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Research article

The existence, uniqueness, and stability analyses of the generalized Caputo-type fractional boundary value problems

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Abstract: In this article, we derive some novel results of the existence, uniqueness, and stability of the solution of generalized Caputo-type fractional boundary value problems (FBVPs). The Banach contraction principle, along with necessary features of fixed point theory, is used to establish our results. An example is illustrated to justify the validity of the theoretical observations.

Keywords: generalized Caputo derivative; fractional boundary value problem; existence; uniqueness, Ulam-Hyers stability
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1. Introduction

In the present scenario, fractional calculus has been used as an empirical tool for exploring the features of real-life phenomena. In recent years, there has been considerable growth in ordinary and partial differential equations involving fractional derivative operators. The most well-known derivative operators are Riemann-Liouville, Caputo, Hilfer, Hadamard, Katugampola, etc. [1, 2]. Consequently, this led to studying various fractional differential equations (FDEs) defined by several fractional operators. Recent research on the existence and uniqueness of solutions for generalized FDEs is found in [3–5].

In particular, many novel results on the existence and stability of solutions for boundary value problems (BVPs) using different categories of FDEs have been discussed. In [6], the authors derived some results on the existence and uniqueness of solutions for a class of nonlinear FBVPs. In [7], the authors used ψ-Hilfer fractional derivative on b-metric spaces to derive the existence of positive solutions of BVPs. In [8], some stability analysis for BVPs with generalized nonlocal conditions via Hilfer-Katugampola fractional derivative was proposed. The authors in [9] analyzed a Hadamard-type integral BVP of a coupled system of FDEs. In [10], the authors derived positive solutions for BVPs of nonlinear FDEs. In [11], the authors considered ψ-Caputo FDEs with multi-point boundary conditions. In [12], the researchers derived positive solutions for FBVPs under a generalized operator. In [13], Khuddush et al. derived existence, uniqueness, and stability analyses of tempered thermistor FBVPs. In [14], Matar et al. investigated the p-Laplacian nonperiodic nonlinear BVP via generalized Caputo-type fractional derivatives. In [15], the authors derived existence and stability analysis for a system of BVPs. In [16], some analyses of the solution of a fractional BVP were derived. In [17], the authors simulated BVP for Hilfer fractional differential inclusions with nonlocal integral boundary conditions. In [18], Erturk et al. proposed some existence and stability results for nonlocal fractional BVP. In [19], Bekri et al. investigated the existence and uniqueness of a nonlinear q-difference fractional BVP. In [20], the authors derived some novel analyses of two different Caputo-type FBVPs. In [21], the authors studied FBVPs of Caputo-type along with Ulam-type stability. In [22], the researchers analyzed the stability of multipoint FBVPs with non-instantaneous integral impulse. In [23], the authors derived some novel results on the existence and Ulam-Hyers stability for a system of coupled generalized Liouville-Caputo Langevin equations with multipoint boundary conditions. In [24], the authors derived a BVP of Riemann-Liouville FDEs in the variable exponent Lebesgue spaces $L^{p(.)}$ given by

$$\begin{cases} D_a^{\sigma} y(t) = f(t, y(t)), \qquad t \in I := [0, T], \\ \gamma y(0) + \mu y(b) = c, \end{cases} \tag{1.1}$$

and proved the existence and Hyers-Ulam stability of the solution.

For the same BVP (1.1), the existence and stability of solutions in terms of generalized Caputo-type derivative still need to be studied. Therefore, we deal with the existence and stability of the solution in $L^{p(.)}(I, \mathbb{R})$ for the following generalized Caputo-type FBVP:

$$\begin{cases} D_a^{(\rho,\alpha)} y(t) = f(t, y(t)), \qquad t \in I := [0, T], \\ \gamma y(0) + \mu y(b) = c, \end{cases} \tag{1.2}$$

where $0 < \alpha < 1$, $\rho > 0$, $f(., y(.)) \in L^{p(.)}(I \times \mathbb{R}, \mathbb{R})$ and $y \in L^{p(.)}(I, \mathbb{R}), c, T, \mu, \gamma$, with $\mu \neq 0$ are real constants. The motivation behind using the generalized Caputo derivative in the proposed problem is an advanced feature of this fractional derivative where an extra parameter ρ is present with the fractional order α, which adds an extra degree of freedom in the system. Therefore, the proposed generalized Caputo problem (1.2) looks realistic compared to the previous Eq (1.1).

The Generalized Caputo fractional derivative for $y(t) \in L^{p(.)}$ is defined by [25]

$$D_a^{(\rho,\alpha)} y(t) = \frac{\rho^{1-\alpha}}{\Gamma(1-\alpha)} \int_{T_{j-1}}^{t} (t^\rho - s^\rho)^{-\alpha} y'(s) ds, \tag{1.3}$$

where $\Gamma(.)$is the Gamma function.

Relatively, the generalized Caputo fractional integral of order α where $\rho > 0$ for the function of $y(t)$ in $L^{p(.)}$ is defined by [25]

$$I_{a}^{(\rho,\alpha)}y(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int\limits_{T_{j-1}}^{t} (t^{\rho} - s^{\rho})^{\alpha-1} s^{\rho-1} y(s) ds. \tag{1.4}$$

The novel contributions of this work are the analyses of the existence and uniqueness of the solution and the Ulam-Hyers stability, along with novel examples. The organization of the article is as follows: In Section 3, we investigate the existence and uniqueness of solutions for the given BVP involving generalized Caputo fractional derivative. In Section 4, the Ulam-Hyers (UH) stability of the obtained solution is proved. In Section 5, an illustrative example is investigated to clarify the validity of the theoretical results. In Section 6, we conclude our findings.

2. Preliminaries

Here we remind the following lemma and definition.

