5.5}$$

Phase II.

(a) Aggregate all the expert matrices M^p to single matrix M by applying Pythagorean double hierarchy linguistic weighted averaging operators.

$$M=[A_{B_{ij}}]_{m\times n}=\begin{matrix} u_1 \\ \\ u_2 \\ \vdots \\ u_m \end{matrix}\begin{pmatrix} \Psi_1 & \Psi_2 & ... & \Psi_n \\ A_{B_{11}} & A_{B_{12}} & ... & A_{B_{1n}} \\ A_{B_{21}} & A_{B_{22}} & ... & A_{B_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ A_{B_{m1}} & A_{B_{m2}} & ... & A_{B_{mn}} \end{pmatrix}$$

where $i=1,2,...,m$ and $j=1,2,...,n$.

(b) To find the criteria weights ω_j first we evaluate the score of aggregated matrix then apply Renyi entropy (E_j) [51] to find the entropy measure as follows:

$$E_j=\frac{1}{1-\varphi}\log\sum_{i=1}^{m} Sc\left\langle A_{B_{ij}}\right\rangle^{\varphi}. \tag{5.6}$$

The criteria weights are evaluated as.

$$\omega_j=\frac{\left(1-E_j\right)}{\sum_{j=1}^{m}\left(1-E_j\right)}. \tag{5.7}$$

Phase III.
CONDITIONAL PROBABILITY:
(a) Calculate the positive ideal solution (PIS) and negative ideal solution (NIS) of PyDHLTSs, i.e. $U^+ = (u_1^+, u_2^+, ..., u_n^+)$ and $U^- = (u_1^-, u_2^-, ..., u_n^-)$ by Eqs (5.8) and (5.9).

$$u_j^+ = \max_i Sc\left(S_{\alpha_{ij}\langle O_{k_{ij}}\rangle}, S_{\beta_{ij}\langle O_{l_{ij}}\rangle}\right), \tag{5.8}$$

and

$$u_j^- = \min_i Sc\left(S_{\alpha_{ij}\langle O_{k_{ij}}\rangle}, S_{\beta_{ij}\langle O_{l_{ij}}\rangle}\right) \tag{5.9}$$

where $(j = 1, ..., n)$. If the TWDs, PIS and NIS are added together, they are equal to the set of states, A and A^c.
(b) Calculate the grey relational coefficient (GRC) on the jth criterion among u_i and the $PIS\ U^+$, $PIS\ U^-$ by Eqs (5.10)–(5.14) are as follows:

$$g_{ij}^+ = \frac{\min_i\left(\min_j\right) d_{ij}^+ + \vartheta \max_i\left(\max_j\right) d_{ij}^+}{d_{ij}^+ + \vartheta \max_i\left(\max_j\right) d_{ij}^+} \tag{5.10}$$

and

$$G_i^+ = \sum_{j=1}^{m} \omega_j g_{ij}^+. \tag{5.11}$$

Similarly

$$g_{ij}^- = \frac{\min_i\left(\min_j\right) d_{ij}^- + \vartheta \max_i\left(\max_j\right) d_{ij}^-}{d_{ij}^- + \vartheta \max_i\left(\max_j\right) d_{ij}^-} \tag{5.12}$$

and

$$G_i^- = \sum_{j=1}^{m} \omega_j g_{ij}^- \tag{5.13}$$

where $d_{ij}^+ = d\left(u_{ij}, u_j^+\right)$, $d_{ij}^- = d\left(u_{ij}, u_j^-\right)$ and $\vartheta = 0.5$ $(i = 1, 2, ..., m)$ $(j = 1, 2, ..., n)$.
(c) The relative rational degree (RRD) is denoted by F_i.

$$F_i = \frac{G_i^+}{G_i^+ + G_i^-}. \tag{5.14}$$

(d) Where F_i is considered to be the conditional probability of an object lies in state A, that is

$$P_r(A/x_i) = F_i \tag{5.15}$$

such that $0 \le P_r(A/x_i) \le 1$.
Phase IV.
DECISION MAKING based on DHLDTRS Model with Pythagorean fuzzy set:

As from the definition of PyDHLTS, it is a combination of a two sets, such as the first hierarchy and second hierarchy linguistic term sets, which can describe uncertainty and ambiguity more flexibly than a single term set. Here, we address the loss functions in TWDs by using Pythagorean double hierarchy linguistic numbers (PyDHLTNs) as well as how to build a new DTRS model for PyDHLTNs. This model consists of two states $\{A, A^c\}$ which express whether an element belongs to A or not, with regard to three actions like $\left\{a_p, a_b, a_n\right\}$. Where a_p, a_b and a_n shows the action which is applied for determining the objects u_i, that is, a_p indicates u_i belong to positive region $POS(A)$, a_b indicates u_i belong to boundary region $BND(A)$ and a_n shows u_i is in negative region $NEG(A)$ respectively. The overall situation of an object is classified by the states of a set, while the judgment is represented by the action. Here we determined the loss function for PyDHLTSs, which are given in Table 1.

Table 1. Loss function.

U	$A\,(P)$	$A^c\,(N)$
a_p	$\hbar_{\rho_{pp}} = \left\langle S_{\alpha_{pp}\langle O_{k_{pp}}\rangle}, S_{\beta_{pp}\langle O_{\iota_{pp}}\rangle}\right\rangle$	$\hbar_{\rho_{pn}} = \left\langle S_{\alpha_{pn}\langle O_{k_{pn}}\rangle}, S_{\beta_{pn}\langle O_{\iota_{pn}}\rangle}\right\rangle$
a_b	$\hbar_{\rho_{bp}} = \left\langle S_{\alpha_{bp}\langle O_{k_{bp}}\rangle}, S_{\beta_{bp}\langle O_{\iota_{bp}}\rangle}\right\rangle$	$\hbar_{\rho_{bn}} = \left\langle S_{\alpha_{bn}\langle O_{k_{bn}}\rangle}, S_{\beta_{bn}\langle O_{\iota_{bn}}\rangle}\right\rangle$
a_n	$\hbar_{\rho_{np}} = \left\langle S_{\alpha_{np}\langle O_{k_{np}}\rangle}, S_{\beta_{np}\langle O_{\iota_{np}}\rangle}\right\rangle$	$\hbar_{\rho_{nn}} = \left\langle S_{\alpha_{nn}\langle O_{k_{nn}}\rangle}, S_{\beta_{nn}\langle O_{\iota_{nn}}\rangle}\right\rangle$

From Table 1, we see that the determined LF are in the form PyDHTN, and $\hbar_{\rho_{pp}}, \hbar_{\rho_{bp}}$ and $\hbar_{\rho_{np}}$ are the loss degrees generated by taking actions of a_p, a_b and a_n for u given state A, with PyDHTN settings. Similarly, $\hbar_{\rho_{pn}}, \hbar_{\rho_{bn}}$ and $\hbar_{\rho_{nn}}$ reflect the loss degrees generated by conducting the same actions on u specific to state A^c. Hence, here $\hbar_\rho$ is non empty. According to the definition of PyDHTN and DTRSs [43, 52] the acceptable relation are as follows:

$$\hbar_{\rho_{pp}} \leq \hbar_{\rho_{bp}} < \hbar_{\rho_{np}}, \tag{5.16}$$

$$\hbar_{\rho_{nn}} \leq \hbar_{\rho_{bn}} < \hbar_{\rho_{pn}}. \tag{5.17}$$

That is, the loss degrees of incorrect decision are larger than the loss degrees of delayed decision, and both of these loss degrees are larger than the loss degrees of correct judgment.

Conditional probabilities are an important part of the Bayesian decision-making methods [34, 35].

