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Research article

Discussion on iterative process of nonlocal controllability exploration for Hilfer neutral impulsive fractional integro-differential equation

Kanagaraj Muthuselvan[1], **Baskar Sundaravadivoo**[1], **Kottakkaran Sooppy Nisar**[2,3,*], **and Suliman Alsaeed**[4,5]

[1] Department of Mathematics, Alagappa University, Karaikudi-630004, Tamil Nadu, India
[2] Department of Mathematics, College of Science and Humanities in Alkharj, Prince Sattam bin Abdulaziz University, Alkharj, 11942, Saudi Arabia
[3] School of Technology, Woxsen University-Hyderabad-502345, Telangana State, India
[4] Applied Sciences Collage, Department of Mathematical Sciences, Umm Al-Qura University P.O. Box 715, 21955, Makkah, Saudi Arabia
[5] Mathematics Department, College of Sciences and Humanities, Prince Sattam bin Abdulaziz University, Saudi Arabia

* **Correspondence:** Email: n.sooppy@psau.edu.sa; Tel: +966563456976

Abstract: This manuscript primarily focuses on the nonlocal controllability results of Hilfer neutral impulsive fractional integro-differential equations of order $0 \leq w \leq 1$ and $0 < g < 1$ in a Banach space. The outcomes are derived from the strongly continuous operator, Wright function, linear operator, and bounded operator. First, we explore the existence and uniqueness of the results of the mild solution of Hilfer's neutral impulsive fractional integro-differential equations using Schauder's fixed point theorem and an iterative process. In order to determine nonlocal controllability, the Banach fixed point technique is used. We employed some specific numerical computations and applications to examine the effectiveness of the results.

Keywords: Hilfer fractional derivative; nonlocal controllability; impulsive function; fixed point theorems; iterative process
Mathematics Subject Classification:34A08, 34A12, 34A37, 45J05, 93B05

1. Introduction

Fractional calculus (FC) is the theory of differential and integral operators of non-integer order. In recent years, it has attracted numerous researchers, engineers, and scientists who have developed innovative models involving fractional differential equations (FDE). In the field of mechanics, the theories of viscoelasticity and viscoplasticity, modelling of proteins and polymers, modelling of ultrasound waves, and modelling of human tissue under mechanical loads have been successfully applied. In the following research articles and books, readers will find applications of FC ([1–6]). Recently, Y. Cao et al. [7] discussed the global Mittag-Leffler stability of the delayed fractional-coupled reaction-diffusion system on networks without strong connectedness. Most recently, Y. Kao et al. [8,9] established the application of FDE in the fields of Mittag-Leffler synchronization of delayed fractional memristor neural networks via adaptive control and global Mittag-Leffler synchronization of coupled delayed fractional reaction-diffusion Cohen-Grossberg neural networks via sliding mode control. In recent times, G. Li et al. [10] discussed the stability analysis of multi-point boundary conditions for fractional differential equations with non-instantaneous integral impulses. In [11], R. Rao et al. discussed the synchronization of epidemic systems with the Neumann boundary value under delayed impulse. Most recently, Y. Zhao et al. [12] investigated the practical exponential stability of an impulsive stochastic food chain system with time-varying delays. The main feature of FC is that it can handle the required rate of evolution in accordance with the needs of the occasion.

When impulsive differential equations (IDE) are used, abrupt changes and discontinuous jumps occur in an extremely short period of time. There are many good monographs on the IDE (see, [13–20]). There are many processes in the applied sciences that are represented by differential equations. A wide variety of physical phenomena exhibit sudden changes in their states, including biological systems with blood flow, popular dynamics, natural disasters, climate change, chemistry, control theory, and engineering. FDE differs from IDE primarily due to the discontinuous and continuous parts of the solution.

The evolution of a physical phenomenon over time is described by its local and nonlocal conditions. In many real-life situations, nonlocal conditions provide a greater benefit than local ones. Since these problems apply to many different areas, such as science and mathematics, the study of initial value problems (IVP) with nonlocal conditions is of paramount importance. A new form of fractional derivative has been developed by Hilfer that combines Riemann-Liouville fractional derivatives (RLFD) and Caputo fractional derivatives (CFD).

Control theory plays a vital role in ensuring system stability. A wide range of applied and pure mathematics problems are addressed in this field. It has the potential to influence the behavior of a dynamical system in a manner that achieves the desired result. In recent years, many scientists and researchers have been working in the field of controllability in Hilfer fractional derivatives (HFD) with different domains such as non-densely domain (NDD), neutral functional differential equations (NFDE), delay differential equations (DDE), and impulsive differential equation (IDE). They may refer to the following monographs ([21–23]). In [24], P. Bedi et al. are demonstrate the exact controllability of HFD. In [25] J. Du et al. investigated exact controllability for HFD inclusion involving nonlocal conditions. In [26], X. Liu et al. investigated the finite approximate controllability for Hilfer fractional evolution systems. In [27], D. Luo et al. established the result on the averaging principle of stochastic Hilfer-type fractional systems involving non-Lipschitz coefficients. In [28], K. S. Nisar et al. established the controllability of HFD with a nondense domain. In [29], Y. Zhou et al. discussed the HFD on a semi-infinite interval. Recently, M. Zhou et al. [30] established the Hilfer fractional evolution equations with almost sectorial operators. From the above referred articles, no manuscript deals with the nonlocal controllability exploration for Hilfer neutral type fractional integro-differential equations (HNFrIDE) with impulsive conditions through the application of a filter system.

As a result, we will demonstrate this concept and consider the following form of IHFrNIDE:

$$\begin{aligned} {}^{H}D^{w,g}(\nu(t)-\Theta(t,\nu(t))) &= Q\nu(t)+\mathcal{P}u(t)+\phi(t,\nu(t),\int_0^t \chi(t,s,\nu(s))ds), \\ & \quad t\in\mathcal{J}^* := [0,T]\backslash\{t_1,t_2,...,t_\rho\}, \\ \Delta\nu(t) &= \nu(t_\varepsilon^+)-\nu(t_\varepsilon^-) = I_\varepsilon(\nu(t_\varepsilon^-)), \varepsilon = 1,2,3,...,\rho, \\ I_{0^+}^{1-\eta}\nu(0) &= \nu_0-\mathcal{G}(\nu). \end{aligned} \tag{1.1}$$

Where, ${}^{H}D^{w,g}$ denotes HFD of order $0 \le w \le 1$ and $0 < g < 1$ and $I_{0^+}^{1-\eta}$ is generalized fractional derivatives of order $1-\eta = (1-w)(1-g)$. The neutral term $\Theta : \mathcal{J}^* \times \Xi \to \Xi$ is continuous. Let Q is a closed, linear, and bounded operator in Ξ. The control function u(t) is given in $L^2(\mathcal{J}^*, \Xi)$ a Banach space of admissible control functions with Ξ as a Banach space. The bounded linear operator $\mathcal{P} : \Xi \to \Xi$ is continuous. Consider the functions $\phi : \mathcal{J}^* \times \Xi \times \Xi \to \Xi$ and $\chi : \mathcal{J}^* \times \Xi \times \Xi \to \Xi$ are continuous. The nonlocal term $\mathcal{G} : C(\mathcal{J}^*, \Xi) \to \Xi$ is a given continuous function. I_ε is an impulse operator. Where, $\nu(t_\varepsilon^+) = \lim_{\zeta\to 0^+}(t_\varepsilon+\zeta)$ and $\nu(t_\varepsilon^-) = \lim_{\zeta\to 0^-} = (t_\varepsilon-\zeta)$ represents right and left limit of $\nu(t)$ at $t = t_\varepsilon$ and the discontinuous points are, $0 = t_0 < t_1 < t_2 < ... < t_\rho < t_{\rho+1} = T < \infty$.