Lemma 2.1. *[26] If $\alpha \in \mathbb{R}^{+}, \rho > 0$, then $D_{\varrho^{+}}^{(\alpha,\rho)}\Theta = 0$, has a unique solution $\Theta(t) = \beta_1(t^{\rho} - \varrho^{\rho})^{\alpha-1} + \beta_2(t^{\rho} - \varrho^{\rho})^{\alpha-2} + ... + \beta_n(t^{\rho} - \varrho^{\rho})^{\alpha-n}, \beta_j \in \mathbb{R}, 1 \leq j \leq n$, here $n = [\alpha] + 1$.*

Lemma 2.2. *[26] Let $\varrho > 0$, $\Theta \in L^{1}(I, \mathbb{R}), D_{\varrho^{+}}^{(\alpha,\rho)}\Theta \in L^{1}(I, \mathbb{R})$, then $I_{\varrho^{+}}^{(\alpha,\rho)}D_{\varrho^{+}}^{(\alpha,\rho)}\Theta(t) = \Theta(t) + \beta_1(t^{\rho} - \varrho^{\rho})^{\alpha-1} + \beta_2(t^{\rho} - \varrho^{\rho})^{\alpha-2} + ... + \beta_n(t^{\rho} - \varrho^{\rho})^{\alpha-n}, \beta_j \in \mathbb{R}, 1 \leq j \leq n$, here $n = [\alpha] + 1$.*

Definition 2.1. *The FBVP* (1.2) *is Ulam-Hyers stable (UH) if there exists $c_g > 0$, such that for any $\epsilon > 0$ and for every solution $x \in L^{p}(I, \mathbb{R})$ of the equation*

$$|D_{0}^{(\rho,\alpha)}x(t) - f(t, x(t))| \leq \epsilon, \ t \in I, \tag{2.1}$$

there exists a solution $y \in L^{p}(I, \mathbb{R})$ of problem (2) with

$$|x(t) - y(t)| \leq c_g \epsilon, \ t \in I. \tag{2.2}$$

3. Existence and uniqueness analysis

Here we derive our main results to prove the existence of a unique solution of the proposed problem Eq (1.2). Firstly, we discuss the following assumptions.

(H1) Let $n \in \mathbb{N}$ and $(T_j^{\rho})_{j=0}^{n}$ such that $0 = T_0^{\rho} < T_j^{\rho} < T_n^{\rho} = T^{\rho}, \quad j = 1, 2, ..., n-1$. If $I_j = (T_{j-1}^{\rho}, T_j^{\rho}], j = 1, 2, ..., n$ then $\mathcal{P} = \cup_{j=1}^{n} I_j$ is a partition of the interval I. The Banach space of measurable function $E_j = L^{p_j}(I_j, \mathbb{R})$ with $j = 1, 2, ..., n$. from I_j into $\mathbb{R}$ equipped with

$$\|y\|_{E_j} = \Big(\int\limits_{I_j} |y|^{p_j} dx \Big)^{\frac{1}{p_j}} < \infty, \tag{3.1}$$

where $1 \leq j \leq n$.

Consider $p(t) : I \to [1,\infty)$ be a piecewise constant function with respect to $\mathcal{P}$, such that, $p(t) = \sum_{j=1}^{n} p_j I_j(t)$ where $1 \le p_j < \infty$ are constant and I_j is the indicator of the interval I_j, $j = 1, 2, ..., n$,

$$I_j(t) = \begin{cases} 1, & \text{for } t \in I_j \\ 0, & \text{elsewhere.} \end{cases}$$

Therefore, for any $t \in I_j, 1 \le j \le n$, the generalized Caputo derivative (1.3) can be represented as a sum

$$\begin{aligned} (D_a^{(\rho,\alpha)} y)(t) &= \frac{\rho^\alpha}{\Gamma(1-\alpha)} \int_0^t (t^\rho - s^\rho)^{-\alpha} y'(s) ds \\ &= \frac{\rho^\alpha}{\Gamma(1-\alpha)} \Big(\sum_{i=1}^{j-1} \int_{T_{i-1}}^{T_i} (t^\rho - s^\rho)^{-\alpha} y'(s) ds + \int_{T_{j-1}}^{t} (t^\rho - s^\rho)^{-\alpha} y'(s) ds \Big). \end{aligned} \tag{3.2}$$

Thus, Eq (1.2) has the form

$$= \frac{\rho^\alpha}{\Gamma(1-\alpha)} \Big(\sum_{i=1}^{j-1} \int_{T_{i-1}}^{T_i} (t^\rho - s^\rho)^{-\alpha} y'(s) ds + \int_{T_{j-1}}^{t} (t^\rho - s^\rho)^{-\alpha} y'(s) ds \Big) = f(t, y(t)), \tag{3.3}$$

where $t \in I_j, 1 \le j \le n$. Assume that $\tilde{x} \in L^{p_j}(I_j)$ be such that $\tilde{x} = 0$ on $t \in [1, T_{j-1}]$ and it satisfies inequality (3.3), then (3.3) reduce to

$$D_{T_{j-1}}^{(\alpha,\rho)} \tilde{x}(t) = g(t, \tilde{x}(t)), t \in I_j.$$

Let us consider the auxiliary BVP for generalized Caputo FDEs

$$\begin{cases} D_a^{(\rho,\alpha)} y(t) = f(t, y(t)), & t \in I_j, 1 \le j \le n, \\ \gamma y(T_{j-1}) + \mu y(T_j) = c. \end{cases} \tag{3.4}$$

Lemma 3.1. *Let $j \in N \cap [1, n], 0 < \alpha < 1$, and $g \in L^{p_j}(I_j \times \mathbb{R}, \mathbb{R})$. A function $y_j \in E_j$ is a solution of* (3.4) *if and only if $y_j \in E_j$ solves*

$$\begin{aligned} y(t) \;\; = \;\; & \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{T_{j-1}}^{t} (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} f(s, y_j(s)) ds - \frac{(T_j^\alpha - T_{j-1}^\alpha)^{1-\alpha} (t^\alpha - T_{j-1}^\alpha)^{\alpha-1} (\rho^{1-\alpha})}{\Gamma(\alpha)} \\ & \times \int_{T_{j-1}}^{T_j} (T_j^\rho - s^\rho)^{\alpha-1} s^{\rho-1} f(s, y_j(s)) ds + \frac{c(T_j^\rho - T_{j-1}^\alpha)^{1-\alpha} (t^\rho - T_{j-1}^\alpha)^{\alpha-1}}{\mu}. \end{aligned} \tag{3.5}$$

Proof. Let y_j solves the Eq (3.4), then Lemma 2.2 gives that

$$y_j(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{T_{j-1}}^{t} (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} f(s, y_j(s)) ds - \beta (t^\alpha - T_{j-1}^\alpha)^{\alpha-1}.$$

By the boundary value of (3.4), we get

$$\beta=\frac{(T_j^{\alpha}-T_{j-1}^{\alpha})^{1-\alpha}(\rho^{1-\alpha})}{\Gamma(\alpha)}\int\limits_{T_{j-1}}^{T_j}(T_j^{\rho}-s^{\rho})^{\alpha-1}s^{\rho-1}f(s,y_j(s))ds-\frac{c(T_j^{\rho}-T_{j-1}^{\alpha})^{1-\alpha}}{\mu}.$$

Hence, we get Eq (3.5).