$P_r(A/u_i)$,$P_r(A^c/u_i)$ indicates the conditional probablity belonging to A and A^c respectively, subject to $P_r(A/u_i) + P_r(A^c/u_i) = 1$. The expected loss for the corresponding action $R_r(a_\triangledown/u_i)$ where $(_\triangledown = a, b, n)$ can be computed for a given object u_i as follows:

$$R_r\left(a_p/u_i\right) = P_r(A/u_i) \odot \hbar_{\rho_{pp}} \oplus P_r(A^c/u_i) \odot \hbar_{\rho_{pn}}, \tag{5.18}$$

$$R_r(a_b/u_i) = P_r(A/u_i) \odot \hbar_{\rho_{bp}} \oplus P_r(A^c/u_i) \odot \hbar_{\rho_{bn}}, \tag{5.19}$$

$$R_r(a_n/u_i) = P_r(A/u_i) \odot \hbar_{\rho_{np}} \oplus P_r(A^c/u_i) \odot \hbar_{\rho_{nn}}. \tag{5.20}$$

The minimum loss decision laws can be derived by using the result given [31–33], which are as follows;

(1) Decide u_i belong to $POS(A)$, indicate that the action are acceptable if

$$Sc\left(R_r\left(a_p/u_i\right)\right) < Sc(R_r(a_b/u_i)) < Sc(R_r(a_n/u_i)). \tag{5.21}$$

(2) Decide u_i belong to $BND(A)$, shows the action are delayed if

$$Sc\left(R_r\left(a_b/u_i\right)\right) < Sc\left(R_r\left(a_p/u_i\right)\right) < Sc\left(R_r\left(a_n/u_i\right)\right). \tag{5.22}$$

(3) Decide u_i belong to $NEG(A)$, represents the action is rejected if

$$Sc\left(R_r\left(a_n/u_i\right)\right) < Sc\left(R_r\left(a_p/u_i\right)\right) < Sc\left(R_r\left(a_b/u_i\right)\right). \tag{5.23}$$

The graphical framework of the proposed method is given in Figure 1.

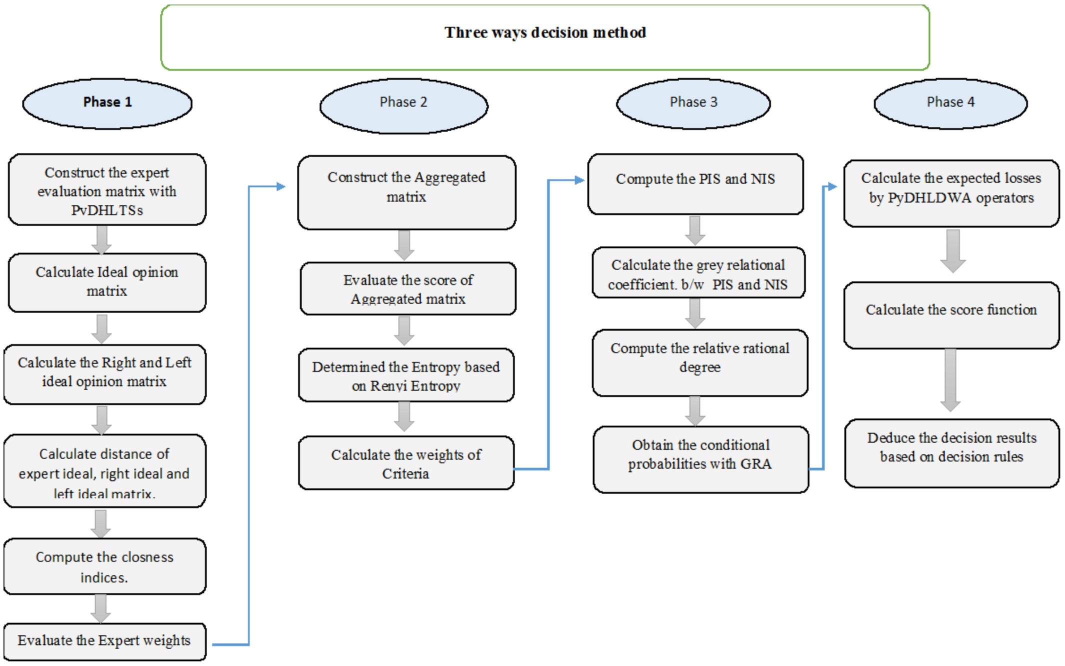

Figure 1. Graphical framework of proposed method.

6. Application of proposed method for selecting sustainable transportation

A practical DM problem concerning the selection of sustainable transportation is considered as an example in this section to validate the applicability and practicality of the developed methodology.

6.1. Case study

In this section, we apply the proposed techniques the real-world transportation problems [28] to demonstrate their effectiveness in solving the problem of sustainable transportation investment decision-making based on TWDs in the form of PyDHLTS. To address the problem, let there be three experts represented by a set $\{E_1, E_2, E_3\}$ to assess four transport investments considered as alternatives denoted by a set $U = \{u_1, u_2, u_3, u_4\}$ for detail description [58] based on four criteria represented by a set $C = \{\Psi_1, \Psi_2, \Psi_3, \Psi_4\}$.

The criteria used for evaluation are based on a comprehensive review of some of the recent studies on this subject [58] which are given in Table 2.

Table 2. Criteria evaluation of three factors in sustainability.

Economic	Ψ_1= Reduction in costs
Social	Ψ_2= Basic accessibility Ψ_3= Safety
Environmental	Ψ_4= Efficiency resources

Due to the confidentiality of the information, only limited project details are presented. Assuming that the experts' weight vectors ϖ_p and the criteria weights ω_j are totally unknown, the evaluated value of candidates while considering criteria are directly provided by judgments of decision experts. Now we apply the above problem to TWDs based on PyDHLTSS setting. The stepwise details are as follows.

Phase I.

(a) Construct the experts evaluation matrix in the form of double hierarchy Pythagorean linguistic term sets, so the linguistic term set is denoted by $S = \{s_0 = medium, s_1 = low, s_2 = sightly\ low, s_3 = very\ low, s_4 = high\ , s_5 = slightly\ high, s_6 = very\ high\}$ and $O = \{o_0 = right, o_1 = only\ right, o_2 = much, o_3 = very\ much, o_4 = little,\ o_5 = just\ little,\ o_6 = extermely\ little\}$ are defined based on the following set as follows in Tables 3–5.

Table 3. Expert matrix E_1.

U	Ψ_1	Ψ_2	Ψ_3	Ψ_4
u_1	$\langle S_{1\langle O_2\rangle}, S_{4\langle O_3\rangle}\rangle$	$\langle S_{4\langle O_2\rangle}, S_{0\langle O_3\rangle}\rangle$	$\langle S_{6\langle O_2\rangle}, S_{0\langle O_2\rangle}\rangle$	$\langle S_{1\langle O_5\rangle}, S_{4\langle O_0\rangle}\rangle$
u_2	$\langle S_{2\langle O_3\rangle}, S_{4\langle O_2\rangle}\rangle$	$\langle S_{5\langle O_3\rangle}, S_{1\langle O_2\rangle}\rangle$	$\langle S_{2\langle O_4\rangle}, S_{3\langle O_1\rangle}\rangle$	$\langle S_{3\langle O_1\rangle}, S_{3\langle O_2\rangle}\rangle$
u_3	$\langle S_{4\langle O_2\rangle}, S_{1\langle O_3\rangle}\rangle$	$\langle S_{2\langle O_4\rangle}, S_{2\langle O_1\rangle}\rangle$	$\langle S_{3\langle O_2\rangle}, S_{1\langle O_4\rangle}\rangle$	$\langle S_{4\langle O_2\rangle}, S_{1\langle O_0\rangle}\rangle$
u_4	$\langle S_{6\langle O_4\rangle}, S_{0\langle O_1\rangle}\rangle$	$\langle S_{4\langle O_5\rangle}, S_{1\langle O_0\rangle}\rangle$	$\langle S_{5\langle O_2\rangle}, S_{1\langle O_3\rangle}\rangle$	$\langle S_{0\langle O_3\rangle}, S_{4\langle O_2\rangle}\rangle$

Table 4. Expert matrix E_2.