Foremost, the primary key factors of our proposed work are as follows:

- The strongly continuous operator, the linear operator, and the bounded operators are used to obtain the solution representation of our system.
- An iterative process is a means of generating sequences that can approximate the solution of equations describing real-life problems.
- Define a unique control function for our given system.
- Existence solutions are explored by Schauder's fixed point theorem, and the Arzela-Ascoli theorem.
- Uniqueness results are attained from the Banach fixed point theorem.
- Nonlocal controllability is examined with the defined control function, contraction mapping, and iterative process.
- The novelty of this proposed work is that it establishes new assumptions for our system. When compared to prior studies ([31–34]), it helps to reduce the complexity of the result outcomes. Moreover, we also discuss the applications of our problem through a filter system as well as numerical computations. We have presented graphical representations of the given problem. It is used to obtain the existence and uniqueness results for a given system with different parameters at an instant time.

This manuscript is organized into five sections. In Section 2, we introduce some preliminary definitions, remarks, and lemmas that can be used to prove the proposed work. In Section 3, we examine the nonlocal controllability result using the necessary and sufficient conditions we have assumed. In Section 4, we provide the applications of our suggested work with numerical computations and a filter system. At the end of this manuscript, we discuss the conclusion.

2. Fundamental materials and solution representation

Finding our main results will be of significant assistance. It is pertinent to note that the following notation will be used throughout the paper: $\|\nu(t)\| = \sup_{t\in\mathcal{J}^*}\{|\nu(t)|, \nu(t) \in C(\mathcal{J}^*, \Xi)\}$.

Definition 2.1. *[35] The Riemann-Liouville fractional integral (RLFI) of order $w \in \mathbb{R}^+$ (the set of positive real numbers) and the function $\phi(t)$ is defined as*

$$I^{w}_{o^+}\phi(t) = \frac{1}{\Gamma(w)}\int_0^t (t-s)^{w-1}\phi(s)ds, \ t > 0.$$

Definition 2.2. *[35] The RLFD of order $n-1 \le w < n$, $n \in \mathbb{N}$ for the function $\phi : [w, +\infty) \to \mathbb{R}$ is defined by*

$$D^{w}_{0+}\phi(t) = \frac{1}{\Gamma(n-w)}\frac{d^n}{dt^n}\int_0^t \frac{\phi(s)}{(t-s)^{w-n+1}}ds, \ t > 0.$$

Remark 2.1. *[35] A subset Λ in $C(\mathcal{J}^*, \Xi)$ is relatively compact if and only if it is uniformly bounded and equicontinuous on $\mathcal{J}^*$.*

Definition 2.3. *[36] The HFD is the generalized the RLFD of order $0 \le w \le 1$ and $0 < g < 1$, with lower limit '0' is defined as*

$$D^{w,g}_{0+}\phi(t) = I^{w(1-g)}_{0+}\frac{d}{dt}I^{(1-w)(1-g)}_{0+}\phi(t).$$

where, I represents the Riemann-Liouville fractional integral.

Definition 2.4. *[37] A system is said to be nonlocal controllable on $\mathcal{J}^*$ if every pair of vector $v_0, v_t \in \Xi$ there exists a control $u \in L^2(\mathcal{J}^*, \Xi)$ such that the mild solution v which satisfies $v(t) = v_t - \mathcal{G}(v)$.*

Theorem 2.1. *[38] Let Ξ be a real Banach space, $\varphi \subset \Xi$ a nonempty closed bounded convex subset and $\Lambda : \varphi \to \varphi$ is compact. Then Λ has a fixed point.*

Lemma 2.1. *[39] The operator $S_{w,g}(t)$ and $\Psi_g(t)$ have the following properties:*

- *$\{\Psi_g(t) : t > 0\}$ is continuous in the uniform operator topology.*
- *For any fixed $t > 0, S_{w,g}(t)$ and $\Psi_g(t)$ are linear and bounded operators*

$$\|\Psi_g(t)\| \le \frac{\mathcal{K}t^{g-1}}{\Gamma(g)}, \ \text{and} \ \|S_{w,g}(t)\| \le \frac{\mathcal{K}t^{(w-1)(g-1)}}{\Gamma(w(1-g)+g)}. \tag{2.1}$$

- *$\{\Psi_g(t) : t > 0\}$ and $\{S_{w,g}(t) : t > 0\}$ are strongly continuous.*

Lemma 2.2. *Let* $0 \leq w \leq 1$, $0 < g < 1$ *then the equation (1.1) can be equivalent in the form of*

$$v(t) = \begin{cases} S_{w,g}(t)(v_0 - \mathcal{G}(v) - \Theta(0, v(0)) + \Theta(t, v(t))) + \int_0^t Q\Psi_g(t-s)\Theta(s, v(s))ds \\ \quad + \int_0^t \Psi_g(t-s)(\mathcal{P}u(s) + \phi(s, v(s), \int_0^s \chi(s, m, v(m))dm))ds \\ \quad + S_{w,g}(t-t_1)I_1(v(t_1^-)), & t \in [0, t_1]; \\ S_{w,g}(t)(v_0 - \mathcal{G}(v) - \Theta(0, v(0)) + \Theta(t, v(t))) + \int_0^t Q\Psi_g(t-s)\Theta(s, v(s))ds \\ \quad + \int_0^t \Psi_g(t-s)(\mathcal{P}u(s) + \phi(s, v(s), \int_0^s \chi(s, m, v(m))dm))ds \\ \quad + \sum_{\varepsilon=1}^{2} S_{w,g}(t-t_\varepsilon)I_\varepsilon(v(t_\varepsilon^-)), & t \in (t_1, t_2]; \\ . \\ . \\ . \\ S_{w,g}(t)(v_0 - \mathcal{G}(v) - \Theta(0, v(0)) + \Theta(t, v(t))) + \int_0^t Q\Psi_g(t-s)\Theta(s, v(s))ds \\ \quad + \int_0^t \Psi_g(t-s)(\mathcal{P}u(s) + \phi(s, v(s), \int_0^s \chi(s, m, v(m))dm))ds \\ \quad + \sum_{\varepsilon=1}^{\rho} S_{w,g}(t-t_\varepsilon)I_\varepsilon(v(t_\varepsilon^-)), & t \in (t_\rho, T]; \end{cases}$$

where, $S_{w,g} = I_{0^+}^{w(1-g)}\Psi_g(t)$, $\Psi_g(t) = t^{g-1}\mathcal{T}_g(t)$ *and* $\mathcal{T}_g(t) = \int_0^\infty g\sigma\mathcal{R}_g(\sigma)S(t^g\sigma)d\sigma$.

$$\mathcal{R}_g(\sigma) = \sum_{n=1}^{\infty} \frac{(-\sigma)^{n-1}}{(n-1)!\Gamma(1-ng)}, \sigma \in (0, \infty),$$

where, $\mathcal{R}_g(\sigma)$ *is a function of Wright type which satisfies* $\int_0^\infty \sigma^\delta \mathcal{R}_g(\sigma)d\sigma = \frac{\Gamma(1+\delta)}{\Gamma(1+g\delta)}, \sigma \geq 0$.