Besides, obviously y_j solves the FBVP (3.4). □

Theorem 3.1. *If Lemma 3.1 is satisfied and there exist $M>0$ such that $|g(t,y_1)-g(t,y_2)|\leq M|y_1-y_2|$ for any $y_1,y_2\in L^{p_j}(I_j)$ and $t\in I_j$, and the inequality*

$$W_{\alpha,M,p_j,T_{j-1},T_j}<1, \tag{3.6}$$

holds, where

$$\begin{aligned}W_{\alpha,M,p_j,T_{j-1},T_j} &= \Bigg(\frac{1}{\rho^{\alpha p_j+1}}\Big[\frac{2M}{\rho^{\alpha}\Gamma(\alpha)(q_j(\alpha-1)+1)^{\frac{1}{q_j}}}\Big]^{p_j}\frac{(T_j^{\rho}-T_{j-1}^{\rho})^{\frac{p_j(q_j(\alpha-1)+1)}{q_j}+1}}{(\frac{p_j(q_j(\alpha-1)+1)}{q_j}+1)\rho T_j^{\rho-1}}\\ &\quad+\frac{1}{\rho^{\alpha p_j+1}}\Big[\frac{2M(T_j^{\rho}-T_{j-1}^{\rho})^{(1-\alpha)+\frac{q_j(\alpha-1)+1}{q_j}}}{\rho^{\alpha}\Gamma(\alpha)(q_j(\alpha-1)+1)^{\frac{1}{q_j}}}\Big]^{p_j}\frac{(T_j^{\rho}-T_{j-1}^{\rho})^{p_j(\alpha-1)+1}}{p_j(\alpha-1)+1)\rho T_j^{\rho-1}}\Bigg)^{\frac{1}{p_j}}.\end{aligned}$$

Then, for every $1\leq j\leq n$ there exist a unique solution I_j for FBVP (3.4).

Proof. Let us convert the Eq (3.4) into a fixed point problem.

Consider the operator $S:L^{p_j}(I_j,\mathbb{R})\to L^{p_j}(I_j,\mathbb{R})$ derived by

$$\begin{aligned}S(f)(t) &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int\limits_{T_{j-1}}^{t}(t^{\rho}-s^{\rho})^{\alpha-1}s^{\rho-1}f(s,f(s))ds-\frac{(T_j^{\alpha}-T_{j-1}^{\alpha})^{1-\alpha}(t^{\alpha}-T_{j-1}^{\alpha})^{\alpha-1}(\rho^{1-\alpha})}{\Gamma(\alpha)}\\ &\quad\times\int\limits_{T_{j-1}}^{T_j}(T_j^{\rho}-s^{\rho})^{\alpha-1}s^{\rho-1}f(s,f(s))ds+\frac{c(T_j^{\rho}-T_{j-1}^{\alpha})^{1-\alpha}(t^{\rho}-T_{j-1}^{\alpha})^{\alpha-1}}{\mu}.\end{aligned}$$

Using Banach contraction principle, we will prove that S has a unique fixed point.

Let $y_1,y_2\in L^{p_j}(I_j)$,

$$\begin{aligned}&\Big\|S(y_1(t))-S(y_2(t))\Big\|_{E_j}^{p_j}\\ &=\Bigg\|\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int\limits_{T_{j-1}}^{t}(t^{\rho}-s^{\rho})^{\alpha-1}s^{\rho-1}(f(s,y_1(s))-f(s,y_2(s)))ds-\frac{(T_j^{\alpha}-T_{j-1}^{\alpha})^{1-\alpha}(t^{\alpha}-T_{j-1}^{\alpha})^{\alpha-1}}{\Gamma(\alpha)}\\ &\times\int\limits_{T_{j-1}}^{T_j}(T_j^{\rho}-s^{\rho})^{\alpha-1}s^{\rho-1}(f(s,y_1(s))-f(s,y_2(s))ds\Bigg\|_{E_j}^{p_j}\end{aligned}$$

$$= \int_{T_{j-1}}^{T_j} \Big| \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{T_{j-1}}^{t} (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} (f(s, y_1(s)) - f(s, y_2(s))) ds - \frac{(T_j^\alpha - T_{j-1}^\alpha)^{1-\alpha} (t^\alpha - T_{j-1}^\alpha)^{\alpha-1} (\rho^{1-\alpha})}{\Gamma(\alpha)}$$

$$\times \int_{T_{j-1}}^{T_j} (T_j^\rho - s^\rho)^{\alpha-1} s^{\rho-1} (f(s, y_1(s)) - f(s, y_2(s)) ds) \Big|^{p_j} dt$$

$$\leq \frac{2^{p_j}}{(\Gamma(\alpha))^{p_j}} \int_{T_{j-1}}^{T_j} |\rho^{1-\alpha} \int_{T_{j-1}}^{t} (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} (f(s, y_1(s)) - f(s, y_2(s)) ds)|^{p_j} dt$$

$$+ \frac{2^{p_j} (T_j^\rho - T_{j-1}^\rho)^{(1-\alpha)p_j}}{(\Gamma(\alpha))^{p_j}} \int_{T_{j-1}}^{T_j} |\rho^{1-\alpha} (t^\rho - T_{j-1}^\rho)^{\alpha-1} \int_{T_{j-1}}^{T_j} (T_j^\rho - s^\rho)^{\alpha-1} s^{\rho-1} (f(s, y_1(s)) - f(s, y_2(s)) ds)|^{p_j} dt$$

$$\leq \frac{(2M)^{p_j}}{(\Gamma(\alpha))^{p_j}} \int_{T_{j-1}}^{T_j} (\rho^{1-\alpha} \int_{T_{j-1}}^{t} (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} |y_1(s) - y_2(s)| ds)^{p_j} dt$$

$$+ \frac{(2M)^{p_j} (T_j^\rho - T_{j-1}^\rho)^{(1-\alpha)p_j}}{(\Gamma(\alpha))^{p_j}} \int_{T_{j-1}}^{T_j} (t^\rho - T_{j-1}^\rho)^{\alpha-1} (\rho^{1-\alpha} \int_{T_{j-1}}^{T_j} (T_j^\rho - s^\rho)^{\alpha-1} s^{\rho-1} |y_1(s) - y_2(s)| ds)^{p_j} dt$$

$$\leq \frac{(2M)^{p_j}}{(\Gamma(\alpha))^{p_j}} \int_{T_{j-1}}^{T_j} \Big([(\rho^{1-\alpha} \int_{T_{j-1}}^{t} (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} ds)^{q_j}]^{\frac{1}{q_j}} [\int_{T_{j-1}}^{t} |y_1(s)) - y_2(s)|^{p_j} ds]^{\frac{1}{p_j}} \Big)^{p_j} dt$$