U	Ψ_1	Ψ_2	Ψ_3	Ψ_4
u_1	$\langle S_{0\langle O_3\rangle}, S_{2\langle O_2\rangle}\rangle$	$\langle S_{1\langle O_3\rangle}, S_{0\langle O_2\rangle}\rangle$	$\langle S_{2\langle O_2\rangle}, S_{0\langle O_3\rangle}\rangle$	$\langle S_{1\langle O_4\rangle}, S_{3\langle O_0\rangle}\rangle$
u_2	$\langle S_{0\langle O_1\rangle}, S_{2\langle O_2\rangle}\rangle$	$\langle S_{1\langle O_2\rangle}, S_{4\langle O_3\rangle}\rangle$	$\langle S_{1\langle O_2\rangle}, S_{1\langle O_3\rangle}\rangle$	$\langle S_{2\langle O_1\rangle}, S_{2\langle O_4\rangle}\rangle$
u_3	$\langle S_{3\langle O_1\rangle}, S_{2\langle O_4\rangle}\rangle$	$\langle S_{2\langle O_3\rangle}, S_{2\langle O_2\rangle}\rangle$	$\langle S_{1\langle O_3\rangle}, S_{2\langle O_1\rangle}\rangle$	$\langle S_{0\langle O_1\rangle}, S_{6\langle O_1\rangle}\rangle$
u_4	$\langle S_{5\langle O_1\rangle}, S_{0\langle O_4\rangle}\rangle$	$\langle S_{1\langle O_2\rangle}, S_{2\langle O_4\rangle}\rangle$	$\langle S_{2\langle O_3\rangle}, S_{1\langle O_3\rangle}\rangle$	$\langle S_{1\langle O_3\rangle}, S_{2\langle O_1\rangle}\rangle$

Table 5. Expert matrix E_3.

U	Ψ_1	Ψ_2	Ψ_3	Ψ_4
u_1	$\langle S_{2\langle O_1\rangle}, S_{2\langle O_3\rangle}\rangle$	$\langle S_{0\langle o_1\rangle}, S_{2\langle O_1\rangle}\rangle$	$\langle S_{1\langle o_2\rangle}, S_{3\langle O_2\rangle}\rangle$	$\langle S_{0\langle o_3\rangle}, S_{2\langle O_2\rangle}\rangle$
u_2	$\langle S_{3\langle O_4\rangle}, S_{3\langle O_2\rangle}\rangle$	$\langle S_{2\langle O_3\rangle}, S_{1\langle O_2\rangle}\rangle$	$\langle S_{3\langle O_4\rangle}, S_{2\langle O_1\rangle}\rangle$	$\langle S_{3\langle O_4\rangle}, S_{1\langle O_0\rangle}\rangle$
u_3	$\langle S_{2\langle O_3\rangle}, S_{2\langle O_1\rangle}\rangle$	$\langle S_{3\langle O_2\rangle}, S_{3\langle O_2\rangle}\rangle$	$\langle S_{2\langle O_3\rangle}, S_{2\langle O_0\rangle}\rangle$	$\langle S_{2\langle O_6\rangle}, S_{2\langle O_0\rangle}\rangle$
u_4	$\langle S_{1\langle O_3\rangle}, S_{1\langle O_2\rangle}\rangle$	$\langle S_{3\langle O_4\rangle}, S_{1\langle O_2\rangle}\rangle$	$\langle S_{4\langle O_2\rangle}, S_{1\langle O_0\rangle}\rangle$	$\langle S_{3\langle O_4\rangle}, S_{3\langle O_1\rangle}\rangle$

(b) Calculate the ideal opinion matrix IO by Eq (5.1) as shown in Table 6.

Table 6. Ideal opinion matrix.

U	Ψ_1	Ψ_2	Ψ_3	Ψ_4
u_1	$\langle S_{1.31.\langle O_{2.20}\rangle}, S_{2.52\langle O_{2.62}\rangle}\rangle$	$\langle S_{2.58\langle O_{2.20}\rangle}, S_{0.00\langle O_{1.82}\rangle}\rangle$	$\langle S_{6.00\langle O_{1.99}\rangle}, S_{0.00\langle O_{2.29}\rangle}\rangle$	$\langle S_{0.82\langle O_{4.23}\rangle}, S_{2.89\langle O_{0.00}\rangle}\rangle$
u_2	$\langle S_{2.13\langle O_{3.06}\rangle}, S_{2.89\langle O_{2.00}\rangle}\rangle$	$\langle S_{3.59\langle O_{2.72}\rangle}, S_{1.59\langle O_{2.29}\rangle}\rangle$	$\langle S_{2.20\langle O_{3.55}\rangle}, S_{1.82\langle O_{1.44}\rangle}\rangle$	$\langle S_{2.72\langle O_{2.64}\rangle}, S_{1.82\langle O_{0.00}\rangle}\rangle$
u_3	$\langle S_{3.18\langle O_{2.20}\rangle}, S_{1.59\langle O_{2.29}\rangle}\rangle$	$\langle S_{2.40\langle O_{3.18}\rangle}, S_{2.29\langle O_{1.59}\rangle}\rangle$	$\langle S_{2.20\langle O_{2.72}\rangle}, S_{1.59\langle O_{0.00}\rangle}\rangle$	$\langle S_{2.75\langle O_{6.00}\rangle}, S_{2.29\langle O_{0.00}\rangle}\rangle$
u_4	$\langle S_{6.00\langle O_{3.06}\rangle}, S_{0.00\langle O_{2.00}\rangle}\rangle$	$\langle S_{3.06\langle O_{4.10}\rangle}, S_{1.26\langle O_{0.00}\rangle}\rangle$	$\langle S_{4.10\langle O_{2.40}\rangle}, S_{1.00\langle O_{0.00}\rangle}\rangle$	$\langle S_{1.90\langle O_{3.40}\rangle}, S_{2.89\langle O_{1.26}\rangle}\rangle$

(c) By Eqs (5.2) and (5.3) the right and left ideal opinion matrixs RIO, LIO are evaluated in Tables 7 and 8 as follows:

Table 7. Right ideal matrix.

U	Ψ_1	Ψ_2	Ψ_3	Ψ_4
u_1	$\langle S_{0\langle O_3\rangle}, S_{2\langle O_2\rangle}\rangle$	$\langle S_{4\langle O_2\rangle}, S_{0\langle O_3\rangle}\rangle$	$\langle S_{6\langle O_2\rangle}, S_{0\langle O_2\rangle}\rangle$	$\langle S_{1\langle O_5\rangle}, S_{4\langle O_0\rangle}\rangle$
u_2	$\langle S_{3\langle O_4\rangle}, S_{3\langle O_2\rangle}\rangle$	$\langle S_{5\langle O_3\rangle}, S_{1\langle O_2\rangle}\rangle$	$\langle S_{3\langle O_4\rangle}, S_{2\langle O_1\rangle}\rangle$	$\langle S_{3\langle O_4\rangle}, S_{1\langle O_0\rangle}\rangle$
u_3	$\langle S_{4\langle O_2\rangle}, S_{1\langle O_3\rangle}\rangle$	$\langle S_{2\langle O_4\rangle}, S_{2\langle O_1\rangle}\rangle$	$\langle S_{2\langle O_3\rangle}, S_{2\langle O_0\rangle}\rangle$	$\langle S_{2\langle O_6\rangle}, S_{2\langle O_0\rangle}\rangle$
u_4	$\langle S_{6\langle O_4\rangle}, S_{0\langle O_1\rangle}\rangle$	$\langle S_{4\langle O_5\rangle}, S_{1\langle O_0\rangle}\rangle$	$\langle S_{5\langle O_2\rangle}, S_{1\langle O_3\rangle}\rangle$	$\langle S_{3\langle O_4\rangle}, S_{3\langle O_1\rangle}\rangle$

Table 8. Left ideal matrix.