3. Discussion on nonlocal controllability

The fundamental objective of this section is to examine the nonlocal controllability results of Eq (1.1) with iterative type. Before that, we have to define the function $\mathcal{F} : C(\mathcal{J}^*, \Xi) \to C(\mathcal{J}^*, \Xi)$ is follows:

$$\mathcal{F}(v(t)) = \begin{cases} S_{w,g}(t)(v_0 - \mathcal{G}(v) - \Theta(0, v(0)) + \Theta(t, v(t))) + \int_0^t Q\Psi_g(t-s)\Theta(s, v(s))ds \\ \quad + \int_0^t \Psi_g(t-s)(\mathcal{P}u(s) + \phi(s, v(s), \int_0^s \chi(s, m, v(m))dm))ds \\ \quad + S_{w,g}(t-t_1)I_1(v(t_1^-)), \quad t \in [0, t_1]; \\ S_{w,g}(t)(v_0 - \mathcal{G}(v) - \Theta(0, v(0)) + \Theta(t, v(t))) + \int_0^t Q\Psi_g(t-s)\Theta(s, v(s))ds \\ \quad + \int_0^t \Psi_g(t-s)(\mathcal{P}u(s) + \phi(s, v(s), \int_0^s \chi(s, m, v(m))dm))ds \\ \quad + \sum_{\varepsilon=1}^{2} S_{w,g}(t-t_\varepsilon)I_\varepsilon(v(t_\varepsilon^-)), \quad t \in (t_1, t_2]; \\ . \\ . \\ . \\ S_{w,g}(t)(v_0 - \mathcal{G}(v) - \Theta(0, v(0)) + \Theta(t, v(t))) + \int_0^t Q\Psi_g(t-s)\Theta(s, v(s))ds \\ \quad + \int_0^t \Psi_g(t-s)(\mathcal{P}u(s) + \phi(s, v(s), \int_0^s \chi(s, m, v(m))dm))ds \\ \quad + \sum_{\varepsilon=1}^{\rho} S_{w,g}(t-t_\varepsilon)I_\varepsilon(v(t_\varepsilon^-)), \quad t \in (t_\rho, t_{\rho+1}]; \end{cases} \tag{3.1}$$

Our considerations are based by the following assumptions:

($\mathcal{H}1$) The impulsive operator is function from Ξ *to* Ξ then there exists a constants $K_\varepsilon^* > 0, \mathcal{M}_\varepsilon^* > 0$ such that

(i) $\sum_{\varepsilon=1}^{\rho} \|I_\varepsilon \nu_n(t_\varepsilon^-) - I_\varepsilon \nu(t_\varepsilon^-)\| \leq \sum_{\varepsilon=1}^{\rho} K_\varepsilon^* \|\nu_n - \nu\|$.

(ii) $\sum_{\varepsilon=1}^{\rho} \|I_\varepsilon \nu(t_\varepsilon^-)\| \leq \sum_{\varepsilon=1}^{\rho} \mathcal{M}_\varepsilon^*$.

($\mathcal{H}2$) A map $\mathcal{G} : \Xi \to \Xi$ be a continuous function and it is satisfy the following condition

$$\|\mathcal{G}(\nu_n) - \mathcal{G}(\nu)\| \leq \eta^* \|\nu_n - \nu\|, \ \eta^* > 0.$$

($\mathcal{H}3$) The functions $\phi : \mathcal{J}^* \times \Xi \times \Xi \to \Xi$ and $\chi : \mathcal{J}^* \times \Xi \times \Xi \to \Xi$ are both continuous with respect to t on $\mathcal{J}^*$ and there exists a constants $\mathcal{L}_\chi^* > 0, \mathcal{N}_\chi^* > 0, \gamma_\chi > 0, \mathcal{K}_1 > 0, \mathcal{K}_2 > 0$ and $\mathcal{S}_\chi > 0$ such that

(i) $\|\phi(t, \nu(t), \Omega(t))\| \leq \mathcal{L}_\chi^* \|\nu\| + \mathcal{N}_\chi^* \|\Omega\| + \gamma_\chi$.

(ii) $\|\phi(t, \nu(t), \Omega(t)) - \phi(t, \nu^*(t), \Omega^*(t))\| \leq \mathcal{K}_1 \|\nu(t) - \nu^*(t)\| + \mathcal{K}_2 \|\Omega(t) - \Omega^*(t)\|$.

(iii) $\|\Omega(t) - \Omega^*(t)\| \leq \mathcal{S}_\chi \|\nu(t) - \nu^*(t)\|$.

Where, $\Omega(t) = \int_0^t \chi(t, s, \nu(s))ds$, and $\Omega^*(t) = \int_0^t \chi(t, s, \nu^*(s))ds$.

($\mathcal{H}4$) A map $\Theta : \mathcal{J}^* \times \Xi \to \Xi$ be a continuous function with respect to t on $\mathcal{J}^*$ then there exists a constants $\mathcal{W}_\xi > 0, \ \lambda^* > 0$ such that

(i) $\|\Theta(t, \nu(t))\| \leq \mathcal{W}_\xi$.

(ii) $\|\Theta(t, \nu_n(t)) - \Theta(t, \nu(t))\| \leq \lambda^* \|\nu_n - \nu\|$.

($\mathcal{H}5$) The linear operator $\mathcal{B} : L^2(\mathcal{J}^*, \Xi) \to \Xi$ is defined as follows:

$$\mathcal{B}u = \int_0^t \Psi_g(t-s)\mathcal{P}u(s)ds. \tag{3.2}$$

Equation (3.2) is invertible and it is denoted by $\mathcal{B}^{-1}$. Where, $\mathcal{B}^{-1}$ takes value from $\frac{L^2(\mathcal{J}^*,\Xi)}{ker\mathcal{B}}$ then there exists a $\varrho > 0$ such that $\|\mathcal{B}^{-1}\| \leq \varrho$. Here we define the control term u(t) for every $t \in (t_\rho, T]$ as follows:

$$u(t) = \mathcal{B}^{-1}\Big[\nu_t - \mathcal{G}(\nu) - s_{g,w}(t)(\nu_0 - \mathcal{G}(\nu) - \Theta(0, \nu(0)) + \Theta(t, \nu(t))) - \int_0^t \mathcal{Q}\psi_g(t-s)\Theta(s, \nu(s))ds$$
$$- \int_0^t \psi_g(t-s)\phi(s, \nu(s), \Omega(s))ds - \sum_{\varepsilon=1}^{\rho} S_{w,g}(t - t_\varepsilon)I_\varepsilon(\nu(t_\varepsilon^-))\Big],$$

$$\|u(t)\| = \sup_{t\in\mathcal{J}^*}\Big|\mathcal{B}^{-1}\Big[\nu_t - \mathcal{G}(\nu) - s_{g,w}(t)(\nu_0 - \mathcal{G}(\nu) - \Theta(0, \nu(0)) + \Theta(t, \nu(t)))$$
$$- \int_0^t \mathcal{Q}\psi_g(t-s)\Theta(s, \nu(s))ds - \int_0^t \psi_g(t-s)\phi(s, \nu(s), \Omega(s))ds$$
$$- \sum_{\varepsilon=1}^{\rho} S_{w,g}(t - t_\varepsilon)I_\varepsilon(\nu(t_\varepsilon^-))\Big]\Big|,$$
$$\leq \mathcal{D}_m\Big[\|\nu_t - \mathcal{G}(\nu)\| - \frac{\mathcal{K}t^{(w-1)(g-1)}}{\Gamma(w(1-g)+g)}\Big(C_\nu + \mathcal{W}_\xi\Big) - \frac{\mathcal{K}t^{g-1}}{\Gamma(g)}$$
$$(\|\mathcal{Q}\|\mathcal{W}_\xi + \mathcal{L}_\chi^*\|\nu(t)\| + \mathcal{N}_\chi^*\|\Omega(t)\| + \gamma_\chi) - \frac{\sum_{\varepsilon=1}^{\rho}\mathcal{M}_\varepsilon^*\mathcal{K}(t - t_\varepsilon)^{(w-1)(g-1)}}{\Gamma(w(1-g)+g)}\Big],$$
$$\leq \mathcal{D}_m\beta_\gamma^*.$$