$$+ \frac{(2M)^{p_j} (T_j^\rho - T_{j-1}^\rho)^{(1-\alpha)p_j}}{(\Gamma(\alpha))^{p_j}} \int_{T_{j-1}}^{T_j} (t^\rho - T_{j-1}^\rho)^{(\alpha-1)p_j} \Big([(\rho^{1-\alpha} \int_{T_{j-1}}^{T_j} (T_j^\rho - s^\rho)^{\alpha-1} s^{\rho-1} ds)^{q_j}]^{\frac{1}{q_j}}$$

$$\times (\int_{T_{j-1}}^{T_j} |y_1(s) - y_2(s)|^{p_j} ds)^{\frac{1}{p_j}} \Big)^{p_j} dt$$

$$\leq \frac{(2M)^{p_j}}{\rho^{\alpha p_j} (\Gamma(\alpha))^{p_j}} \int_{T_{j-1}}^{T_j} \frac{(t^\rho - T_{j-1}^\rho)^{\frac{p_j(q_j(\alpha-1)+1)}{q_j}}}{(q_j(\alpha-1)+1)^{\frac{p_j}{q_j}}} \Big(\int_{T_{j-1}}^{t} |y_1(s) - y_2(s)|^{p_j} ds \Big) dt + \frac{(2M)^{p_j} (T_j^\rho - T_{j-1}^\rho)^{(1-\alpha)p_j}}{\rho^{\alpha p_j} (\Gamma(\alpha))^{p_j}}$$

$$\times \int_{T_{j-1}}^{T_j} (t^\rho - T_{j-1}^\rho)^{(\alpha-1)p_j} \quad \frac{(T_j^\rho - T_{j-1}^\rho)^{\frac{p_j(q_j(\alpha-1)+1)}{q_j}}}{(q_j(\alpha-1)+1)^{\frac{p_j}{q_j}}} \|y_1 - y_2\|_{E_j}^{p_j} dt$$

$$\leq \Big(\frac{2M}{\rho^\alpha \Gamma(\alpha) (q_j(\alpha-1)+1)^{\frac{1}{q_j}}} \Big)^{p_j} \int_{T_{j-1}}^{T_j} (t^\rho - T_{j-1}^\rho)^{\frac{p_j(q_j(\alpha-1)+1)}{q_j}} \Big(\int_{T_{j-1}}^{t} |y_1(s) - y_2(s)|^{p_j} ds \Big) dt$$

$$+\Big(\frac{2M(T_j^\rho - T_{j-1}^\rho)^{(1-\alpha)+\frac{q_j(\alpha-1)+1}{q_j}}}{\rho^\alpha\Gamma(\alpha)(q_j(\alpha-1)+1)^{\frac{1}{q_j}}}\Big)^{p_j}\int_{T_{j-1}}^{T_j}(t^\rho - T_{j-1}^\rho)^{(\alpha-1)p_j}\,\|y_1-y_2\|_{E_j}^{p_j}dt.$$

We put

$$\int_{T_{j-1}}^{T_j}(t^\rho - T_{j-1}^\rho)^{\frac{p_j(q_j(\alpha-1)+1)}{q_j}}\Big(\int_{T_{j-1}}^{t}|y_1(s)-y_2(s)|^{p_j}ds\Big)dt = \int_{T_{j-1}}^{T_j}(t^\rho - T_{j-1}^\rho)^{\theta_j}\,r_j(t)dt = I,$$

where

$$\theta_j = \frac{p_j(q_j(\alpha-1)+1)}{q_j},\quad r_j(t) = \int_{T_{j-1}}^{t}|y_1(s)-y_2(s)|^{p_j}ds.$$

Integrating by parts, we have

$$\begin{aligned} I &= \frac{(T_j^\alpha - T_{j-1}^\alpha)^{\theta_j+1}}{(\theta_j+1)\rho T_j^{\rho-1}}r_j(T_j) - \int_{T_{j-1}}^{T_j}\frac{(T_j^\rho - T_{j-1}^\rho)^{\theta_j+1}}{(\theta_j+1)\rho T_j^{\rho-1}}\,r_j'(t)dt \\ &= \frac{(T_j^\alpha - T_{j-1}^\alpha)^{\theta_j+1}}{(\theta_j+1)\rho T_j^{\rho-1}}\|y_1-y_2\|_{E_j}^{p_j} - \int_{T_{j-1}}^{T_j}\frac{(T_j^\rho - T_{j-1}^\rho)^{\theta_j+1}}{(\theta_j+1)\rho T_j^{\rho-1}}\,r_j'(t)dt. \end{aligned}$$

Since, the integral

$$\int_{T_{j-1}}^{T_j}\frac{(T_j^\rho - T_{j-1}^\rho)^{\theta_j+1}}{(\theta_j+1)\rho T_j^{\rho-1}}\,r_j'(t)dt \geq 0,$$

then,

$$\begin{aligned} &\|S(y_1(t)) - S(y_2(t))\|_{E_j}^{p_j} \\ &\leq \Big[\frac{2M}{\rho^\alpha\Gamma(\alpha)(q_j(\alpha-1)+1)^{\frac{1}{q_j}}}\Big]^{p_j}\frac{(T_j^\rho - T_{j-1}^\rho)^{\theta_j+1}}{(\theta_j+1)\rho T_j^{\rho-1}}\|y_1-y_2\|_{E_j}^{p_j} \\ &\quad+\Big[\frac{2M(T_j^\rho - T_{j-1}^\rho)^{(1-\alpha)+\frac{q_j(\alpha-1)+1}{q_j}}}{\rho^\alpha\Gamma(\alpha)(q_j(\alpha-1)+1)^{\frac{1}{q_j}}}\Big]^{p_j}\frac{(T_j^\rho - T_{j-1}^\rho)^{p_j(\alpha-1)+1}}{(p_j(\alpha-1)+1)\rho T_j^{\rho-1}}\|y_1-y_2\|_{E_j}^{p_j} \\ &\leq \Big(\Big[\frac{2M}{\rho^\alpha\Gamma(\alpha)(q_j(\alpha-1)+1)^{\frac{1}{q_j}}}\Big]^{p_j}\frac{(T_j^\rho - T_{j-1}^\rho)^{\frac{p_j(q_j(\alpha-1)+1)}{q_j}+1}}{(\frac{p_j(q_j(\alpha-1)+1)}{q_j}+1)\rho T_j^{\rho-1}} \\ &\quad+\Big[\frac{2M(T_j^\rho - T_{j-1}^\rho)^{(1-\alpha)+\frac{q_j(\alpha-1)+1}{q_j}}}{\rho^\alpha\Gamma(\alpha)(q_j(\alpha-1)+1)^{\frac{1}{q_j}}}\Big]^{p_j}\frac{(T_j^\rho - T_{j-1}^\rho)^{p_j(\alpha-1)+1}}{(p_j(\alpha-1)+1)\rho T_j^{\rho-1}}\Big)\|y_1-y_2\|_{E_j}^{p_j}, \end{aligned}$$