U	Ψ_1	Ψ_2	Ψ_3	Ψ_4
u_1	$\langle S_{1\langle O_2\rangle}, S_{4\langle O_3\rangle}\rangle$	$\langle S_{0\langle O_1\rangle}, S_{2\langle O_1\rangle}\rangle$	$\langle S_{1\langle O_2\rangle}, S_{3\langle O_2\rangle}\rangle$	$\langle S_{0\langle O_3\rangle}, S_{2\langle O_2\rangle}\rangle$
u_2	$\langle S_{0\langle O_1\rangle}, S_{2\langle O_2\rangle}\rangle$	$\langle S_{1\langle O_2\rangle}, S_{4\langle O_3\rangle}\rangle$	$\langle S_{1\langle O_2\rangle}, S_{1\langle O_3\rangle}\rangle$	$\langle S_{2\langle O_1\rangle}, S_{2\langle O_4\rangle}\rangle$
u_3	$\langle S_{3\langle O_1\rangle}, S_{2\langle O_4\rangle}\rangle$	$\langle S_{3\langle O_2\rangle}, S_{3\langle O_2\rangle}\rangle$	$\langle S_{3\langle O_2\rangle}, S_{1\langle O_4\rangle}\rangle$	$\langle S_{0\langle O_1\rangle}, S_{6\langle O_1\rangle}\rangle$
u_4	$\langle S_{1\langle O_3\rangle}, S_{1\langle O_2\rangle}\rangle$	$\langle S_{1\langle O_2\rangle}, S_{2\langle O_4\rangle}\rangle$	$\langle S_{2\langle O_3\rangle}, S_{1\langle O_3\rangle}\rangle$	$\langle S_{0\langle O_3\rangle}, S_{4\langle O_2\rangle}\rangle$

(d–f) Based on Eq (4.1) we determine the distance measure denoted by $dIO_i^{(p)}$, $dRIO_i^{(p)}$ and $dLIO_i^{(p)}$ and, using Eq (5.4), we find the closeness index CI_p. Then by using (5.5), the weights of experts are computed as follows:

$$\varpi_1 = .34, \varpi_2 = .31, \varpi_3 = .35.$$

Phase II.

(a) Aggregate all the expert matrices M^P to single matrix M by using PyDHLTWA operators as follows in Table 9.

Table 9. Aggregated matrix M.

U	Ψ_1	Ψ_2	Ψ_3	Ψ_4
u_1	$\left\langle S_{1.34\langle O_{2.16}\rangle}, S_{2.53\langle O_{2.65}\rangle} \right\rangle$	$\left\langle S_{2.60\langle O_{2.16}\rangle}, S_{0.00\langle O_{1.80}\rangle} \right\rangle$	$\left\langle S_{6.00\langle O_{2.00}\rangle}, S_{0.00\langle O_{2.27}\rangle} \right\rangle$	$\left\langle S_{0.81\langle O_{4.23}\rangle}, S_{2.87\langle O_{0.00}\rangle} \right\rangle$
u_2	$\left\langle S_{2.17\langle O_{3.11}\rangle}, S_{2.92\langle O_{2.00}\rangle} \right\rangle$	$\left\langle S_{3.62\langle O_{2.75}\rangle}, S_{1.54\langle O_{2.27}\rangle} \right\rangle$	$\left\langle S_{2.24\langle O_{3.59}\rangle}, S_{1.85\langle O_{1.41}\rangle} \right\rangle$	$\left\langle S_{2.75\langle O_{2.69}\rangle}, S_{1.80\langle O_{0.00}\rangle} \right\rangle$
u_3	$\left\langle S_{3.18\langle O_{2.24}\rangle}, S_{1.58\langle O_{2.23}\rangle} \right\rangle$	$\left\langle S_{2.42\langle O_{3.18}\rangle}, S_{2.30\langle O_{1.58}\rangle} \right\rangle$	$\left\langle S_{2.22\langle O_{2.72}\rangle}, S_{1.58\langle O_{0.00}\rangle} \right\rangle$	$\left\langle S_{2.78\langle O_{6.00}\rangle}, S_{2.22\langle O_{0.00}\rangle} \right\rangle$
u_4	$\left\langle S_{6.00\langle O_{3.09}\rangle}, S_{0.00\langle O_{1.96}\rangle} \right\rangle$	$\left\langle S_{3.09\langle O_{4.14}\rangle}, S_{1.24\langle O_{0.00}\rangle} \right\rangle$	$\left\langle S_{4.14\langle O_{2.38}\rangle}, S_{1.00\langle O_{0.00}\rangle} \right\rangle$	$\left\langle S_{1.93\langle O_{3.42}\rangle}, S_{2.92\langle O_{1.27}\rangle} \right\rangle$

(b) Calculate the score function of the aggregated matrix M, and then apply Eq (5.6) to determine entropy measure and calculate criteria weights by using Eq (5.7) as follows:

$$\omega_1 = .29, \omega_2 = .17, \omega_3 = .26, \omega_4 = .28.$$

Phase III.

(a) By applying Eqs (5.8) and (5.9), the PIS and NIS solutions are calculated in Table 10.

Table 10. Positive and negative ideal solution.

	Ψ_1	Ψ_2	Ψ_3	Ψ_4
U^+	$\left\langle S_{6.00\langle O_{3.09}\rangle}, S_{0.00\langle O_{1.96}\rangle} \right\rangle$	$\left\langle S_{3.62\langle O_{2.75}\rangle}, S_{0.00\langle O_{1.80}\rangle} \right\rangle$	$\left\langle S_{6.00\langle O_{2.00}\rangle}, S_{0.00\langle O_{2.27}\rangle} \right\rangle$	$\left\langle S_{2.78\langle O_{6.00}\rangle}, S_{1.80\langle O_{0.00}\rangle} \right\rangle$
U^-	$\left\langle S_{1.34\langle O_{2.16}\rangle}, S_{2.30\langle O_{1.58}\rangle} \right\rangle$	$\left\langle S_{2.42\langle O_{3.18}\rangle}, S_{0.00\langle O_{1.80}\rangle} \right\rangle$	$\left\langle S_{2.22\langle O_{2.72}\rangle}, S_{1.85\langle O_{1.41}\rangle} \right\rangle$	$\left\langle S_{0.81\langle O_{4.23}\rangle}, S_{2.92\langle O_{1.27}\rangle} \right\rangle$

(b–d) Combined, the TWDs, PIS and NIS are equivalent to state sets, i.e., A and A^c. The relative rational degree RRD of u_i, the PIS and NIS represented by F_i and conditional probability $P_r(A/u_i)$ based on GRA that the object belong to the state A, are computed by the Eqs (5.10)–(5.15) in Table 11.

Table 11. GRC and RRD and its Conditional probablity.