Here we used the following notation of above equation

$$
\begin{aligned}
Where,\ \beta^*_\gamma &= \Bigg[\|v_t - \mathcal{G}(v)\| - \frac{\mathcal{K}t^{(w-1)(g-1)}}{\Gamma(w(1-g)+g)}\Big(\mathcal{C}_v + \mathcal{W}_\xi\Big) - \frac{\mathcal{K}t^{g-1}}{\Gamma(g)} \\
&\quad (\|\mathcal{Q}\|\mathcal{W}_\xi + \mathcal{L}^*_\chi\|v(t)\| + \mathcal{N}^*_\chi\|\Omega(t)\| + \gamma_\chi) \\
&\quad -\frac{\sum_{\varepsilon=1}^{\rho}\mathcal{M}^*_\varepsilon\mathcal{K}(t-t_\varepsilon)^{(w-1)(g-1)}}{\Gamma(w(1-g)+g)}\Bigg]. \\
\|v_0 - \mathcal{G}(v) - \Theta(0, v(0))\| &\leq \mathcal{C}_v. \\
\|\Theta(s, v(s))\| &\leq \mathcal{W}_\xi. \\
\|\phi(t, v(t), \Theta(t))\| &\leq \mathcal{L}^*_\chi\|v(t)\| + \mathcal{N}^*_\chi\|\Omega(t)\| + \gamma_\chi. \\
\|\mathcal{B}^{-1}\| &\leq \mathcal{D}_m. \\
\sum_{\varepsilon=1}^{\rho}\|S_{w,g}(t-t_\varepsilon)I_\varepsilon(v(t^-_\varepsilon))\| &\leq \frac{\sum_{\varepsilon=1}^{\rho}\mathcal{M}^*_\varepsilon\mathcal{K}(t-t_\varepsilon)^{(w-1)(g-1)}}{\Gamma(w(1-g)+g)}.
\end{aligned}
$$

Theorem 3.1. *The hypothesis* $(\mathcal{H}1)-(\mathcal{H}3)(i)$ *and Lemma 2.1 are hold then (1.1) is uniformly bounded for every* $t \in [0, T]$ *and provided that*

$$\|\mathcal{F}(v(t))\| \leq \mathcal{Z}^*. \tag{3.3}$$

$$
\begin{aligned}
Where,\ \|\mathcal{F}(v(t))\| \leq\ & \frac{\mathcal{K}(t-t_\varepsilon)^{(w-1)(g-1)}}{\Gamma(w(1-g)+g)}\Big(\mathcal{C}_v + \mathcal{W}_\xi\Big) + \|\mathcal{Q}\|\frac{\mathcal{K}\mathcal{W}_\xi t^{g-1}}{\Gamma(g)} \\
&+\frac{\mathcal{K}t^{g-1}\mathcal{L}^*_\chi\|v(t)\| + \mathcal{N}^*_\chi\|\Omega(t)\| + \gamma_\chi}{\Gamma(g)} + \frac{\sum_{\varepsilon=1}^{\rho}\mathcal{M}^*_\varepsilon\mathcal{K}(t-t_\varepsilon)^{(w-1)(g-1)}}{\Gamma(w(1-g)+g)}.
\end{aligned}
$$

Proof. We want to show that the Eq (1.1) is uniformly bounded for every $t \in [0, T]$. First we prove that a function $\mathcal{F}(v(t))$ is bounded on $[0, t_1]$ and we get the following inequality

$$
\begin{aligned}
\|\mathcal{F}(v(t))\| &= \sup_{t\in\mathcal{J}^*}\Bigg\{S_{w,g}(t)(v_0 - \mathcal{G}(v) - \Theta(0, v(0)) + \Theta(t, v(t))) + \int_0^t \mathcal{Q}\Psi_g(t-s)\Theta(s, v(s))ds \\
&+ \int_0^t \Psi_g(t-s)(\mathcal{P}u(s) + \phi(s, v(s), \int_0^s \chi(s, m, v(m))dm))ds + S_{w,g}(t-t_1)I_1(v(t^-_1))\Bigg\}, \\
&\leq \frac{\mathcal{K}t^{(w-1)(g-1)}}{\Gamma(w(1-g)+g)}\Big(\mathcal{C}_v + \mathcal{W}_\xi\Big) + \|\mathcal{Q}\|\frac{\mathcal{K}\mathcal{W}_\xi t^{g-1}}{\Gamma(g)} \\
&+ \frac{\mathcal{K}t^{g-1}}{\Gamma(g)}\Big(\|\mathcal{P}\|\mathcal{D}_m\beta^* + \mathcal{L}^*_\chi\|v(t)\| + \mathcal{N}^*_\chi\|\Omega(t)\| + \gamma_\chi\Big) + \frac{\mathcal{M}^*_1\mathcal{K}(t-t_1)^{(w-1)(g-1)}}{\Gamma(w(1-g)+g)}, \\
&\leq \mathcal{Z}^*_1.
\end{aligned} \tag{3.4}
$$

Proceeding in similar way we define the function $\mathcal{F}$ for every $t \in (t_\rho, T]$,

$$
\begin{aligned}
\|\mathcal{F}(v(t))\| \leq\ & \frac{\mathcal{K}t^{(w-1)(g-1)}}{\Gamma(w(1-g)+g)}\Big(\mathcal{C}_v + \mathcal{W}_\xi\Big) + \|\mathcal{Q}\|\frac{\mathcal{K}\mathcal{W}_\xi t^{g-1}}{\Gamma(g)} \\
&+ \frac{\mathcal{K}t^{g-1}}{\Gamma(g)}\Big(\|\mathcal{P}\|\mathcal{D}_m\beta^* + \mathcal{L}^*_\chi\|v(t)\| + \mathcal{N}^*_\chi\|\Omega(t)\| + \gamma_\chi\Big)
\end{aligned}
$$

$$+\frac{\sum_{\varepsilon=1}^{\rho}\mathcal{M}_{\varepsilon}^{*}\mathcal{K}(t-t_{\varepsilon})^{(w-1)(g-1)}}{\Gamma(w(1-g)+g)},$$
$$\leq \mathcal{Z}_{\rho}^{*}. \tag{3.5}$$

From the inequality (3.4) and (3.5) then we have

$$\begin{aligned}\|\mathcal{F}(\nu(t))\| &\leq sup\{\mathcal{Z}_1^*,\mathcal{Z}_2^*,...,\mathcal{Z}_\rho^*\},\\ &= \mathcal{Z}^*,\\ \|\mathcal{F}(\nu(t))\| &\leq \mathcal{Z}^*. \end{aligned}\tag{3.6}$$

From the inequality (3.6) we can say that $\mathcal{F}(\nu(t))$ is uniformly bounded for every $t\in[0,T]$. □

Theorem 3.2. *Assume that the hypothesis* $(\mathcal{H}1),(\mathcal{H}3)(ii)$*,* $(\mathcal{H}3)(iii)$ *and Lemma (2.1) are holds then prove that the function* $\mathcal{F}$ *has atleast one solution on* $C(\mathcal{J}^*,\Xi)$.

Proof. **Step 1:** We want to show that $\mathcal{F}$ is continuous on $C(\mathcal{J}^*,\Xi)$. Let $\{\nu_n(t)\}$ be a sequence on $C(\mathcal{J}^*,\Xi)$ such that $\{\nu_n(t)\}\to\nu$ as $n\to\infty$. For every $t\in[0,t_1]$ and then we have

$$\begin{aligned}\|\mathcal{F}(\nu_n(t))-\mathcal{F}(\nu(t))\| \leq\ & \|S_{w,g}(t)\|\{\|(\mathcal{G}(\nu_n)-\mathcal{G}(\nu))\|+\|(\Theta(t,\nu_n(t))-\Theta(t,\nu(t)))\|\}\\ &+\int_0^t\|Q\|\|\Psi_g(t-s)\|\times\|\Theta(s,\nu_n(s))-\Theta(s,\nu(s))\|ds\\ &+\int_0^t\|\Psi_g(t-s)\|\times\|\phi(s,\nu_n(s),\Omega_n(s))-\phi(s,\nu(s),\Omega(s))\|\\ &+\|S_{w,g}(t-t_1)\|\times\|I_1\nu_n(t_1^-)-I_1\nu(t_1^-)\|,\\ \leq\ & \Bigg\{\frac{\mathcal{K}t^{(w-1)(g-1)}(\eta+\lambda^*)}{\Gamma(w(1-g)+g)}+\frac{\mathcal{K}t^{g-1}(\|Q\|\lambda^*+(\mathcal{K}_1+\mathcal{K}_2\mathcal{S}_\chi))}{\Gamma(g)}\\ &+\frac{K_1^*\mathcal{K}(t-t_1)^{(w-1)(g-1)}}{\Gamma(w(1-g)+g)}\Bigg\}\times\|\nu_n-\nu\|.\end{aligned}$$

Since, as $n\to\infty,\ \nu_n\to\nu\Rightarrow\|(\mathcal{F}(\nu_n)(t)-(\mathcal{F})(\nu)(t)\|\to0$ for every $[0,t_1]$.