$$
\begin{aligned}
&\|S(y_1(t)) - S(y_2(t))\|_{E_j} \\
&\leq \Big(\frac{1}{\rho^{\alpha p_j+1}}\Big[\frac{2M}{\Gamma(\alpha)(q_j(\alpha-1)+1)^{\frac{1}{q_j}}}\Big]^{p_j}\frac{(T_j^{\rho}-T_{j-1}^{\rho})^{\frac{p_j(q_j(\alpha-1)+1)}{q_j}+1}}{(\frac{p_j(q_j(\alpha-1)+1)}{q_j}+1)T_j^{\rho-1}} \\
&\quad+\frac{1}{\rho^{\alpha p_j+1}}\Big[\frac{2M(T_j^{\rho}-T_{j-1}^{\rho})^{(1-\alpha)+\frac{q_j(\alpha-1)+1}{q_j}}}{\Gamma(\alpha)(q_j(\alpha-1)+1)^{\frac{1}{q_j}}}\Big]^{p_j}\frac{(T_j^{\rho}-T_{j-1}^{\rho})^{p_j(\alpha-1)+1}}{(p_j(\alpha-1)+1)T_j^{\rho-1}}\Big)^{\frac{1}{p_j}}\|y_1-y_2\|_{E_j}.
\end{aligned}
$$

Therefore, from inequality (3.6), the operator S is a contraction. Using Banach contraction principle, S exists a unique fixed point $\tilde{y}_i \in L^{p_j}(I_j)$, which is a unique solution of the FBVP (3.4). □

Now we will derive the existence proof for FBVP (1.2). We consider the following assumption:
(H2) For the function f, there exist constant $M > 0$, such that,

$$|f(t, y_1) - f(t, y_2)| \leq M|y_1 - y_2|, \; for\ any\ y_1, y_2 \in L^{p(.)}(I)\ and\ t \in I.$$

Theorem 3.2. *Let (H1), (H2) and inequality* (3.6) *hold for all* $1 \leq \rho \leq n$. *Then, the FBVP* (1.2) *exists a unique solution in* $L^{p(.)}(I)$.

Proof. From Theorem 3.1 for $1 \leq j \leq n$, the Generalized Caputo FBVP (3.4) posses a unique solution $\tilde{y} \in E_j$.

For $1 \leq j \leq n$, we derive the function

$$y_j = \begin{cases} 0 & \text{if } t \in [0,\ T_{j-1}], \\ \tilde{y} & \text{if } t \in I_j. \end{cases}$$

Thus, the function $y_j \in L^p([0, T_{j-1}], \mathbb{R})$ solves the integral equation (3.3) for $t \in I_j$.

Then,

$$y(t) = \begin{cases} y_1(t) \in L^{p_1}(I_1, \mathbb{R}), \\ y_2(t) \in L^{p_2}(I_2, \mathbb{R}), \\ . \\ . \\ . \\ y_n(t) \in L^{p_n}(I_n, \mathbb{R}), \end{cases}$$

is a unique solution of the FBVP (1.2) in $L^{p(.)}(I)$. □

4. Ulam-Hyers (UH) stability

Theorem 4.1. *Let (H1), (H2), and expression* (3.6) *holds. Then, FBVP* (1.2) *is UH stable.*

Proof. Let $\epsilon > 0$ be an arbitrary number and the function $z(t)$ from $z \in L^{p_j}(I_j, \mathbb{R})$ satisfies inequality (2.1). For any $j \in 1, 2, ..., n$, we define the function $z_1 \equiv z(t), t \in [0,\ T_1]$ and for $j = 2, 3, ..., n$,

$$z_j(t) = \begin{cases} 0 & \text{if } t \in [0, T_{j-1}], \\ z(t) & \text{if } t \in I_j. \end{cases} \tag{4.1}$$

From Eq (3.2) for any $j \in 1, 2, ..., n$ and $t \in I_j$, we have

$$D^{\rho,\alpha}_{0^+} z_j(t) = \frac{\rho^{1-\alpha}}{\Gamma(1-\alpha)} \int_{T_{j-1}}^{t} (t^\rho - s^\rho)^{m-\alpha-1} s^{\rho-1} z(s) ds.$$

Taking $I^\alpha_{T^+_{j-1}}$ on both sides of the inequality (2.1), we obtain

$$\begin{aligned} &\Bigg| z_j(t) - \frac{\rho^{1-\alpha}}{\Gamma\alpha} \int_{T_{j-1}}^{t} (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} f(s, z_j(s)) ds \\ &+ \frac{\rho^{1-\alpha}(T_j^\rho - T_{j-1}^\rho)^{1-\alpha}(t^\rho - T_{j-1}^\rho)^{\alpha-1}}{\Gamma(\alpha)} \int_{T_{j-1}}^{T_j} (T_j^\rho - s^\rho)^{\alpha-1} s^{\rho-1} f(s, z_j(s)) ds - \frac{c(T_j^\rho - T_{j-1}^\rho)^{1-\alpha}(t^\rho - T_{j-1}^\rho)^{\alpha-1}}{\mu} \Bigg| \\ &\leq \epsilon \frac{(T_j^\rho - T_{j-1}^\rho)^\alpha}{\rho^\alpha \Gamma(\alpha+1)(T_{j-1}^{\rho-1})}. \end{aligned}$$

According to Theorem 3.2, the FBVP (1.2) exists a unique solution $y \in L^{p(\cdot)}(I)$ given by

$$y(t) = y_j(t)\ for\ t \in I_j,\ \ j = 1, 2, ..., n.$$

Here

$$y_j = \begin{cases} 0 & \text{if } t \in [0, T_{j-1}], \\ \tilde{y}_j & \text{if } t \in I_j. \end{cases} \tag{4.2}$$

$\tilde{y}_j \in E_j$ is a unique solution of the FBVP (3.4).