G_i^+	.579	.418	.596	.702
G_i^-	.690	.659	.729	.636
F_i	.456	.388	.450	.525
$P_r(A/u_i)$	.456	.388	.450	.525

Phase IV.

(a) We constructed the loss functions matrix in the form of PyDHLTNs, given in Table 12 as follows:

Table 12. Loss function.

	$A(P)$	$A^c(N)$
a_p	$\langle S_{1.34\langle O_{2.16}\rangle}, S_{2.53\langle O_{2.65}\rangle}\rangle$	$\langle S_{2.53\langle O_{2.65}\rangle}, S_{1.34\langle O_{2.16}\rangle}\rangle$
a_b	$\langle S_{1.93\langle O_{3.42}\rangle}, S_{2.92\langle O_{1.27}\rangle}\rangle$	$\langle S_{2.92\langle O_{1.27}\rangle}, S_{1.93\langle O_{3.42}\rangle}\rangle$
a_n	$\langle S_{2.17\langle O_{3.11}\rangle}, S_{2.92\langle O_{2.00}\rangle}\rangle$	$\langle S_{2.92\langle O_{2.00}\rangle}, S_{2.17\langle O_{3.11}\rangle}\rangle$

(b) Based on PyDHLTA operational laws, we derived the expected loss of each action by applying Eqs (5.18)–(5.20) as follows in Table 13.

Table 13. Expected losses.

U	u_1	u_2	u_3	u_4
$Sc\left(R_r\left(a_p/u_i\right)\right)$	.0423	.0669	.0447	.0166
$Sc\left(R_r\left(a_b/u_i\right)\right)$	.0808	.0596	.0789	.0987
$Sc\left(R_r\left(a_n/u_i\right)\right)$	.0342	.0258	.0334	.0411

(c) Determine the decision result for each object further using the decision rules (1)–(3) based on the minimum loss principle. Thus according to rules (1)–(3), the final result of each object's decision is determined as $POS(A) = \{u_4\}$, $NEG(A) = \{u_1, u_2, u_3\}$ and $BND(A) = \phi$. From the above result we analyze that u_4 are considered to be selected and u_1, u_2 and u_3 are assumed to be rejected. The graphical representation of the expected loss function is given in Figure 2.

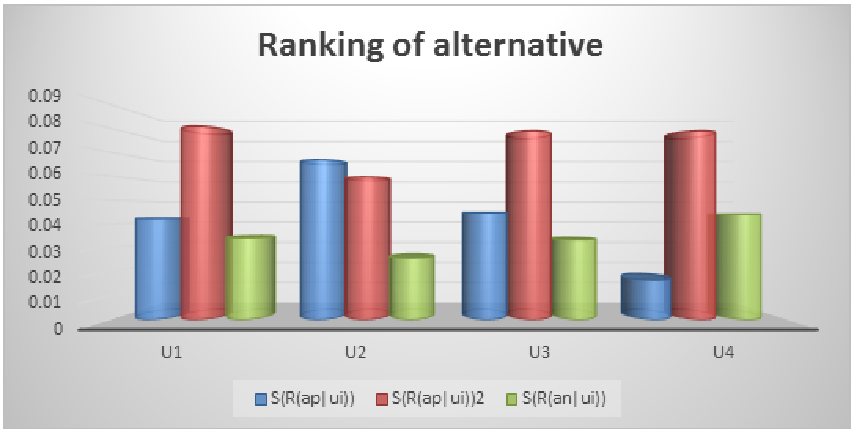

Figure 2. Graphical representation of the expected loss function.

6.2. Selection of third party logistic provider based on proposed method

Here, we provide a real-world problem on selecting a third-party logistics provider to verify the applicability and practicality of the developed method. Logistics management is essential to any business sector as it is an integral part of the supply chain. It is the process of organizing, implementing and supervising the smooth and efficient flow of goods, services and related information from the point of origin to the point of consumption, with the aim of satisfying the needs of consumers [60]. Logistics activities consist of many activities, most of which focus on transportation and storage. Transportation is moving resources from one location to another location. The role of transport in the supply chain is to focus on the movement of product from the seller to the consumer, which must be satisfied with the right time and the quality of the goods. The storage is managing the area and resources related to keeping the product and material in perfect condition before delivery to the customer with the lowest cost. Many organizations usually choose an outsourcing company to manage their logistics activities. Here, we consider a case company that offers logistics services. The case company, which was founded in 1988 and is located in Hangzhou, is one of China's top 3 producers of medical equipment and devices [61]. The company has been working to help those with hearing loss for more than 30 years, and it mostly concentrates on hearing industry devices. They outsource logistics service providers, and their shipping goods include devices, equipment, accessories, and marketing items, among other things. For this, three experts have been invited by the company to give an evaluation report for four different third party logistic providers (3PLPs) as alternatives, each with its own desired characteristics, such as u_1 = (delivery reliability), u_2 = (transportation cost), u_3 = (Rethinking risk) and u_4 = (damage rate). When choosing logistics partners, since the company's major products are expensive, fragile and small in size, it is understandable that the following criteria should be considered.

(1) Ψ_1 = Cost/price,

(2) Ψ_2 = Performance ,

(3) Ψ_3 = Quality,

(4) Ψ_4 = Service capacity.

To select the best alternative we used the proposed method to make the decision easier.
The stepwise details are as follows.

Phase I.

(a) Construct experts evaluation matrix in the form of double hierarchy Pythagorean linguistic term sets, as given in Tables 14–16.

Table 14. Expert matrix E_1.

U	Ψ_1	Ψ_2	Ψ_3	Ψ_4
u_1	$\langle S_{1\langle O_5\rangle}, S_{4\langle O_0\rangle}\rangle$	$\langle S_{1\langle O_2\rangle}, S_{4\langle O_3\rangle}\rangle$	$\langle S_{6\langle O_2\rangle}, S_{0\langle O_2\rangle}\rangle$	$\langle S_{4\langle O_2\rangle}, S_{0\langle O_3\rangle}\rangle$
u_2	$\langle S_{3\langle O_1\rangle}, S_{3\langle O_2\rangle}\rangle$	$\langle S_{2\langle O_3\rangle}, S_{4\langle O_2\rangle}\rangle$	$\langle S_{2\langle O_4\rangle}, S_{3\langle O_1\rangle}\rangle$	$\langle S_{5\langle O_3\rangle}, S_{1\langle O_2\rangle}\rangle$
u_3	$\langle S_{4\langle O_2\rangle}, S_{1\langle O_0\rangle}\rangle$	$\langle S_{4\langle O_2\rangle}, S_{1\langle O_3\rangle}\rangle$	$\langle S_{3\langle O_2\rangle}, S_{1\langle O_4\rangle}\rangle$	$\langle S_{2\langle O_4\rangle}, S_{2\langle O_1\rangle}\rangle$
u_4	$\langle S_{0\langle O_3\rangle}, S_{4\langle O_2\rangle}\rangle$	$\langle S_{6\langle O_4\rangle}, S_{0\langle O_1\rangle}\rangle$	$\langle S_{5\langle O_2\rangle}, S_{1\langle O_3\rangle}\rangle$	$\langle S_{4\langle O_5\rangle}, S_{1\langle O_0\rangle}\rangle$

Table 15. Expert matrix E_2.