Proceeding like this, we define for every $t\in(t_\rho,T]$ and then we obtained

$$\begin{aligned}\|\mathcal{F}(\nu_n(t))-\mathcal{F}(\nu(t))\| \leq\ & \Bigg\{\frac{\mathcal{K}t^{(w-1)(g-1)}(\eta+\lambda^*)}{\Gamma(w(1-g)+g)}+\frac{\mathcal{K}t^{g-1}(\|Q\|\lambda^*+(\mathcal{K}_1+\mathcal{K}_2\mathcal{S}_\chi))}{\Gamma(g)}\\ &+\frac{\sum_{\varepsilon=1}^{\rho}K_\varepsilon^*\mathcal{K}(t-t_\varepsilon)^{(w-1)(g-1)}}{\Gamma(w(1-g)+g)}\Bigg\}\times\|\nu_n-\nu\|.\end{aligned}$$

Since, as $n\to\infty,\ \nu_n\to\nu\Rightarrow\|(\mathcal{F}(\nu_n)(t)-(\mathcal{F})(\nu)(t)\|\to0$ for every $(t_\rho,T]$.

Step 2: Next, we have to show that $\mathcal{F}$ is equicontinuous on $C(\mathcal{J}^*,\Xi)$. Let us consider the two arbitrary elements $\theta_1,\theta_2\in[0,t_1]$ and relation between θ_1,θ_2 is $\theta_1<\theta_2$.

$$\begin{aligned}
\|(\mathcal{F}\nu)(\theta_2)-(\mathcal{F}\nu)(\theta_1)\| &= \sup_{t\in\mathcal{J}^*}\Bigg|\mathcal{S}_{w,g}(t)\Theta(\theta_2,\nu(\theta_2))+\int_0^{\theta_2}\mathcal{Q}\Psi_g(\theta_2-s)\Theta(s,\nu(s))ds \\
&\quad+\int_0^{\theta_2}\Psi_g(\theta_2-s)\times(\mathcal{P}u(s)+\phi(s,\nu(s),\Omega(s)))ds \\
&\quad-\mathcal{S}_{w,g}(t)\Theta(\theta_1,\nu(\theta_1))-\int_0^{\theta_1}\mathcal{Q}\Psi_g(\theta_1-s)\Theta(s,\nu(s))ds \\
&\quad-\int_0^{\theta_1}\Psi_g(\theta_1-s)(\mathcal{P}u(s)+\phi(s,\nu(s),\Omega(s)))ds\Bigg|, \\
&\leq \frac{\mathcal{K}t^{(w-1)(g-1)}\lambda^*\|\theta_2-\theta_1\|}{\Gamma(w(1-g)+g)}+\|\mathcal{Q}\|\left(\frac{\mathcal{K}\theta_2^{g-1}}{\Gamma(g)}-\frac{\mathcal{K}\theta_1^{g-1}}{\Gamma(g)}\right)\times\mathcal{W}_\xi \\
&\quad+\left(\frac{\mathcal{K}\theta_2^{g-1}}{\Gamma(g)}-\frac{\mathcal{K}\theta_1^{g-1}}{\Gamma(g)}\right)\|\mathcal{P}\|\mathcal{D}_m\beta_{\gamma^*}+\mathcal{L}_\chi^*\|\nu(t)\|+\mathcal{N}_\chi^*\|\Omega(t)\|+\gamma_\chi.
\end{aligned} \tag{3.7}$$

As $\theta_2 \to \theta_1$ in (3.7) then we have $\|(\mathcal{F}\nu)(\theta_2)-(\mathcal{F}\nu)(\theta_1)\| \to 0$ and therefore $(\mathcal{F}\nu)(t)$ is equicontinuous on $[0,t_1]$. In similar manner, we prove the function $\mathcal{F}$ is equicontinuous on every $t \in (t_\rho, T]$,

$$\begin{aligned}
|(\mathcal{F}\nu)(\theta_{\rho+1})-(\mathcal{F}\nu)(\theta_\rho)\| &\leq \frac{\mathcal{K}t^{(w-1)(g-1)}\lambda^*\|\theta_{\rho+1}-\theta_\rho\|}{\Gamma(w(1-g)+g)}+\|\mathcal{Q}\|\left(\frac{\mathcal{K}\theta_2^{g-1}}{\Gamma(g)}-\frac{\mathcal{K}\theta_\rho^{g-1}}{\Gamma(g)}\right)\times\mathcal{W}_\xi \\
&\quad+\left(\frac{\mathcal{K}\theta_{\rho+1}^{g-1}}{\Gamma(g)}-\frac{\mathcal{K}\theta_\rho^{g-1}}{\Gamma(g)}\right)\|\mathcal{P}\|\mathcal{D}_m\beta_{\gamma^*}+\mathcal{L}_\chi^*\|\nu(t)\|+\mathcal{N}_\chi^*\|\Omega(t)\|+\gamma_\chi.
\end{aligned} \tag{3.8}$$

Since $\theta_{\rho+1} \to \theta_\rho$ in (3.8) which implies $\|(\mathcal{F}\nu)(\theta_{\rho+1})-(\mathcal{F}\nu)(\theta_\rho)\| \to 0$ and therefore $(\mathcal{F}\nu)$ is equicontinuous on $(t_\rho, T]$ then using Arzela-Ascoli theorem and remark (2.1), we get $(\mathcal{F}\nu)$ is compact on $\mathcal{J}^*$. Using the Theorem 3.1, Steps 1 and 2 in conjunction with the Schauder fixed point theorem, we attained a solution for Eq (1.1) on $\mathcal{J}^*$. □

Theorem 3.3. *The hypothesis* $(\mathcal{H}1), (\mathcal{H}2), (\mathcal{H}3)(ii), (\mathcal{H}3)(iii), (\mathcal{H}4)(ii), (\mathcal{H}5)$*, and Lemma 2.1 are satisfied then the Eq (1.1) has a unique solution and nonlocal controllable on* $\mathcal{J}^*$.