From Lemma 3.1, the integral equation

$$\begin{aligned} \tilde{y}_j(t) =& \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{T_{j-1}}^{t} (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} f(s, \tilde{y}_j(s)) ds - \frac{\rho^{1-\alpha}(T_j^\rho - T_{j-1}^\rho)^{1-\alpha}(t^\rho - T_{j-1}^\rho)^{\alpha-1}}{\Gamma(\alpha)} \\ &\times \int_{T_{j-1}}^{T_j} (T_j^\rho - s^\rho)^{\alpha-1} s^{\rho-1} f(s, \tilde{y}_j(s)) ds + \frac{c(T_j^\rho - T_{j-1}^\rho)^{1-\alpha}(t^\rho - T_{j-1}^\rho)^{\alpha-1}}{\mu} \end{aligned} \tag{4.3}$$

holds.

For $t \in I_j$, $j = 1, 2, ..., n$, by (4.1) and (4.2), we have

$$|z(t) - y(t)| = |z(t) - y_j(t)| = |z_j(t) - \tilde{y}_j(t)|.$$

Then, by (4.3), we get

$$
\begin{aligned}
&\|z-y\|_{E_j}^{p_j}=\|z-y_j\|_{E_j}^{p_j}=\|z_j-\tilde{y}_j\|_{E_j}^{p_j}\\
&=\Big\|z_j(t)-\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{T_{j-1}}^{t}(t^\rho-s^\rho)^{\alpha-1}s^{\rho-1}f(s,\tilde{y}_j(s))ds+\frac{\rho^{1-\alpha}(T_j^\rho-T_{j-1}^\rho)^{1-\alpha}(t^\rho-T_{j-1}^\rho)^{\alpha-1}}{\Gamma(\alpha)}\\
&\quad\times\int_{T_{j-1}}^{T_j}(T_j^\rho-s^\rho)^{\alpha-1}s^{\rho-1}f(s,\tilde{y}_j(s))ds-\frac{c(T_j^\rho-T_{j-1}^\rho)^{1-\alpha}(t^\rho-T_{j-1}^\rho)^{\alpha-1}}{\mu}\Big\|_{E_j}^{p_j}\\
&\leq 2^{p_j}\Big\|z_j(t)-\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{T_{j-1}}^{t}(t^\rho-s^\rho)^{\alpha-1}s^{\rho-1}f(s,z_j(s))ds+\frac{\rho^{1-\alpha}(T_j^\rho-T_{j-1}^\rho)^{1-\alpha}(t^\rho-T_{j-1}^\rho)^{\alpha-1}}{\Gamma(\alpha)}\\
&\quad\times\int_{T_{j-1}}^{T_j}(T_j^\rho-s^\rho)^{\alpha-1}s^{\rho-1}f(s,z_j(s))ds-\frac{c(T_j^\rho-T_{j-1}^\rho)^{1-\alpha}(t^\rho-T_{j-1}^\rho)^{\alpha-1}}{\mu}\Big\|_{E_j}^{p_j}\\
&\quad+2^{p_j}\Big\|\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{T_{j-1}}^{t}(t^\rho-s^\rho)^{\alpha-1}s^{\rho-1}(f(s,z_j(s))-f(s,\tilde{y}_j(s)))ds\Big\|_{E_j}^{p_j}\\
&\quad+2^{p_j}\Big\|\frac{(T_j^\rho-T_{j-1}^\rho)^{1-\alpha}(t^\rho-T_{j-1}^\rho)^{\alpha-1}(\rho^{1-\alpha})}{\Gamma(\alpha)}\int_{T_{j-1}}^{T_j}s^{\rho-1}(T_j^\rho-s^\rho)^{\alpha-1}\big(f(s,z_j(s))-f(s,\tilde{y}_j(s))ds\big)\Big\|_{E_j}^{p_j}\\
&\leq\frac{2^{p_j}\epsilon^{p_j}(T_j^\rho-T_{j-1}^\rho)^{\alpha p_j+1}}{\rho^{\alpha p_j}\Gamma^{p_j}(\alpha+1)(T_{j-1}^{\rho-1})^{p_j}}\\
&\quad+\frac{2^{p_j}}{\Gamma^{p_j}(\alpha)}\int_{T_{j-1}}^{T_j}\Big(\rho^{1-\alpha}\int_{T_{j-1}}^{t}(t^\rho-s^\rho)^{\alpha-1}s^{\rho-1}|f(s,z_j(s))-f(s,\tilde{y}_j(s))|ds\Big)^{p_j}dt\\
&\quad+\frac{2^{p_j}(T_j^\rho-T_{j-1}^\rho)^{(1-\alpha)p_j}}{\Gamma^{p_j}(\alpha)}\int_{T_{j-1}}^{T_j}(t^\rho-T_{j-1}^\rho)^{(\alpha-1)p_j}\Big(\rho^{1-\alpha}\int_{T_{j-1}}^{T_j}(T_j^\rho-s^\rho)^{\alpha-1}s^{\rho-1}|f(s,z_j(s))-f(s,\tilde{y}_j(s))|ds\Big)^{p_j}dt\\
&\leq\frac{2^{p_j}\epsilon^{p_j}(T_j^\rho-T_{j-1}^\rho)^{\alpha p_j+1}}{\rho^{\alpha p_j}\Gamma^{p_j}(\alpha+1)(T_{j-1}^{\rho-1})^{p_j}}\\
&\quad+\frac{2^{p_j}}{\Gamma^{p_j}(\alpha)}\times\int_{T_{j-1}}^{T_j}\Big[\Big(\rho^{(1-\alpha)q_j}\int_{T_{j-1}}^{t}(t^\rho-s^\rho)^{q_j(\alpha-1)}s^{(\rho-1)q_j}ds\Big)^{\frac{1}{q_j}}\Big(\int_{T_{j-1}}^{t}|f(s,z_j(s))-f(s,\tilde{y}_j(s))|^{p_j}ds\Big)^{\frac{1}{p_j}}\Big]^{p_j}dt\\
&\quad+\frac{2^{p_j}(T_j^\rho-T_{j-1}^\rho)^{(1-\alpha)p_j}}{\Gamma^{p_j}(\alpha)}\int_{T_{j-1}}^{T_j}(t^\rho-T_{j-1}^\rho)^{(\alpha-1)p_j}\Big[\Big(\rho^{(1-\alpha)q_j}\int_{T_{j-1}}^{T_j}(T_j^\rho-s^\rho)^{(\alpha-1)q_j}\,s^{(\rho-1)q_j}
\end{aligned}
$$