U	Ψ_1	Ψ_2	Ψ_3	Ψ_4
u_1	$\langle S_{1\langle O_4\rangle}, S_{3\langle O_0\rangle}\rangle$	$\langle S_{0\langle O_3\rangle}, S_{2\langle O_2\rangle}\rangle$	$\langle S_{2\langle O_2\rangle}, S_{0\langle O_3\rangle}\rangle$	$\langle S_{1\langle O_3\rangle}, S_{0\langle O_2\rangle}\rangle$
u_2	$\langle S_{2\langle O_1\rangle}, S_{2\langle O_4\rangle}\rangle$	$\langle S_{0\langle O_1\rangle}, S_{2\langle O_2\rangle}\rangle$	$\langle S_{1\langle O_2\rangle}, S_{1\langle O_3\rangle}\rangle$	$\langle S_{1\langle O_2\rangle}, S_{4\langle O_3\rangle}\rangle$
u_3	$\langle S_{0\langle O_1\rangle}, S_{6\langle O_1\rangle}\rangle$	$\langle S_{3\langle O_1\rangle}, S_{2\langle O_4\rangle}\rangle$	$\langle S_{1\langle O_3\rangle}, S_{2\langle O_1\rangle}\rangle$	$\langle S_{2\langle O_3\rangle}, S_{2\langle O_2\rangle}\rangle$
u_4	$\langle S_{1\langle O_3\rangle}, S_{2\langle O_1\rangle}\rangle$	$\langle S_{5\langle O_1\rangle}, S_{0\langle O_4\rangle}\rangle$	$\langle S_{2\langle O_3\rangle}, S_{1\langle O_3\rangle}\rangle$	$\langle S_{1\langle O_2\rangle}, S_{2\langle O_4\rangle}\rangle$

Table 16. Expert matrix E_3.

U	Ψ_1	Ψ_2	Ψ_3	Ψ_4
u_1	$\langle S_{2\langle O_1\rangle}, S_{2\langle O_3\rangle}\rangle$	$\langle S_{0\langle o_1\rangle}, S_{2\langle O_1\rangle}\rangle$	$\langle S_{1\langle o_2\rangle}, S_{3\langle O_2\rangle}\rangle$	$\langle S_{0\langle o_3\rangle}, S_{2\langle O_2\rangle}\rangle$
u_2	$\langle S_{3\langle O_4\rangle}, S_{3\langle O_2\rangle}\rangle$	$\langle S_{2\langle O_3\rangle}, S_{1\langle O_2\rangle}\rangle$	$\langle S_{3\langle O_4\rangle}, S_{2\langle O_1\rangle}\rangle$	$\langle S_{3\langle O_4\rangle}, S_{1\langle O_0\rangle}\rangle$
u_3	$\langle S_{2\langle O_3\rangle}, S_{2\langle O_1\rangle}\rangle$	$\langle S_{3\langle O_2\rangle}, S_{3\langle O_2\rangle}\rangle$	$\langle S_{2\langle O_3\rangle}, S_{2\langle O_0\rangle}\rangle$	$\langle S_{2\langle O_6\rangle}, S_{2\langle O_0\rangle}\rangle$
u_4	$\langle S_{1\langle O_3\rangle}, S_{1\langle O_2\rangle}\rangle$	$\langle S_{3\langle O_4\rangle}, S_{1\langle O_2\rangle}\rangle$	$\langle S_{4\langle O_2\rangle}, S_{1\langle O_0\rangle}\rangle$	$\langle S_{3\langle O_4\rangle}, S_{3\langle O_1\rangle}\rangle$

Phase II.

(a) By applying the proposed PyDHLTWA aggregation operator and considering the same expert and criteria weight as we calculated in above example like $\varpi = (.34, .31, .35)^T$ aggregatate all the individual decision experts matrix as given in Table 17.

Table 17. Aggregated matrix.

U	Ψ_1	Ψ_2	Ψ_3	Ψ_4
u_1	$\left\langle S_{1.44\langle O_{4.01}\rangle}, S_{2.87\langle O_0\rangle}\right\rangle$	$\left\langle S_{0.58\langle O_{2.16}\rangle}, S_{2.53\langle O_{1.80}\rangle}\right\rangle$	$\left\langle S_{6.00\langle O_{2.00}\rangle}, S_{0.00\langle O_{2.27}\rangle}\right\rangle$	$\left\langle S_{2.60\langle O_{2.71}\rangle}, S_{0.00\langle O_{2.29}\rangle}\right\rangle$
u_2	$\left\langle S_{2.74\langle O_{2.68}\rangle}, S_{2.64\langle O_{2.47}\rangle}\right\rangle$	$\left\langle S_{1.67\langle O_{2.59}\rangle}, S_{1.98\langle O_{2.00}\rangle}\right\rangle$	$\left\langle S_{2.23\langle O_{3.58}\rangle}, S_{1.85\langle O_{1.40}\rangle}\right\rangle$	$\left\langle S_{3.79\langle O_{3.22}\rangle}, S_{1.53\langle O_{0.00}\rangle}\right\rangle$
u_3	$\left\langle S_{2.77\langle O_{2.23}\rangle}, S_{2.22\langle O_0\rangle}\right\rangle$	$\left\langle S_{3.40\langle O_{1.76}\rangle}, S_{1.82\langle O_{2.84}\rangle}\right\rangle$	$\left\langle S_{2.22\langle O_{2.71}\rangle}, S_{1.58\langle O_{0.00}\rangle}\right\rangle$	$\left\langle S_{2.00\langle O_{6.00}\rangle}, S_{2.00\langle O_{0.00}\rangle}\right\rangle$
u_4	$\left\langle S_{0.81\langle O_{3.00}\rangle}, S_{1.98\langle O_{1.61}\rangle}\right\rangle$	$\left\langle S_{6.00\langle O_{3.49}\rangle}, S_{0.00\langle O_{1.95}\rangle}\right\rangle$	$\left\langle S_{4.13\langle O_{2.37}\rangle}, S_{1.00\langle O_{0.00}\rangle}\right\rangle$	$\left\langle S_{3.09\langle O_{4.13}\rangle}, S_{1.82\langle O_{0.00}\rangle}\right\rangle$

Phase III.

(a) Utilizing Eqs (5.8) and (5.9) to calculate PIS and NIS solutions as shown in Table 18.

Table 18. Positive and negative ideal solution.

	Ψ_1	Ψ_2	Ψ_3	Ψ_4
U^+	$\left\langle S_{1.44\langle O_{4.01}\rangle}, S_{2.87\langle O_0\rangle} \right\rangle$	$\left\langle S_{6.00\langle O_{3.49}\rangle}, S_{0.00\langle O_{1.95}\rangle} \right\rangle$	$\left\langle S_{6.00\langle O_{2.00}\rangle}, S_{0.00\langle O_{2.27}\rangle} \right\rangle$	$\left\langle S_{2.00\langle O_{6.00}\rangle}, S_{2.00\langle O_{0.00}\rangle} \right\rangle$
U^-	$\left\langle S_{2.74\langle O_{2.68}\rangle}, S_{2.64\langle O_{2.47}\rangle} \right\rangle$	$\left\langle S_{0.58\langle O_{2.16}\rangle}, S_{2.53\langle O_{1.80}\rangle} \right\rangle$	$\left\langle S_{2.22\langle O_{2.71}\rangle}, S_{1.58\langle O_{0.00}\rangle} \right\rangle$	$\left\langle S_{2.60\langle O_{2.71}\rangle}, S_{0.00\langle O_{2.29}\rangle} \right\rangle$

(c–d) According to the Eqs (5.10)–(5.15) and criteria weights $\omega = (.29, .17, .26, .28)^T$ determined relative rational degree RRD of u_i, the PIS and NIS represented by F_i and determined conditional probability $P_r(A/u_i)$ by GRA that the object belong to the state A, as shown in Table 19.

Table 19. GRC and RRD and its Conditional probablity.