Proof. In order to satisfy the Banach contraction, we consider two solutions of given system (1.1) namely, $\nu(t)$ and $\mu(t)$ in Ξ and define the contraction mapping $\mathcal{F}:\Xi\to\Xi$ by $\|\mathcal{F}(\nu)(t)-\mathcal{F}(\mu)(t)\| \leq \Upsilon\|\nu(t)-\mu(t)\|$ for every $t \in (t_\rho, T]$ and $0 \leq \Upsilon < 1$ and then prove the uniqueness and nonlocal controllability of IHFrNIDE (1.1). Initially, prove the contraction mapping for every $t \in [0,t_1]$ by using above hypothesis,

$$\begin{aligned}
\|\mathcal{F}(\nu)(t)-\mathcal{F}(\mu)(t)\| &\leq \sup_{t\in\mathcal{J}^*}\left|S_{w,g}(t)\right|(|\mathcal{G}(\nu)-\mathcal{G}(\mu)|+|\Theta(t,\nu(t))-\Theta(t,\mu(t))|) \\
&\quad+\int_0^t|\mathcal{Q}||\Psi_g(t-s)|\times|\Theta(s,\nu(s))-\Theta(s,\mu(s))|ds
\end{aligned}$$

$$
\begin{aligned}
& +\int_0^t |\Psi_g(t-s)| \times |\phi(s,\nu(s),\Omega(s)) - \phi(s,\mu(s),\Omega^*(s))| \\
& +|S_{w,g}(t-t_1)| \times |I_1(\nu(t_1^-)) - I_1(\mu(t_1^-))|, \\
\leq\ & \left\{ \frac{\mathcal{K} t^{(w-1)(g-1)}(\eta^* + \lambda^*)}{\Gamma(w(1-g)+g)} + \frac{\mathcal{K} t^{(g-1)}}{\Gamma(g)} \times \left\{ \|Q\|\lambda^* + (\mathcal{K}_1 + \mathcal{K}_2 \mathcal{S}_\chi) \right\} \right. \\
& \left. + \frac{\mathcal{K}(t-t_1)^{(w-1)(g-1)} K_1^*}{\Gamma(w(1-g)+g)} \right\} \times \|\nu(t) - \mu(t)\|, \\
\leq\ & \Upsilon_1 \|\nu(t) - \mu(t)\|. \qquad (3.9)
\end{aligned}
$$

Similarly, next prove the contraction mapping for every $\in (t_1, t_2]$ and we get the following inequality:

$$
\begin{aligned}
\|\mathcal{F}(\nu)(t) - \mathcal{F}(\mu)(t)\| \leq\ & \left\{ \frac{\mathcal{K} t^{(w-1)(g-1)}(\eta^* + \lambda^*)}{\Gamma(w(1-g)+g)} + \frac{\mathcal{K} t^{(g-1)}}{\Gamma(g)} \times \left\{ \|Q\|\lambda^* + (\mathcal{K}_1 + \mathcal{K}_2 \mathcal{S}_\chi) \right\} \right. \\
& \left. + \frac{\sum_{\varepsilon=1}^{2} \mathcal{K}(t-t_\varepsilon)^{(w-1)(g-1)} K_\varepsilon^*}{\Gamma(w(1-g)+g)} \right\} \times \|\nu(t) - \mu(t)\|, \\
\leq\ & \Upsilon_2 \|\nu(t) - \mu(t)\|. \qquad (3.10)
\end{aligned}
$$

Proceeding similar way, we prove the contraction mapping for every $t \in (t_\rho, t_{\rho+1}]$ and obtained the following inequality:

$$
\begin{aligned}
\|\mathcal{F}(\nu)(t) - \mathcal{F}(\mu)(t)\| \leq\ & \left\{ \frac{\mathcal{K} t^{(w-1)(g-1)}(\eta^* + \lambda^*)}{\Gamma(w(1-g)+g)} + \frac{\mathcal{K} t^{(g-1)}}{\Gamma(g)} \times \left\{ \|Q\|\lambda^* + (\mathcal{K}_1 + \mathcal{K}_2 \mathcal{S}_\chi) \right\} \right. \\
& \left. + \frac{\sum_{\varepsilon=1}^{\rho} \mathcal{K}(t-t_\varepsilon)^{(w-1)(g-1)} K_\varepsilon^*}{\Gamma(w(1-g)+g)} \right\} \times \|\nu(t) - \mu(t)\|, \\
\leq\ & \Upsilon_\rho \|\nu(t) - \mu(t)\|. \qquad (3.11)
\end{aligned}
$$

From the inequality (3.9), (3.10) and (3.11) we get contraction mapping for every $t \in \mathcal{J}^*$

$$
\begin{aligned}
\|\mathcal{F}(\nu)(t) - \mathcal{F}(\mu)(t)\| \leq\ & sup\{\Upsilon_1, \Upsilon_2, \Upsilon_3, ..., \Upsilon_\rho\} \|\nu(t) - \mu(t)\|, \\
=\ & \Upsilon \|\nu(t) - \mu(t)\|. \qquad (3.12)
\end{aligned}
$$

Hence, from Eq (3.12) we get the contraction mapping $\|\mathcal{F}(\nu)(t) - \mathcal{F}(\mu)(t)\| \leq \Upsilon \|\nu(t) - \mu(t)\|$ for every t on $\mathcal{J}^*$. Since $\Upsilon < 1$ and as a consequence of Banach fixed point theorem, we say that the IHFrNIDE (1.1) has a unique solution on $\mathcal{J}^*$ then using the hypothesis ($\mathcal{H}5$) and definition (2.4) such that

$$
\begin{aligned}
\mathcal{F}(\nu)(t) =\ & S_{w,g}(t)(\nu_0 - \mathcal{G}(\nu) - \Theta(0,\nu(0)) + \Theta(t,\nu(t))) + \int_0^t Q\Psi_g(t-s)\Theta(s,\nu(s))ds \\
& + \int_0^t \Psi_g(t-s)(\mathcal{P}u(s) + \phi(s,\nu(s), \int_0^s \chi(s,m,\nu(m))dm))ds \\
& + \sum_{\varepsilon=1}^{\rho} S_{w,g}(t-t_\varepsilon) I_\varepsilon(\nu(t_\varepsilon^-)), \\
=\ & S_{w,g}(t)(\nu_0 - \mathcal{G}(\nu) - \Theta(0,\nu(0)) + \Theta(t,\nu(t))) + \int_0^t Q\Psi_g(t-s)\Theta(s,\nu(s))ds
\end{aligned}
$$

$$
\begin{aligned}
&+\mathcal{B}\mathcal{B}^{-1}\Big[\nu_t - \mathcal{G}(\nu) - S_{g,w}(t)(\nu_0 - \mathcal{G}(\nu) - \Theta(0,\nu(0)) + \Theta(s,\nu(s))) \\
&-\int_0^t Q\psi_g(t-s)\Theta(s,\nu(s))ds - \int_0^t \psi_g(t-s)\phi(s,\nu(s),\Omega(s))ds \\
&-\sum_{\varepsilon=1}^{\rho} S_{w,g}(t-t_\varepsilon)I_\varepsilon(\nu(t_\varepsilon^-))\Big] + \int_0^t \psi_g(t-s)\phi(s,\nu(s),\Omega(s))ds \\
&+\sum_{\varepsilon=1}^{\rho} S_{w,g}(t-t_\varepsilon)I_\varepsilon(\nu(t_\varepsilon^-)), \\
\mathcal{F}(\nu)(t) \quad &= \quad \nu_t - \mathcal{G}(\nu). \qquad (3.13)
\end{aligned}
$$

From Eq (3.13), we attained the nonlocal controllable result for IHFrNIDE (1.1) with respect to t on $\mathcal{J}^*$. □

4. Applications

Application 1. *Consider the following impulsive Hilfer fractional neutral integro-differential (IHFrNIDE) system*

$$
\begin{aligned}
{}^H D^{\frac{2}{3},\frac{3}{5}}\left(\nu(t) - \int_0^5 e^{-2t}\sin(\nu(t))dt\right) &= Q\nu(t) + \mathcal{P}u(t) + \frac{1}{\Gamma(\pi)}\int_0^5 \frac{\sqrt{(e^{-t}\sin(\nu(t))}}{1+9\sec(\nu(t))}dt, \\
& \quad t \in \mathcal{J}^* := [0,5]\backslash\{1,2,3,4\}, \\
\Delta\nu(t) &= \frac{1}{\pi\sqrt{(\sin(\nu(t_\varepsilon^-)))}}, \varepsilon = 1,2,3,4. \qquad (4.1) \\
I_{0^+}^{0.1333}\nu(0) &= \nu_0 - \frac{3\pi}{43}\sin(\nu(t)).
\end{aligned}
$$