$$\Big(\int_{T_{j-1}}^{T_j} |f(s,z_j(s)) - f(s,\tilde{y}_j(s))|^{p_j} ds\Big)^{\frac{1}{p_j}}\Big]^{p_j} dt$$

$$\leq \frac{2^{p_j}\epsilon^{p_j}(T_j^{\rho} - T_{j-1}^{\rho})^{\alpha p_j+1}}{\rho^{\alpha p_j}\Gamma^{p_j}(\alpha+1)(T_{j-1}^{\rho-1})^{p_j}}$$

$$+\frac{2^{p_j}}{\rho^{p_j}\Gamma^{p_j}(\alpha)}\int_{T_{j-1}}^{T_j}\Big[\Big(\frac{(t^{\rho} - T_{j-1}^{\rho})^{q_j(\alpha-1)+1}}{q_j(\alpha-1)+1}\Big)^{\frac{1}{q_j}}\Big(\int_{T_{j-1}}^{t}|f(s,z_j(s)) - f(s,\tilde{y}_j(s))|^{p_j}ds\Big)^{\frac{1}{p_j}}\Big]^{p_j}dt$$

$$+\frac{2^{p_j}(T_j^{\rho} - T_{j-1}^{\rho})^{(1-\alpha)p_j}}{\Gamma^{p_j}(\alpha)\rho^{\alpha p_j}}$$

$$\times\int_{T_{j-1}}^{T_j}(t^{\rho} - T_{j-1}^{\rho})^{(\alpha-1)p_j}\Big[\Big(\frac{(T_j^{\rho} - T_{j-1}^{\rho})^{(\alpha-1)q_j+1}}{(\alpha-1)q_j+1}\Big)^{\frac{1}{q_j}}\Big(\int_{T_{j-1}}^{T_j}|f(s,z_j(s)) - f(s,\tilde{y}_j(s))|^{p_j}ds\Big)^{\frac{1}{p_j}}\Big]^{p_j}dt$$

$$\leq \frac{2^{p_j}\epsilon^{p_j}(T_j^{\rho} - T_{j-1}^{\rho})^{\alpha p_j+1}}{\rho^{\alpha p_j}\Gamma^{p_j}(\alpha+1)(T_{j-1}^{\rho-1})^{p_j}}$$

$$+\frac{2^{p_j}}{\Gamma^{p_j}(\alpha)\rho^{p_j}}\int_{T_{j-1}}^{T_j}\frac{(t^{\rho} - T_{j-1}^{\rho})^{\frac{p_j(q_j(\alpha-1)+1)}{q_j}}}{(q_j(\alpha-1)+1)^{\frac{p_j}{q_j}}}\Big(\int_{T_{j-1}}^{t}|f(s,z_j(s)) - f(s,\tilde{y}_j(s))|^{p_j}ds\Big)dt$$

$$+\frac{2^{p_j}(T_j^{\rho} - T_{j-1}^{\rho})^{(1-\alpha)p_j}}{\Gamma^{p_j}(\alpha)\rho^{\alpha p_j}}\int_{T_{j-1}}^{T_j}(t^{\rho} - T_{j-1}^{\rho})^{(\alpha-1)p_j}\frac{(T_j^{\rho} - T_{j-1}^{\rho})^{\frac{p_j(q_j(\alpha-1)+1)}{q_j}}}{(q_j(\alpha-1)+1)^{\frac{p_j}{q_j}}}\Big|f(s,z_j(s)) - f(s,\tilde{y}_j(s))\Big|_{E_j}^{p_j}dt$$

$$\leq 2^{p_j}\epsilon^{p_j}\frac{(T_j^{\rho} - T_{j-1}^{\rho})^{\alpha p_j+1}}{\rho^{\alpha p_j}\Gamma^{p_j}(\alpha+1)(T_{j-1}^{\rho-1})^{p_j}}$$

$$+\frac{2^{p_j}}{\rho^{p_j+1}\Gamma^{p_j}(\alpha)\,(q_j(\alpha-1)+1)^{\frac{p_j}{q_j}}}\frac{(T_j^{\rho} - T_{j-1}^{\rho})^{\frac{p_j(q_j(\alpha-1)+1)}{q_j}+1}}{(\frac{p_j(q_j(\alpha-1)+1)}{q_j}+1)T_j^{\rho-1}}(M^{p_j}\|z_j - \tilde{y}_j\|_{E_j}^{p_j})$$

$$+\frac{2^{p_j}(T_j^{\rho} - T_{j-1}^{\rho})^{(1-\alpha)p_j}}{\rho^{\alpha p_j+1}\ \Gamma^{p_j}(\alpha)}\frac{(T_j^{\rho} - T_{j-1}^{\rho})^{\frac{p_j(q_j(\alpha-1)+1)}{q_j}}}{(q_j(\alpha-1)+1)^{\frac{p_j}{q_j}}}\frac{(T_j^{\rho} - T_{j-1}^{\rho})^{p_j(1-\alpha)+1}}{(p_j(1-\alpha)+1)T_j^{\rho-1}}(M^{p_j}\|z_j - \tilde{y}_j\|_{E_j}^{p_j})$$

$$\leq 2^{p_j}\epsilon^{p_j}\frac{(T_j^{\rho} - T_{j-1}^{\rho})^{\alpha p_j+1}}{\rho^{\alpha p_j}\Gamma^{p_j}(\alpha+1)(T_{j-1}^{\rho-1})^{p_j}}$$

$$+\Big[\frac{2^{p_j}M^{p_j}}{\rho^{p_j+1}\Gamma^{p_j}(\alpha)\,(q_j(\alpha-1)+1)^{\frac{p_j}{q_j}}}\frac{(T_j^{\rho} - T_{j-1}^{\rho})^{\frac{p_j(q_j(\alpha-1)+1)}{q_j}+1}}{(\frac{p_j(q_j(\alpha-1)+1)}{q_j}+1)T_j^{\rho-1}}$$

$$+\frac{2^{p_j}M^{p_j}(T_j^\rho - T_{j-1}^\rho)^{(1-\alpha)p_j}}{\rho^{\alpha p_j+1}\,\Gamma^{p_j}(\alpha)} \frac{(T_j^\rho - T_{j-1}^\rho)^{\frac{p_j(q_j(\alpha-1)+1)}{q_j}}}{(q_j(\alpha-1)+1)^{\frac{p_j}{q_j}}} \frac{(T_j^\rho - T_{j-1}^\rho)^{p_j(1-\alpha)+1}}{(p_j(1-\alpha)+1)T_j^{\rho-1}}\Big]\|z_j - \tilde{y}_j\|_{E_j}^{p_j}$$