G_i^+	.713	.421	.602	.615
G_i^-	.702	.776	.663	.550
F_i	.503	.351	.475	.528
$P_r(A/u_i)$	.503	.351	.475	.528

Phase IV.

(a) The loss functions matrix in the form of PyDHLTNs is calculated in Table 20.

Table 20. Loss function.

	$A(P)$	$A^c(N)$
a_p	$\langle S_{1.44\langle O_{4.01}\rangle}, S_{2.87\langle O_{3.00}\rangle}\rangle$	$\langle S_{2.53\langle O_{2.65}\rangle}, S_{1.34\langle O_{2.16}\rangle}\rangle$
a_b	$\langle S_{1.93\langle O_{4.50}\rangle}, S_{2.92\langle O_{1.00}\rangle}\rangle$	$\langle S_{2.92\langle O_{1.27}\rangle}, S_{1.93\langle O_{3.42}\rangle}\rangle$
a_n	$\langle S_{2.17\langle O_{3.00}\rangle}, S_{2.92\langle O_{2.90}\rangle}\rangle$	$\langle S_{2.92\langle O_{3.00}\rangle}, S_{2.17\langle O_{2.00}\rangle}\rangle$

(b) Based on PyDHLTA operational laws, we derived the expected loss of each action by applying Eqs (5.18)–(5.20) as follows in Table 21.

Table 21. Expected losses.

U	u_1	u_2	u_3	u_4
$Sc\left(R_r\left(a_p/u_i\right)\right)$	.1662	.1842	.1702	.1070
$Sc\left(R_r\left(a_b/u_i\right)\right)$	.2767	.1913	.2631	.2877
$Sc\left(R_r\left(a_n/u_i\right)\right)$	.1151	.1620	.1242	.1624

(c) According to the minimum loss principles (1)–(3), the final result of each object is $POS(A) = \{u_4\}$,

$NEG(A) = \{u_1, u_2, u_3\}$, and $BND(A) = \phi$. Hence, u_4 are considered to be selected and u_1, u_2 and u_3 are assumed to be rejected.

7. Comparision section

In this section, we discussed the advantages and implementation of the proposed method by comparing it with the TOPSIS method and TODIM method.

7.1. Comparision with TOPSIS method

Here, we apply TOPSIS method proposed by Liang et al. [47] to determine the conditional probability. Hence, this comparison is made by considering the same weights for PIS and NIS we calculated in our proposed method. Then, we used the TOPSIS method to find out the conditional probability as shown in Table 22.

Table 22. Condinatioal probablity with topsis method.

$d(u_i, u^+)$	.177	.247	.165	.112
$d(u_i, u^-)$	.112	.090	.094	.146
$F(u_i)$	.388	.265	.365	.563
$P_r(A/u_i)$	.388	.265	.365	.563

Next, we consider the same LF as obtained in Table 20, and the expected loss are calculated in Table 23.

Table 23. Score of expected losses.

U	u_1	u_2	u_3	u_4
$Sc\left(R_r\left(a_p/u_i\right)\right)$	.067	.109	.075	.001
$Sc(R_r(a_b/u_i))$	.059	.011	.051	.107
$Sc(R_r(a_n/u_i))$	.025	.006	.022	.004

Hence, the final result of each object's decision can be determined according to the minimum loss principle, (1)–(3), that is $POS(A) = \{u_4\}$, $NEG(A)\{u_1, u_2, u_3\}$ and $BND(A) = \phi$, which is same as those of our proposed method. From Table 23 it is clear that if we apply our proposed method to any other approach like TOPSIS, the result will be the same. Hence it is analyzed that our proposed method is efficient and practical to solve the ambiguity and uncertainty in DM problems.

7.2. Comparision with TODIM method

In this section, to demonstrate the effectiveness of the developed decision-making technique, we compare it with the existing TODIM [54] technique. Therefore, this comparison is made by considering the same weights and evaluation matrices of the decision experts as we have calculated

in our proposed method. The detailed steps of the TODIM method are as follows.
Step 1. Determine the relative weight ω_{jr} of the criteria Ψ_j to Ψ_r by the given formula.

$$\omega_{jr} = \frac{\omega_j}{\omega_r}, \text{ where } \omega_r = \max\left\{\omega_j, j = 1, 2, ..., j\right\}.$$

Step 2. Calculate the dominance degree of the alternative u_i over each alternative according to each criterion as follows.

$$\phi_j^p(u_i, u_t) = \begin{cases} \sqrt{\frac{\omega_{jr} d\left(A^p_{B_{ij}} - A^p_{B_{tj}}\right)}{\sum_{j=1}^n \omega_{jr}}} & if \quad A^p_{B_{ij}} > A^p_{B_{tj}} \\ 0 & if \quad A^p_{B_{ij}} = A^p_{B_{tj}} \\ -\frac{1}{\theta}\sqrt{\frac{\sum_{j=1}^n \omega_{jr} d\left(A^p_{B_{ij}} - A^p_{B_{tj}}\right)}{\omega_{jr}}} & if \quad A^p_{B_{ij}} < A^p_{B_{tj}} \end{cases}.$$

Step 3. The overall dominance degree of the alternative u_i over each alternative u_t according to the decision experts matrix is evaluated by the given formula.

$$\delta^p(u_i, u_t) = \sum_{j=1}^n \phi_j^p(u_i, u_t)$$

where $(i, t = 1, 2, .., m; p = 1, 2, 3.., e)$.
Step 4. According to decision experts weights evaluate the collective overall dominance degree of the alternative u_i of each u_t as follows.

$$\delta(u_i, u_t) = \sum_{j=1}^n \delta^p(u_i, u_t).$$

Step 5. Evaluate the overall value of the alternative u_i by the given equation as follows.

$$\delta(u_i) = \frac{\sum_{t=1}^m \delta(u_i, u_t) - \min_i\left\{\sum_{t=1}^m \delta(u_i, u_t)\right\}}{\max_i\left\{\sum_{t=1}^m \delta(u_i, u_t)\right\} - \min_i\left\{\sum_{t=1}^m \delta(u_i, u_t)\right\}}.$$

Step 6. Rank all the alternatives by the overall values of $\delta(u_i)$. The bigger $\delta(u_i)$ is, the better the alternative.

The evaluation steps are as follows.
Step 1. The weights of critria are $\omega_j = (.29, .17, .26, .28)^T$ as we calculated in phase II. The relative weight ω_{jr} of the criteria Cj to Cr are determined as.

$$\begin{aligned} \omega_r &= \max\{.29, .17, .26, .28\} = .29 \\ \omega_{jr} &= (1.000, .586, .897, .966)^T. \end{aligned}$$

Step 2–3. Calculate the dominance degree of the alternative u_i over each alternative u_t with respect to M^p under the criteria Cj, $\theta = 2.4$, and the overall dominance degree of the alternative over each alternative is determined as.

$$\delta^1(u_i, u_t) = \begin{bmatrix} 0 & -.0019 & -.4418 & .5029 \\ -.6349 & 0 & -.8815 & -1.205 \\ -1.058 & -.2593 & 0 & .2480 \\ -.0119 & -.6027 & .3363 & 0 \end{bmatrix}$$

$$\delta^2(u_i,u_t)=\begin{bmatrix} 0 & .6625 & -.4222 & .7822 \\ -.3016 & 0 & -.3958 & -1.307 \\ .3206 & -.1728 & 0 & -.1483 \\ .1912 & -1.157 & .8152 & 0 \end{bmatrix}$$

$$\delta^3(u_i,u_t)=\begin{bmatrix} 0 & -1.385 & -.6578 & -.7135 \\ -1.417 & 0 & -.3009 & .2596 \\ -1.303 & .1525 & 0 & -.2150 \\ .8207 & .8279 & .7693 & 0 \end{bmatrix}$$

Step 4. The collective overall dominance degree of the alternative u_i over each alternative u_t are computed as follows.

$$\delta(u_i,u_t)=\begin{bmatrix} 0 & -.2801 & -.5113 & .1637 \\ -.8055 & 0 & -.5277 & -.7241 \\ -.7166 & -.0884 & 0 & -.0369 \\ .3424 & -.2740 & .1309 & 0 \end{bmatrix}.$$

Step 5. The overall value of the alternative u_i is evaluated as

$$\delta(u_1)=.6334, \delta(u_2)=0, \delta(u_3)=.5386, \delta(u_4)=1.000.$$

Hence from the overall value of the alternative $\delta(u_4)>\delta(u_1)>\delta(u_3)>\delta(u_2)$, it is clear that the best alternative based on the TODIM method is u_4 which are identical to that of the proposed method. Hence it is analyzed that our proposed method is efficient and practical to solve the ambiguity and uncertainty in the DM problems. The graphical ranking alternative based on TOPSIS and TODIM is shown in Figure 3.