In Table 1, we provided the symbol of assumptions and interpretation of our given application. Let $Q(t) \equiv Q : \mathfrak{D}(Q) \subset \Xi \to \Xi$ is a closed linear bounded operator is defined by $Q\mathcal{E} = \mathcal{E}$ with the domain $\mathcal{D}(Q) = \{\mathcal{E} \in \Xi : \mathcal{E}$ is absolutely continuous, $\mathcal{E}(0) = \mathcal{E}(5) = 0\}$. Let $\mathcal{K} = 0.7, t = 5, g = \frac{3}{5},\ w = \frac{2}{3}$. We assume that the function $\phi : \mathcal{J}^* \times \Xi \to \Xi$ and satisfies the hypothesis $(\mathcal{H}3)$, as follows

$$
\begin{aligned}
&\|\phi(t,\nu(t),\Omega(t)) - \phi(t,\nu^*(t),\Omega^*(t))\| \\
&= \frac{1}{\Gamma(\pi)}\left[\int_0^5 \frac{\sqrt{(e^{-t}\sin(\nu(t)))}}{1+9\sec(\nu(t))}dt - \int_0^5 \frac{\sqrt{(e^{-t}\sin(\nu^*(t)))}}{1+9\sec(\nu^*(t))}dt\right], \\
&\le \frac{1}{\Gamma(\pi)}\|\nu(t) - \nu^*(t)\|.
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
\|\phi(t,\nu(t),\Omega(t))\| &\le \left\|\frac{1}{\Gamma(\pi)}\int_0^5 \frac{\sqrt{(e^{-t}\sin(\nu(t))}}{1+9\sec(\nu(t))}dt\right\|, \\
&\le \frac{1}{\Gamma(\pi)}(\|\nu(t)\| + \|\Omega(t)\| + 1).
\end{aligned}
$$

Let the neutral term $\Theta : \mathcal{J}^* \times \Xi \to \Xi$ and defined by $\Theta(t, \nu(t)) = \int_0^5 e^{-2t}\sin(t)dt$ and $\|\Theta(t, \nu(t))\| \leq 0.2000$ and it is satisfies the hypothesis ($\mathcal{H}4$). The impulsive function $I_\varepsilon : \Xi \to \Xi$ is defined by $I_\varepsilon(\nu(t)) = \frac{1}{\pi\sqrt{(\sin(\nu(t_\varepsilon^-)))}}$ and by employing the hypothesis ($\mathcal{H}1$) we have,

$$\begin{aligned}\left\|\frac{1}{\pi\sqrt{(\sin(\nu(t_\varepsilon^-)))}} - \frac{1}{\pi\sqrt{(\sin(\nu^*(t_\varepsilon^-)))}}\right\| &\leq \frac{1}{\pi}\left\|\frac{1}{\sqrt{(\sin(\nu(t_\varepsilon^-)))}} - \frac{1}{\sqrt{(\sin(\nu^*(t_\varepsilon^-)))}}\right\|, \\ &\leq 0.3183\|\nu(t) - \nu^*(t)\|.\end{aligned}$$

Consider the nonlocal term $\mathcal{G} : \Xi \to \Xi$ and defined by $\mathcal{G}(\nu(t)) = \frac{3\pi}{43}\sin(\nu(t))$ and applying the hypothesis ($\mathcal{H}2$) then obtained the following inequality

$$\begin{aligned}\left\|\frac{3\pi}{43}\sin(\nu(t)) - \frac{3\pi}{43}\sin(\nu^*(t))\right\| &\leq \frac{3\pi}{43}\|\sin(\nu(t)) - \sin(\nu^*(t))\|, \\ &\leq 0.2192\|\sin(\nu(t)) - \sin(\nu^*(t))\|.\end{aligned}$$

Table 1. Symbol of assumptions and interpretation in Application 1.

SI.No	Symbol	Interpretation	Assumptions
1.	$Q(t)$	Closed, linear and bounded operator	$\mathcal{D}(Q) = \{\mathcal{E} \in \Xi : \mathcal{E}(0) = \mathcal{E}(5) = 0\}$
2.	$\Theta(t, \nu(t))$	Netural function	$\int_0^5 e^{-2t}\sin(\nu(t))dt$
3.	$\phi(t, \nu(t), \Omega(t))$	Integro-Differential function	$\frac{1}{\Gamma(\pi)}\int_0^5 \frac{\sqrt{(e^{-t}\sin(\nu(t))}}{1+9\sec(\nu(t))}dt$
4.	$\Delta\nu(t)$	Impulsive function	$\frac{1}{\pi\sqrt{(\sin(\nu(t_\varepsilon^-)))}}$
	$\mathcal{G}(\nu)$ and ν_0	Nonlocal function and initial value	$\frac{3\pi}{43}\sin(\nu(t))$ and $\nu_0 = 0$
6.	w and g	Order of HFD, $0 \leq w \leq 1$ and $0 < g < 1$	$g = \frac{3}{5}$ and $w = \frac{2}{3}$
8.	u(t)	Control function	Square integrable function on $\mathcal{J}^*$ where, $\mathcal{J}^* := [0, 5]\backslash\{1, 2, 3, 4\}$.

Let us consider the map $\mathcal{F} : C(\mathcal{J}^*, \Xi) \to C(\mathcal{J}^*, \Xi)$ and using the Theorem 3.3 as follow the unique solution to Eq (4.1),

$$\begin{aligned}\|\mathcal{F}(\nu(t)) - \mathcal{F}(\nu^*(t))\| \leq &\left\{\left(\frac{0.7 \times 5^{(\frac{2}{3}-1)(\frac{3}{5}-1)}}{\Gamma(\frac{2}{3}(1-\frac{3}{5})+\frac{3}{5})} \times \frac{3\pi}{43}\right) + \left(\frac{0.7 \times 5^{\frac{3}{5}-1}}{\Gamma(\frac{3}{5})} \times \frac{1}{\Gamma(\pi)}\right)\right. \\ &\left. + \frac{0.7 \times \frac{1}{\pi}}{\Gamma(\frac{2}{3}(1-\frac{3}{5})+\frac{3}{5})}\right\} \times \|\nu(t) - \nu^*(t)\|, \\ \leq &\ 0.4844\|\nu(t) - \nu^*(t)\|.\end{aligned}$$

The linear operator $\mathcal{B} : L^2(\mathcal{J}^*, \Xi) \to \Xi$ and defined by $\mathcal{B}u = \int_0^5 \Psi_{\frac{3}{5}}(5-s)\mathcal{P}u(s)ds$, and the inverse linear operator is takes from $\frac{L^2(\mathcal{J}^*,\Xi)}{ker\mathcal{B}}$ and then there exists $\varrho > 0$ such that $\|\mathcal{B}^{-1}\| \leq \varrho$ and manipulating the hypothesis ($\mathcal{H}5$) and definition (2.4) to get the nonlocal controllability for every $t \in [0, 5]$ and our application can be applied to the problem IHFrNIDE (1.1). Figures 1 and 2 are represents the uniqueness of the solution of different parameters with finite time interval for Eq (4.1).

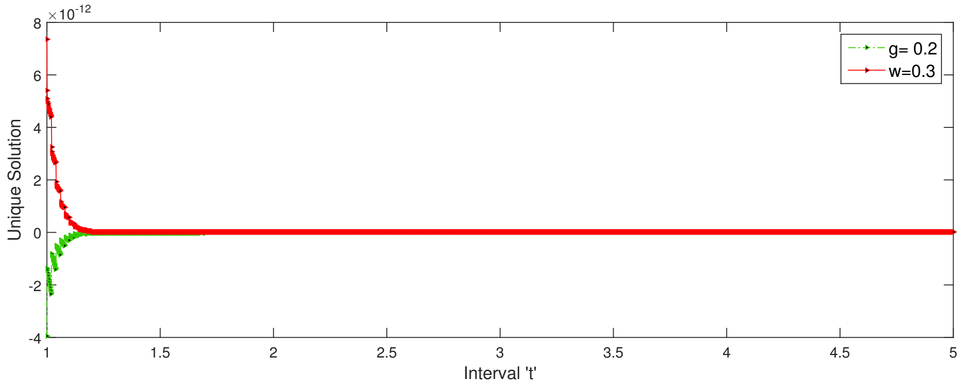


Figure 1. Graphical representation of Hilfer ($w = 0.3, g = 0.2$).