$$\leq \frac{2^{p_j}\epsilon^{p_j}(T_j^\rho - T_{j-1}^\rho)^{\alpha p_j+1}}{\rho^{\alpha p_j}\Gamma^{p_j}(\alpha+1)(T_{j-1}^{\rho-1})^{p_j}} + \vartheta\|z-y\|_{E_j}^{p_j},$$

where

$$\vartheta = \max_{j=1,2,\dots,n}\Big[\frac{2^{p_j}M^{p_j}}{\rho^{p_j+1}\Gamma^{p_j}(\alpha)\,(q_j(\alpha-1)+1)^{\frac{p_j}{q_j}}} \frac{(T_j^\rho - T_{j-1}^\rho)^{\frac{p_j(q_j(\alpha-1)+1)}{q_j}+1}}{(\frac{p_j(q_j(\alpha-1)+1)}{q_j}+1)T_j^{\rho-1}}$$

$$+\frac{2^{p_j}M^{p_j}(T_j^\rho - T_{j-1}^\rho)^{(1-\alpha)p_j+\frac{p_j(q_j(\alpha-1)+1)}{q_j}+p_j(1-\alpha)+1}}{\rho^{\alpha p_j+1}\,\Gamma^{p_j}(\alpha)(q_j(\alpha-1)+1)^{\frac{p_j}{q_j}}\,p_j(1-\alpha)+1}\Big].$$

Then,

$$\|z-y\|_{E_j} \leq \frac{2(T_j^\rho - T_{j-1}^\rho)^{\frac{\alpha p_j+1}{p_j}}}{\rho^\alpha(1-\vartheta)^{\frac{1}{p_j}}\Gamma(\alpha+1)(T_{j-1}^{\rho-1})}\epsilon].$$

We get

$$\|z-y\|_p \leq \frac{1}{\rho^\alpha\Gamma(\alpha+1)}\,(\sum_{j=1}^{j=n}\frac{2(T_j^\rho - T_{j-1}^\rho)^{\frac{\alpha p_j+1}{p_j}}}{(1-\vartheta)^{\frac{1}{p_j}}})\epsilon := c_g\epsilon.$$

Hence, the FBVP (1.2) is UH stable. □

5. Illustrative example

In this section, we verify the above-given results with the help of an example. Here we define the following FBVP.

$$\begin{cases} D^{(0.5,0.5)}\zeta(t) = \frac{|\zeta(t)|}{(9+e^t)(\zeta(t)+1)},\ t\in I := [0,4], \\ \zeta(0)+\zeta(4) = 0. \end{cases} \tag{5.1}$$

Let

$$f(t,\zeta) = \frac{|\zeta(t)|}{(9+e^t)(\zeta(t)+1)}, t\in[0,4].$$

So, we get

$$\begin{aligned} |f(t,\Lambda)-f(t,\zeta)| &= \frac{1}{(9+e^t)}\Big|\frac{\Lambda}{\Lambda+1} - \frac{\zeta}{\zeta+1}\Big| = \frac{|\Lambda-\zeta|}{(9+e^t)(\Lambda+1)(\zeta+1)} \\ &\leq \frac{|\Lambda-\zeta|}{9+e^t} \\ &\leq \frac{1}{9}|\Lambda-\zeta|. \end{aligned}$$

Hence, the inequality (H2) holds with $M = \frac{1}{9}$.

Let

$$\begin{cases} p_1 = 8 \text{ if } & t \in I := [0,2], \\ p_2 = 7 \text{ if } & t \in I := [2,4]. \end{cases} \tag{5.2}$$

Using Eq (3.4), we define two auxiliary Generalized Caputo-type FBVP. The Eq (5.1) is equivalent to,

$$\begin{cases} D^{(0.5,0.5)}\zeta(t) = \frac{|\zeta(t)|}{(9+e^t)(\zeta(t)+1)}, & t \in I := [0,2], \\ \zeta(0) + \zeta(2) = 0, \end{cases} \tag{5.3}$$

and

$$\begin{cases} D^{(0.5,0.5)}\zeta(t) = \frac{|\zeta(t)|}{(9+e^t)(\zeta(t)+1)}, & t \in I := [2,4], \\ \zeta(2) + \zeta(4) = 0. \end{cases} \tag{5.4}$$

Now, we show that condition (3.6) is satisfied for $j = 1, p_1 = 8$.

We have,

$$W_{\alpha,M,p_1,T_0,T_1} = 0.696985982 < 1.$$

So, condition (3.6) is satisfied.

From Theorem 3.1, the FBVP (5.3) exists a unique solution $\tilde{\zeta}_1 \in L^8(I_1, \mathbb{R})$.

Now we check that condition (3.6) is satisfied for $j = 2, p_2 = 7$. We get,

$$W_{\alpha,M,p_1,T_0,T_1} = 0.583666290 < 1.$$

So, condition (3.6) is satisfied.

According to Theorem 3.1, FBVP (5.4) possesses a unique solution $\tilde{\zeta}_2 \in L^7(I_2, \mathbb{R})$.

Then, from Theorem 3.2, the FBVP (5.4) exists a unique solution

$$\zeta(t) = \begin{cases} \tilde{\zeta}(t) \in L^8(I_1, \mathbb{R}), \\ \zeta_2(t) \in L^7(I_2, \mathbb{R}), \end{cases} \tag{5.5}$$

where

$$\zeta_2(t) = \begin{cases} 0, & t \in I_1, \\ \tilde{\zeta}_2(t), & t \in I_2. \end{cases} \tag{5.6}$$

By Theorem 4.1, FBVP (5.1) is UH stable.

While comparing the proposed results to the other literature, the authors in Ref. [24] were restricted to Riemann-Liouville fractional derivative with the same FBVP. We have successfully incorporated a generalized Caputo derivative containing extra parameter ρ along with fractional order α, which adds an extra degree of freedom in the given problem. Therefore, the proposed results are advanced to the previously published analysis of Ref. [24].

6. Conclusions

We have examined the existence, uniqueness, and Ulam-Hyers stability of the solutions for BVP of generalized Caputo-type in the variable exponent Lebesgue spaces. We used the Banach contraction principle along with the necessary features of fixed point theory to establish the desired results. An example has been illustrated to justify the validity of the theoretical observations. In the future, the proposed problem can be further studied by incorporating any other fractional derivative. Also, the problem can be generalized by considering other kinds of boundary conditions.

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Conflict of interest

The authors declare no conflict of interest.

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