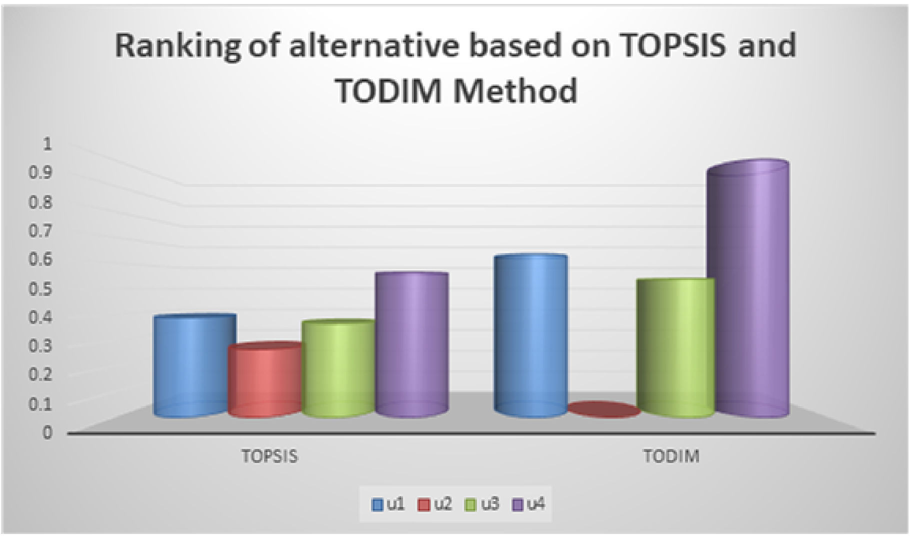

Figure 3. Graphical ranking alternative based on TOPSIS and TODIM.

There are more results, rankings and conditional probabilities we calculated based on different methods and aggregation operators and as a result, u_4 is our best result as shown in Tables 24 and 25.

Table 24. Conditional probablities of existence methods.

U	$P_r(A/u_1)$	$P_r(A/u_2)$	$P_r(A/u_3)$	$P_r(A/u_4)$
Topsis method [47]	.388	.265	.365	.563
Weighted aggregations method [42]	.387	.264	.364	.565
BP techinque [53,55]	.479	.452	.482	.506
Suggested method	.456	.388	.450	.525

Table 25. Conditional probablities wise rankings.

Approaches	Ranking
Topsis method [47]	$u_4 > u_1 > u_3 > u_2$
Weighted aggregations [42]	$u_4 > u_1 > u_3 > u_2$
BP techinque [53,55]	$u_4 > u_3 > u_1 > u_2$
Suggested method	$u_4 > u_1 > u_3 > u_2$

7.3. Conditional probability calculation using existing techniques

As a key component of TWD, conditional probabilities can also be used in ranking schemes. Conditional probabilities are computed using the weighted aggregation method in the TWDs model proposed by Yao et al. [43]. Bidirectional projective (BP) [53,55] technique discusses the connection between the scheme and the ideal solution, improving the objectivity of conditional probability measurement. Considering the same weights of attributes, the conditional probability is obtained in Table 24, based on several existing methods. Table 25 shows the ordering of alternatives based on conditional probability. Hence, from Tables 24 and 25 we determined that the best result is u_4 obtained from other existing methods which are similar to that of the proposed method, which shows the practicability of the proposed methods.

7.4. Discussion

The loss produced by the actions in different states can be displayed by the DTRS model, one of the TWD elements, and the action with the minimum loss principle is chosen. The other elements of the TWD method is conditional probability. In most existing TWD models, the conditional probabilities are given directly by the DM [56,57], making the decision outcome seem less difficult. We use the Renyi entropy to calculate the weight of each attribute, using the RRD of the object determined by the GRA method as the conditional probability. The linguistic terms used to describe the qualitative problem more closely follow human expression habits. In the case of single linguistic term sets, PyDHLTS provides a more flexible ability to express qualitative data. A new method of conveying evaluation data in TWDs has been made possible by the appearance of PyDHLTSs. When DM evaluates project attribute data, it can provide DHLTS with evaluation values more intuitively, which greatly reduces decision-making time. In the DHLT environment, our proposed model is constructed.

A new research direction is the TWD model based on PyDHLT information system. As a result, it has a relatively high research value. In this paper, some desirable properties of the PyDHLHWA operator are demonstrated, making decision-making problems more practical.

The following are the main advantages of the proposed method:

(1) The PyDHLN, which consists of the FHLT and SHLT, can express the evaluation of DMs more flexibly in the TWD process. Therefore, for dealing with decision-making problems, the DHLE-based TWD method is a useful tool.

(2) The GRA method is used to calculate the conditional probability, which replaces hamming distance with a weighted grey relational degree as a distance measure to improve the TOPSIS model. Furthermore, the PyDHL operator takes into account the different decision-making attitudes of DMs when aggregating LFs. They make the decision-making process more rational.

8. Conclusions

In DM problems, factual knowledge about a given fact is usually unknown, which makes the decision-making process more difficult and complex. PyFSs and DHLTS are general mathematical tools that can easily handle uncertain and imprecise knowledge. TWDs are consistent with people's thinking and have an important role in decision-making problems, especially when more conflict criteria exist in DM problems. In this paper, we examined the novel concept of PyDHLTSs by extending the concept of DHLTSs to handle the uncertainty in the DM problem. In addition, we put forward the concept of PyDHLWA, PyDHLOWA and PyDHLHA aggregation operators. The basic desirable characteristics of the investigated operator were given in detail. The Renyi entropy measure was considered to find the weights of criteria. A GRA method was used to calculate the conditional probability based on RRC, which replaces the hamming distance with the weighted GRC as a distance measure to improve the TOPSIS model. The aggregation of loss functions, by the aggregation operators, takes into account different decision attitudes of DMs. They make the process of TWDs more reasonable. A step-by-step description of TWDs was given under the PyFDHLT environment by considering evaluation values and loss functions. Finally, to show the practicability and effectiveness of the proposed methodology we compared and applied it to real-world problems for selecting the best optimal results.

Further research work will focus specifically on (a) Dombi aggregation operators, (b) Hamacher aggregation operators, (c) Fermetean double hierachy lingusitic term sets, (d) q-Rung Orthopair Fuzzy double hierachy lingusitic term sets. It will also focus on incomplete information processing in decision-making application fields such as online project recommendation, resumption of work and production, investment decision-making, online medical selection, disease prediction, etc.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declared that there is no conflict of interest regarding the publication of this paper.

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