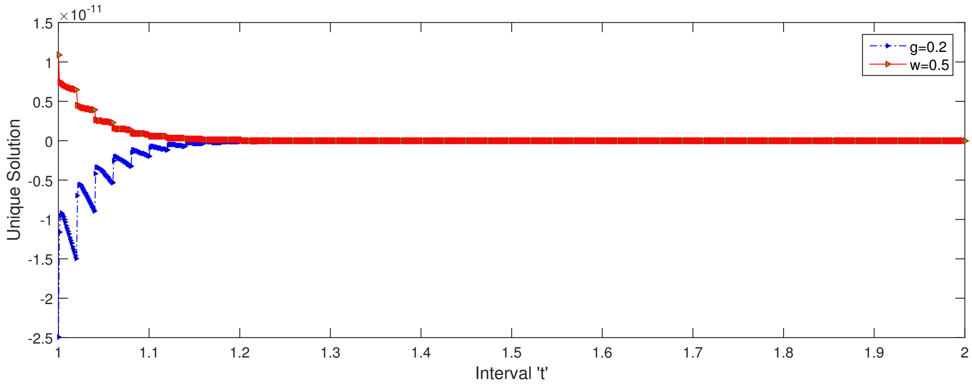
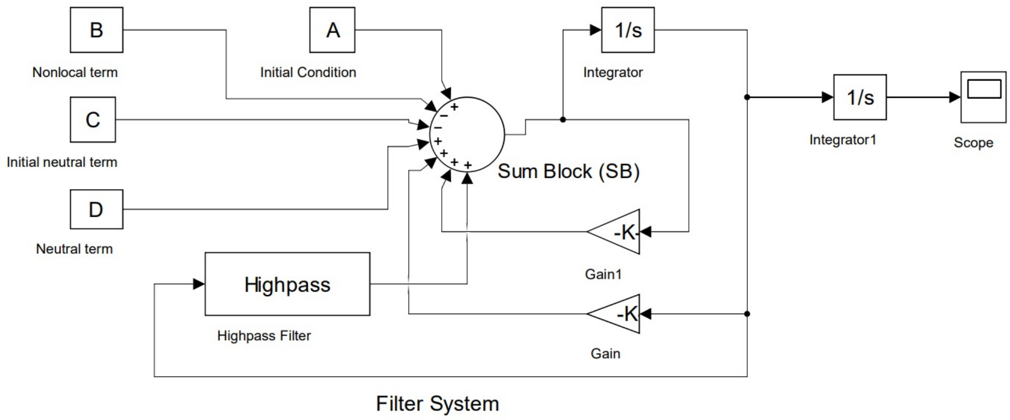


Figure 2. Graphical representation of Hilfer ($w = 0.5, g = 0.2$).

Application 2. (High pass impulsive response filter system)

Filters are an essential component of all signal processing and communication systems. An advantage of a filter system (FS) is that it is used to restrict a signal to a specific frequency band, as in a low-pass filter (LPF), a high-pass filter (HPF), and a band-pass filter (BPF). The finite duration impulse response (FIR) filter and the infinite duration impulse response (IIR) filter are the primary focus of the digital filter class. FIR filters possess significant benefits, such as bounded input-bounded output (BIBO) stability, that make them suitable for widespread applications. Following monographs explain the filter system ([40–42]). FIR filters can be discrete-time or continuous-time, digital or analog. In our model, the filter system includes the high-pass FIR, low-pass FIR, integrator block, and continuous time. Our filter system depicts a block diagram model, which improves the effectiveness of numerical solutions in less time, and the sum block accepts the input values of A,B, C, D, HPF, Gain ($\Theta(t, v(t))$), and Gain 1 ($\mathcal{P}u(t)$) then the overall resultant is connected to the integrator over the interval [0,5]. Where A=$S_{w,g}(t)v_0$ is the initial condition, B=$S_{w,g}(t)\mathcal{G}(v)$ is a nonlocal term, C=$S_{w,g}(t)\Theta(0, v(0))$ is initial neutral term, and D=$S_{w,g}(t)\Theta(t, v(t))$ is neutral term. The output of integrator is connected to a high-pass filter and merged with integrator 1. Finally, all blocks combine to the scope block, and hence the output $v(t)$ is attained, which is bounded and nonlocally controllable on $\mathcal{J}^*$. The output of our filter system is represented in Figures 3 and 4.



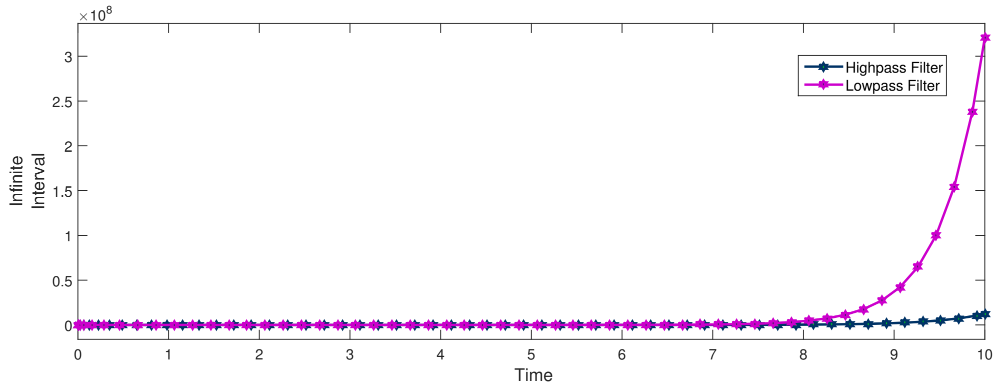


Figure 3. Output of the Filter system.

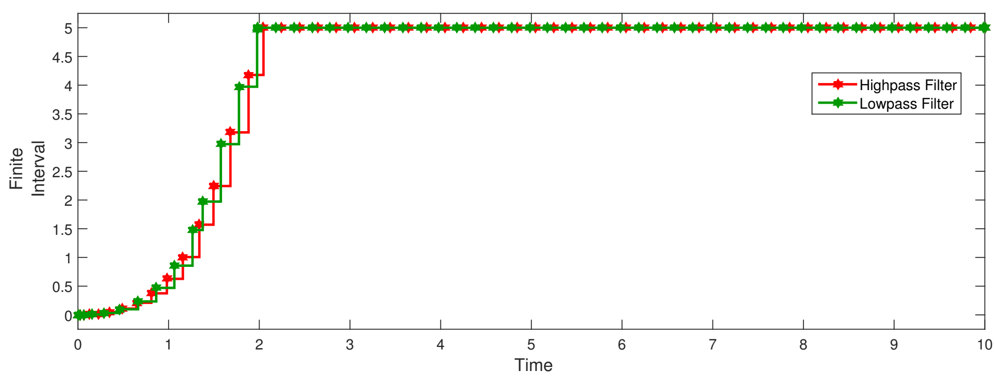


Figure 4. Output of filter system with upper limit '5' and lower limit '0'.

5. Conclusions

In Banach space, we demonstrate the mild solution of the Hilfer neutral impulsive fractional integro-differential equation. The nonlocal controllability results are attained by uniform operator, linear operator, bounded operator, strongly continuous operator, iterative processes, and fixed point techniques. Eventually, an appropriate application was given to enhance the effectiveness and applicability of our proposed work. In the future, we will extend our results to the nonlocal controllability analysis of ψ-Hilfer fractional differential equation with non-instantaneous impulses and state-dependent delay.

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Conflict of interest

The authors declare no conflict of interest.

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